Instantons within the Functional renormalization group

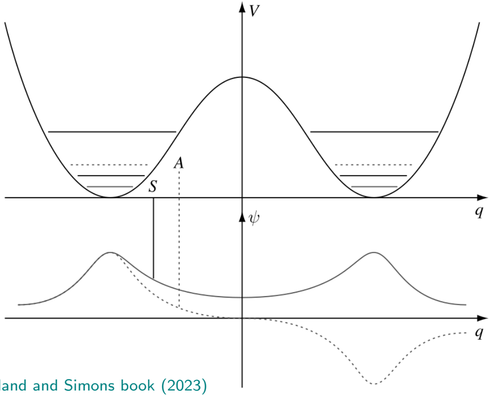
Lucija Nora Farkaš[1], Gilles Tarjus [2], Ivan Balog [1]

[1]Institute of Physics, Zagreb, Croatia
[2]Sorbonne Université, Paris, France

ERG 2024, Les Diablerets - 23.9.2024

HRZZ IP-2022-10-9423 (I.Balog)

Quantum particle in a double well



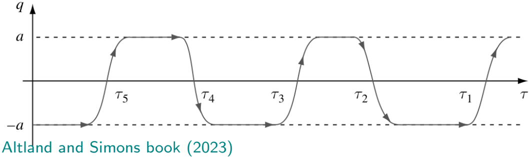
Altland and Simons book (2023)

$$|\psi\rangle = \sum_i a_i |\epsilon_i\rangle \tag{1}$$

Path integral formulation

$$\langle q_f | e^{-i\frac{\hat{H}t}{\hbar}} | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} \mathcal{D}(p,q) e^{\frac{i}{\hbar}\int_0^t dt' \underbrace{(pq - H(p,q))}_{L(q,\dot{q})}} \quad (2)$$

A path in the limit of a deep double well:



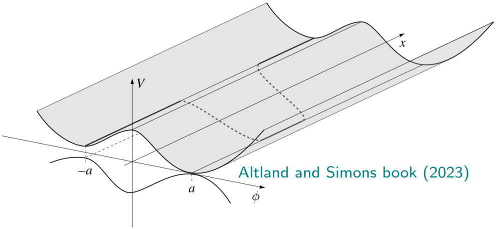
Altland and Simons book (2023)

- "instantons","kinks"

By Wick rotation: $x = it$ (and $\frac{1}{\hbar} \rightarrow \frac{1}{k_B T}$)
Path integral $\Rightarrow$ partition function!

$$Z = \int \mathcal{D}\phi(x) e^{-\frac{1}{k_B T}\int_0^L dx H(\phi(x))} \tag{3}$$

Quantum particle in a double well $\Rightarrow$ 1d elastic string in a double well potential



Altland and Simons book (2023)

$$V(\phi(x)) = -\frac{|r|}{2}\phi(x)^2 + \frac{u}{4!}\phi(x)^4$$

Instanton gas approximation at $d = 1$

Partition function in the instanton gas approximation (Polyakov (1977))

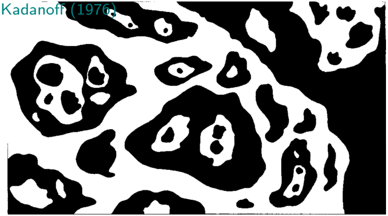
- Evaluate energy cost per instanton (pair)
- Weigh each configuration according to the number of instanton (pairs) treating every instanton as an independent object
- Sum over locations of all instantons

Solution is exact in the $T \to 0$ limit (barrier is large)

- There can never be ordering as $L \to \infty$ at any finite T in 1d (Peierls 1936)
- $G(r) = \langle \phi(r)\phi(0) \rangle \propto e^{-\frac{r}{\xi}}$
- Essential scaling: $\xi \propto e^{\frac{|c|}{T}}$

Generalization of instantons to $d > 1$

- "Droplets" (Bruce and Wallace 1983)
- dedicated renormalization group approach at $d = 1 + \epsilon$



A continuous phase transition at $T_C \propto d - 1$
Power law scaling restored:

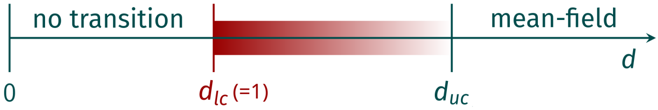
- $\frac{1}{\nu} \propto d - 1$ $(\xi = |T - T_C|^{-\nu})$
- $\beta \propto e^{-\frac{2}{d-1}}$ $(m \propto (T_C - T)^{\beta})$

What is there to add?

- Specifically designed approaches for specific problems
- Polyakov (1977): "We hope that in the functional integral of ... only certain field configurations are important. In this case the hopeless problem of integration over all possible fields is greatly reduced to the reasonably complicated problem."
- Is it hopeless?
- Instantons are nonperturbative objects - a nonperturbative approach required
- Previous attempts by Nonperturbative renormalization group (NPRG) approach to capture the correct behavior have found ambiguous results: Kapoyannis and Tetradis (2000); Zappala (2001); K.-I. Aoki et al. (2002); Weyrauch (2006); Nandori, Marian, Bacso (2014)
- Still NPRG seems the most likely candidate for the task!

Where to look for the instantons?

- Scalar φ^4 theory: $S_B = \int_x \frac{1}{2}(\nabla\phi(x))^2 + \frac{r}{2}\phi(x)^2 + \frac{u}{4!}\phi(x)^4$
- Look for a critical point



Why not put $d = 1$ from the start?

How to look for instantons?

- Derivative expansion approximation:
  $\Gamma[\phi(x)] = \int_x \left(U[\phi(x)] + \frac{1}{2}(\nabla\phi(x))^2 Z[\phi(x)] + \cdots \right)$

- To study criticality express in dimensionless quantities: $\phi = k^{\frac{d-2+\eta}{2}}\varphi$, $U(\phi) = k^d u(\varphi)$, $Z(\phi) = Z_0 z(\varphi)$, such that $Z_0 = const.k^{-\eta}$.

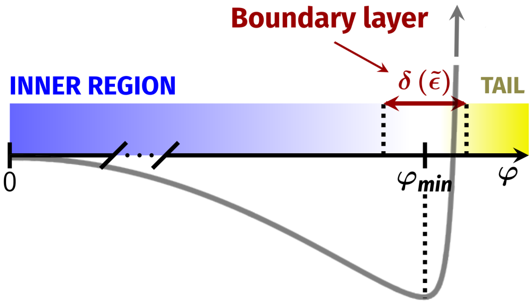
- Solve the fixed point equations:
  $\partial_k u(\varphi) = 0 = -du(\varphi) + D_\varphi \varphi u'(\varphi) + \beta_u[u'', z]$

  $\partial_k z(\varphi) = 0 = \eta z(\varphi) + D_\varphi \varphi z'(\varphi) + \beta_z[u'', u^3, u^{(4)}, z, z', z'']$

- With field rescaling exponent $D_\varphi = \frac{d-2+\eta}{2}$
- η determined selfconsistently
- Lower critical dimension: $d - 2 + \eta \to 0$
- $\tilde{\epsilon} = \frac{d-2+\eta}{2(2-\eta)}$ is the small parameter of the solution

Fixed point solution when $d - 2 + \eta \to 0$

L. N. Farkaš, G. Tarjus, and I. Balog: Phys. Rev. E (2023)



As $\tilde{\epsilon} \to 0$:

- $g(q \to 0; \varphi \to 0) \to \infty$
- $\delta(\tilde{\epsilon}) \propto \tilde{\epsilon}\sqrt{\ln(\frac{1}{\tilde{\epsilon}})}$
- $\varphi_{min} \propto \sqrt{\ln(\frac{1}{\tilde{\epsilon}})}$

Evidence of instantons # 1

- the dependence of φ_{min} is peculiar!
- the effective potential near the minimum $u(\varphi) \approx const.(\varphi - \varphi_{min})^2 = const.\varphi_{min}^2(\frac{\varphi}{\varphi_{min}} - 1)^2$
- $\varphi_{min}^2 \propto \ln(\frac{1}{\tilde{\epsilon}})$ is the Bolzman factor $\propto \frac{1}{T_C}$
- $\Rightarrow \tilde{\epsilon} \propto e^{-\frac{const.}{T_C}}$
- since $\tilde{\epsilon} \propto D_\varphi$ this behavior is analogous to the Bruce and Wallace relation: $D_\varphi \propto e^{-\frac{2}{d-1}}$, where $T_C \propto d - 1$.
- within NPRG $d = 1$ is not a special dimension
- behavior near selfconsistently determined d_l is consistent with instanton physics!

The lower critical dimension

- d_l depends on the approximation and calculation details
- α the regulator magnitude

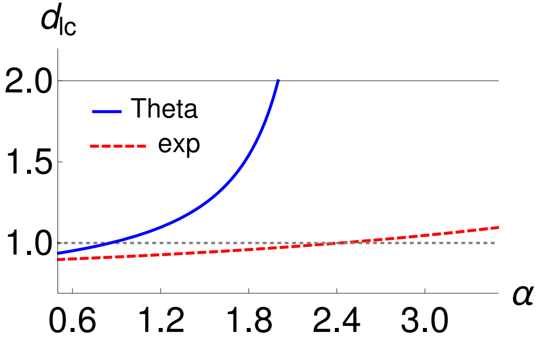


Figure: Analytic results for d_l at LPA' approximation ($z(\varphi) = const.$)

The lower critical dimension

- d_l depends on the approximation and calculation details
- α the regulator magnitude

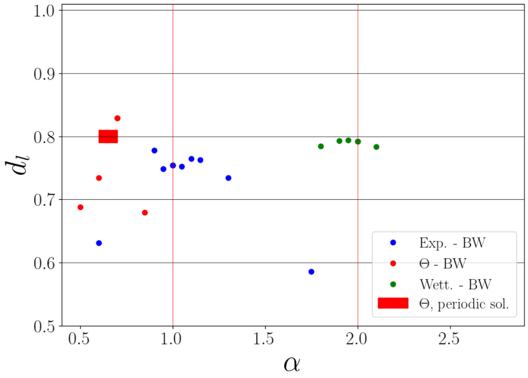


Figure: Extrapolation results for d_l at ∂^2 approximation

Critical exponent $\frac{1}{\nu}$

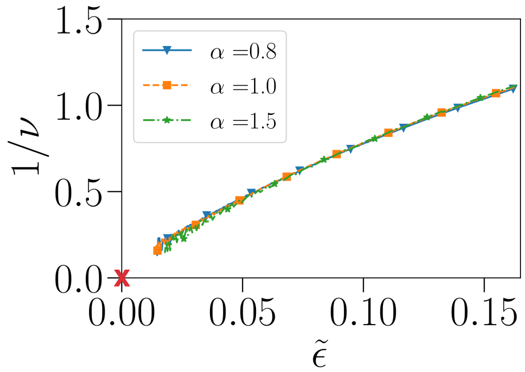


Figure: Results for $\frac{1}{\nu}$ exponent at LPA' approximation ($z(\varphi) = const.$)

Critical exponent $\frac{1}{\nu}$

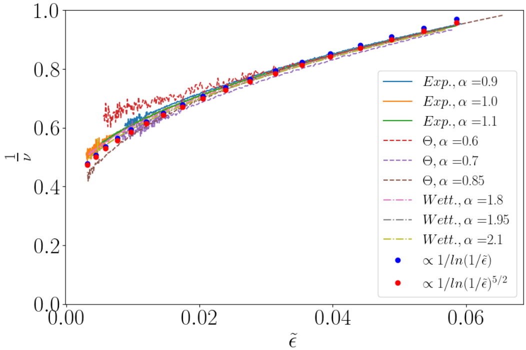


Figure: Results for $\frac{1}{\nu}$ exponent at d_l at ∂^2 approximation

Evidence of instantons # 2

Seemingly increasing the order leeds to $\frac{1}{\nu}$ further away from 0!

Bruce and Wallace prediction:

- $\frac{1}{\nu} \propto d - 1$
- Expressing $d - 1$ in terms of $D_\varphi(\propto \tilde{\epsilon})$:
- $\Rightarrow \frac{1}{\nu} \propto \frac{1}{\ln(\frac{1}{D_\varphi})}$

Semirigorous arguments give:

- $\frac{1}{\nu} \propto \tilde{\epsilon}\sqrt{\ln(\frac{1}{\tilde{\epsilon}})}$ @ LPA'
- $\frac{1}{\nu} \propto \frac{1}{\ln(\frac{1}{\tilde{\epsilon}})^{\frac{5}{2}}}$ @ ∂^2
- Increasing the order of ∂-expansion brings the result closer to BW!

Conclusions

- Identification of the signatures of instantons in the fixed point solution in the limit of $d \to d_l$ of the scalar φ^4 theory.
- Numerical value of d_l dependent on approximation and regulator.
- Nothing special happens at $d = 1$ in the NPRG formalism.
- The results for $\frac{1}{\nu}$ approach exact as the order of $\partial-$expansion increases.

Perspectives

- essential scaling (flow of dimensionfull quantitites)
- lower critical dimension of hysteresis (is it $d = 2$?)
- other types of localized excitations?

Our works:

- Lucija Nora Farkaš, Gilles Tarjus, and Ivan Balog: Phys. Rev. E 108, 054107 (2023) (arXiv: 2307.03578)
- Doctoral thesis: Lucija Nora Farkaš (December 2023)
- new paper soon