Nonperturbative QCD within the functional renormalization group

Wei-jie Fu

Dalian University of Technology

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Based on:
WF, Xiaofeng Luo, Jan M. Pawlowski, Fabian Rennecke, Shi Yin, arXiv: 2308.15508;
Braun, Chen, WF, Gao, Huang, Ihssen, Pawlowski, Rennecke, Sattler, Tan, Wen, and Yin, arXiv:2310.19853;
Yang-yang Tan, Yong-rui Chen, WF, Wei-Jia Li, arXiv: 2403.03503;
WF, Chuang Huang, Jan M. Pawlowski, Yang-yang Tan, arXiv:2209.13120; arXiv:2401.07638;
WF, Chuang Huang, Jan M. Pawlowski, Yang-yang Tan, Li-jun Zhou, in preparation;
Lei Chang, WF, Chuang Huang, Jan M. Pawlowski, Dao-yu Zhang, in preparation;
WF, Jan M. Pawlowski, Robert D. Pisarski, Fabian Rennecke, Rui Wen, and Shi Yin, in preparation;
Yang-yang Tan, Shi Yin, Yong-rui Chen, Chuang Huang, WF, in preparation.

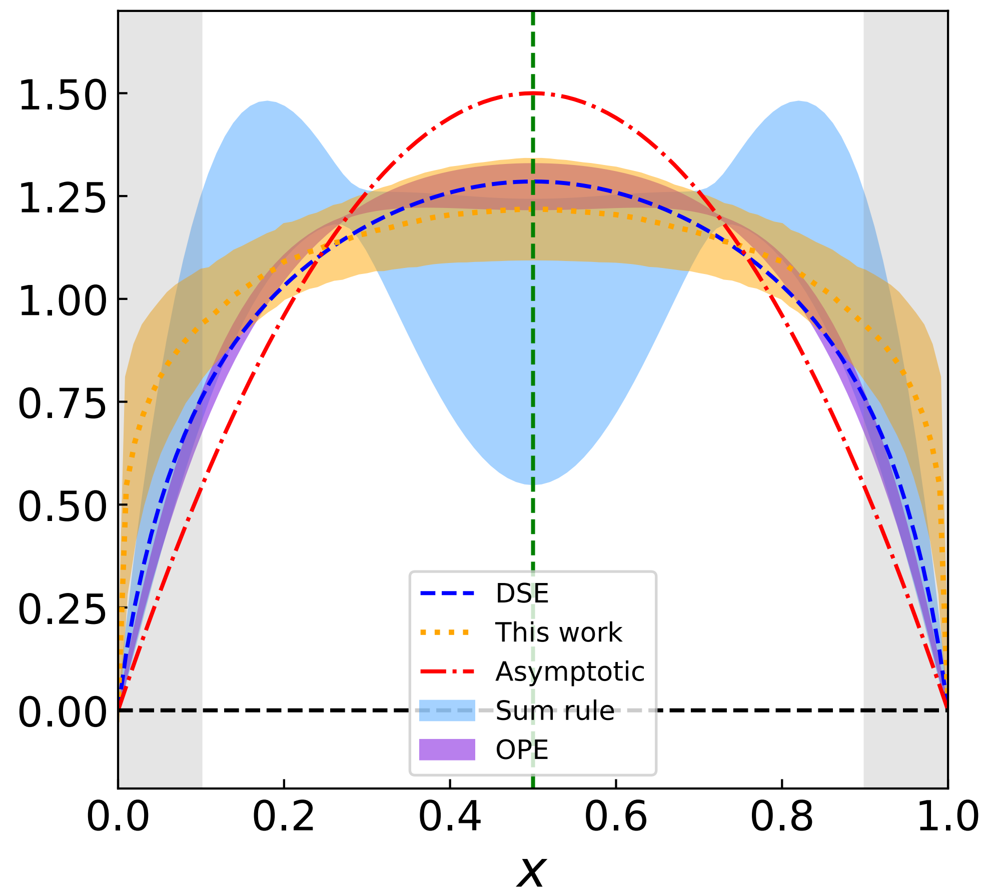
fQCD collaboration:
Braun, Chen, Fu, Gao, Geissel, Huang, Lu, Ihssen, Pawlowski, Rennecke, Sattler, Schallmo, Stoll, Tan, Töpfel, Turnwald, Wessely, Wen, Wink, Yin, Zheng, Zorbach

Basic questions in nuclear physics

Mass generation

Image from BNL website

Distribution amplitudes for pion

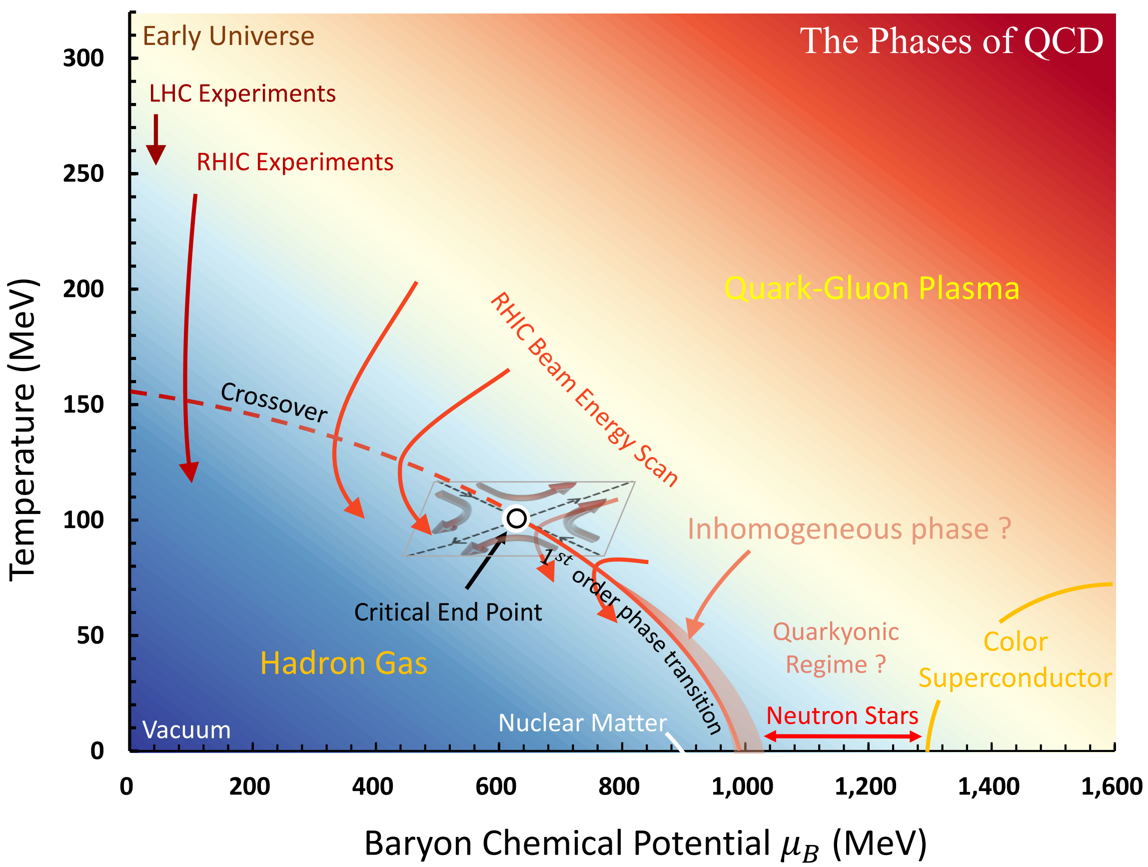


Lattice: J. Hua *et al.* (LPC), *PRL* 129 (2022) 132001;
DSE: C. Roberts *et al.*, *PPNP* 120 (2021) 103883;
Sum rules: P. Ball *et al.*, *JHEP* 08 (2007) 090;
OPE: G. Bali *et al.* (RQCD), *JHEP* 08 (2019) 065; 11 (2020) 37.

- How can we understand mass generation and hadron structure from first-principles QCD?

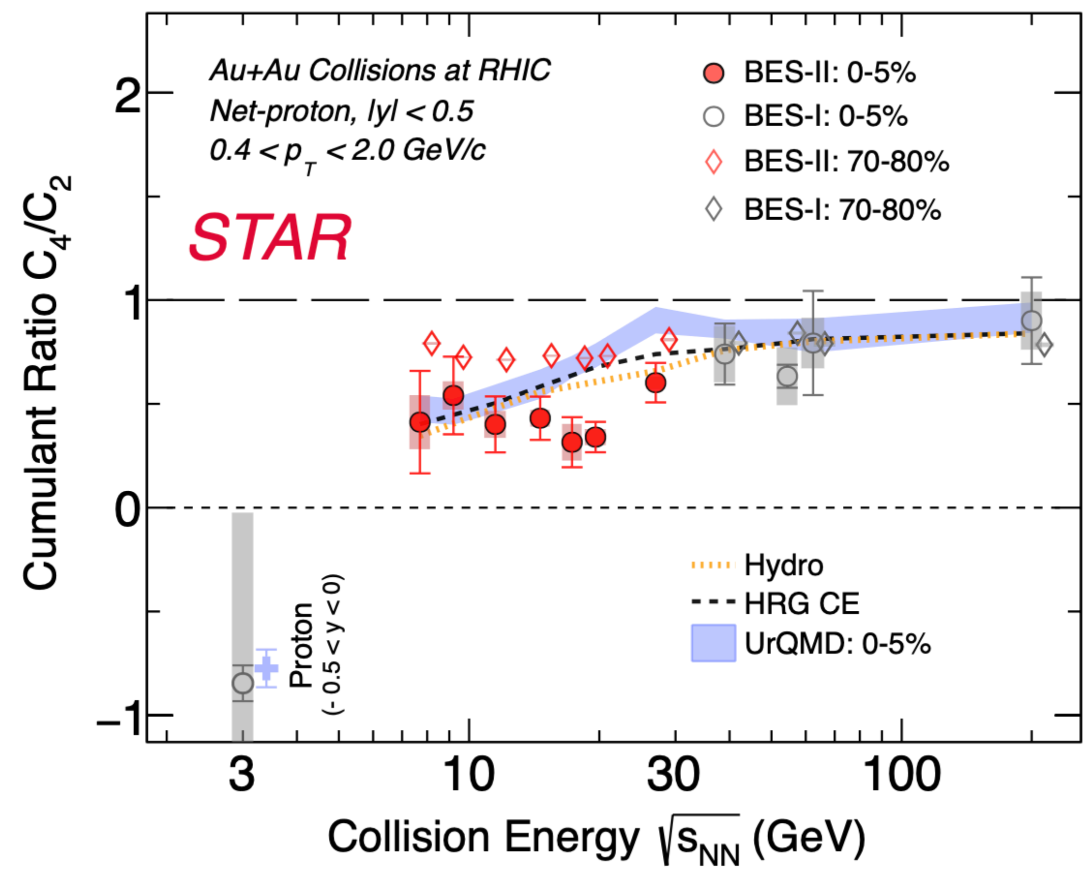
CEP in QCD phase diagram

QCD phase diagram



Non-monotonicity:
M. Stephanov, *PRL* 107 (2011) 052301

Fluctuations measured in BES-II

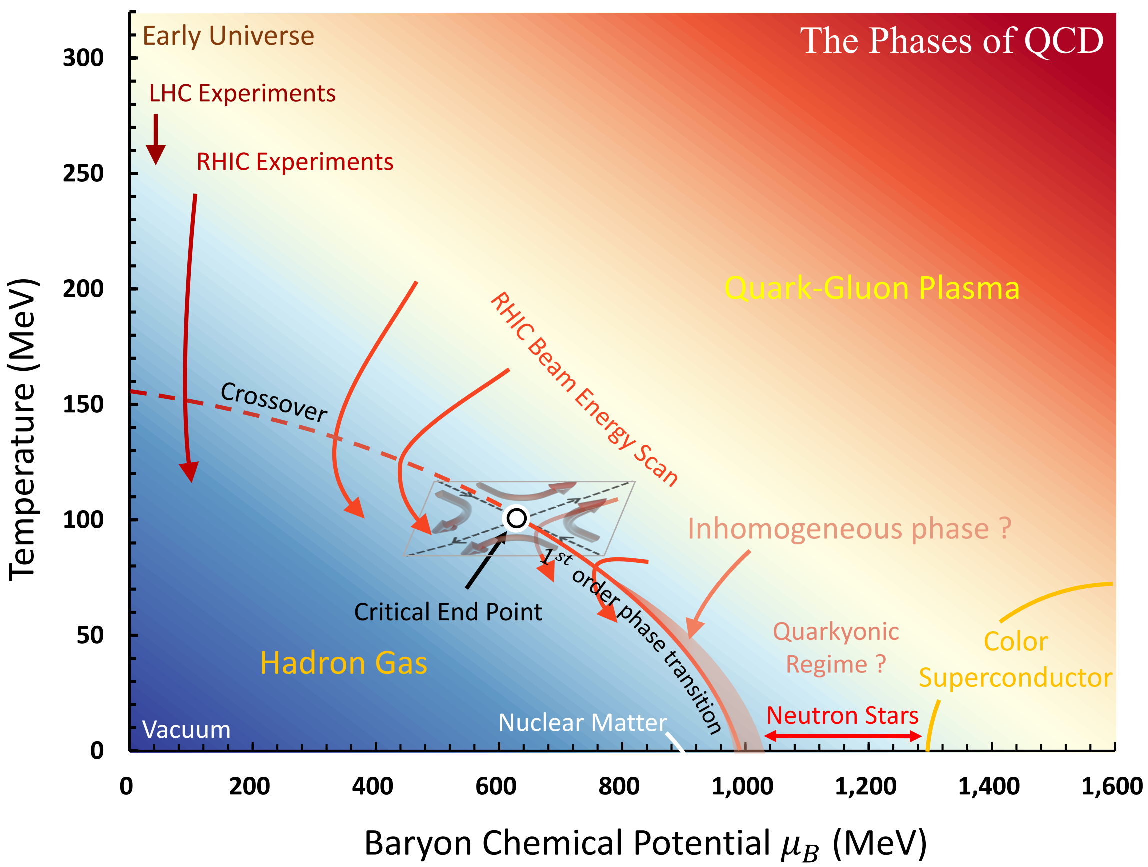


Ashish Pandav for STAR Collaboration in CPOD2024

- Is there a "peak" structure serving as the smoking gun signal for the critical end point in the QCD phase diagram?

CEP in QCD phase diagram

QCD phase diagram



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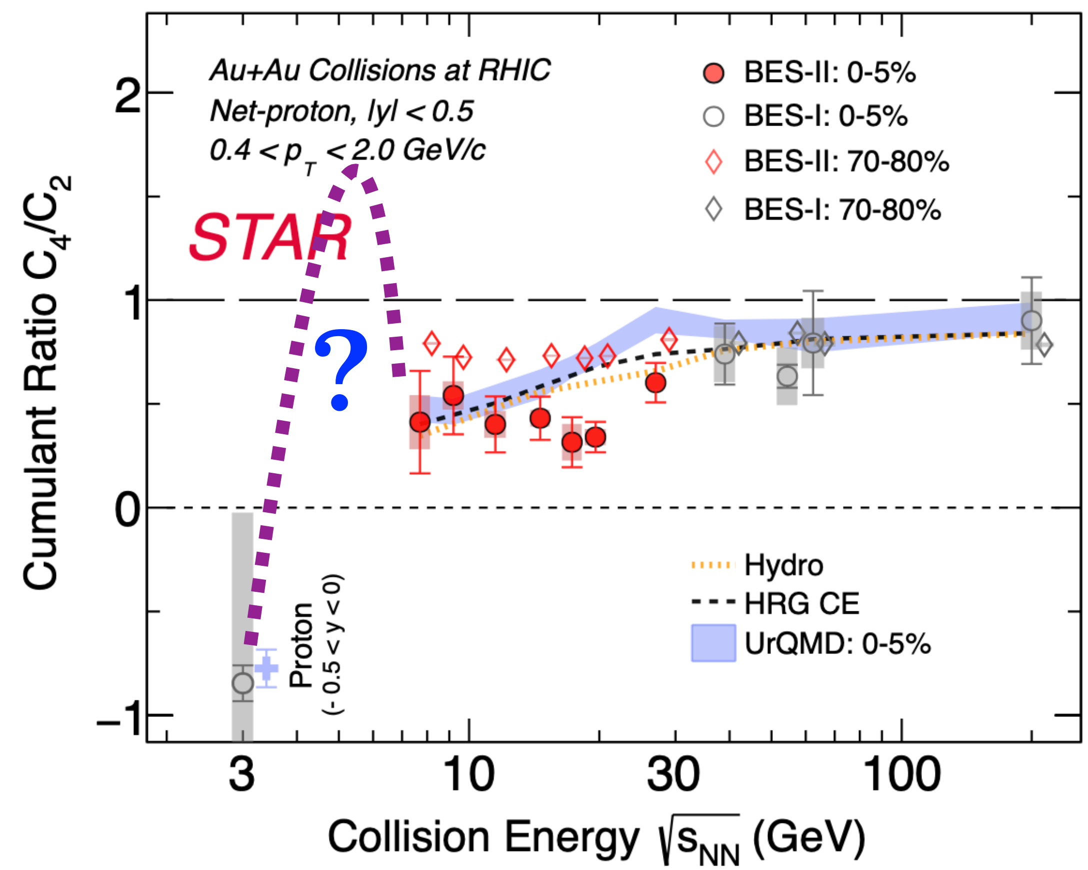

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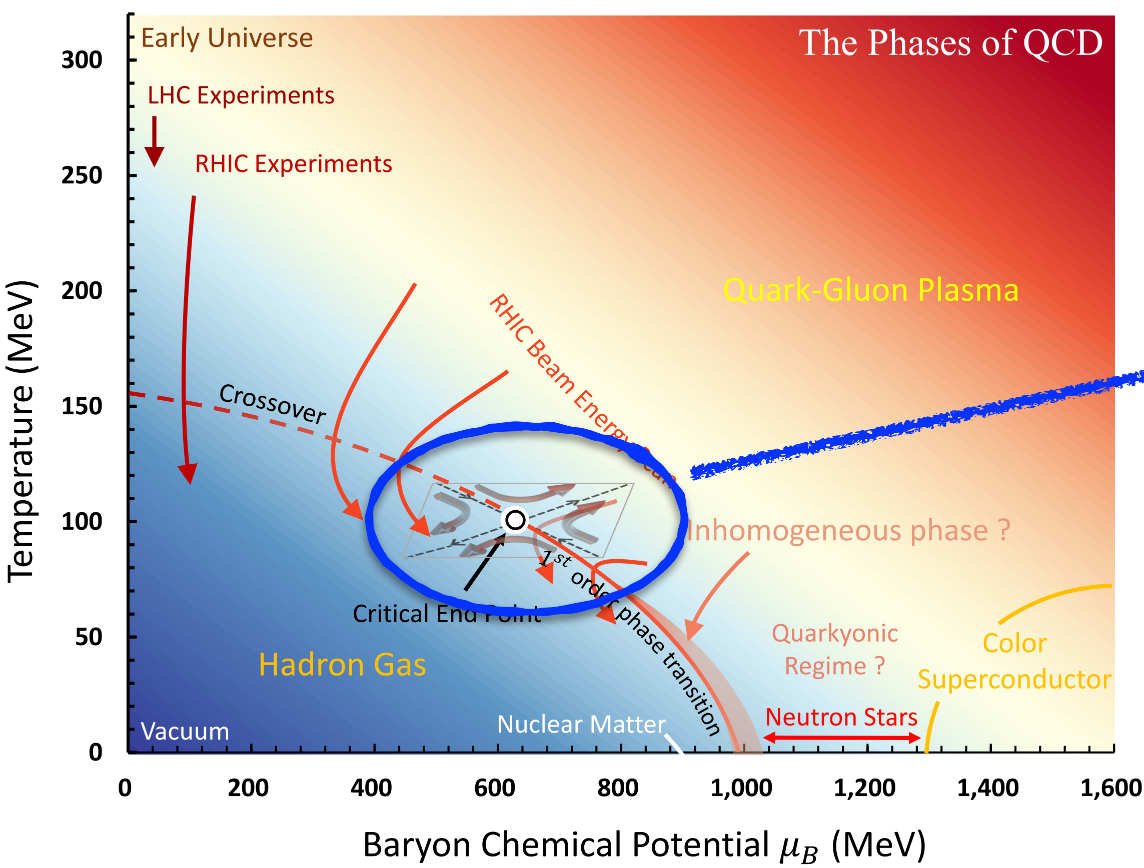
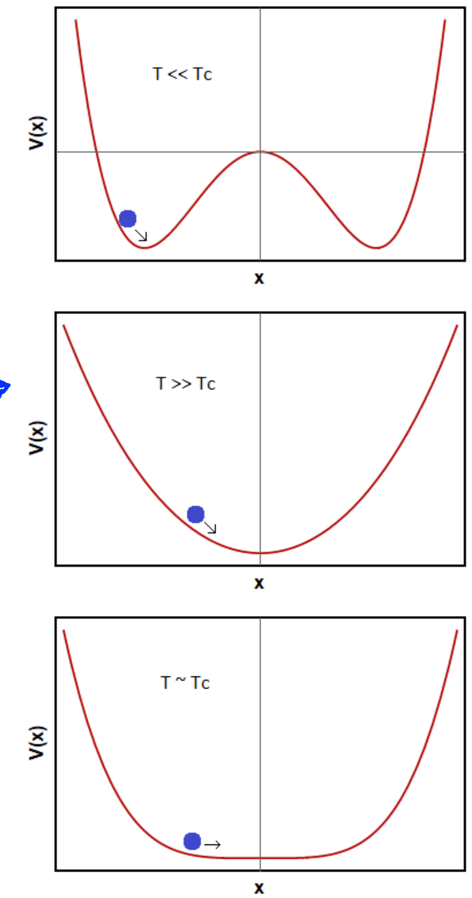
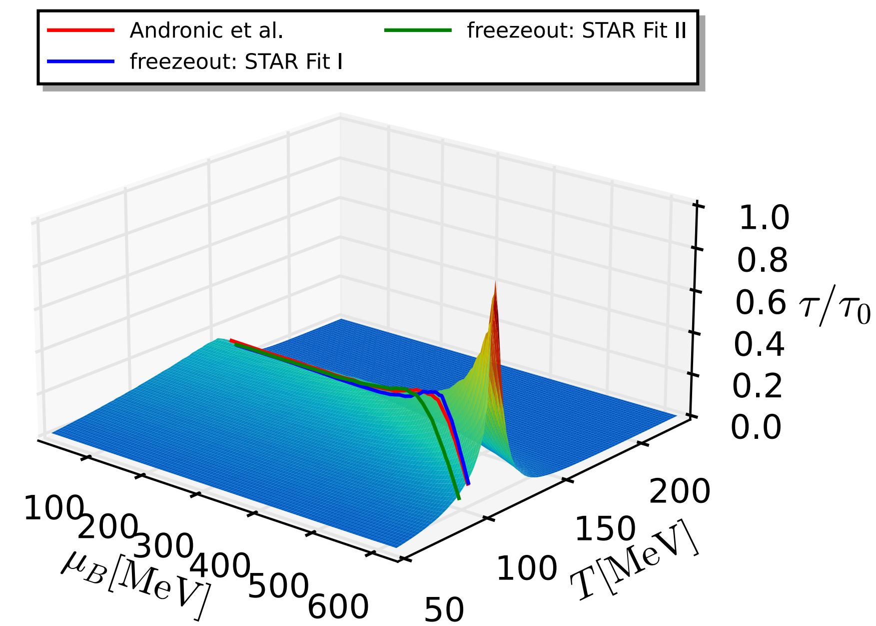
Critical slowing down near QCD critical point







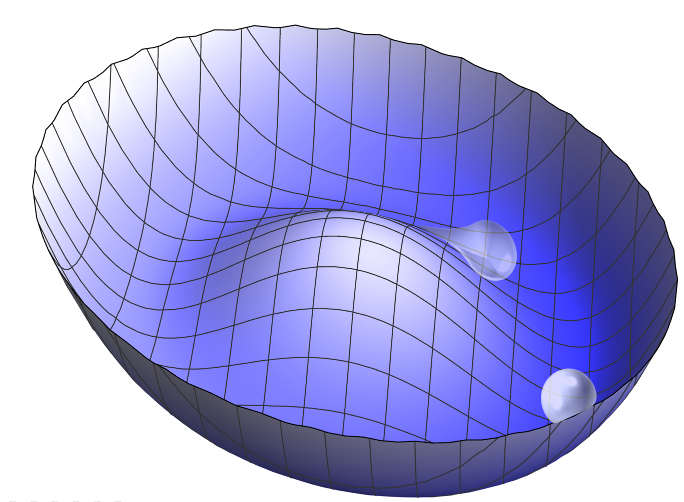
Tan, Yin, Chen, Huang, WF, in preparation

Relaxation time:

$$\tau = \xi^z f(k\xi)$$

z: **dynamic critical exponent**

Goldstone damping



also cf. talk by Lorenz von Smekal

Call for:

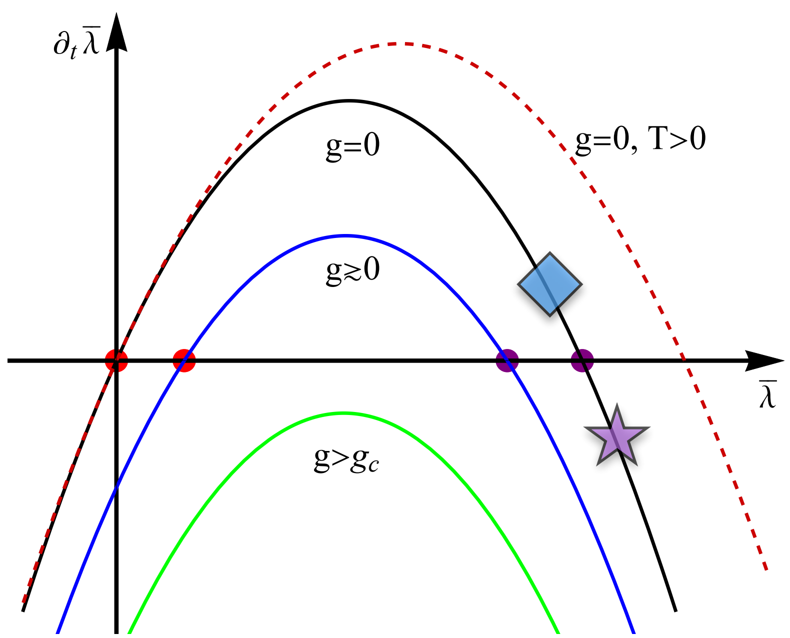
- Real-time description of strongly interacting systems.
- Nonperturbative approach of QCD.

Outline

- Introduction
- QCD in vacuum and hadron structure
- QCD at finite temperature and densities
- Real-time dynamics of QCD
- Summary and outlook

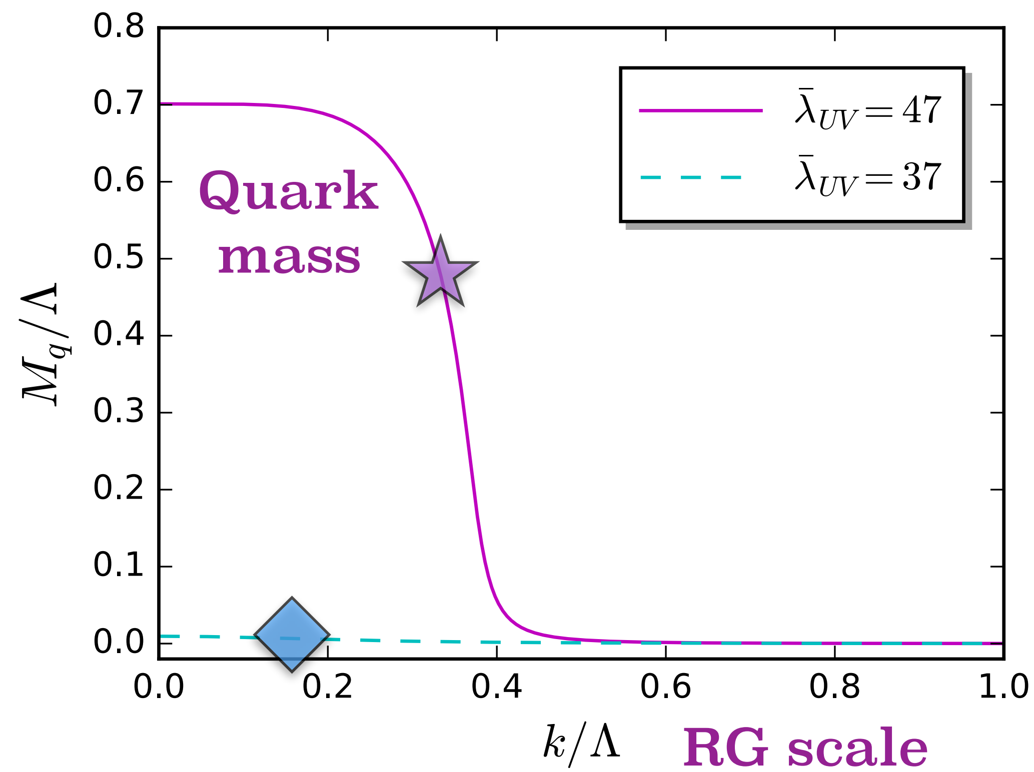
Chiral symmetry breaking and mass generation in RG

- β function of 4-quark coupling:



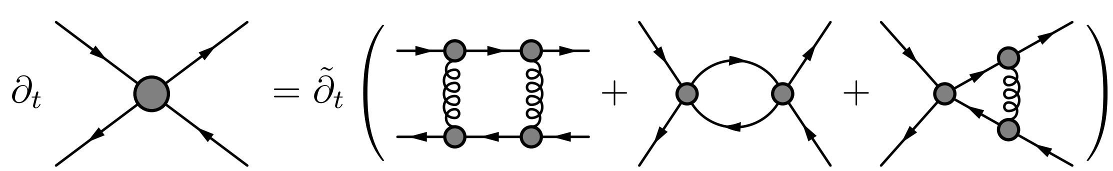
Braun, Gies, *JHEP* 06 (2006) 024.

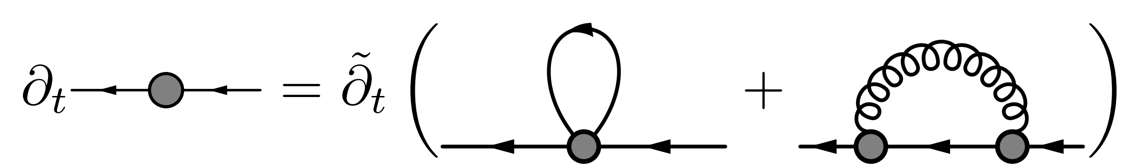
- Quark mass:



WF, Huang, Pawlowski, Tan, *SciPost Phys.* 14 (2023) 069, arXiv:2209.13120

$$\partial_t \bar\lambda = (d-2)\bar\lambda - a\bar\lambda^2 - b\bar\lambda g^2 - c g^4 \,,$$

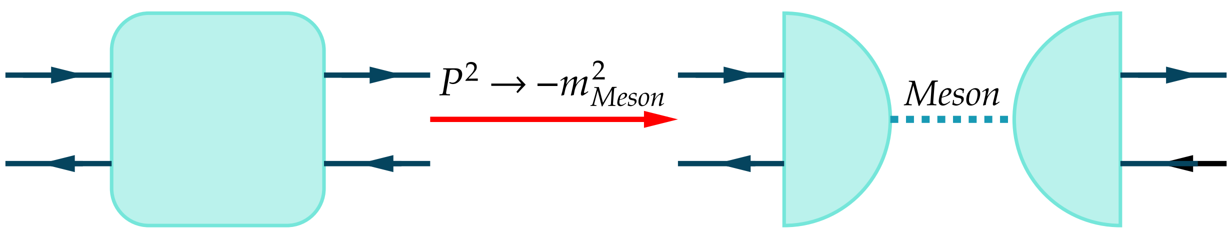




- Flows of two- and four-quark vertices play the same roles of gap and Bethe-Salpeter equations.

Bound states in RG

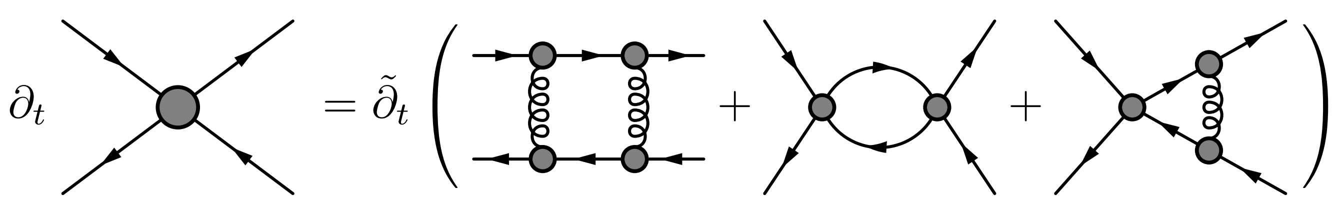
- **Bound states encoded in n-point correlation functions:**



$$\partial_t \lambda_{\pi,k}(P^2) = C_k(P^2)\lambda^2_{\pi,k}(P^2) + A_k(P^2),$$

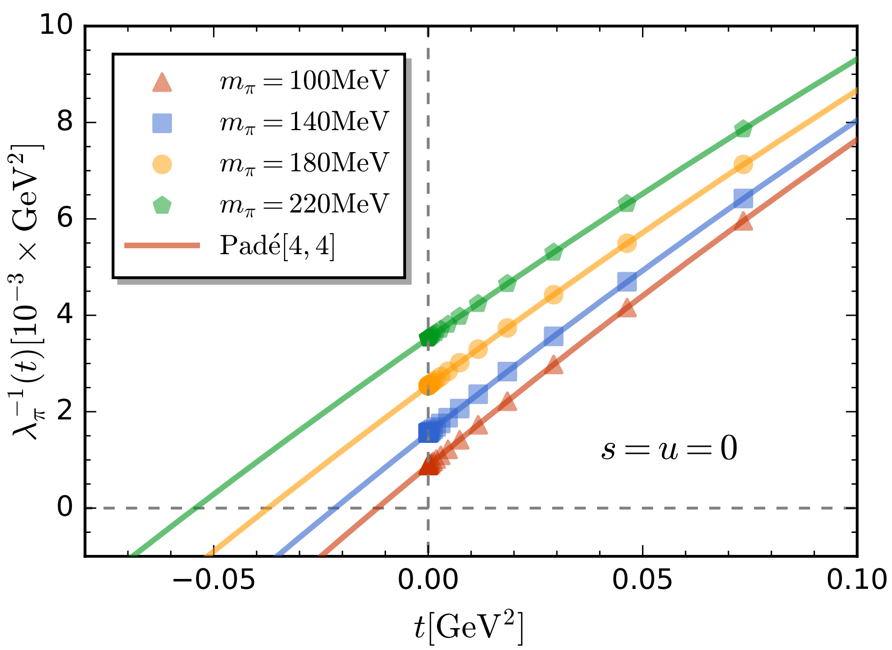
$$\lambda_{\pi,k=0}(P^2) = \frac{\lambda_{\pi,k=\Lambda}}{1 - \lambda_{\pi,k=\Lambda}\int_\Lambda^0 C_k(P^2)\frac{dk}{k}},$$

- **Flow equation of 4-quark interaction:**



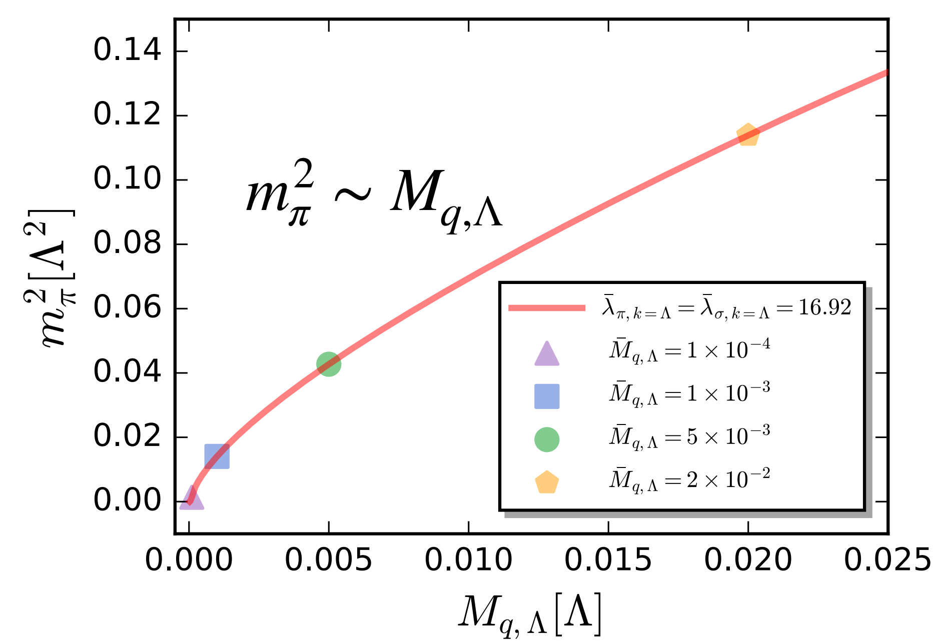
Note: playing the same role as the Bethe-Salpeter equation.

$\frac{1}{\lambda_\pi}$



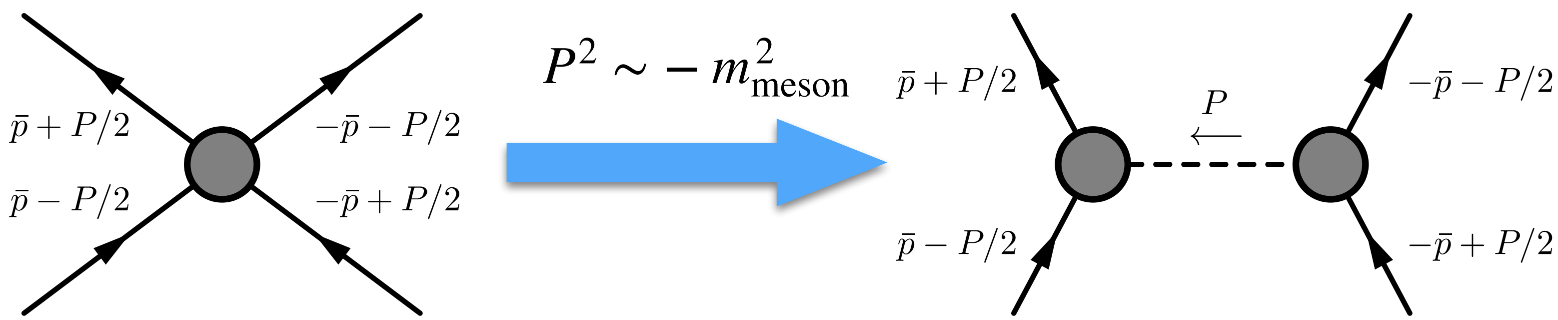
WF, Huang, Pawlowski, Tan, arXiv:2401.07638

Gell-Mann--Oakes--Renner relation



Bethe-Salpeter amplitude

- **Bethe-Salpeter amplitude can be extracted from the 4-quark vertex in the proximity of on-shell momentum of bound states:**



The 4-quark vertex near the on-shell momentum

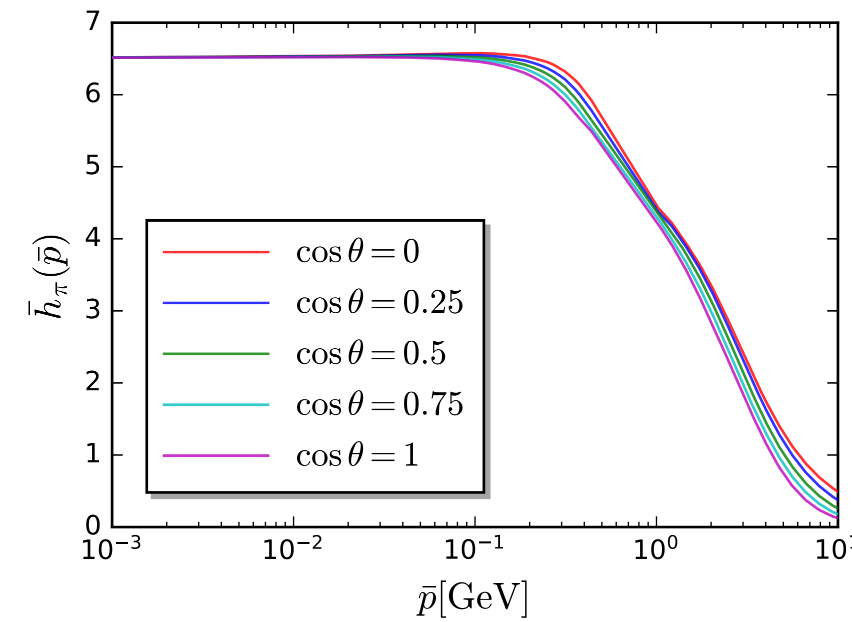
$$\lambda_\pi(P^2, p, \cos\theta) \sim \frac{h_\pi^2(p, \cos\theta)}{P^2 + m_\pi^2}$$

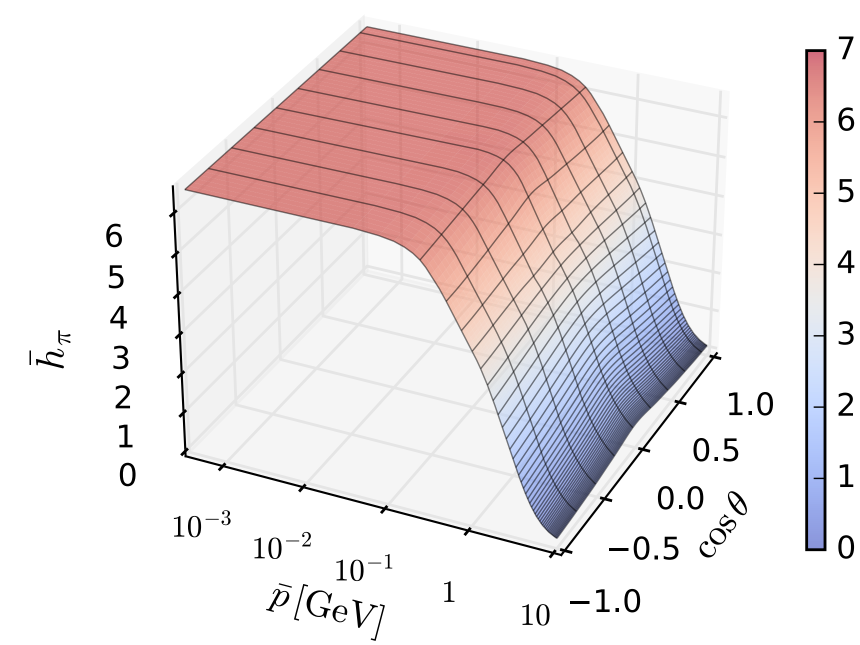
with the BS amplitude

$$h_\pi(p, \cos\theta) = \lim_{P^2 \to -m_\pi^2} \left[\lambda_\pi(P^2, p, \cos\theta)\,(P^2 + m_\pi^2)\right]^{1/2}$$

and

$$P_\mu = \sqrt{P^2}\,\Big(1, 0, 0, 0\Big)$$
$$\bar{p}_\mu = -\bar{p}'_\mu = \sqrt{p^2}\,\Big(\cos\theta,\ \sin\theta, 0, 0\Big)$$

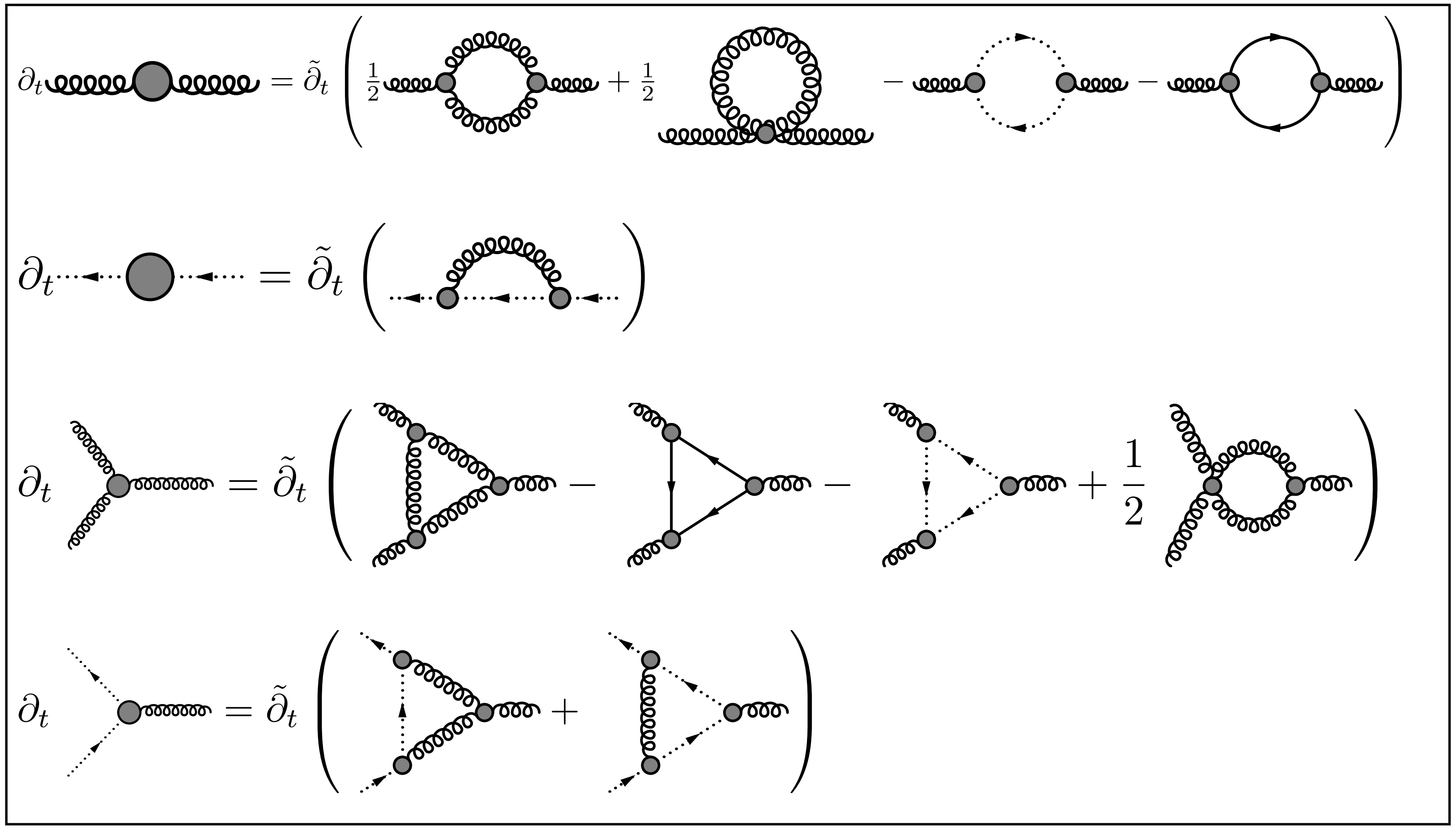




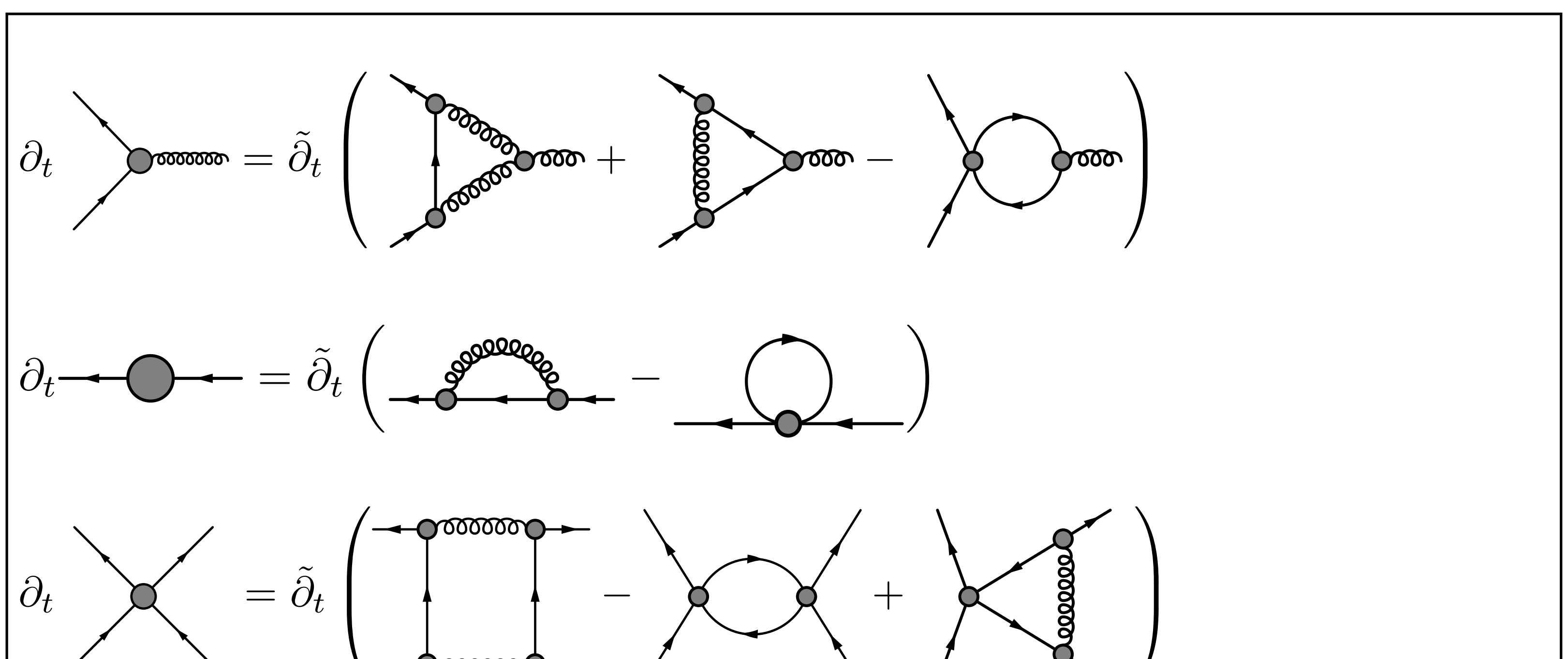
WF, Huang, Pawlowski, Tan, arXiv:2401.07638

QCD within fRG

Glue sector:



Matter sector:



QCD with dynamical hadronization

Introducing a RG scale dependent composite field:

$$\hat{\phi}_k(\hat{\varphi}), \ \textbf{with } \hat{\varphi} = (\hat{A}, \hat{c}, \hat{\bar{c}}, \hat{q}, \hat{\bar{q}}), \qquad \langle \partial_t \hat{\phi}_k \rangle = \dot{A}_k \, \bar{q} \tau q + \dot{B}_k \, \phi + \dot{C}_k \, \hat{e}_\sigma \,,$$

Gies, Wetterich , *PRD* 65 (2002) 065001; 69 (2004) 025001; Pawlowski, *AP* 322 (2007) 2831; Flörchinger, Wetterich, *PLB* 680 (2009) 371

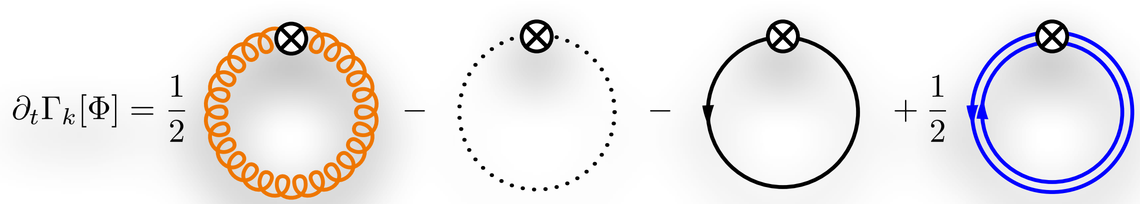
Wetterich equation is modified as

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \mathrm{STr}\big(G_k[\Phi] \, \partial_t R_k\big) + \mathrm{Tr}\left(G_{\phi\Phi_a}[\Phi] \frac{\delta \langle \partial_t \hat{\phi}_k \rangle}{\delta \Phi_a} \, R_\phi \right)$$

$$- \int \langle \partial_t \hat{\phi}_{k,i} \rangle \left(\frac{\delta \Gamma_k[\Phi]}{\delta \phi_i} + c_\sigma \delta_{i\,\sigma} \right) ,$$

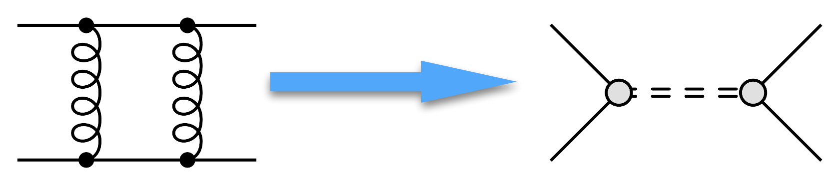
Mitter, Pawlowski, Strodthoff, *PRD* 91 (2015) 054035, arXiv:1411.7978;
Braun, Fister, Pawlowski, Rennecke, *PRD* 94 (2016) 034016, arXiv:1412.1045;
Rennecke, *PRD* 92 (2015) 076012, arXiv:1504.03585;
Cyrol, Mitter, Pawlowski, Strodthoff, *PRD* 97 (2018) 054006, arXiv:1706.06326;
WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

Flow equation:



See also recent work:
Ihssen, Pawlowski, Sattler, Wink, arXiv:2408.08413

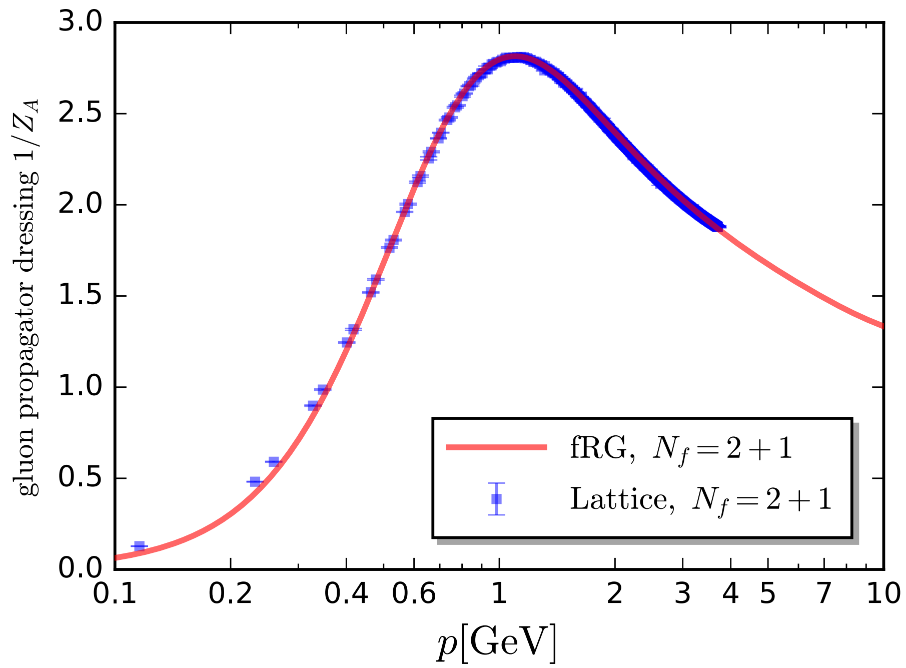
four-quark interaction encoded in Yukawa coupling:



also cf. talks by Franz R. Sattler and Friederike Ihssen

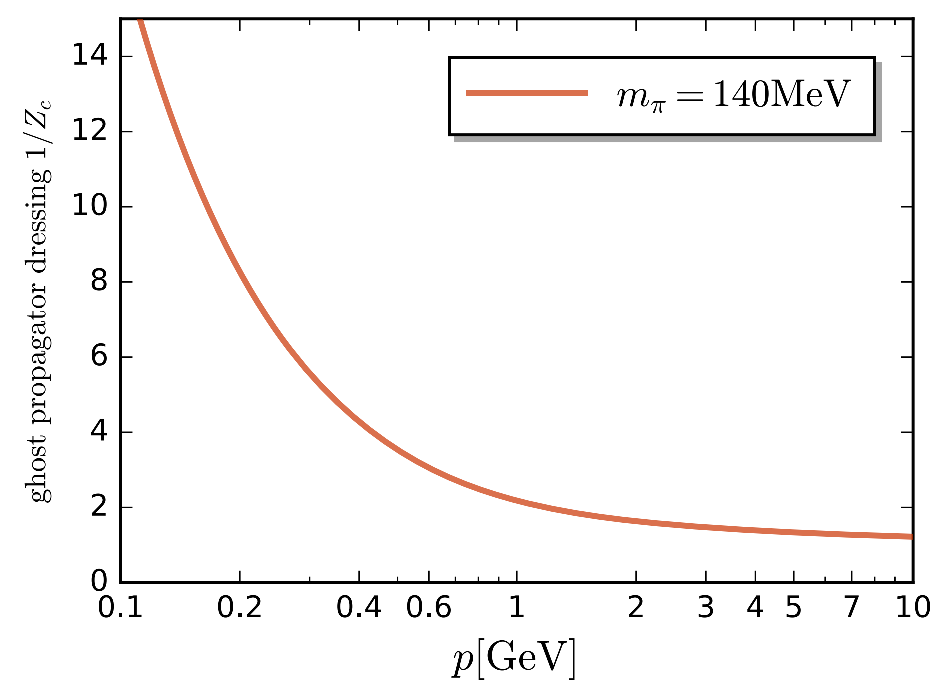
QCD within fRG in vacuum

Gluon dressing:



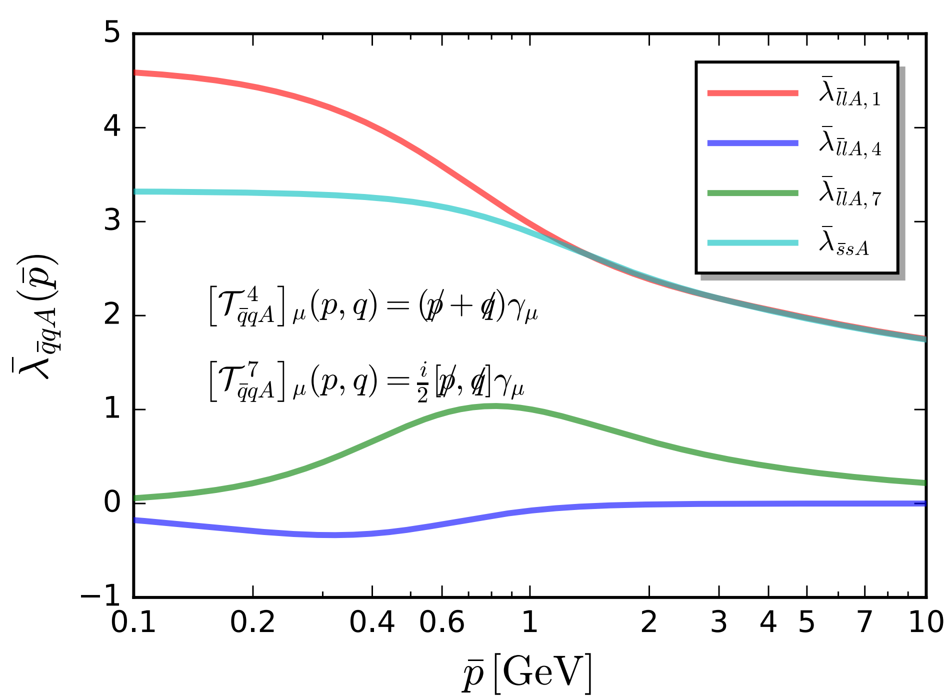
Lattice: Boucaud *et al.*, *PRD* 98 (2018) 114515

Ghost dressing:

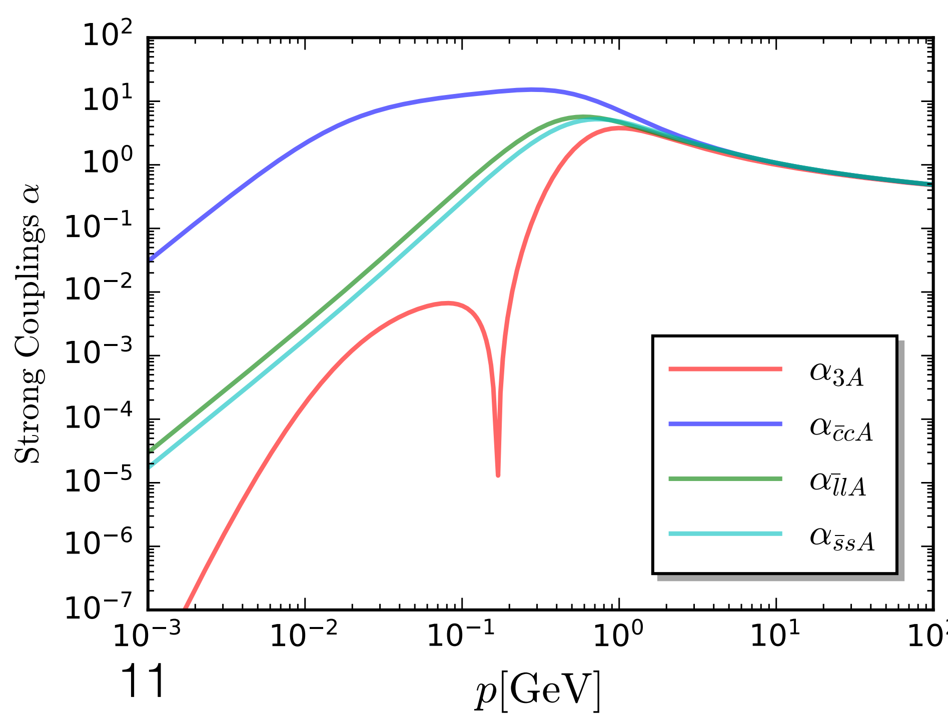


fRG: WF, Huang, Pawlowski, Tan, Zhou, in preparation

Quark-gluon vertex:

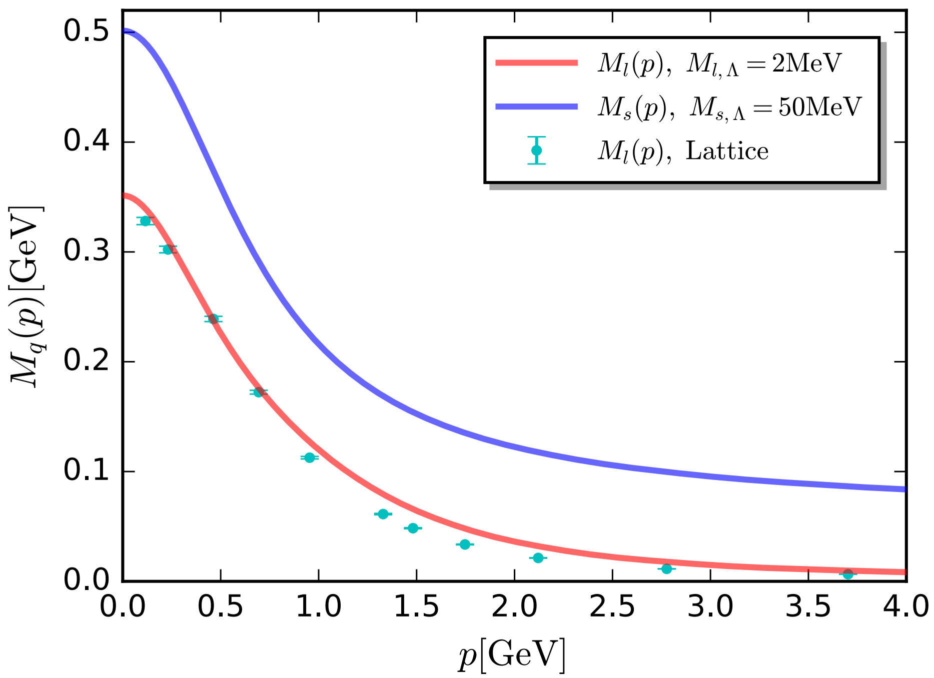


Strong couplings:



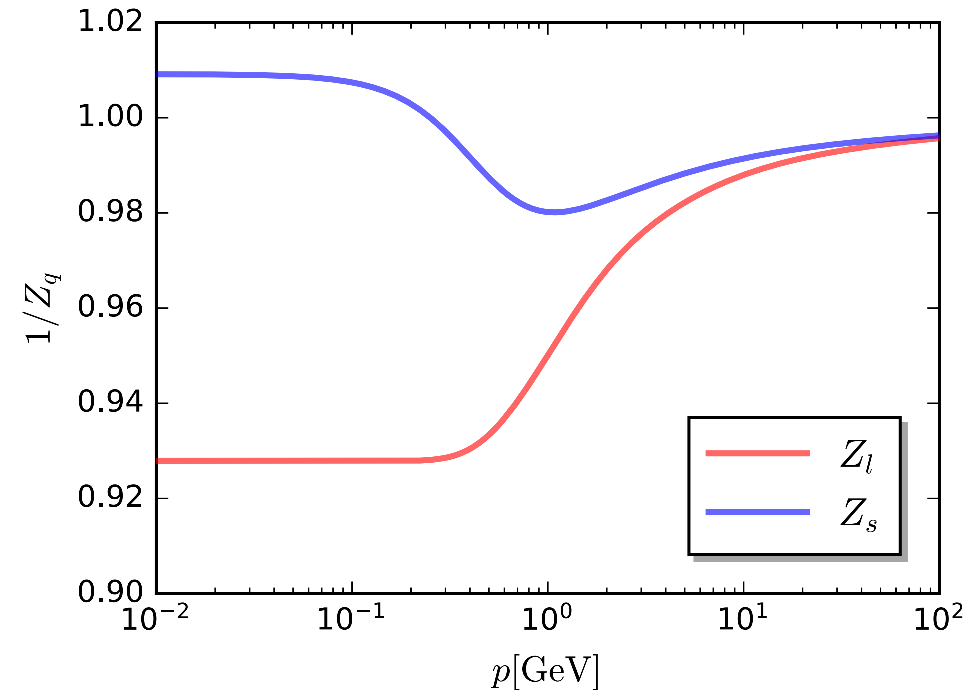
QCD within fRG in vacuum

Quark mass:

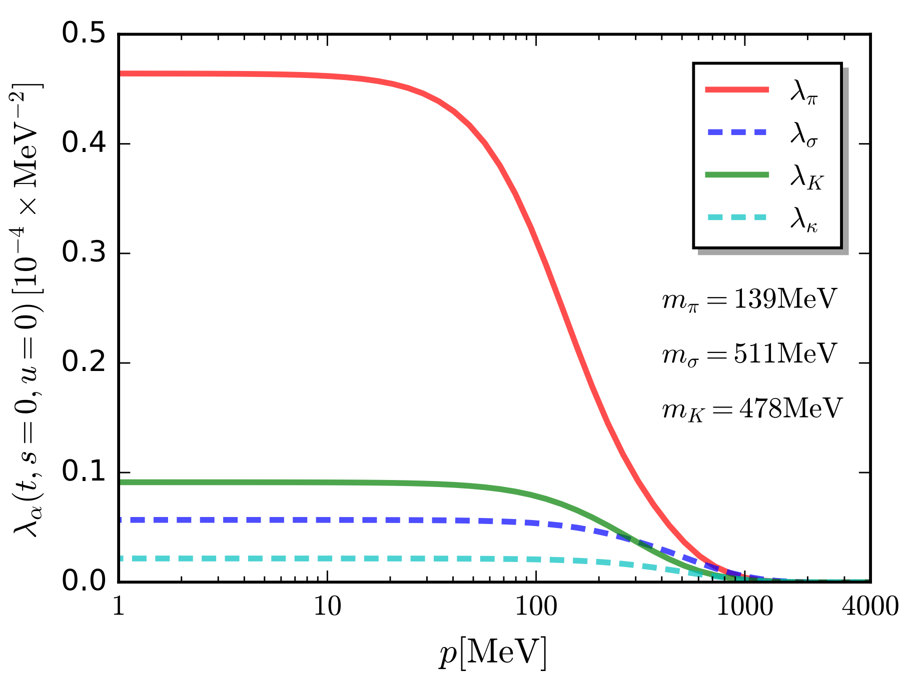


Lattice: Chang *et al.*, *PRD* 104 (2021) 094509

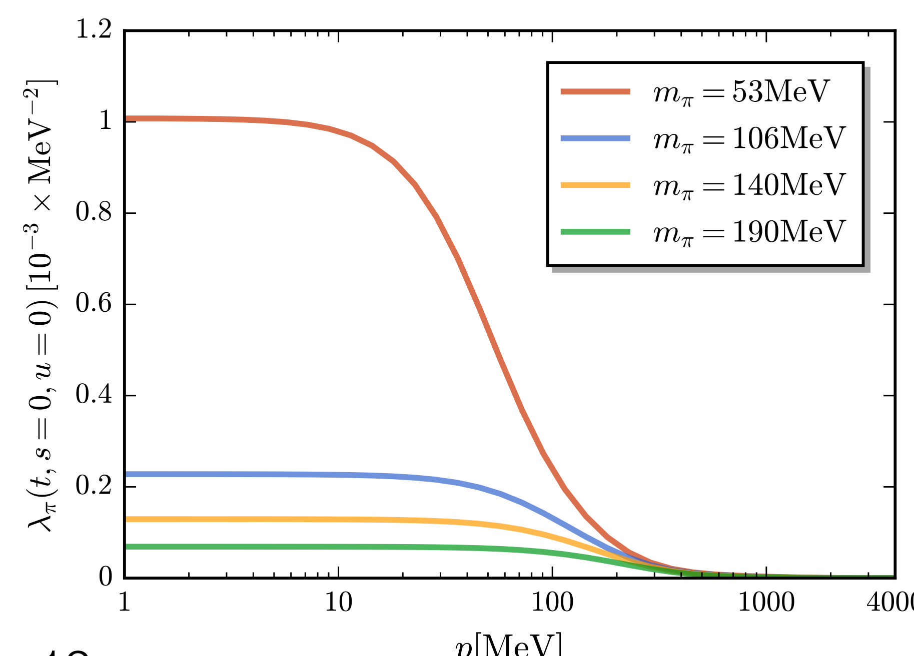
Quark wave function:



Four-quark vertex:



Four-quark vertex (pion channel):



fRG: WF, Huang, Pawlowski, Tan, Zhou, in preparation

Quasi-PDA of pion

- **Bethe-Salpeter amplitude** (unamputated):

$$\chi_\pi(k;P) = G_q(k_+)\Gamma(k;P)G_q(k_-)$$

with

$$\Gamma(k;P) = i\gamma_5 h_\pi(k;P)$$

and $P = (iE_\pi, P_z, 0, 0)$, $E_\pi = \sqrt{P_z^2 + m_\pi^2}$ and $k_\pm = k \pm P/2$

- **Quasi parton distribution amplitude** (qPDA) reads

$$\phi_\pi(x, P_z) = \frac{1}{f_\pi}\mathrm{Tr}_{\mathrm{CD}}\left[\int \frac{d^4k}{(2\pi)^4}\delta(\tilde{n}\cdot k_+ - x\tilde{n}\cdot P)\gamma_5\gamma\cdot\tilde{n}\chi_\pi(k;P)\right]$$

with $\tilde{n} = (0,1,0,0)$. Integrating k_3 firstly by using the delta function, one is led to

$$\phi_\pi(x, P_z) = \frac{1}{f_\pi}\frac{4N_c}{(2\pi)^4}\int d^2k_\perp dk_0\, h_\pi(k;P)P_z\left[xM_q(k_-^2) + (1-x)M_q(k_+^2)\right]$$
$$\times \frac{1}{Z_q(k_+^2)Z_q(k_-^2)}\frac{1}{k_+^2 + M_q^2(k_+^2)}\frac{1}{k_-^2 + M_q^2(k_-^2)}$$

$$k_\mu = \left(k_0, (x-1/2)P_z, k_\perp\right),$$
$$k_{+\mu} = \left(k_0 + iE_\pi/2, xP_z, k_\perp\right),$$
$$k_{-\mu} = \left(k_0 - iE_\pi/2, (x-1)P_z, k_\perp\right)$$

Quasi-PDA and PDA

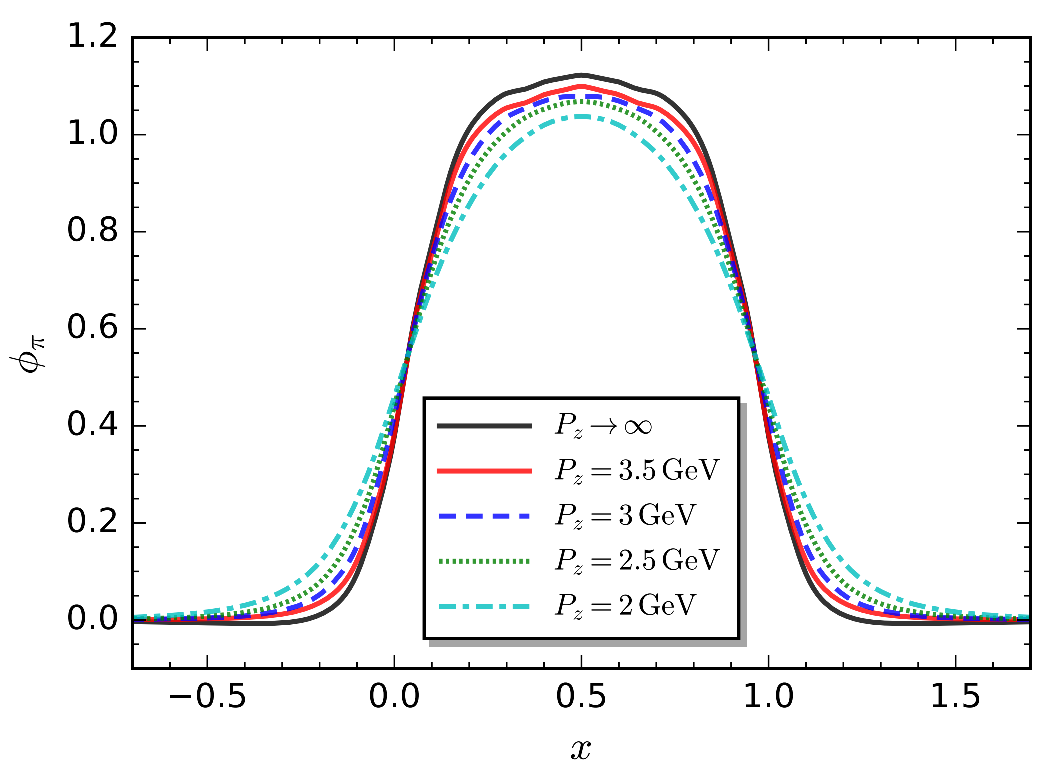
- Quasi-PDA at finite P_z can be used to extrapolate the PDA with $P_z \to \infty$ based on LaMET Ji, *PRL* 110 (2013) 262002

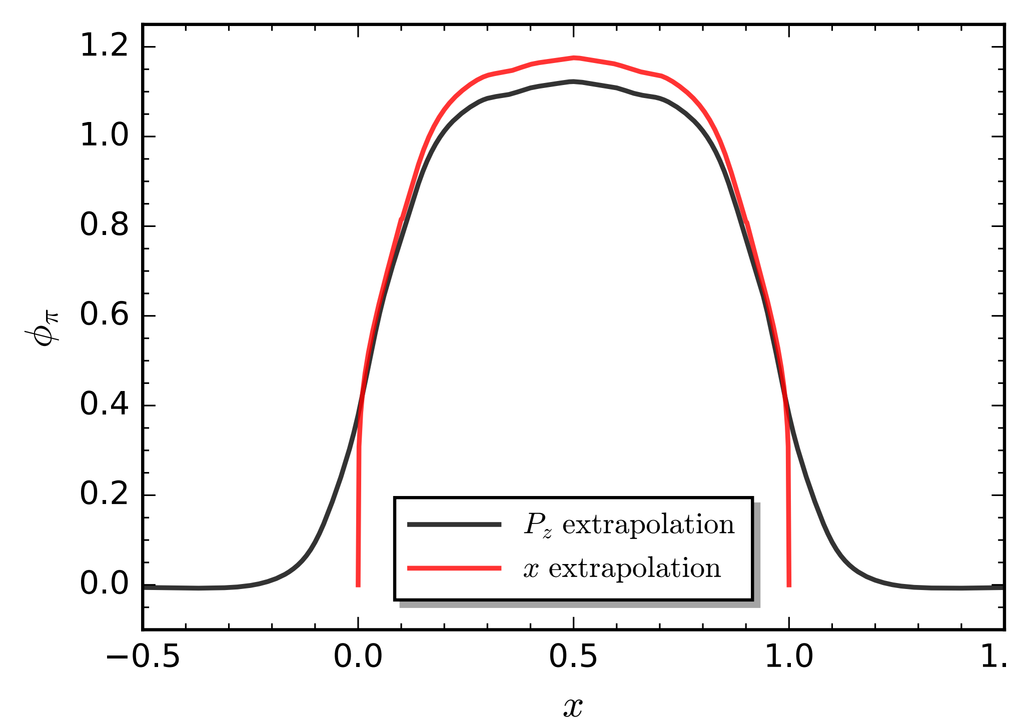
$$\phi_\pi(x, P_z) = \phi_\pi(x, P_z \to \infty) + \frac{c_2(x)}{P_z^2} + \mathcal{O}\left(\frac{1}{P_z^4}\right)$$

- But, in the endpoint region, say $0 < x < 0.1$ and $0.9 < x < 1$, LaMET cannot be reliably used, we adopt a phenomenological extrapolation $x^a(1-x)^a$ J. Hua *et al.* (LPC), *PRL* 129 (2022) 132001

Quasi-PDA:

fRG: Chang, WF, Huang, Pawlowski, Zhang, in preparation



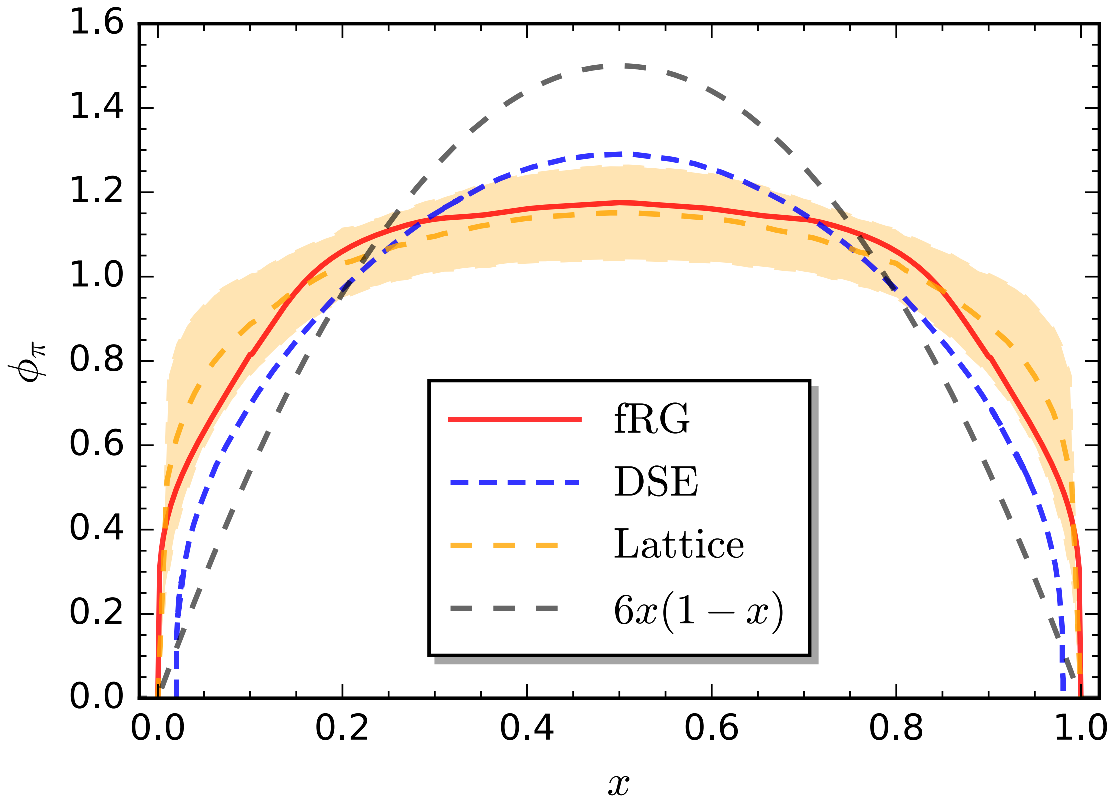


PDA and its moments

- Moments of pion PDA

$$\langle \xi^n \rangle \equiv \int_0^1 dx (2x-1)^n \phi_\pi(x)$$

Pion PDA:



fRG: Chang, WF, Huang, Pawlowski, Zhang, in preparation

Moments:

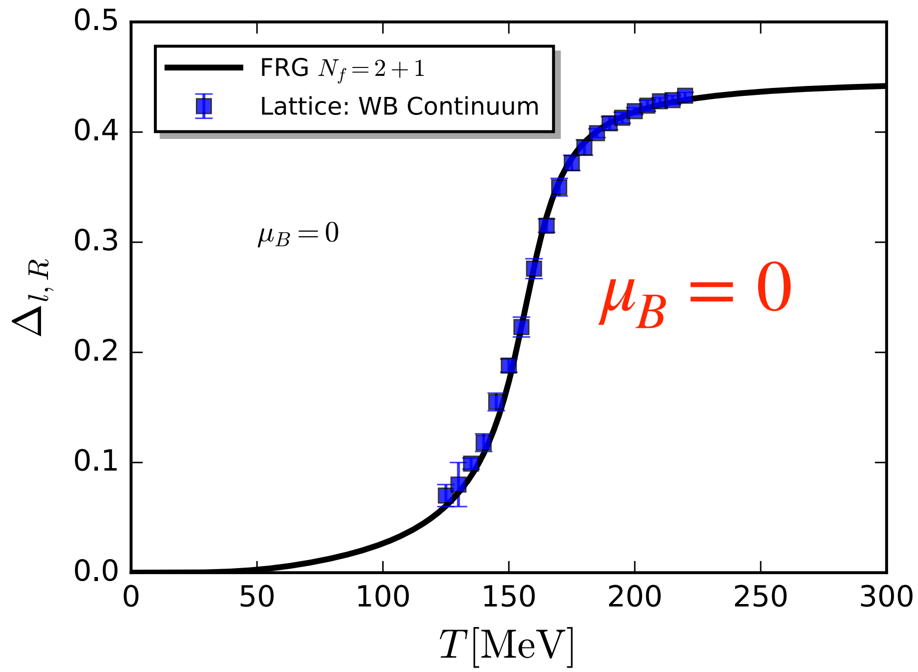
Method	ξ_π^2	ξ_π^4	ξ_π^6
fRG (This Work)	0.271	0.142	0.092
Lattice LaMET (LPC)	0.300(41)	-	-
DSE	0.251	0.128	-
Lattice OPE (RQCD)	$0.234^{+6}_{-6}(4)(4)(2)$	-	-
Lattice OPE (RBC and UKQCD)	0.28(1)(2)	-	-
Sum Rule	0.271(13)	0.138(10)	0.087(6)

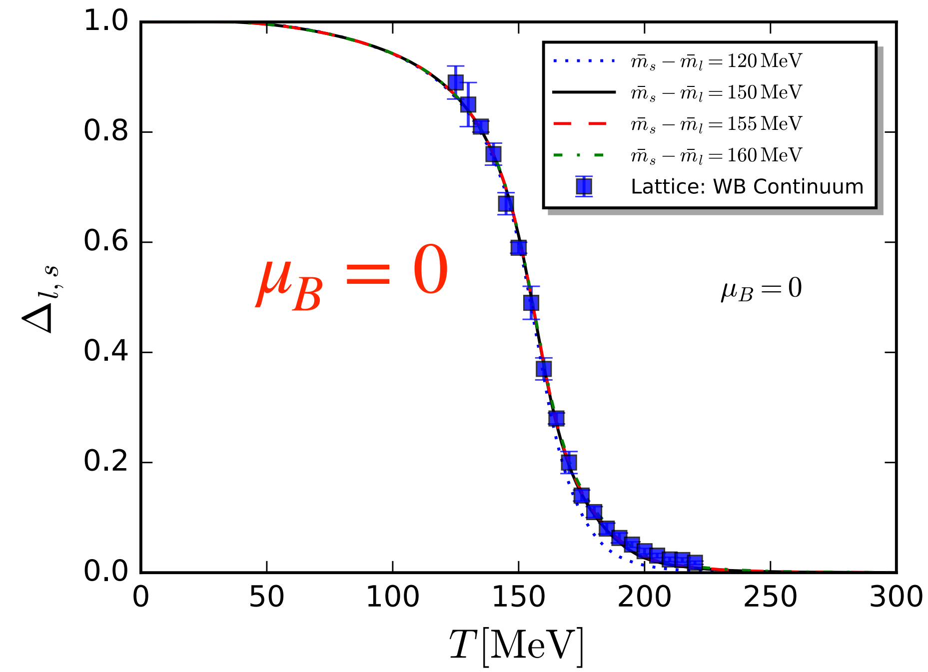
Lattice LaMET: J. Hua *et al.* (LPC), *PRL* 129 (2022) 132001.
DSE: C. Roberts *et al.*, *PPNP* 120 (2021) 103883; Chang *et al.*, *PRL* 110 (2013) 132001.
Lattice OPE: G. Bali *et al.* (RQCD), *JHEP* 08 (2019) 065; 11 (2020) 37.
Lattice OPE: R. Arthur *et al.* (RBC and UKQCD), *PRD* 83 (2011) 074505.
Sum rules: P. Ball *et al.*, *JHEP* 08 (2007) 090; T. Zhong *et al.*, *PRD* 104 (2021) 016021
Asymptotic: $6x(1-x)$

QCD phase transitions

Renormalized light quark condensate:

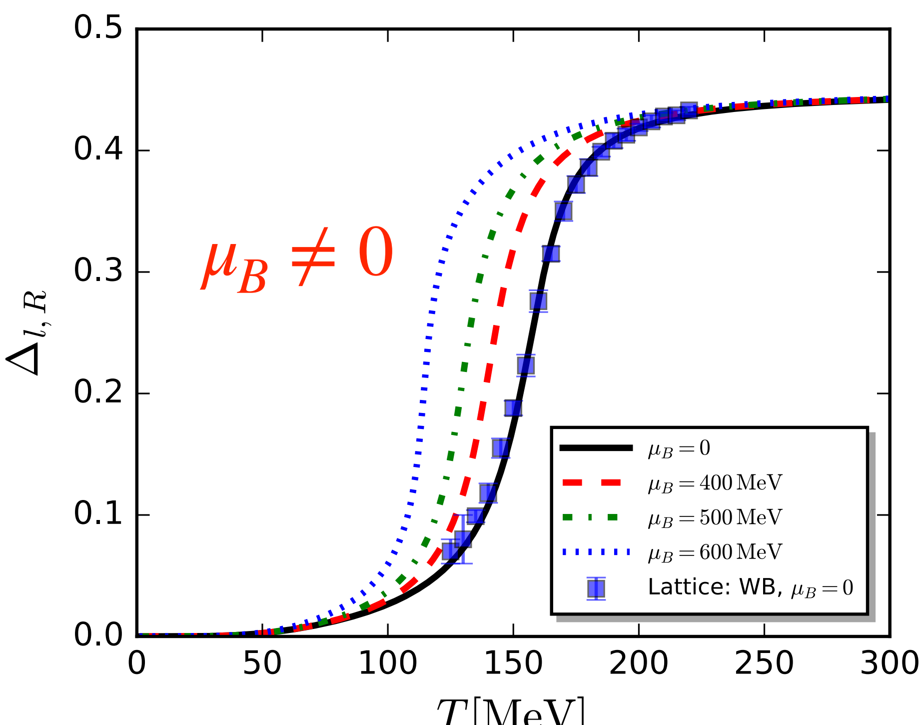
Reduced condensate:

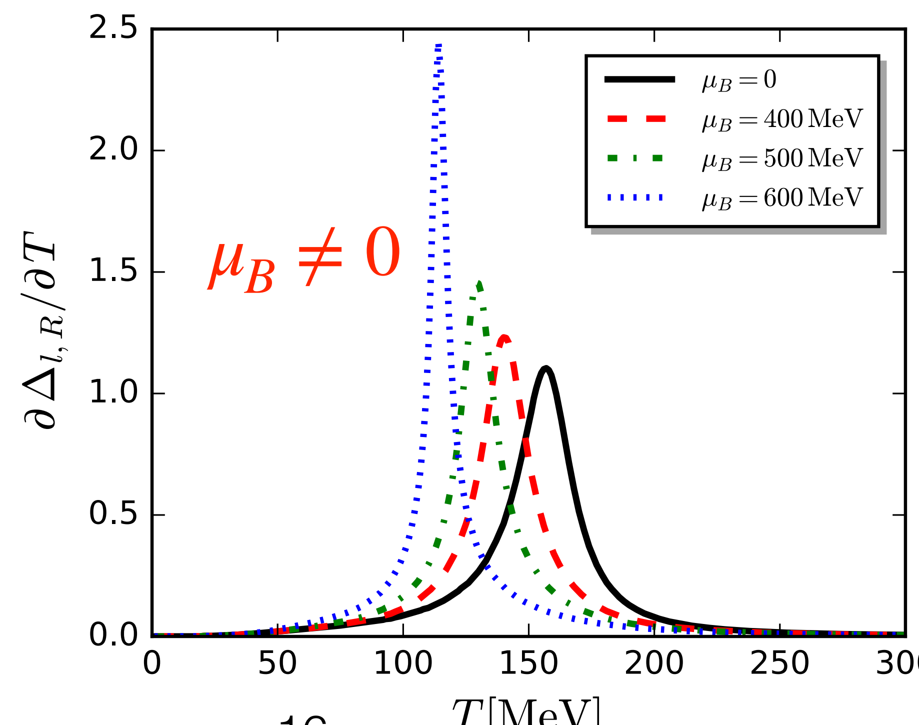




fRG: WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

Lattice: Borsanyi *et al.* (WB), *JHEP* 09 (2010) 073





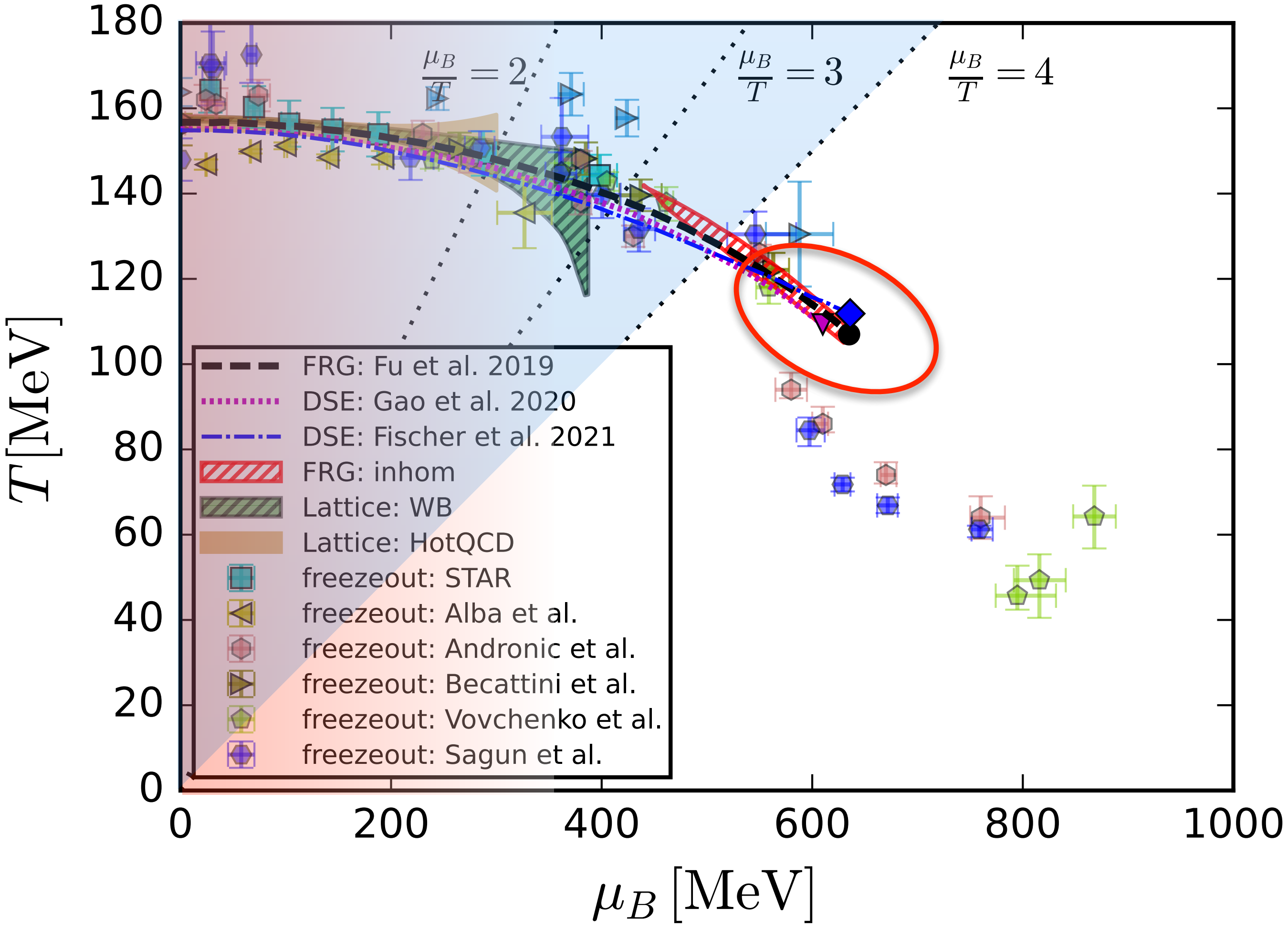
Quark condensate:

$$\Delta_{q_i} \simeq -m_{q_i}^0 T \sum_{n \in \mathbb{Z}} \int \frac{d^3 q}{(2\pi)^3} \mathrm{tr}\, G_{q_i \bar{q}_i}(q)\,,$$

Reduced condensate:

$$\Delta_{l,s}(T,\mu_q) = \frac{\Delta_l(T,\mu_q) - \left(\frac{m_l^0}{m_s^0}\right)^2 \Delta_s(T,\mu_q)}{\Delta_l(0,0) - \left(\frac{m_l^0}{m_s^0}\right)^2 \Delta_s(0,0)}$$

CEP from first-principles functional QCD



Passing through strict benchmark tests in comparison to lattice QCD at vanishing and small μ_B.

Regime of quantitative reliability of functional QCD with $\mu_B/T \lesssim 4$.

also cf. talks by Rui Wen

Estimates of the location of CEP from first-principles functional QCD:

fRG:

● $(T, \mu_B)_{\mathrm{CEP}} = (107, 635)\,\mathbf{MeV}$

fRG: WF, Pawlowski, Rennecke, *PRD* 101 (2020), 054032

DSE:

▼ $(T, \mu_B)_{\mathrm{CEP}} = (109, 610)\,\mathbf{MeV}$

DSE (fRG): Gao, Pawlowski, *PLB* 820 (2021) 136584

◆ $(T, \mu_B)_{\mathrm{CEP}} = (112, 636)\,\mathbf{MeV}$

DSE: Gunkel, Fischer, *PRD* 104 (2021) 5, 054022

- No CEP observed in $\mu_B/T \lesssim 2 \sim 3$ from lattice QCD. Karsch, *PoS* CORFU2018 (2019)163
- Recent studies of QCD phase structure from both fRG and DSE have shown convergent estimate for the location of CEP: $600\ \mathrm{MeV} \lesssim \mu_{B_{\mathrm{CEP}}} \lesssim 650\ \mathrm{MeV}$.

CEP from other approaches

Recent estimates of the location of CEP:

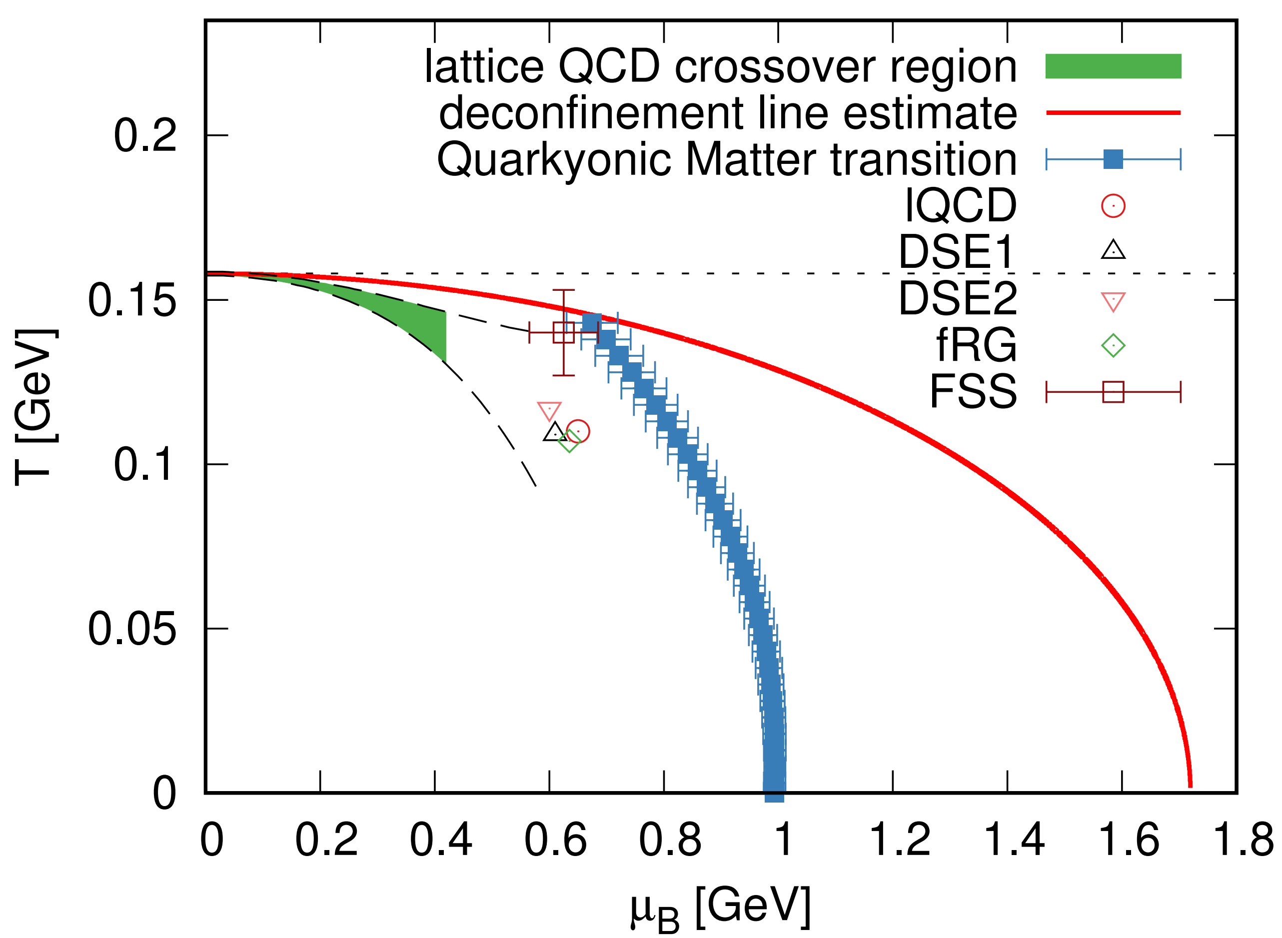


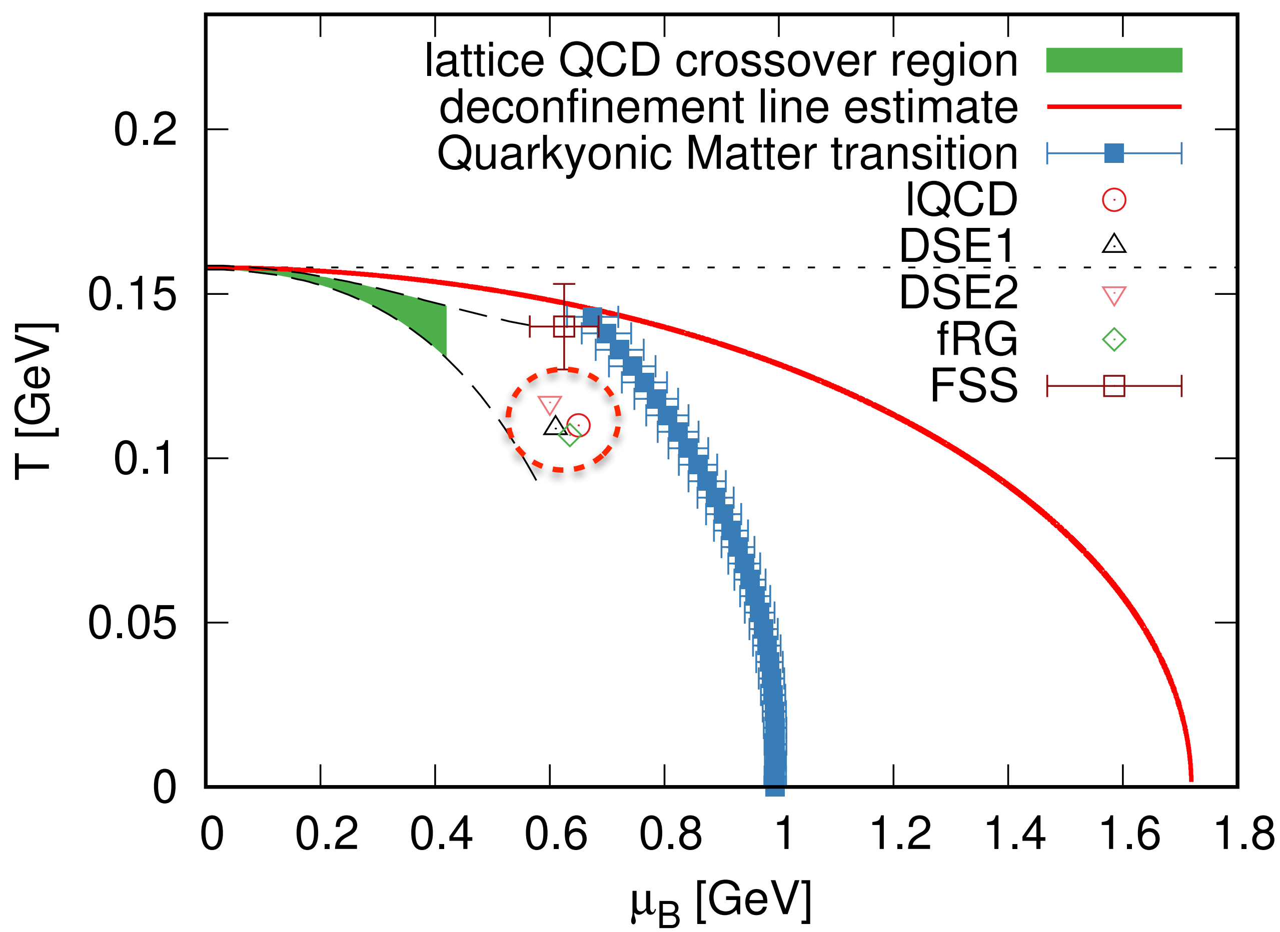
Figure from:
Bluhm, Fujimoto, McLerran, Nahrgang, arXiv:2409.12088

fRG:
WF, Pawlowski, Rennecke, *PRD* 101 (2020), 054032, arXiv:1909.02991.
DSE1:
Gao, Pawlowski, *PLB* 820 (2021) 136584, arXiv:2010.13705.
DSE2:
Gunkel, Fischer, *PRD* 104 (2021) 054022, arXiv:2106.08356.
Lattice extrapolation (Yang-Lee edge singularities):
David A. Clarke *et al.*, arXiv:2405.10196.
Finite-size-scaling analysis:
A. Sorensen, P. Sorensen, arXiv:2405.10278.

- **Estimates of the location of CEP in the QCD phase diagram have arrived at convergence from different approaches.**

CEP from other approaches

Recent estimates of the location of CEP:



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WF, Pawlowski, Rennecke, *PRD* 101 (2020), 054032, arXiv:1909.02991.
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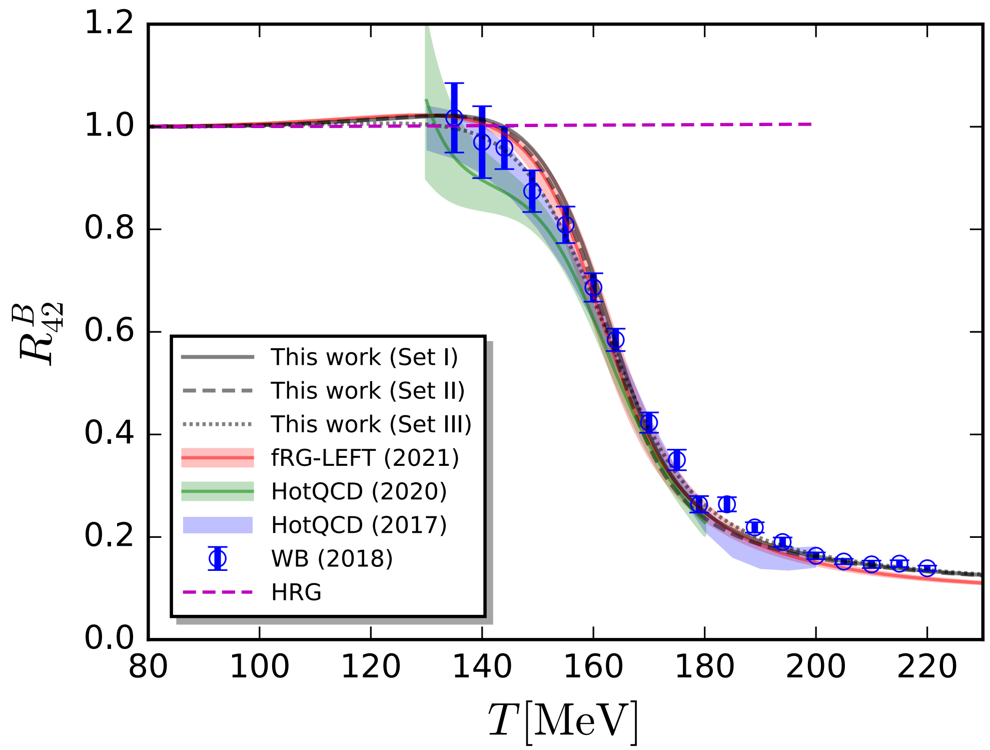
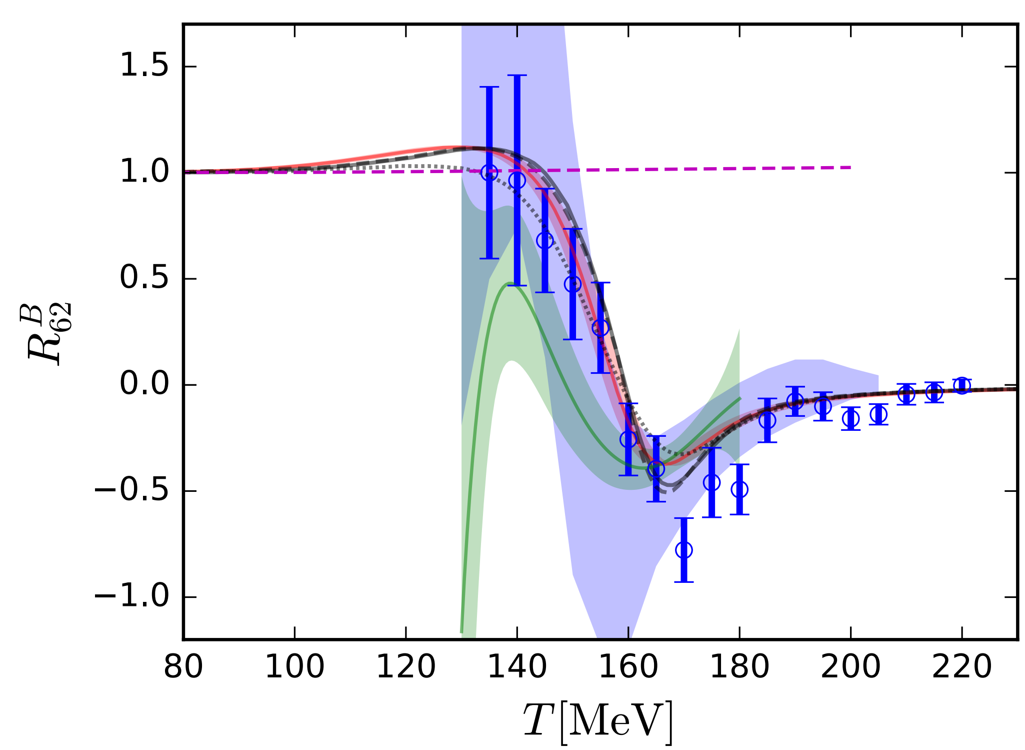
- **Estimates of the location of CEP in the QCD phase diagram have arrived at convergence from different approaches.**

Baryon number fluctuations





$\mu_B = 0$

fRG: WF, Luo, Pawlowski, Rennecke, Yin, arXiv: 2308.15508;
WF, Luo, Pawlowski, Rennecke, Wen, Yin, *PRD* 104 (2021) 094047

HotQCD: A. Bazavov *et al.*, arXiv: *PRD* 95 (2017), 054504; *PRD* 101 (2020), 074502

WB: S. Borsanyi *et al.*, arXiv: *JHEP* 10 (2018) 205

baryon number fluctuations

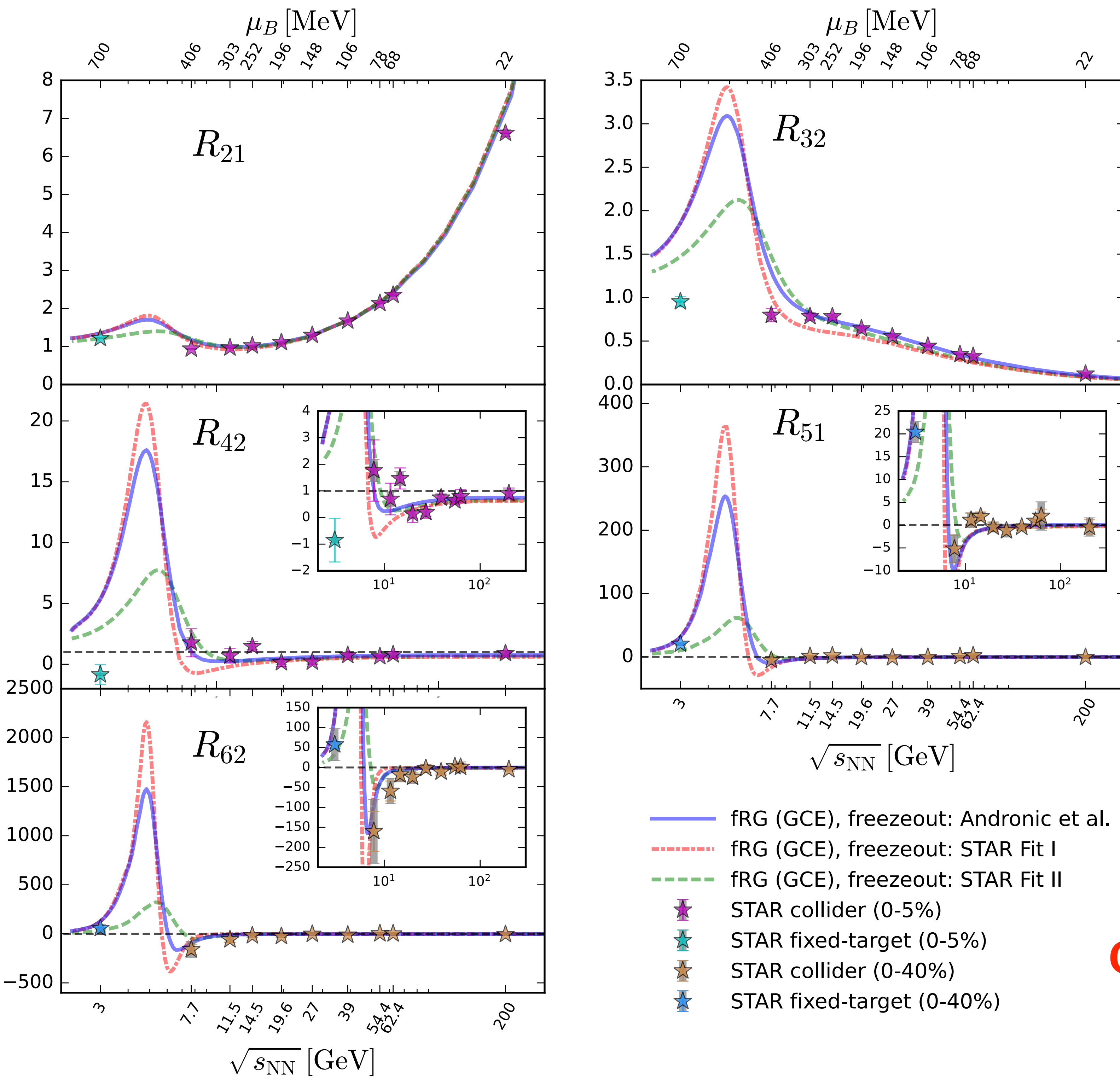
$$\chi_n^B = \frac{\partial^n}{\partial(\mu_B/T)^n}\frac{p}{T^4} \qquad R^B_{nm} = \frac{\chi_n^B}{\chi_m^B}$$

relation to the cumulants

$$\frac{M}{VT^3} = \chi_1^B \,,\ \frac{\sigma^2}{VT^3} = \chi_2^B \,,\ S = \frac{\chi_3^B}{\chi_2^B\sigma} \,,\ \kappa = \frac{\chi_4^B}{\chi_2^B\sigma^2} \,,$$

- **In comparison to lattice results and our former results, the improved results of baryon number fluctuations at vanishing chemical potential in the QCD-assisted LEFT are convergent and consistent.**

Grand canonical fluctuations at the freeze-out



STAR: Adam *et al.* (STAR), *PRL* 126 (2021) 092301;
Abdallah *et al.* (STAR), *PRL* 128 (2022) 202303;
Aboona *et al.* (STAR), *PRL* 130 (2023) 082301

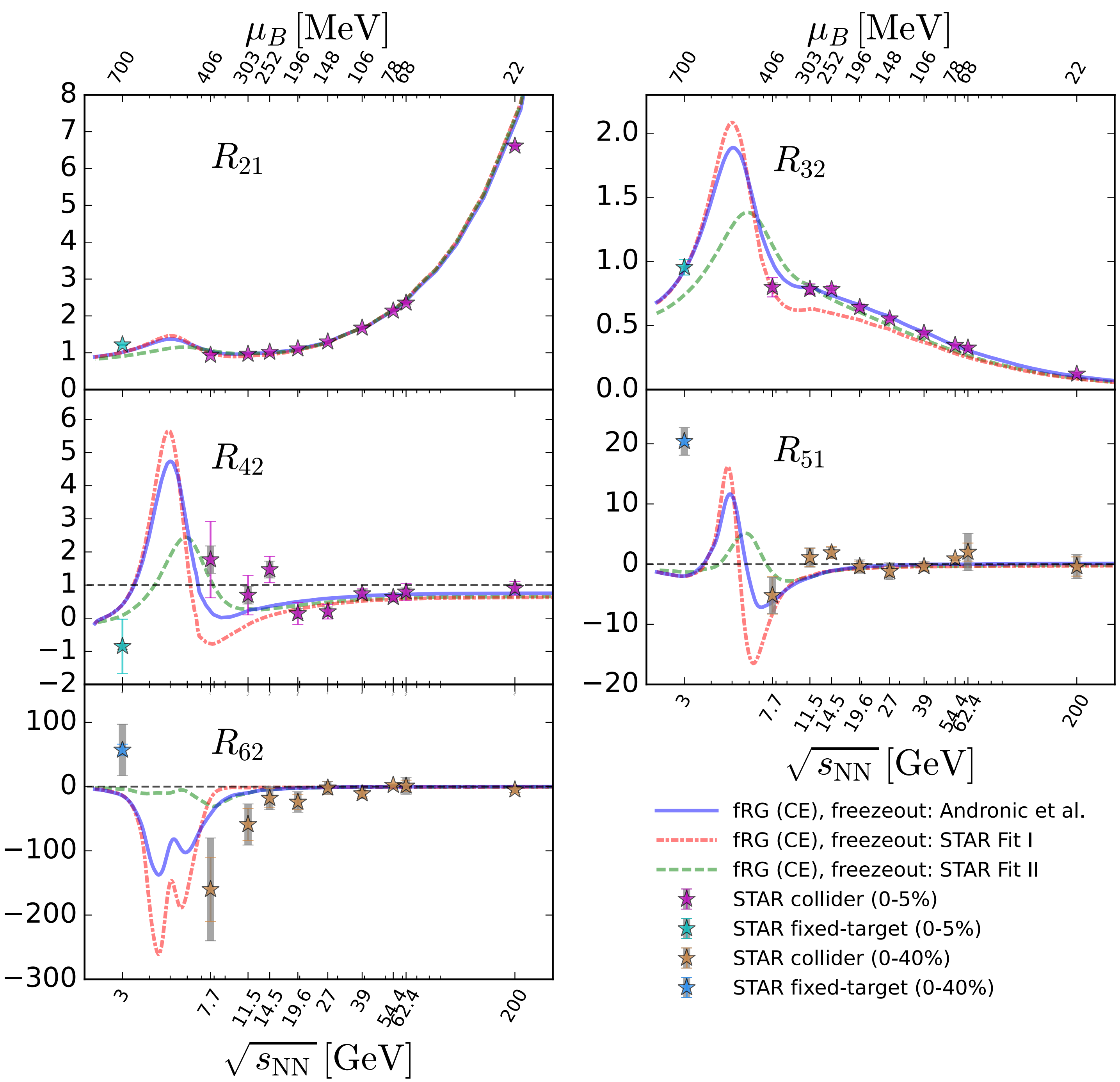
fRG: WF, Luo, Pawlowski, Rennecke, Yin, arXiv: 2308.15508

- **Results in fRG are obtained in the QCD-assisted LEFT with a CEP at $(T_{\rm CEP}, \mu_{B_{\rm CEP}}) = (98{,}643)$ MeV.**
- **Peak structure is found in 3 GeV $\lesssim \sqrt{s_{\rm NN}} \lesssim$ 7.7 GeV.**
- **Agreement between the theory and experiment is worsening with $\sqrt{s_{\rm NN}} \lesssim$ 11.5 GeV.**
- **Effects of global baryon number conservation in the regime of low collision energy should be taken into account.**

Caveat:

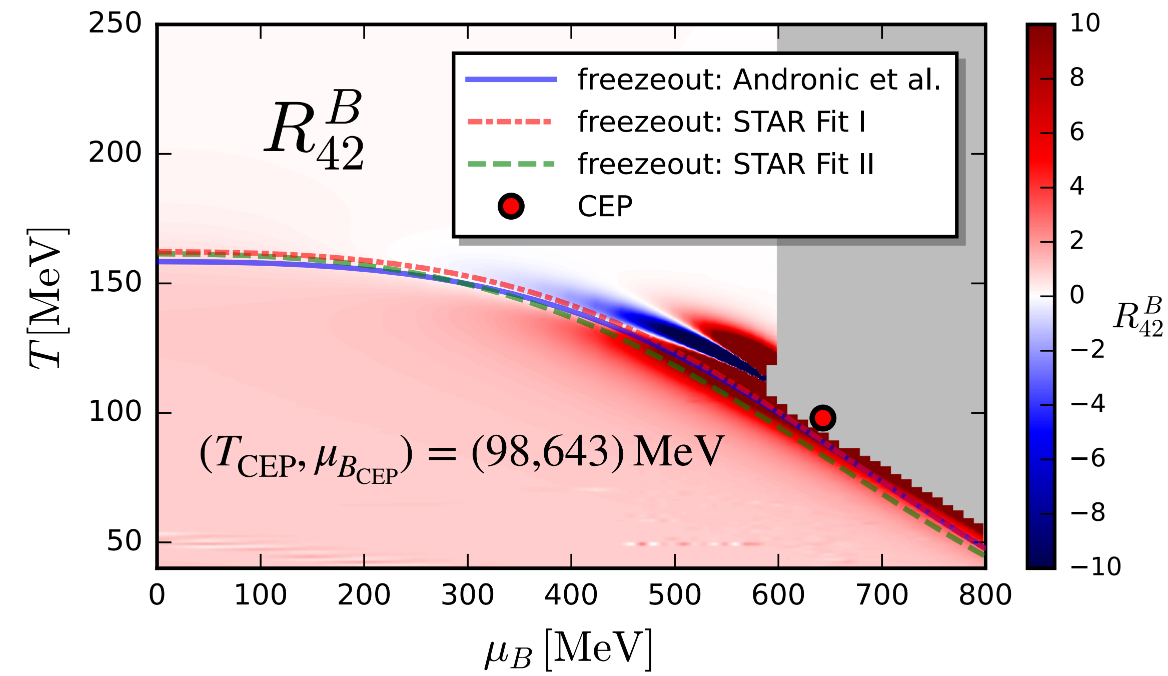
Fluctuations of baryon number in theory are compared with those of proton number in experiments.

Canonical fluctuations at the freeze-out



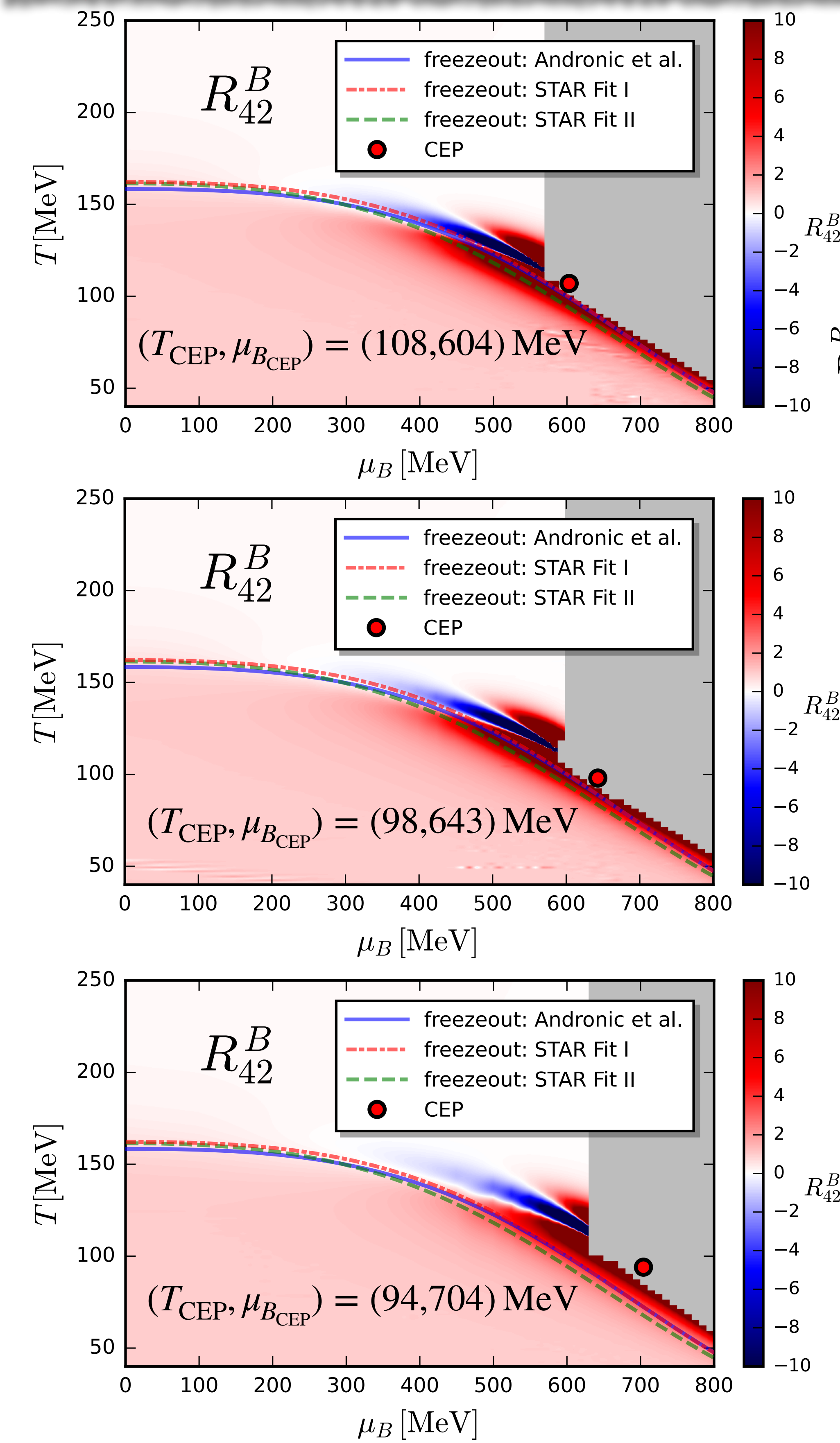
STAR: Adam *et al.* (STAR), *PRL* 126 (2021) 092301; Abdallah *et al.* (STAR), *PRL* 128 (2022) 202303; Aboona *et al.* (STAR), *PRL* 130 (2023) 082301

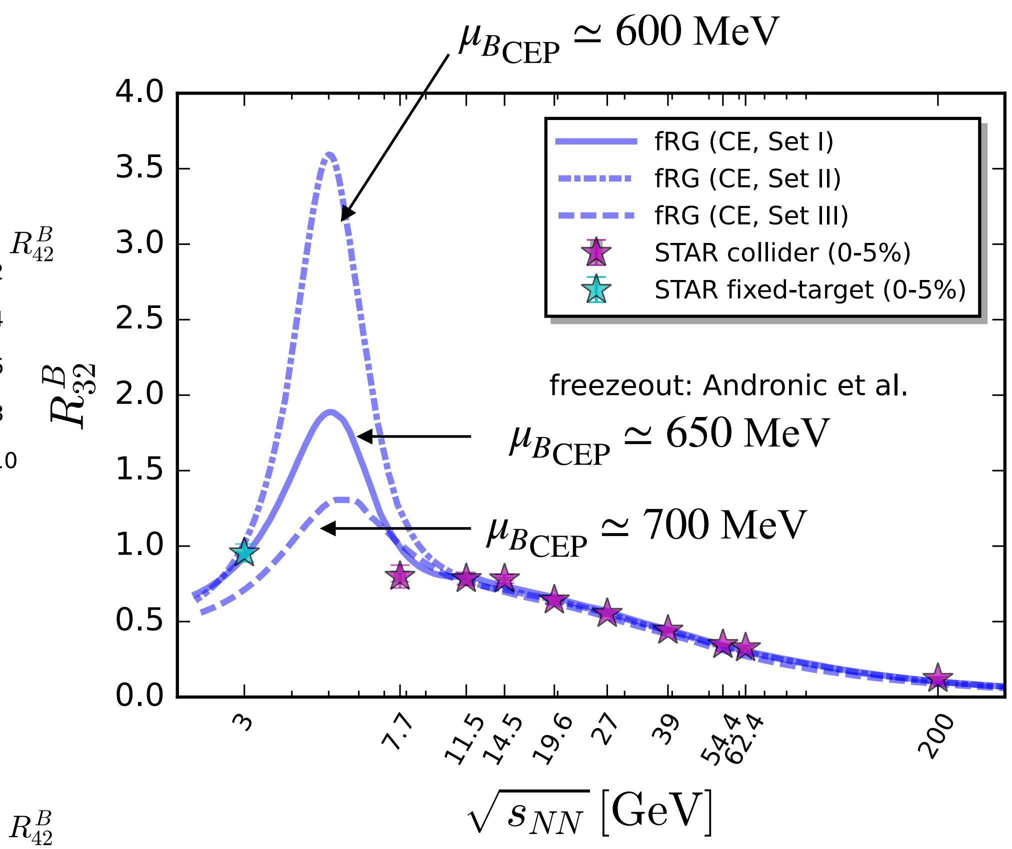
fRG: WF, Luo, Pawlowski, Rennecke, Yin, arXiv: 2308.15508

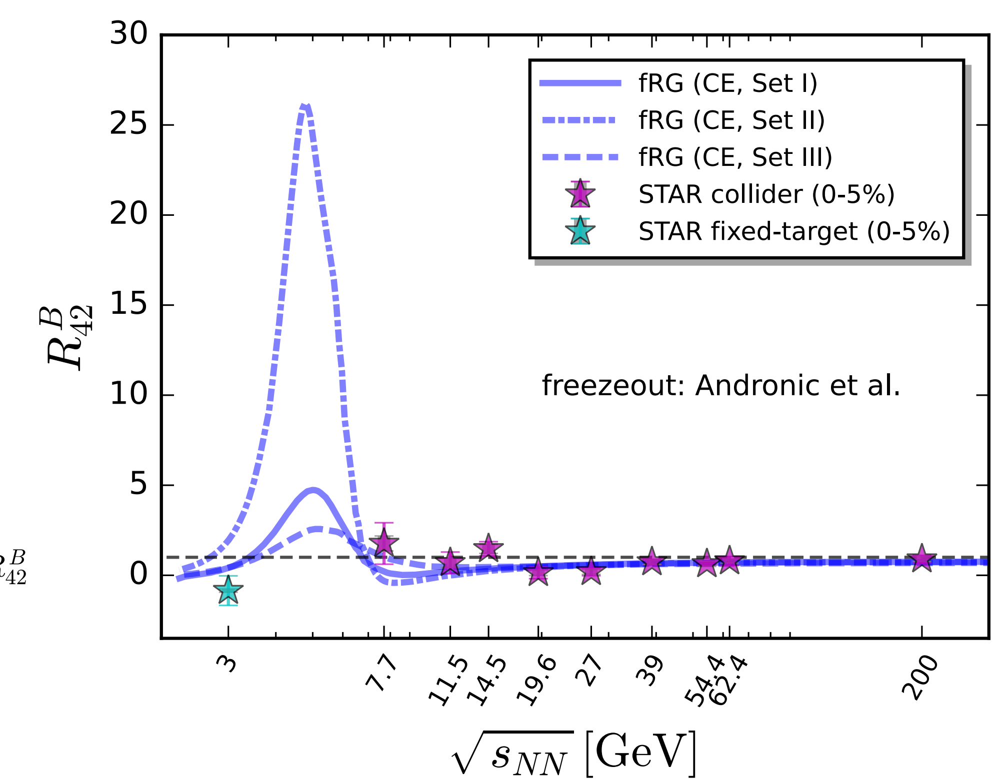


- **Peak structure is found in 3 GeV $\lesssim \sqrt{s_{\rm NN}} \lesssim$ 7.7 GeV.**
- **Position of peak in R_{42} is $\mu_{B_{\rm peak}}$ = 536, 541 and 486 MeV for the three freeze-out curves, significantly smaller than $\mu_{B_{\rm CEP}}$ = 643 MeV.**

Dependence on the location of the CEP







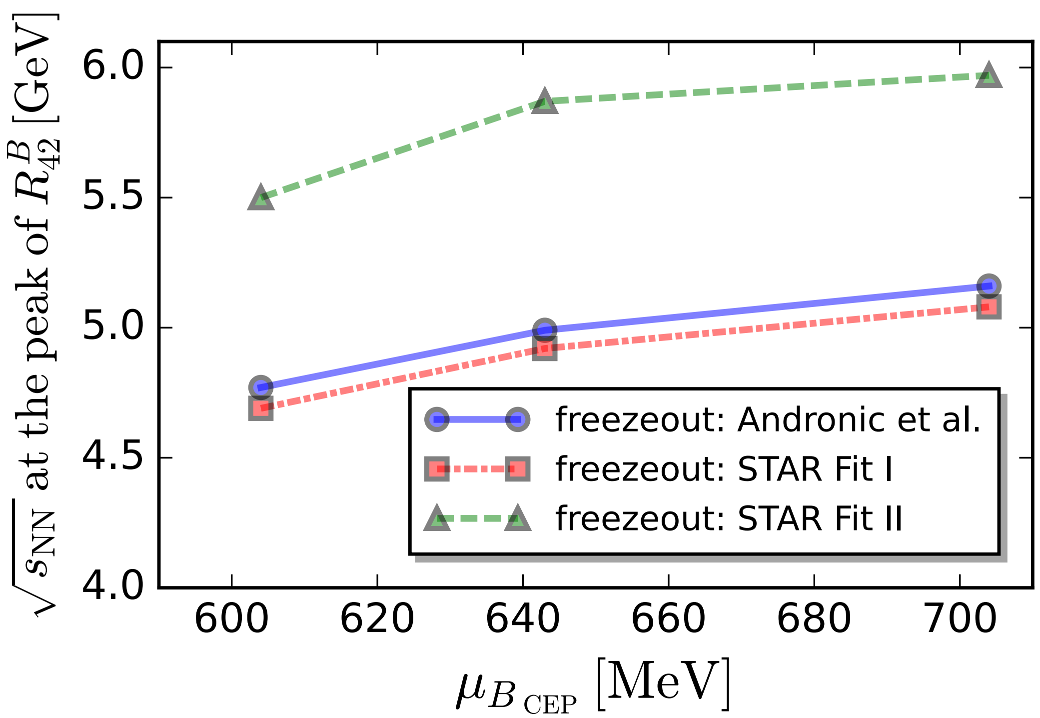
STAR: Adam *et al.* (STAR), *PRL* 126 (2021) 092301

fRG: WF, Luo, Pawlowski, Rennecke, Yin, arXiv: 2308.15508

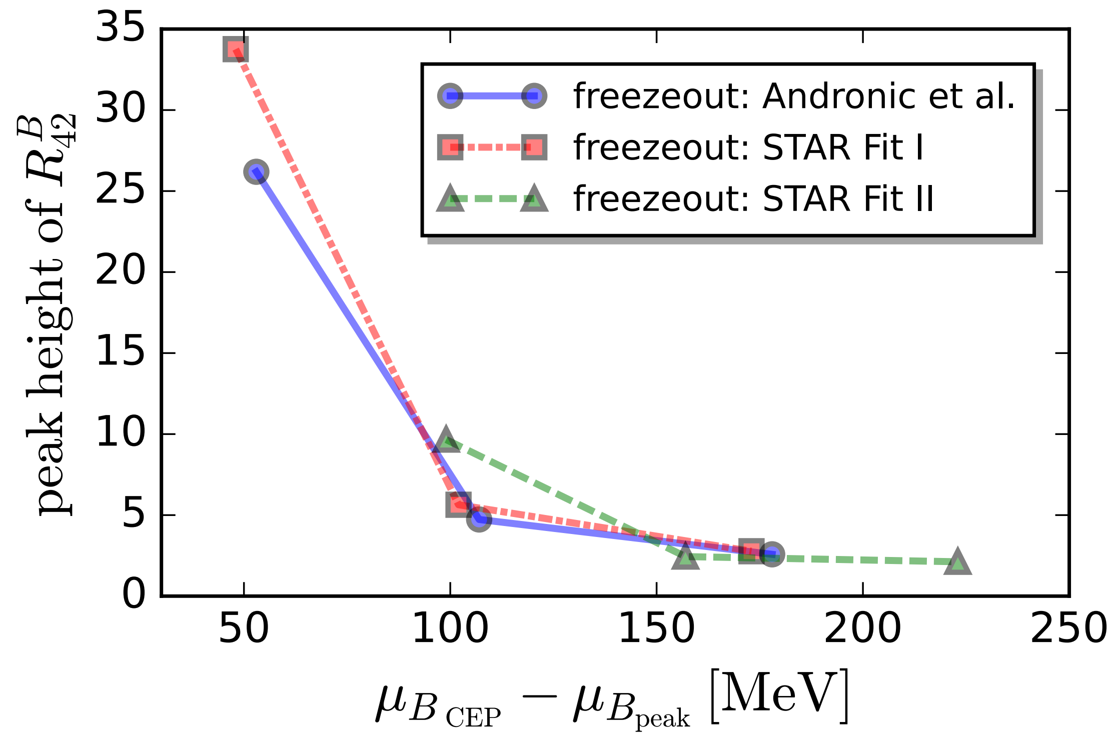
- Position of the peak is insensitive to the location of CEP.
- Height of peak decreases as CEP moves towards larger μ_B.

Ripples of the QCD critical point

Position of peak:



Height of peak:

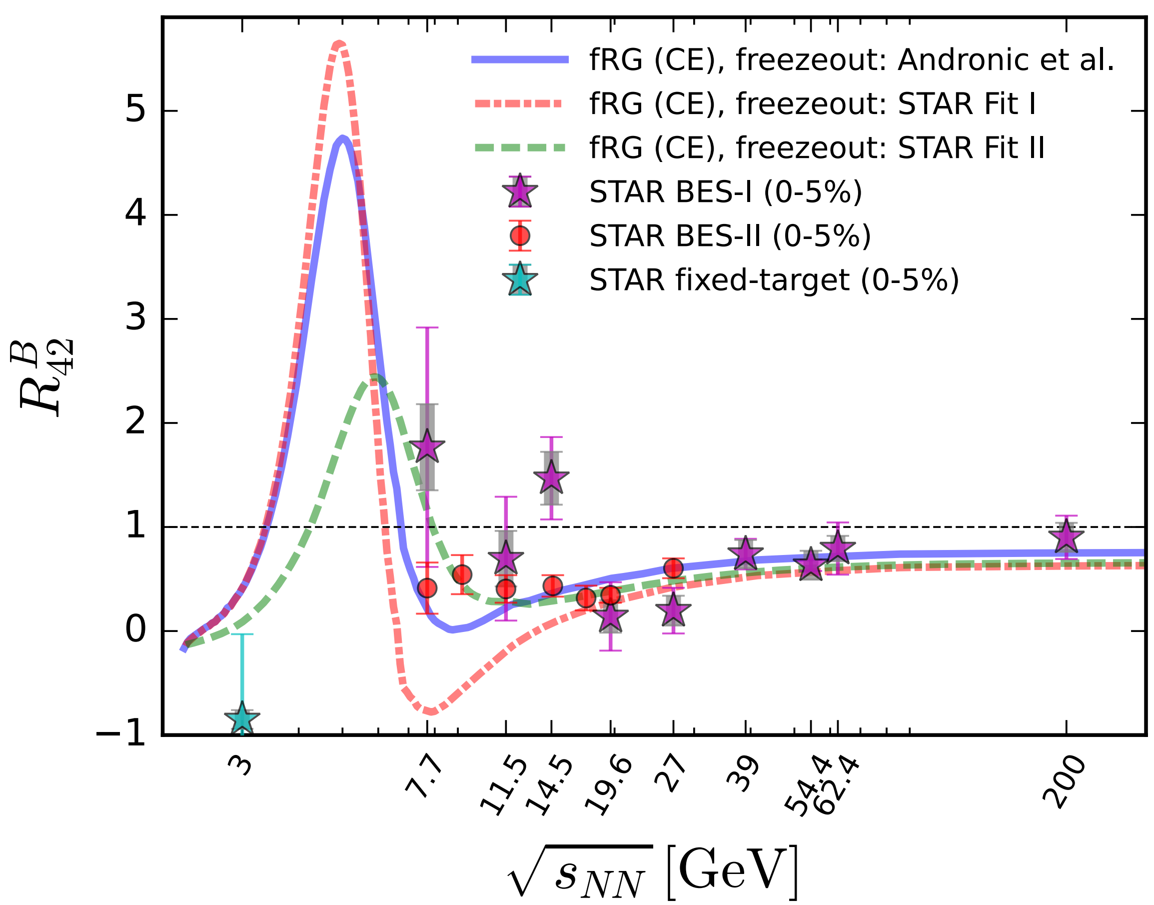


fRG: WF, Luo, Pawlowski, Rennecke, Yin, arXiv: 2308.15508

- Note that the ripples of CEP are far away from the critical region characterized by the universal scaling properties, e.g., the critical slowing down.
- But, the information of CEP, such as its location and properties, etc., is still encoded in the ripples.

Comparison to BES-II

Net baryon (proton) number Kurtosis:



- **In comparison to BES-I, BES-II results are better consistent with the theoretical prediction.**
- **Experimental results in the energy regime of fixed-target experiments, i.e. 3 GeV $\lesssim \sqrt{s_{\rm NN}} \lesssim$ 7.7 GeV, are now very important!! It will finally tell us whether there is a CEP.**

Magnetic equation of state

- The magnetic equation of state (EoS) is obtained via the chiral condensate:

$$\Delta_q = m_q \frac{\partial \Omega\big(T; m_q(T)\big)}{\partial m_q} = m_q \frac{T}{V} \int_x \big\langle \bar{q}(x)\, q(x) \big\rangle$$

- The chiral properties of the magnetic EoS are encoded in the magnetic susceptibility:

$$\chi_M = -\frac{\partial \bar{\Delta}_l}{\partial m_l}, \qquad \text{with} \qquad \bar{\Delta}_l = \frac{\Delta_l}{m_l}$$

- In the critical region, the magnetic EoS can be expressed as a universal scaling function $f_G(z)$ through

$$\bar{\Delta}_l = m_l^{1/\delta} f_G(z)$$

with

$$z = t\, m_l^{-1/\beta\delta}, \qquad \text{and} \qquad t = (T - T_c)/T_c$$

z is the scaling variable and t is the reduced temperature.

- The pseudo-critical temperature T_{pc}, which is defined through the peak location of χ_M, is readily obtained from the scaling function as

$$T_{pc}(m_\pi) \approx T_c + c\, m_\pi^p, \qquad \text{with} \qquad p = 2/(\beta\delta)$$

Critical exponent in fRG for 3d-O(4):

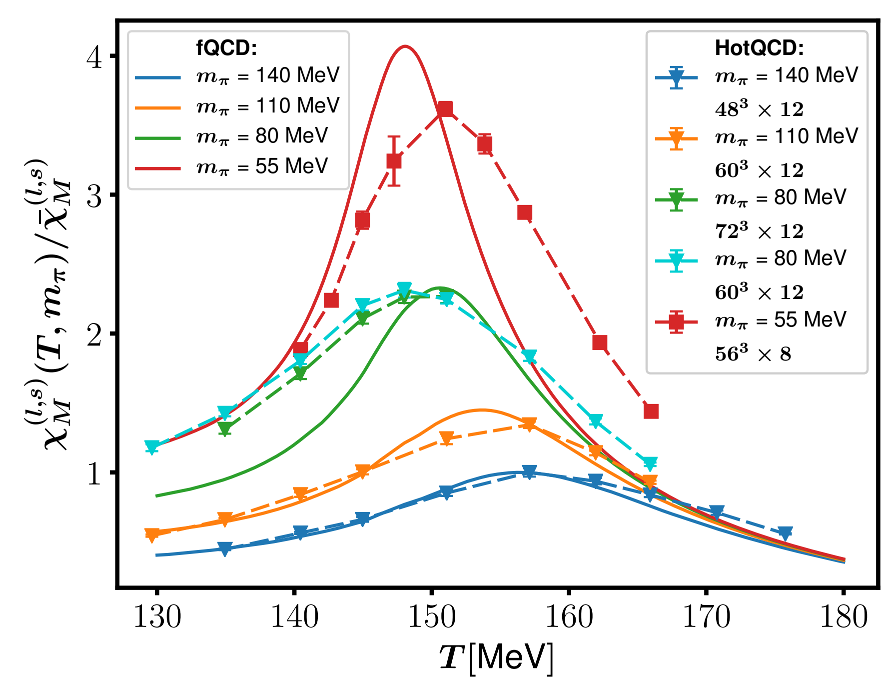
$$\beta = 0.405\,, \quad \delta = 4.784\,, \quad \theta_H = 0.272\,,$$

obtained from the fixed-point equation for the Wilson-Fisher fixed point, which leads us $p_{\mathrm{fRG}} = 1.03$

Critical exponent in mean field:

$$\beta_{\mathrm{MF}} = 1/2\,, \quad \delta_{\mathrm{MF}} = 3\,,$$

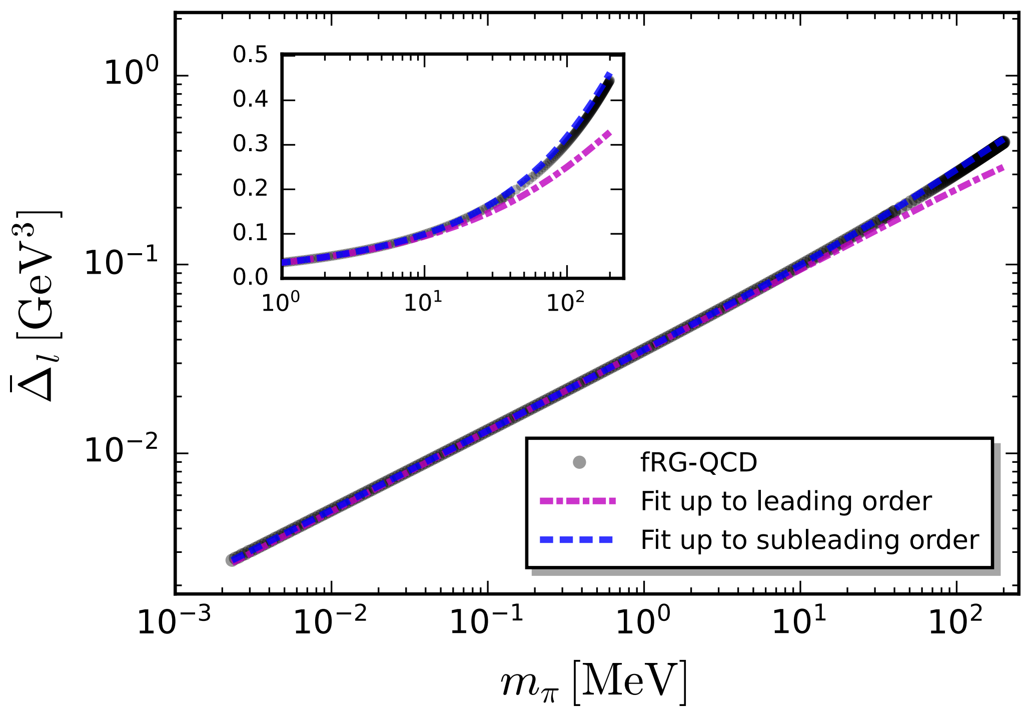
thus, one has $p_{\mathrm{MF}} = 4/3$



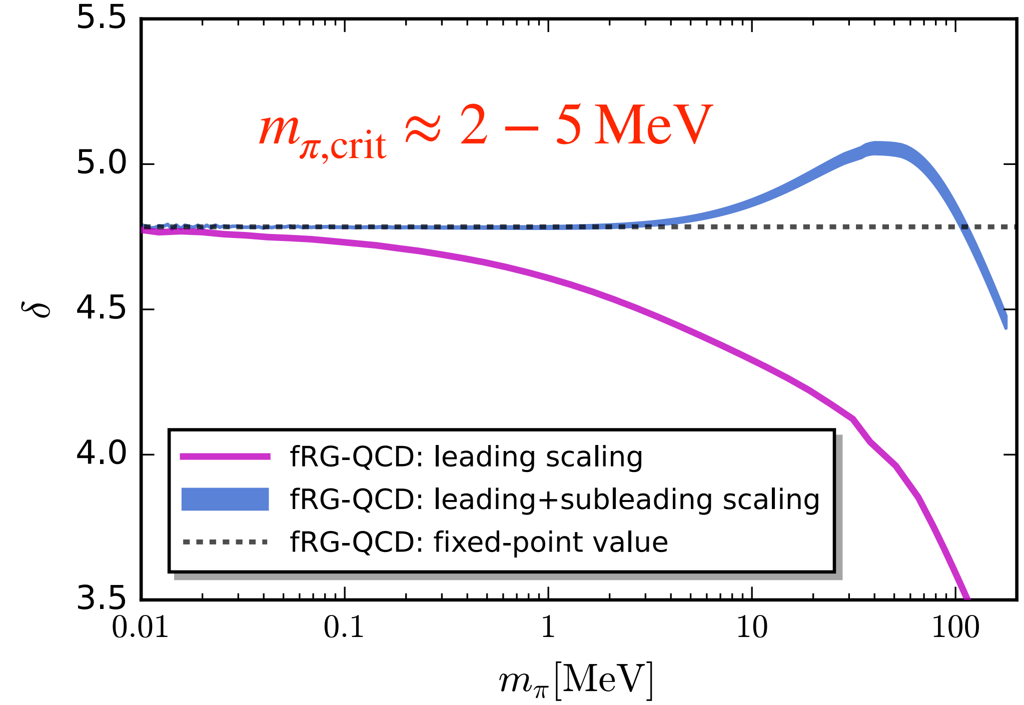
Braun, WF, Pawlowski, Rennecke, Rosenblüh, Yin, *PRD* 102 (2020), 056010.

Critical region in QCD

Scaling in the external field:

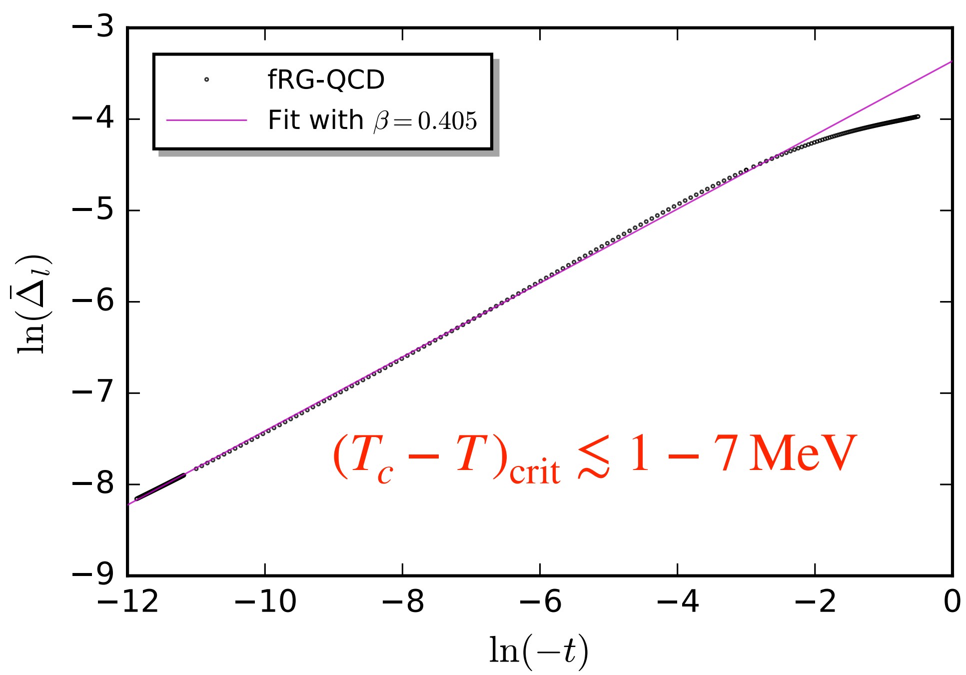


Critical exponent δ:



$$\bar{\Delta}_l^{(\mathrm{crit})}(m_\pi) = B_c\, m_\pi^{2/\delta}\left[1 + a_m m_\pi^{2\theta_H}\right]$$

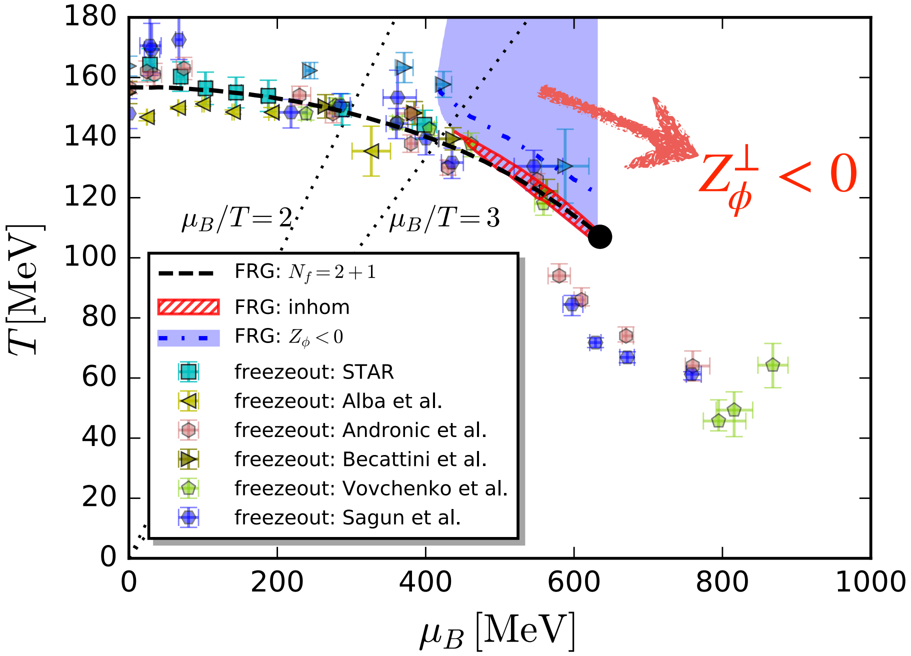
Scaling in the temperature:



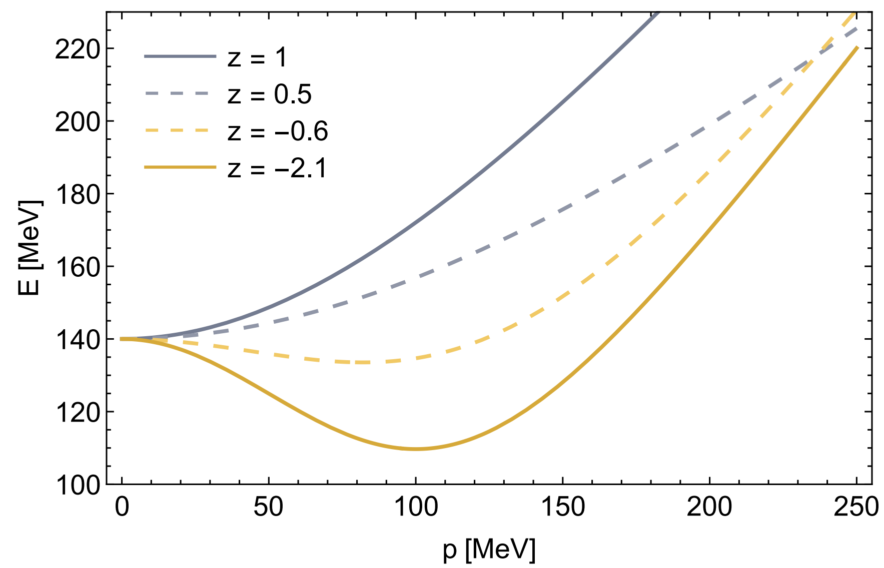
- **QCD at physical light quark mass is far away from the critical region.**
- **The scaling behavior is observed for the first time in the calculations of first-principles QCD.**

Braun, Chen, WF, Gao, Huang, Ihssen, Pawlowski, Rennecke, Sattler, Tan, Wen, and Yin, arXiv:2310.19853.

Moat regime in QCD phase diagam



WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032



Pisarski, Rennecke, *PRL* 127 (2021) 152302;
Rennecke, Pisarski, *PoS* CPOD2021 (2022);
Rennecke, Pisarski, Rischke, *PRD* 107 (2023) 116011

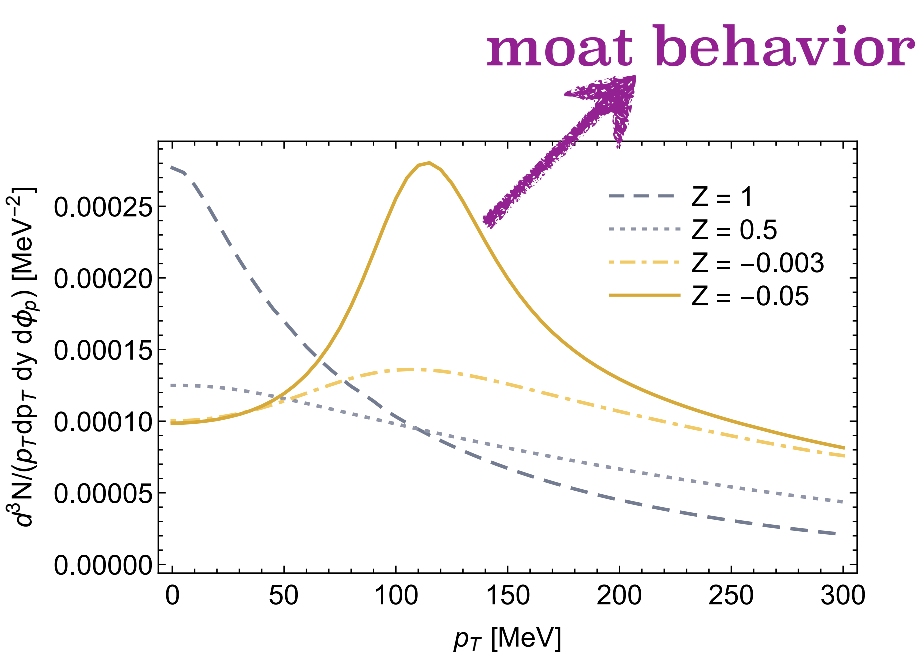
Mesonic two-point correlation function:

$$\Gamma^{(2)}_{\phi\phi}(p) = \left[Z^{\parallel}_{\phi}(p_0, \mathbf{p})\, p_0^2 + Z^{\perp}_{\phi}(p_0, \mathbf{p})\, \mathbf{p}^2\right] + m_\phi^2$$

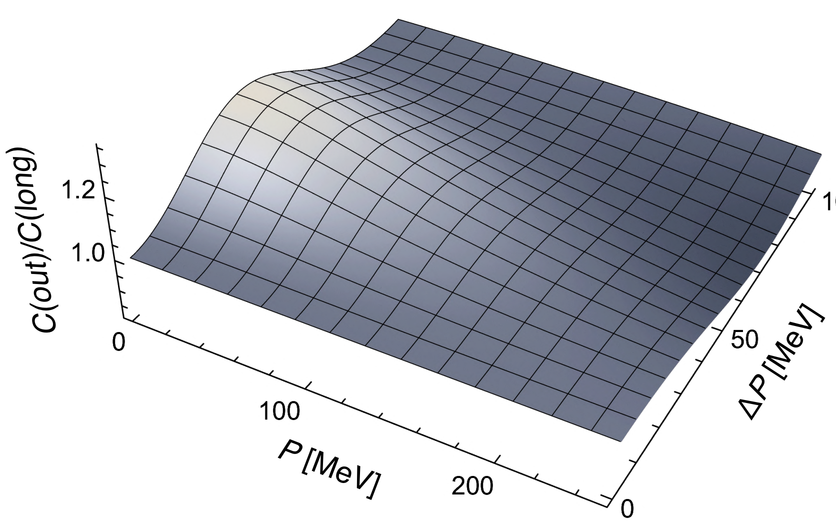
with

$$\Gamma^{(2)}_{\phi\phi,k} = \left.\frac{\delta^2 \Gamma_k[\Phi]}{\delta\phi\,\delta\phi}\right|_{\Phi=\Phi_{\rm EoM}}$$

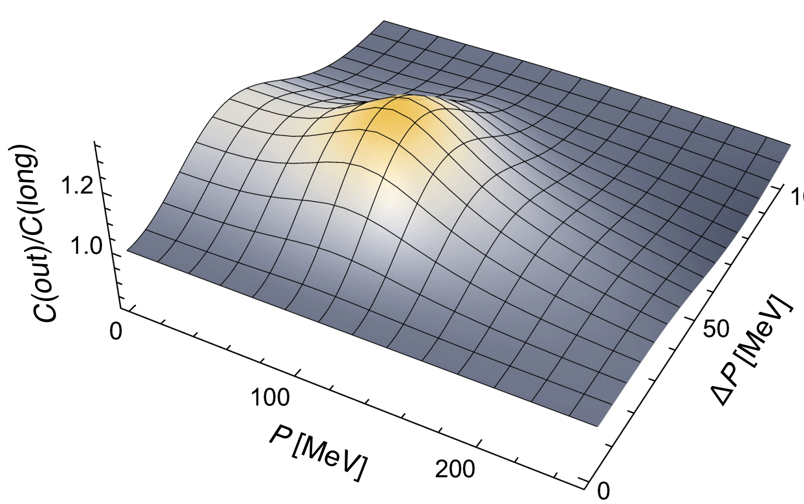
- **Transverse momentum spectrum of one particle:**



- **Two-particle correlation:**



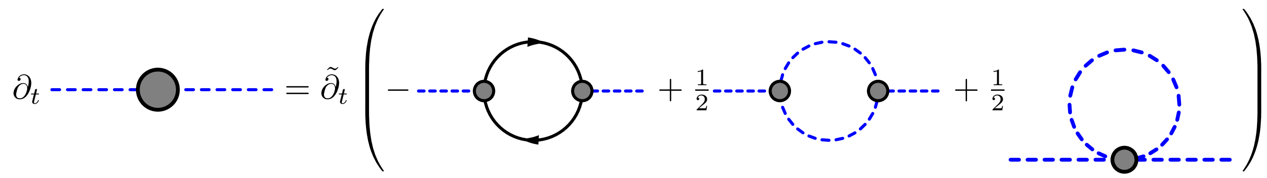
normal phase

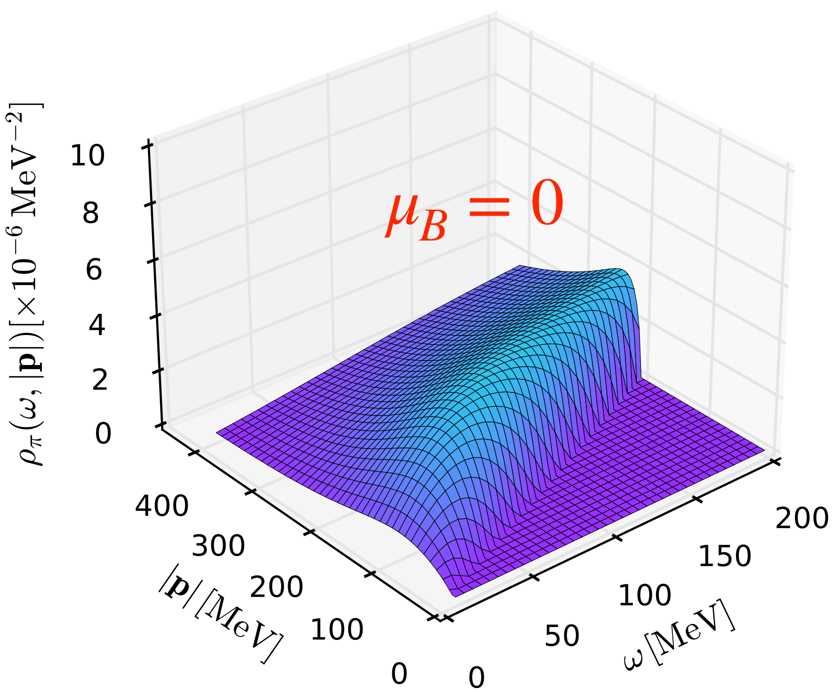


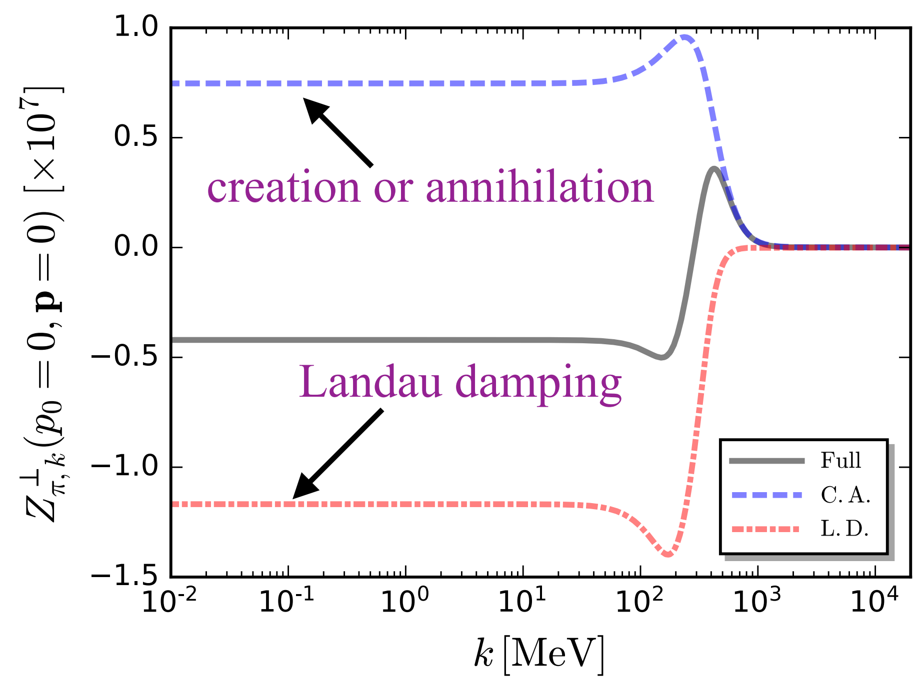
moat regime

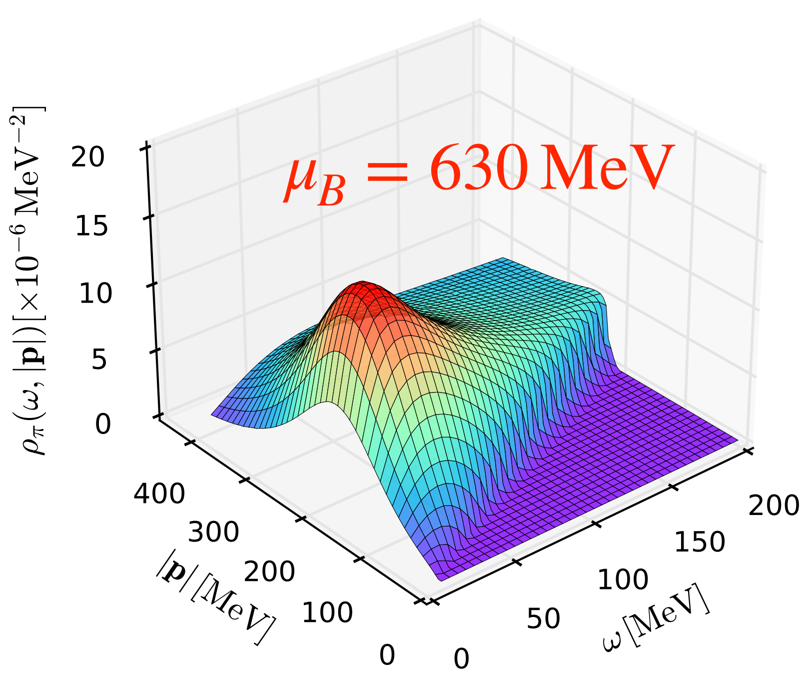
Spectral functions in moat regime

Flow equation for mesonic two-point functions:









WF, Pawlowski, Pisarski, Rennecke, Wen, Yin, in preparation.

- Moat regime is found to be resulted from Landau damping of quarks in thermal bath in the regime of large baryon chemical potential.

Relaxation dynamics of the critical mode

- Langevin dynamics of the critical mode:

$$Z_\phi^{(t)}\partial_t\sigma - Z_\phi^{(i)}\partial_i^2\sigma + U'(\sigma) = \xi$$

with the correlation of the Gaussian white noise

$$\langle \xi(t,\boldsymbol{x})\xi(t',\boldsymbol{x}')\rangle = 2\,Z_\phi^{(t)}\,T\,\delta(t-t')\delta(\boldsymbol{x}-\boldsymbol{x}')$$

- Inputs from first-principles functional QCD: WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

Effective potential:
$$U'(\sigma) = \left.\frac{\delta\Gamma[\Phi]}{\delta\sigma}\right|_{\substack{\sigma(x)=\sigma\\ \tilde{\Phi}=\tilde{\Phi}_{\mathrm{EoM}}}}$$

Spatial wave function:
$$Z_\phi^{(i)} = \left.\frac{\partial\Gamma^{(2)}_{\sigma\sigma}(p_0,\boldsymbol{p})}{\partial\boldsymbol{p}^2}\right|_{\substack{p_0=0\\ \boldsymbol{p}=0}}$$

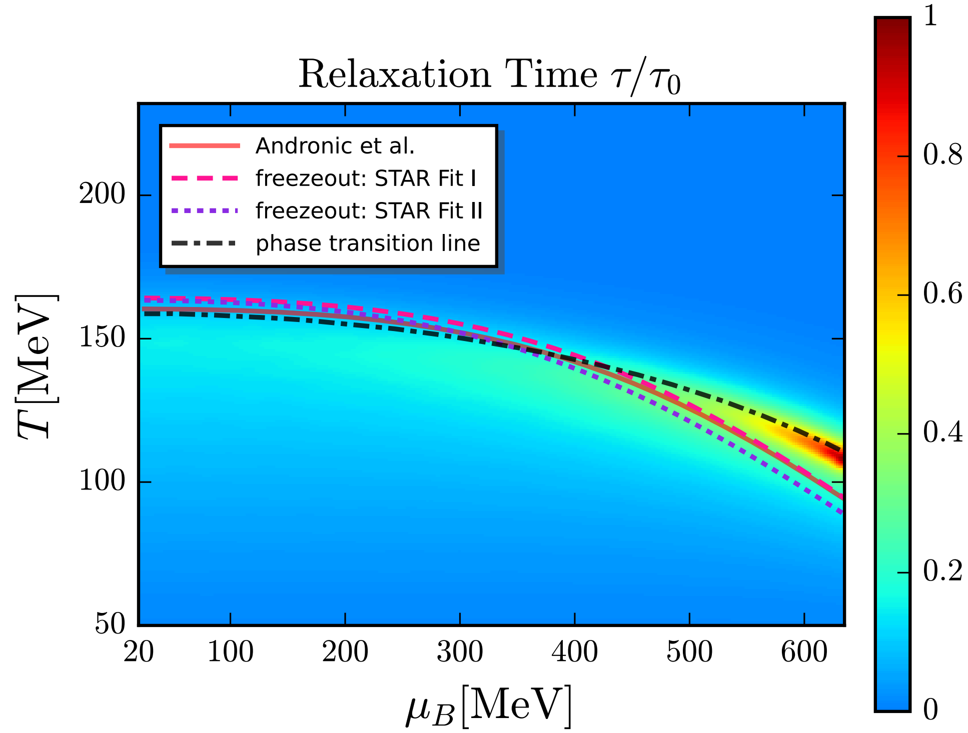
Temporal wave function:
$$Z_\phi^{(t)} = \lim_{|\boldsymbol{p}|\to 0}\lim_{\omega\to 0}\frac{\partial}{\partial\omega}\mathrm{Im}\,\Gamma^{(2)}_{\sigma\sigma,\mathrm{R}}(\omega,\boldsymbol{p})$$

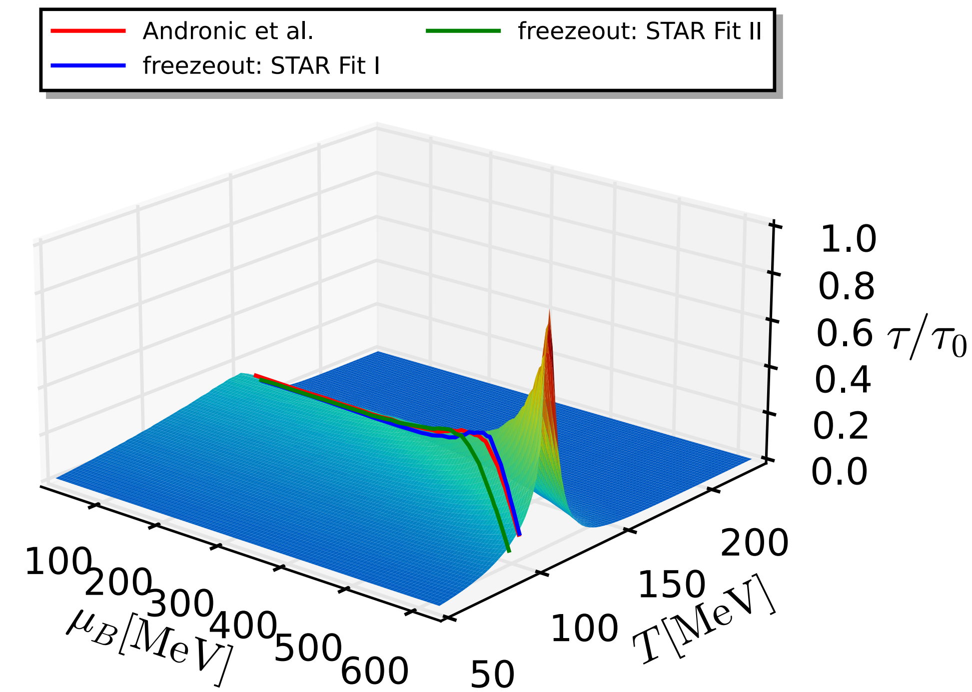
with

$$\Gamma^{(2)}_{\sigma\sigma,\mathrm{R}}(\omega,\boldsymbol{p}) = \lim_{\epsilon\to 0^+}\Gamma^{(2)}_{\sigma\sigma}\big(p_0=-\mathrm{i}(\omega+\mathrm{i}\epsilon),\boldsymbol{p}\big)$$

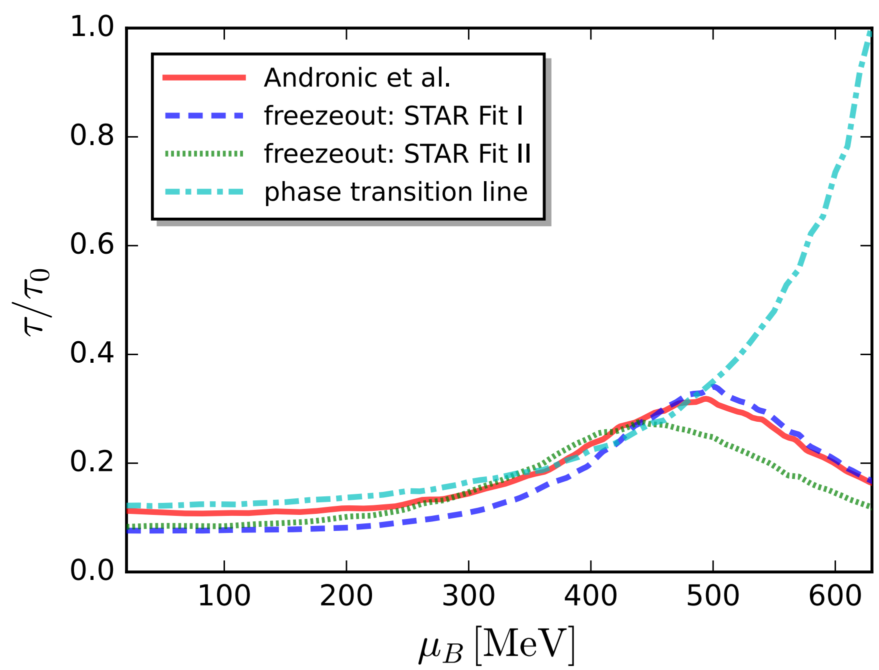
Relaxation time in QCD phase diagram

Relaxation time:





Relaxation time at the freezeout :

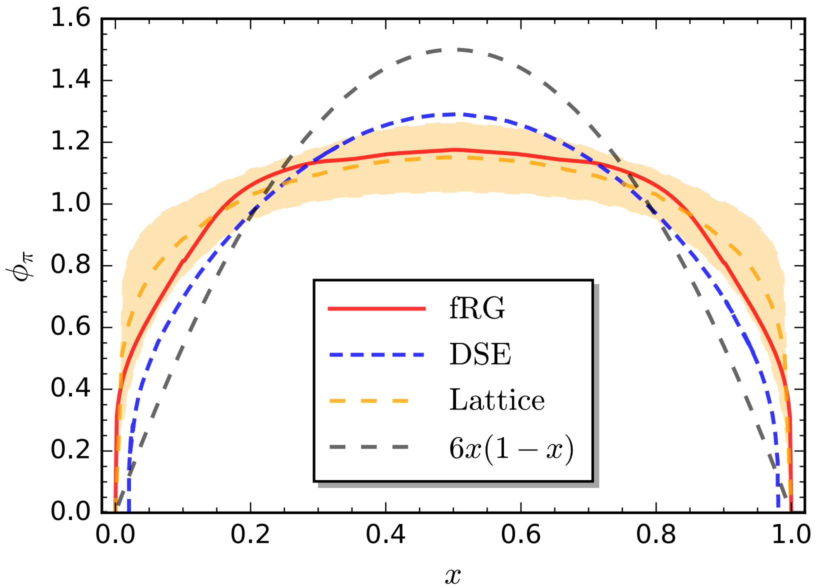


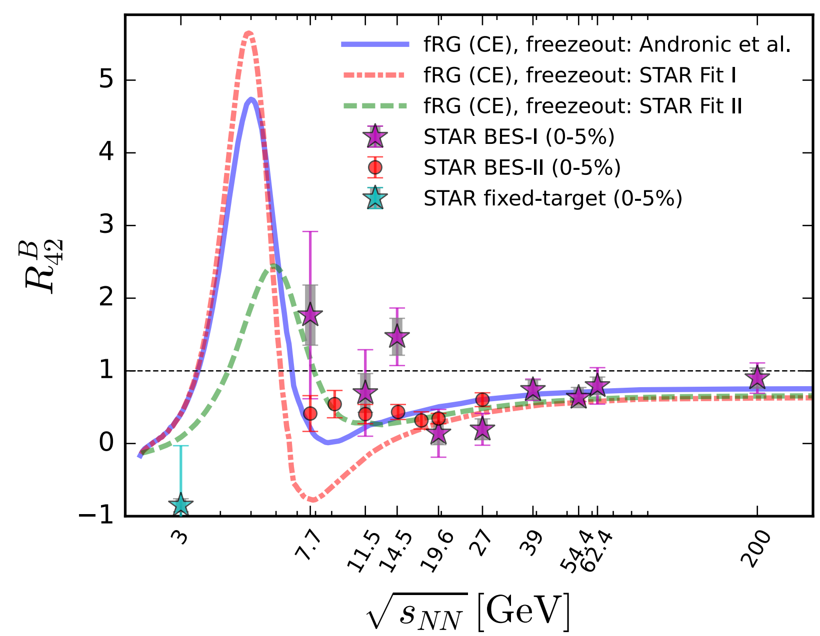
Tan, Yin, Chen, Huang, WF, in preparation

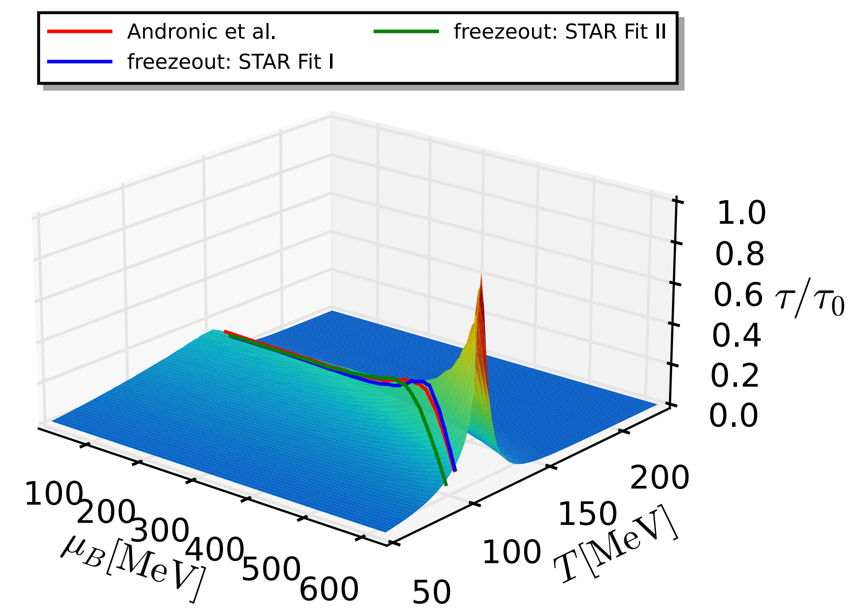
See also:
M. Bluhm *et al.*, *NPA* 982 (2019) 871

- Relaxation time drops quickly once the system is away from the critical regime.

Summary and outlook



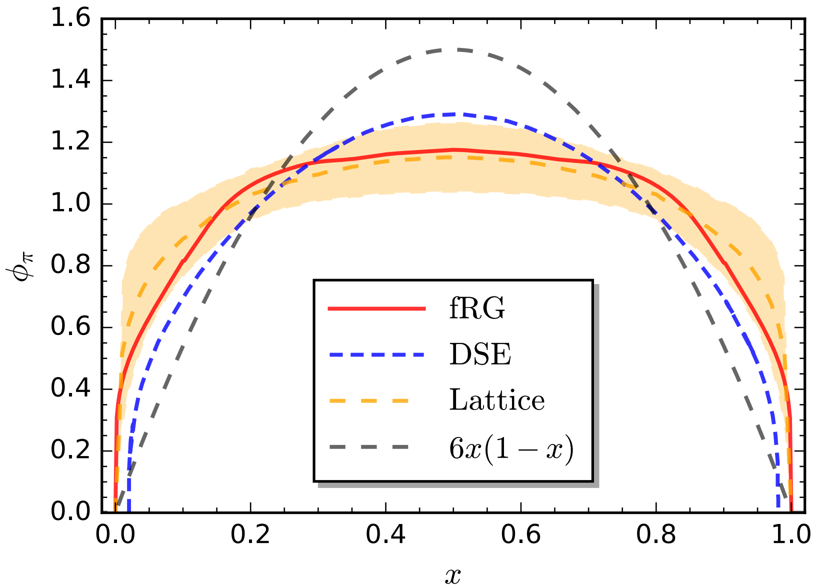


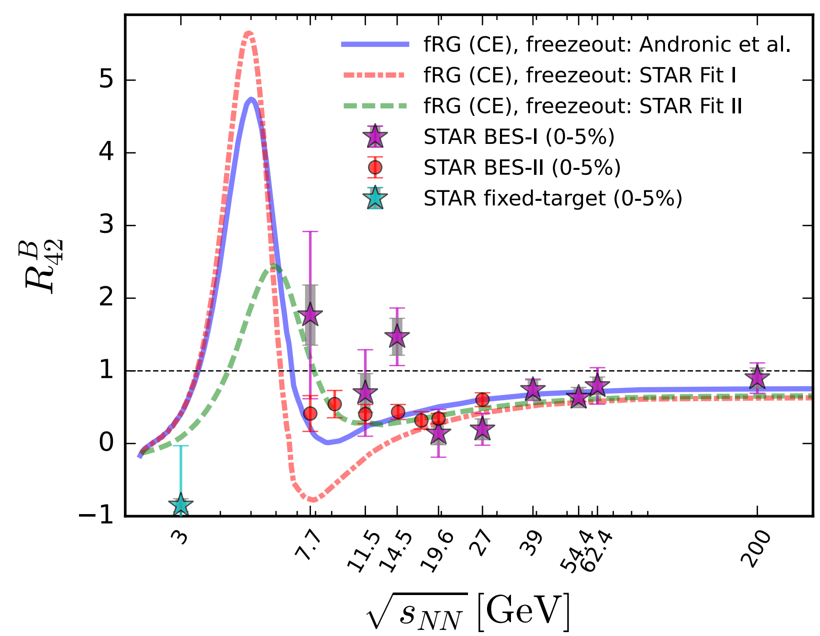


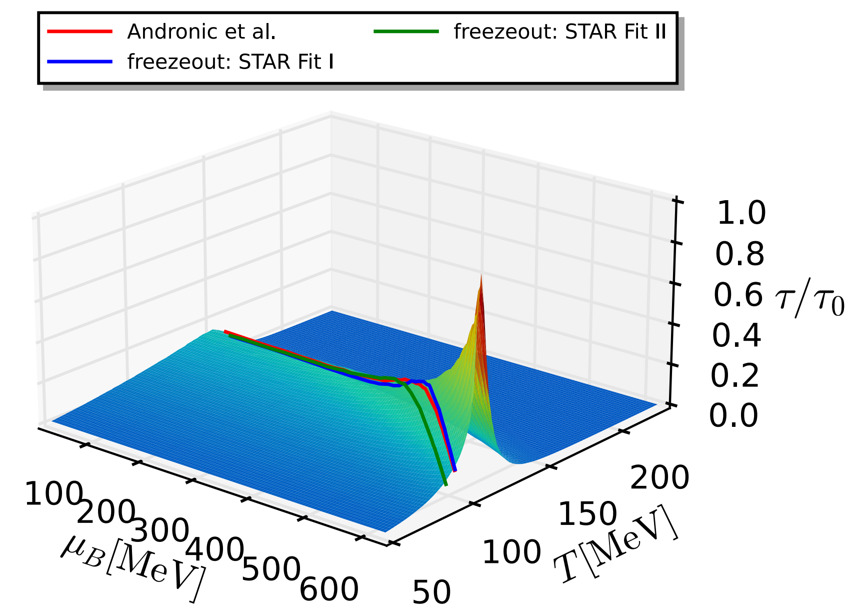
- ★ Functional renormalization group provides us with a powerful approach to study nonperturbative problems, e.g., hadron structure, QCD phase diagram, real-time dynamics, from first-principles QCD.

- ★ There are also challenges and problems to be solved: larger Pz, error controls at large baryon densities, better analytic continuations, etc.

Summary and outlook







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Thank you very much for your attentions!

Backup

Four-quark vertices

- 4-quark effective action:

$$\Gamma_{4q,k} = -\int \frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta(p_1+p_2+p_3+p_4)$$
$$\times \sum_\alpha \lambda_\alpha(\boldsymbol{p})\, \mathcal{T}^{(\alpha)}_{ijlm}(\boldsymbol{p})\, \bar{q}_i(p_1) q_j(p_2) \bar{q}_l(p_3) q_m(p_4)\,,$$

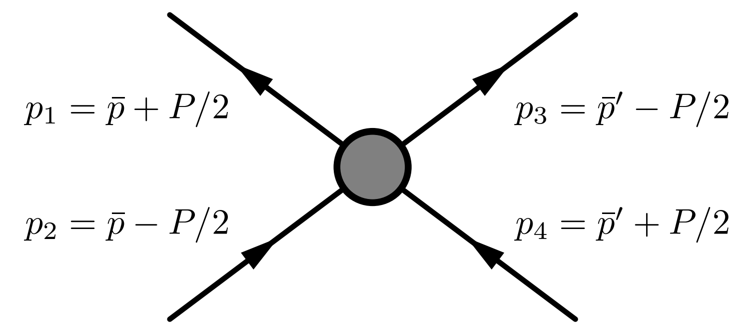
With $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$, $\mathcal{T}^{(\alpha)}(\boldsymbol{p})$ is comprised of 512 tensors. Eichmann, *PRD* 84 (2011) 014014

A basis of the lowest momentum-independent order includes ten elements

$$\alpha \in \Big\{ \sigma\,, \pi\,, a\,, \eta\,, (V \pm A)\,, (V-A)^{\text{adj}}\,,$$
$$(S \pm P)^{\text{adj}}_{-}\,, (S+P)^{\text{adj}}_{+} \Big\}\,,$$

- 4-quark vertex:

$$\Gamma^{(4)}_{\bar{q}_i q_j \bar{q}_l q_m}(\boldsymbol{p}) = \frac{\delta^4 \Gamma_k[q,\bar{q}]}{\delta\bar{q}_i(p_1)\delta q_j(p_2)\delta\bar{q}_l(p_3)\delta q_m(p_4)}$$
$$= -4\,(2\pi)^4\delta(p_1+p_2+p_3+p_4)$$
$$\times \sum_\alpha \Big[\lambda^+_\alpha(\boldsymbol{p})\mathcal{T}^{(\alpha^-)}_{ijlm} + \lambda^-_\alpha(\boldsymbol{p})\mathcal{T}^{(\alpha^+)}_{ijlm}\Big]$$



$$= -\Gamma^{(4)}_{\bar{q}_i q_j \bar{q}_l q_m}(p_1, p_2, p_3, p_4)$$

where we have used 4-quark dressings and tensor structures with definite symmetries, viz.,

$$\lambda^\pm_\alpha(\boldsymbol{p}) \equiv \frac{1}{2}\Big[\lambda_\alpha(p_1,p_2,p_3,p_4) \pm \lambda_\alpha(p_3,p_2,p_1,p_4)\Big]\,,$$

and

$$\mathcal{T}^{(\alpha^\pm)}_{ijlm} \equiv \frac{1}{2}\Big(\mathcal{T}^{(\alpha)}_{ijlm} \pm \mathcal{T}^{(\alpha)}_{ljim}\Big)$$

with the symmetry relations

$$\lambda^+_\alpha(p_1,p_2,p_3,p_4) = \lambda^+_\alpha(p_3,p_2,p_1,p_4)$$
$$= \lambda^+_\alpha(p_1,p_4,p_3,p_2) = \lambda^+_\alpha(p_3,p_4,p_1,p_2)\,,$$
$$\lambda^-_\alpha(p_1,p_2,p_3,p_4) = -\,\lambda^-_\alpha(p_3,p_2,p_1,p_4)$$
$$= -\lambda^-_\alpha(p_1,p_4,p_3,p_2) = \lambda^-_\alpha(p_3,p_4,p_1,p_2)$$

and similar relations for the tensors.

s, t, u-channel truncation

- s, t, u-channel approximation for 4-quark vertices:

$$\lambda_\alpha^\pm(p_1,p_2,p_3,p_4) = \lambda_\alpha^\pm(s,t,u) + \Delta\lambda_\alpha^\pm(p_1,p_2,p_3,p_4) \approx \lambda_\alpha^\pm(s,t,u)$$

with

$$t = (p_1-p_2)^2 = P^2\,,$$
$$u = (p_1-p_4)^2 = (\bar p-\bar p')^2\,,$$
$$s = (p_1+p_3)^2 = (\bar p+\bar p')^2$$

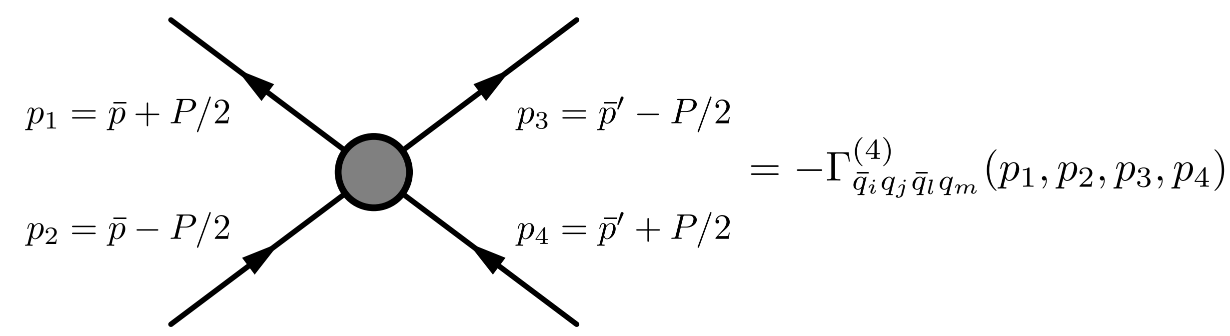
- We choose a subspace of the full momentum of 4-quark vertices as follows

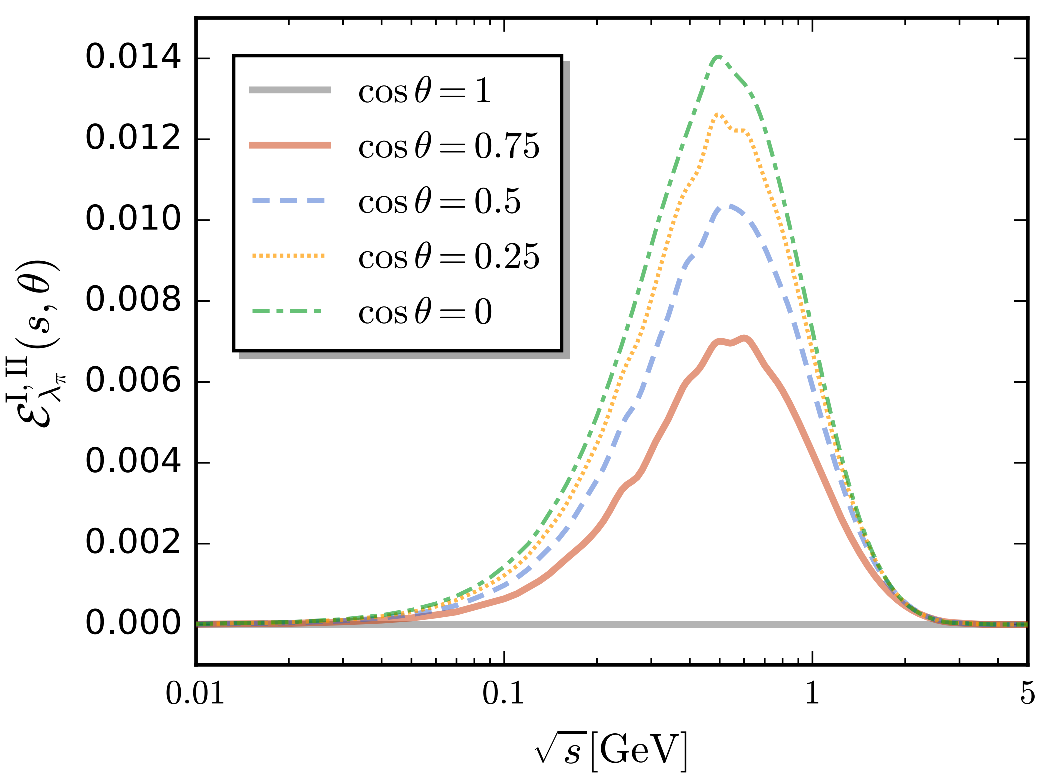
$$P_\mu = \sqrt{P^2}\,\Big(1,0,0,0\Big)\,,$$
$$\bar p_\mu = \sqrt{p^2}\,\Big(1,0,0,0\Big)\,,$$
$$\bar p'_\mu = \sqrt{p^2}\,\Big(\cos\theta,\ \sin\theta, 0, 0\Big)$$

one is led to

$$t = P^2\,, \quad u = 2p^2(1-\cos\theta)\,, \quad s = 2p^2(1+\cos\theta)$$

Here, $\{\sqrt{P^2},\sqrt{p^2},\cos\theta\}$ is in one-by-one correspondence with respect to $\{t,u,s\}$



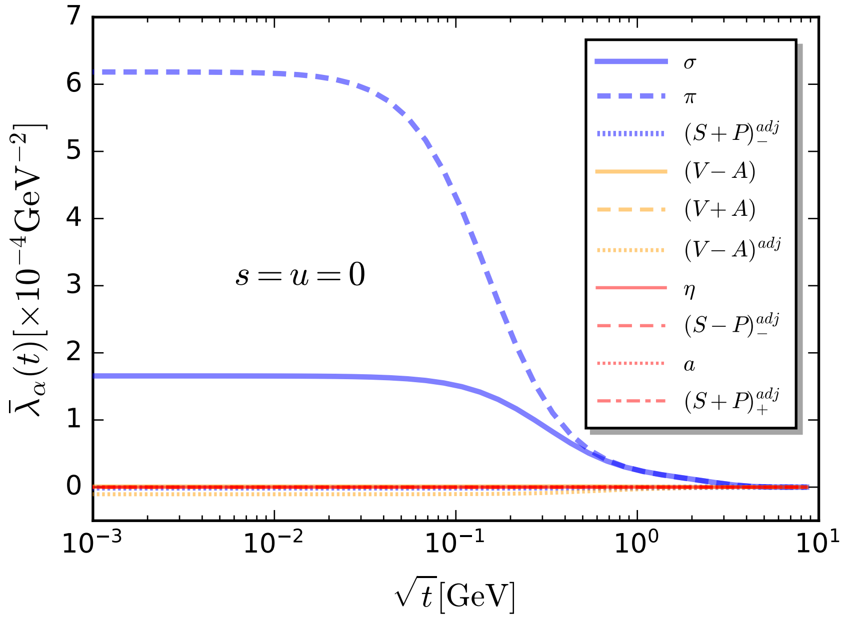


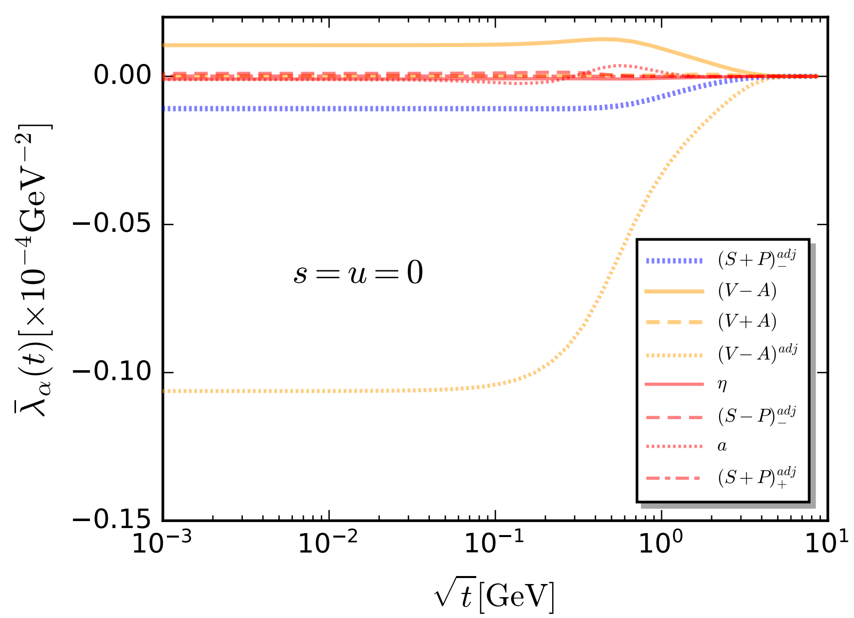
WF, Huang, Pawlowski, Tan, arXiv:2401.07638

The error for the truncation is smaller than 1.5%

Four-quark dressings

Dressings of different tensors:





Symmetry relations

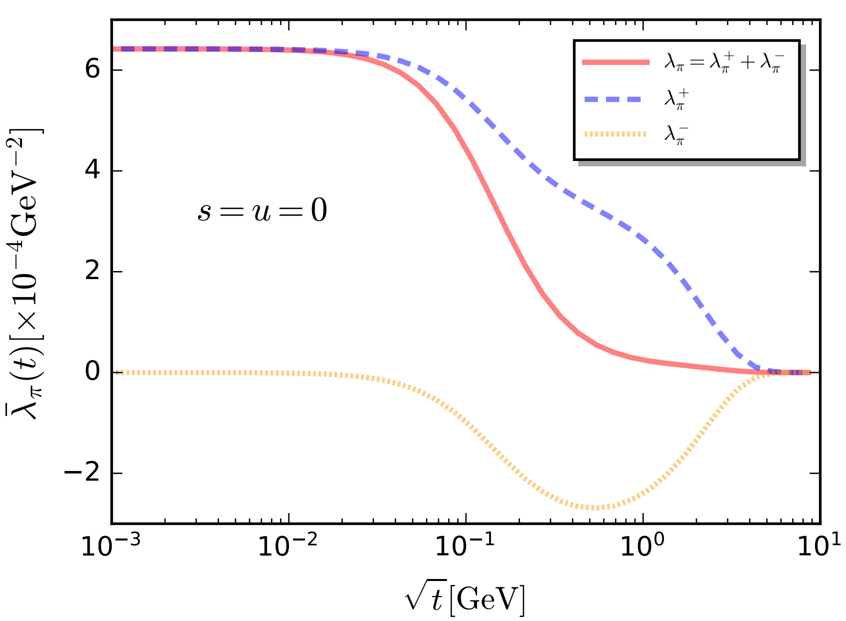
$$\lambda^+_\alpha(s,t,u) = \lambda^+_\alpha(s,u,t)\,,$$
$$\lambda^-_\alpha(s,t,u) = -\lambda^-_\alpha(s,u,t)$$

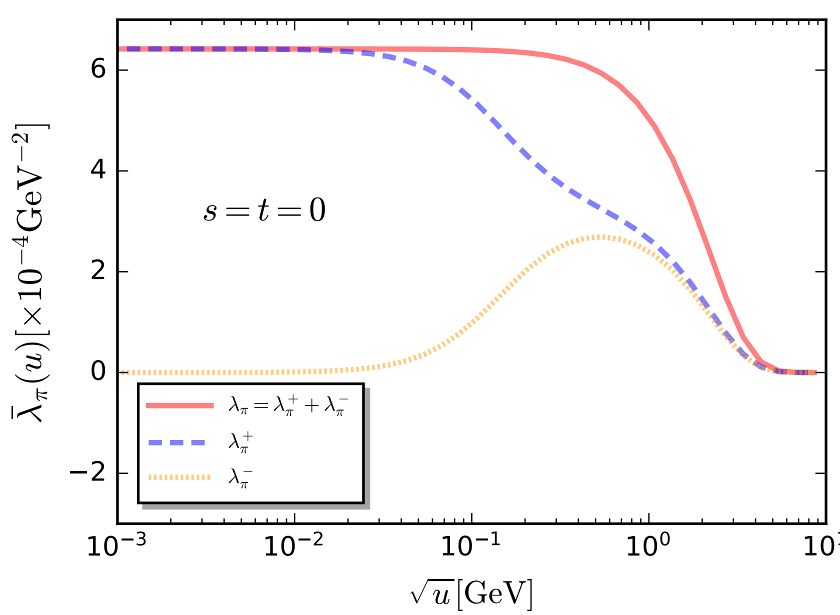
then

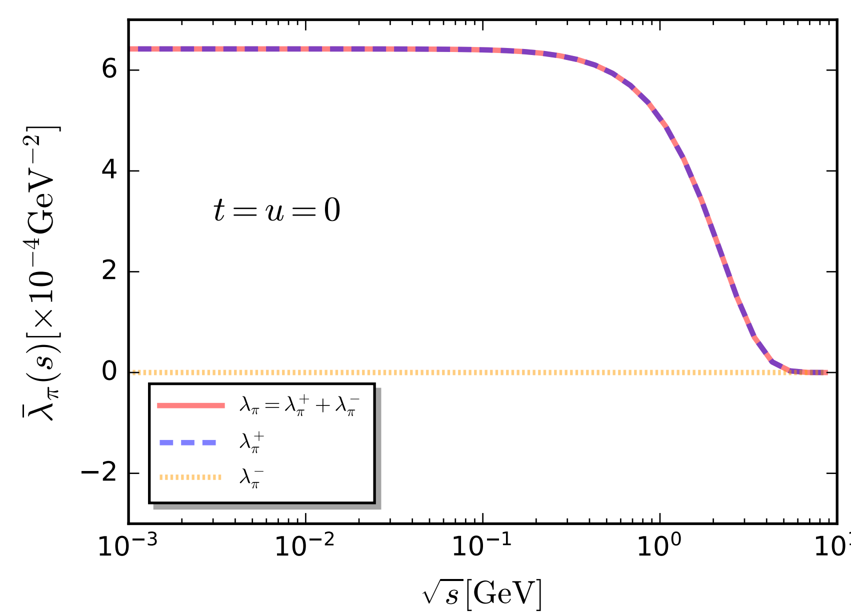
$$\lambda^+_\alpha(0,p^2,0) = \lambda^+_\alpha(0,0,p^2)\,,$$
$$\lambda^-_\alpha(0,p^2,0) = -\lambda^-_\alpha(0,0,p^2)$$

Pion channel:

WF, Huang, Pawlowski, Tan, arXiv:2401.07638







Pion decay constant

The pion weak decay constant is defined as

$$\langle 0 | J^a_{5\mu}(x) | \pi^b \rangle = i P_\mu f_\pi \delta^{ab}$$

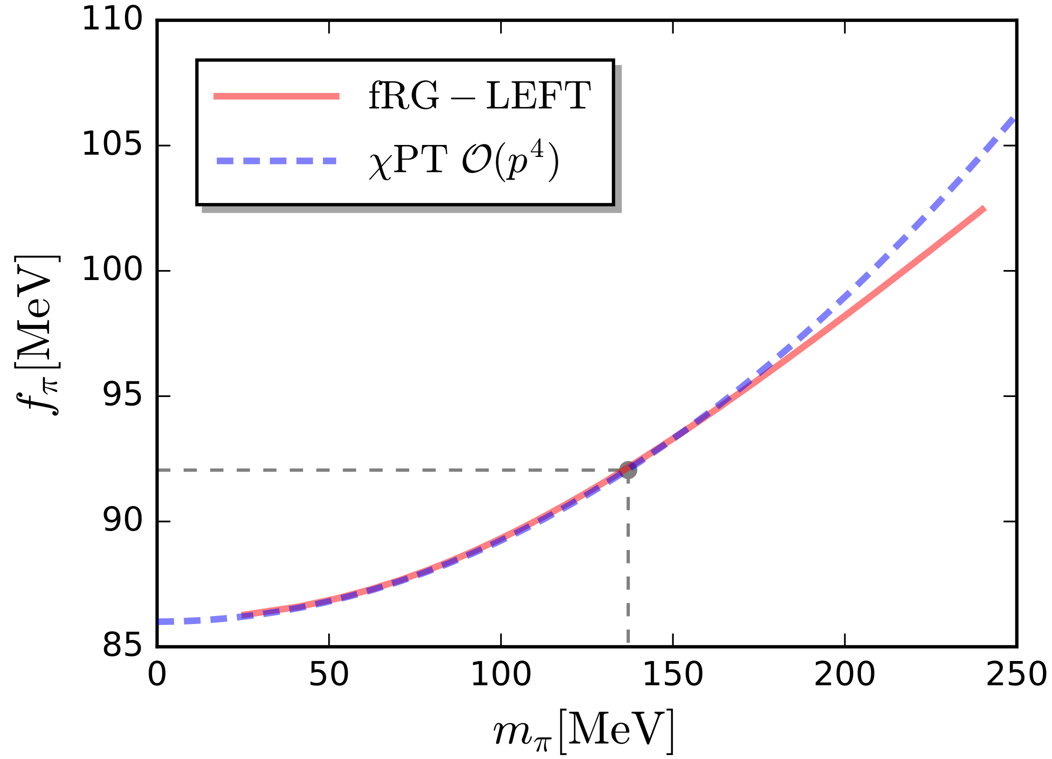
where the left hand side reads

$$\langle 0 | J^a_{5\mu}(x) | \pi^b \rangle$$
$$= \int \frac{d^4q}{(2\pi)^4} \mathrm{Tr} \left[\gamma_\mu \gamma_5 T^a \, \bar{G}_q(q+P) \, \bar{h}_\pi(q) \, \gamma_5 T^b \, \bar{G}_q(q) \right] ,$$

then

$$f_\pi = 2 N_c \int \frac{d^4q}{(2\pi)^4} \frac{\bar{h}_\pi(q) M_q(q)}{\left[q^2 + M_q^2(q)\right]^2}$$

up to leading order in powers of $P^2 = -m_\pi^2$



fRG: WF, Huang, Pawlowski, Tan, arXiv:2401.07638

chiPT: Gasser and Leutwyler, *Annals Phys*. 158 (1984) 142

Contour of k_0 integral

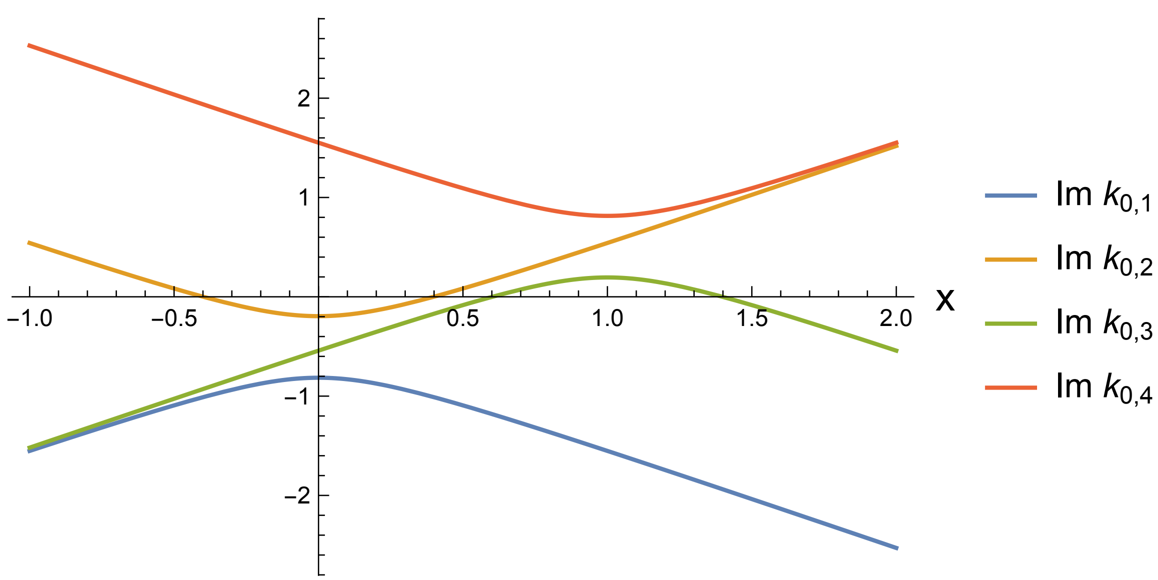
- **Poles of two quark propagators**:

$$k_{0,1} = i\left[-\sqrt{k_\perp^2 + (xP_z)^2 + M_q^2(k_+^2)} - \sqrt{P_z^2 + m_\pi^2}/2 \right],$$

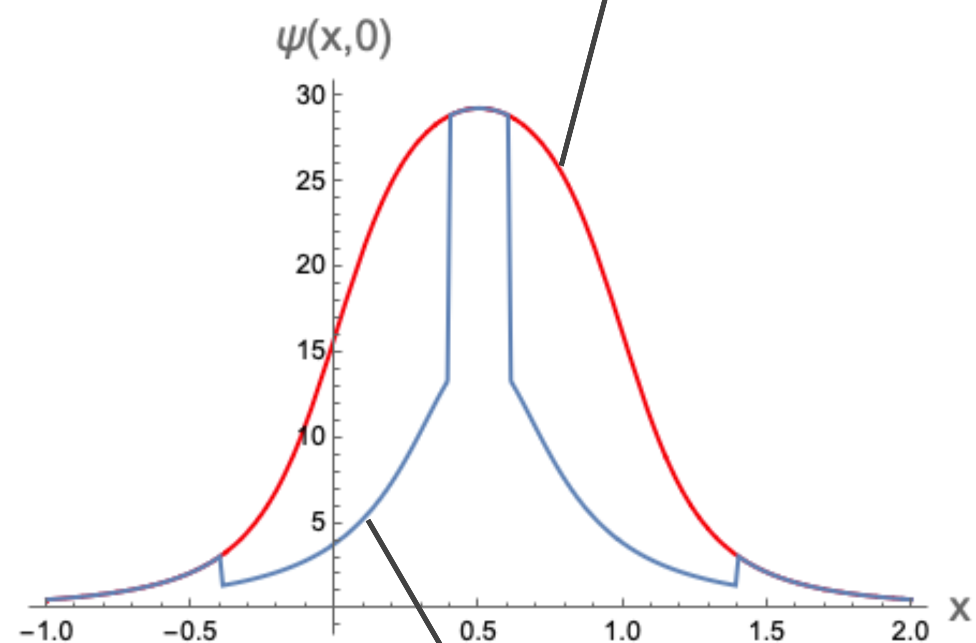
$$k_{0,2} = i\left[\sqrt{k_\perp^2 + (xP_z)^2 + M_q^2(k_+^2)} - \sqrt{P_z^2 + m_\pi^2}/2 \right],$$

$$k_{0,3} = i\left[-\sqrt{k_\perp^2 + (x-1)^2P_z^2 + M_q^2(k_-^2)} + \sqrt{P_z^2 + m_\pi^2}/2 \right],$$

$$k_{0,4} = i\left[\sqrt{k_\perp^2 + (x-1)^2P_z^2 + M_q^2(k_-^2)} + \sqrt{P_z^2 + m_\pi^2}/2 \right]$$



$$\int_{-\infty+(k_{0,2}+k_{0,3})/2}^{\infty+(k_{0,2}+k_{0,3})/2} dk_0$$



$$\int_{-\infty}^{\infty} dk_0$$

With the increase of P_z, $k_{0,2}$ or $k_{0,3}$ cross the x axis, one has to shift the integral of k_0 towards finite imaginary part, such that one can pick up the desired pair of poles, e.g., $k_{0,1}$ and $k_{0,3}$ or $k_{0,2}$ and $k_{0,4}$

Analytic continuation

- We use Taylor expansion to continue h_π, M_q, Z_q in the complex plane of k_0:

$$h_\pi(k^2, P^2, \cos\theta) = h_\pi(\bar{k}^2, P^2, \cos\theta) + \frac{\partial}{\partial k^2} h_\pi \Big|_{k^2=\bar{k}^2} k_0^2 + \cdots$$

and

$$M_q(k_+^2) = M_q(\bar{k}_+^2) + \frac{\partial}{\partial k_+^2} M_q \Big|_{k_+^2=\bar{k}_+^2} (k_0 + iE_\pi/2)^2 + \cdots$$

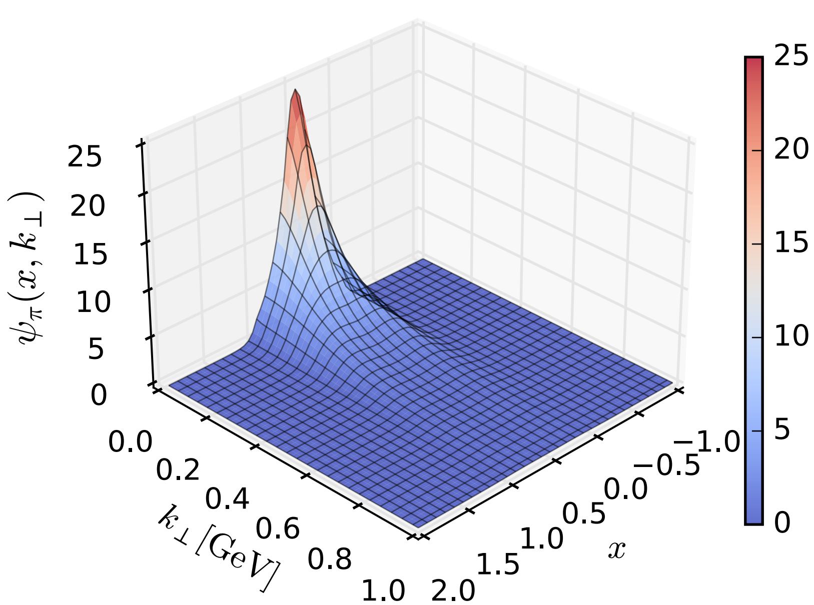
$$M_q(k_-^2) = M_q(\bar{k}_-^2) + \frac{\partial}{\partial k_-^2} M_q \Big|_{k_-^2=\bar{k}_-^2} (k_0 - iE_\pi/2)^2 + \cdots$$

$$k^2 = \bar{k}^2 + k_0^2$$
$$\bar{k}^2 = k_\perp^2 + (x - 1/2)^2 P_z^2$$

$$k_+^2 = \bar{k}_+^2 + (k_0 + iE_\pi/2)^2$$
$$\bar{k}_+^2 = k_\perp^2 + x^2 P_z^2$$

$$k_-^2 = \bar{k}_-^2 + (k_0 - iE_\pi/2)^2$$
$$\bar{k}_-^2 = k_\perp^2 + (x - 1)^2 P_z^2$$

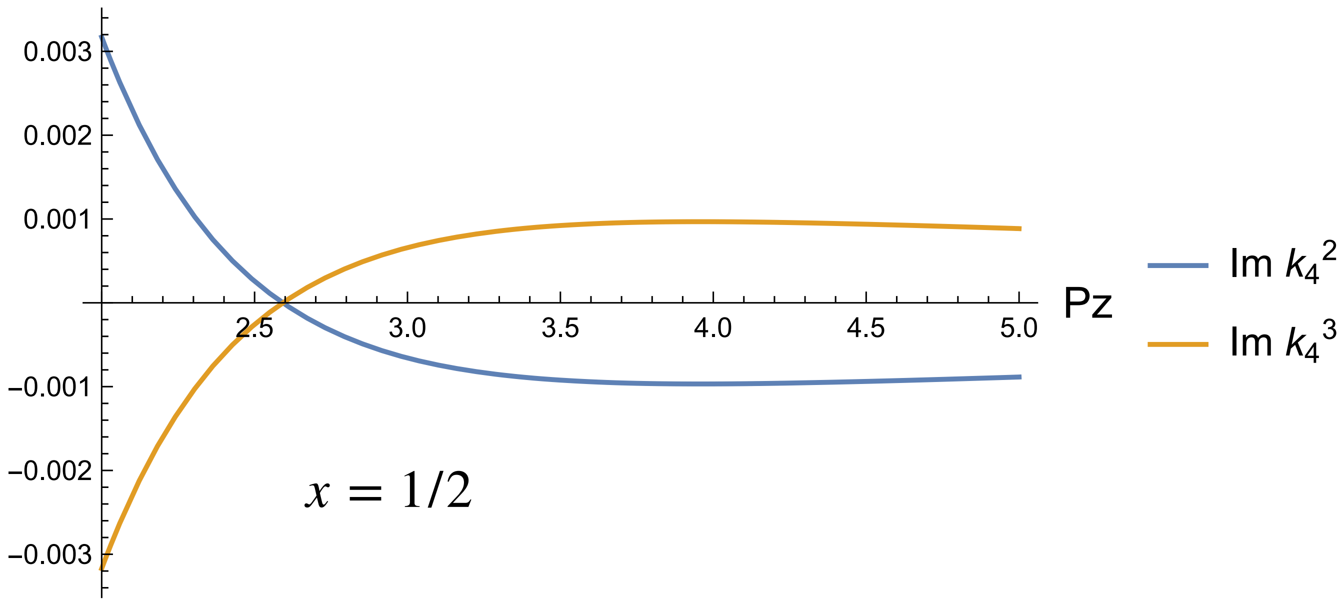
Pion wave function amplitudes:



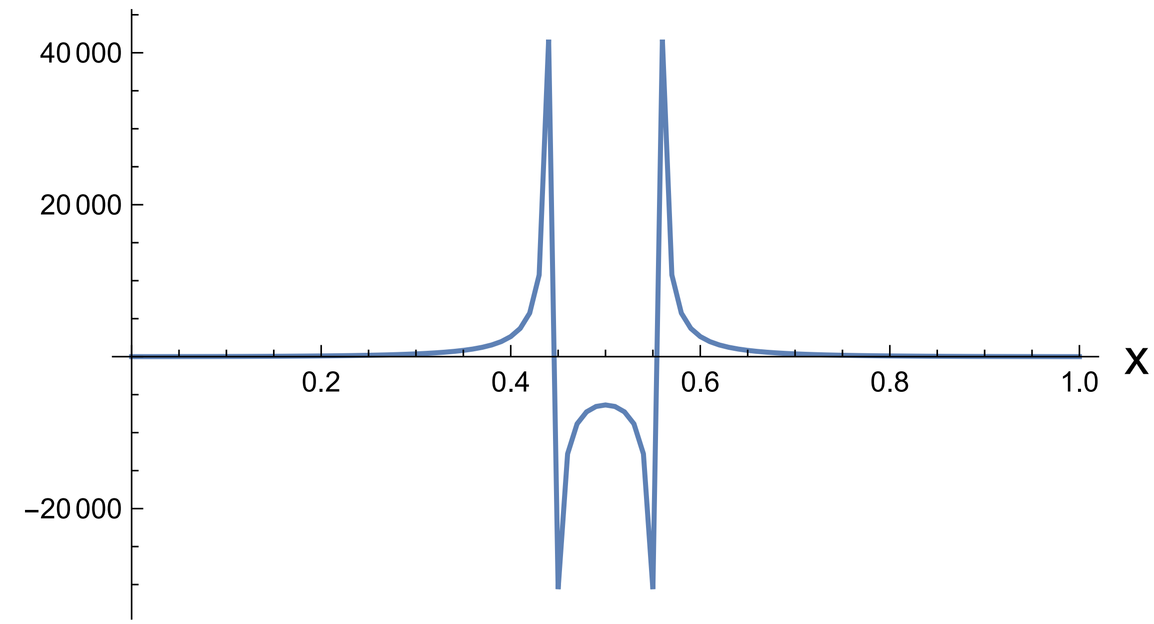
Chang, WF, Huang, Pawlowski, Zhang, in preparation

Larger P_z?

Pole crossing:



Wave function:

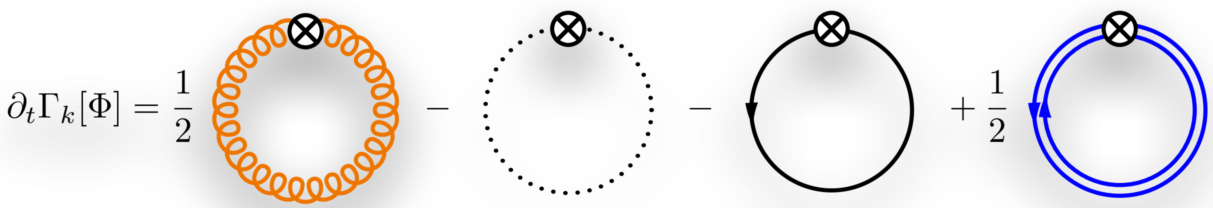


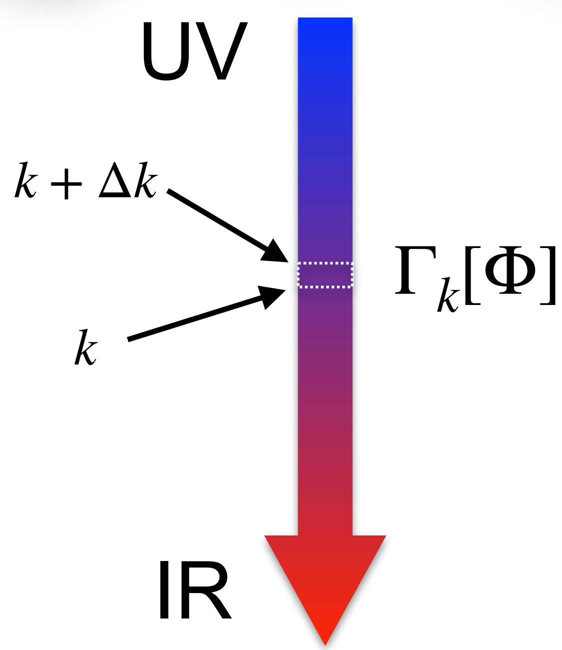
Poles of $k_{0,2}$ and $k_{0,3}$ interchange their potions, when $P_z \gtrsim 3.5$ GeV

fRG: Chang, WF, Huang, Pawlowski, Zhang, in preparation

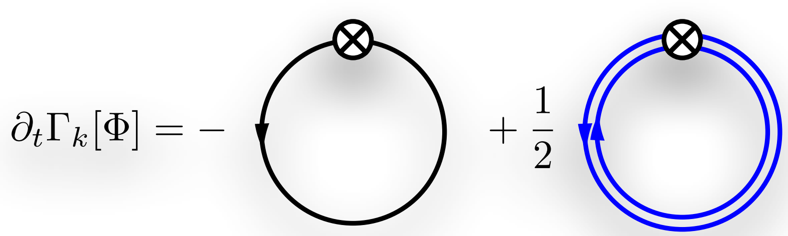
QCD-assisted LEFT

QCD flow equation:



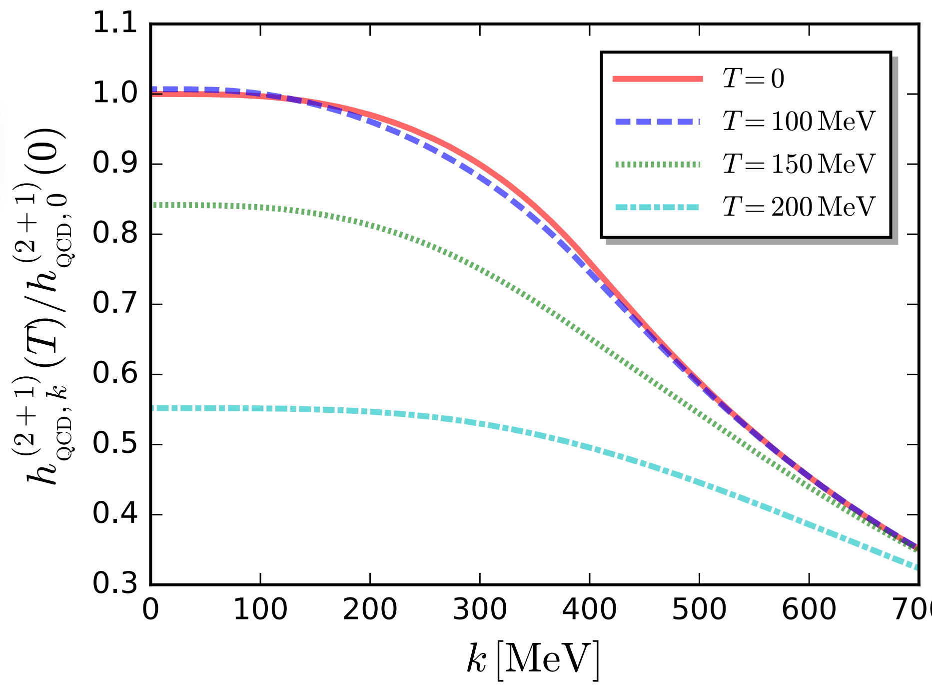


LEFT flow equation:



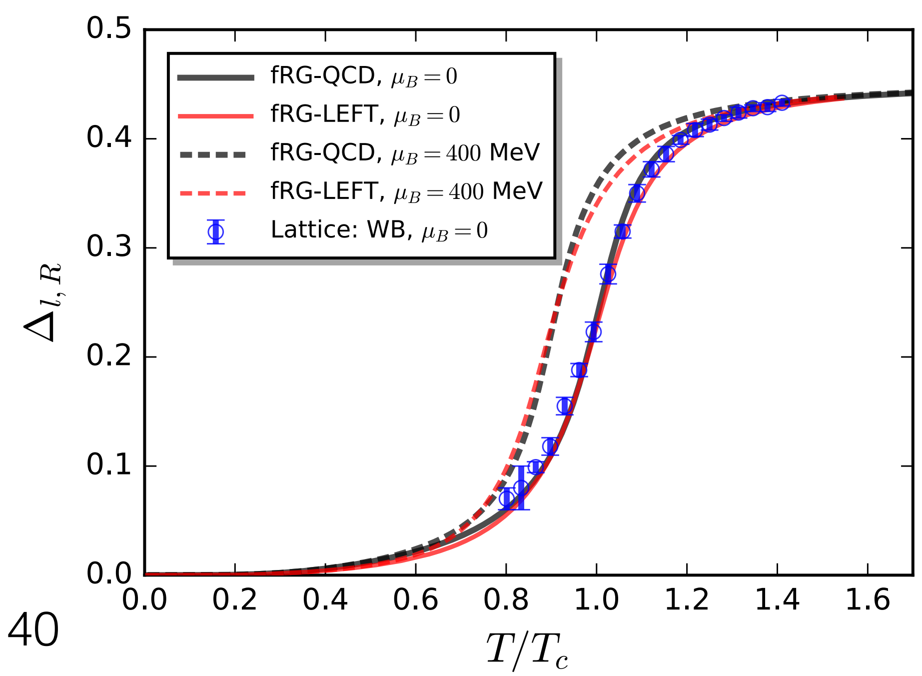
quark meson

- **Yukawa couplings obtained in QCD inputted in QCD-assisted LEFT**



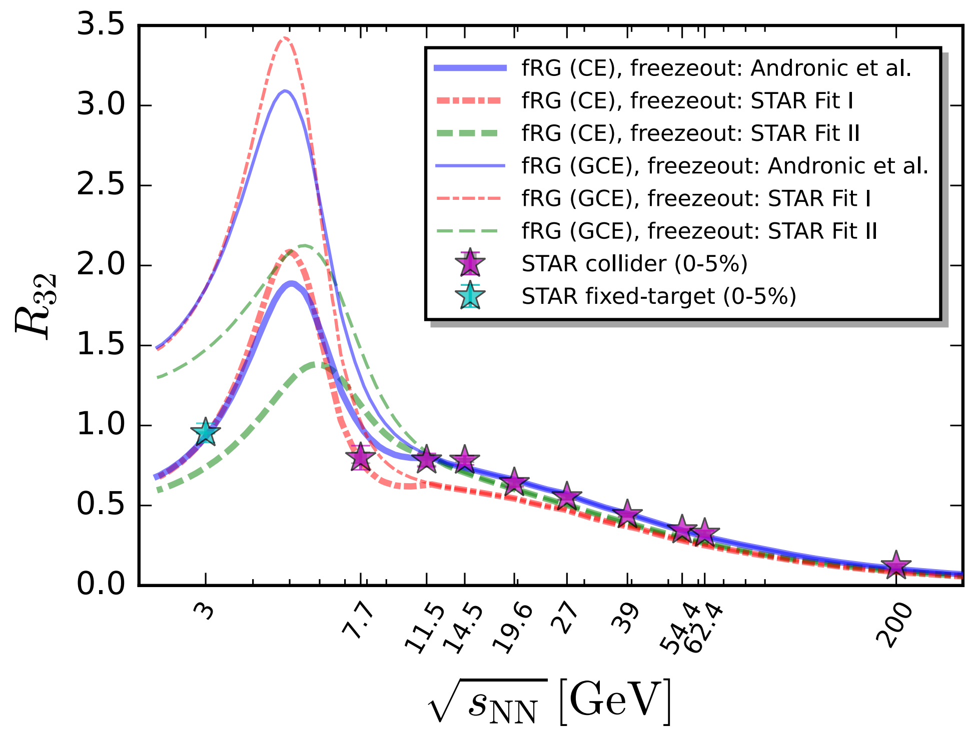
WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

- **Chiral condensates in QCD and QCD-assisted LEFT in agreement**

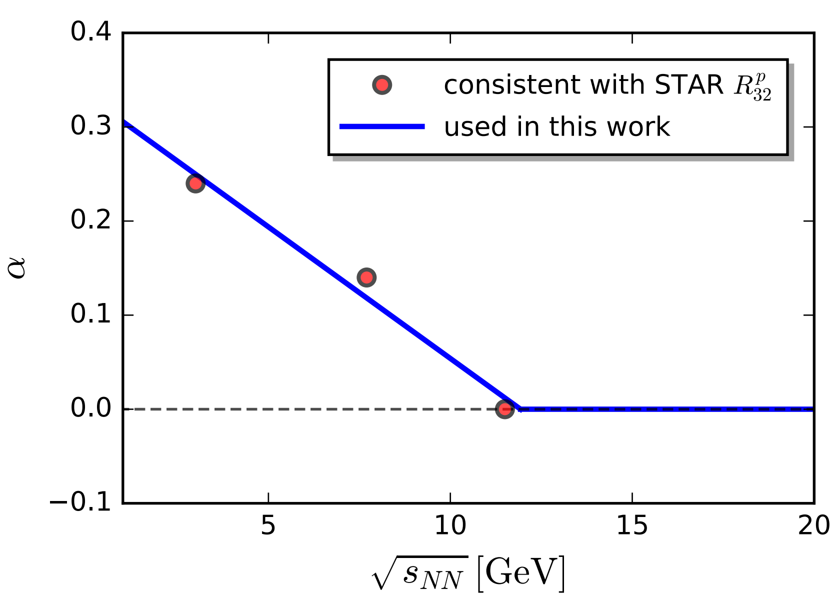


WF, Luo, Pawlowski, Rennecke, Yin, arXiv: 2308.15508

Canonical corrections with SAM



- **Experimental data R_{32} is used to constrain the parameter α in the range $\sqrt{s_{\rm NN}} \lesssim 11.5$ GeV.**
- **We choose the simplest linear dependence**



$$\alpha(\bar{s}) = a\left(1 - \sqrt{\bar{s}}\right)\theta\,(1 - \bar{s})$$

$$a = 0.33\,, \qquad \sqrt{\bar{s}} = \frac{\sqrt{s_{\rm NN}}}{11.9\,\text{GeV}}$$

SAM:

- We adopt the subensemble acceptance method (SAM) to take into account the effects of global baryon number conservation:

$$\alpha = \frac{V_1}{V}$$

V_1: the subensemble volume measured in the acceptance window, V: the volume of the whole system.

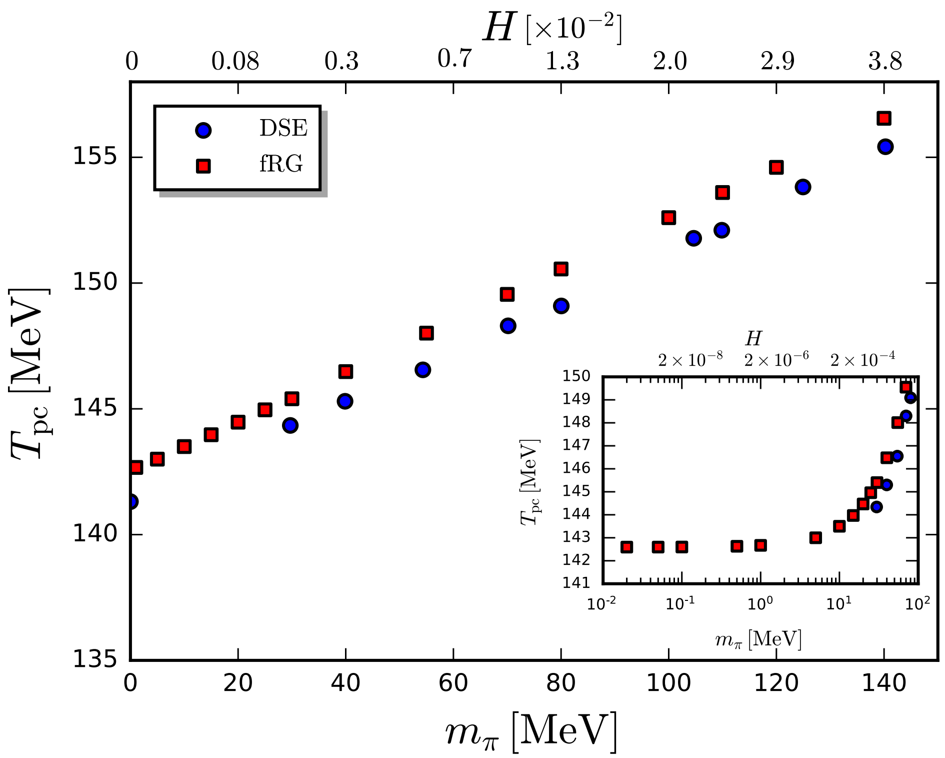
- fluctuations with canonical corrections are related to grand canonical fluctuations as follows:

$$\bar{R}_{21}^B = \beta R_{21}^B, \qquad \bar{R}_{32}^B = (1-2\alpha)R_{32}^B,$$

$$\bar{R}_{42}^B = (1-3\alpha\beta)R_{42}^B - 3\alpha\beta(R_{32}^B)^2$$

SAM: Vovchenko, Savchuk, Poberezhnyuk, Gorenstein, Koch , *PLB* 811 (2020) 135868

Magnetic equation of state



$$T_{\rm pc}(m_\pi) \approx T_c + c\, m_\pi^p$$

Braun, Chen, WF, Gao, Huang, Ihssen, Pawlowski, Rennecke, Sattler, Tan, Wen, and Yin, arXiv:2310.19853.

Lattice (HotQCD):

$$T_c^{\rm lattice} = 132^{+3}_{-6}\,{\rm MeV},$$

Ding *et al.*, *PRL* 123 (2019) 062002.

fRG:

$$T_c^{\rm fRG} \approx 142\,{\rm MeV}\,, \qquad p_{\rm fRG} = 1.024$$

Braun, WF, Pawlowski, Rennecke, Rosenblüh, Yin, *PRD* 102 (2020) 056010.

DSE:

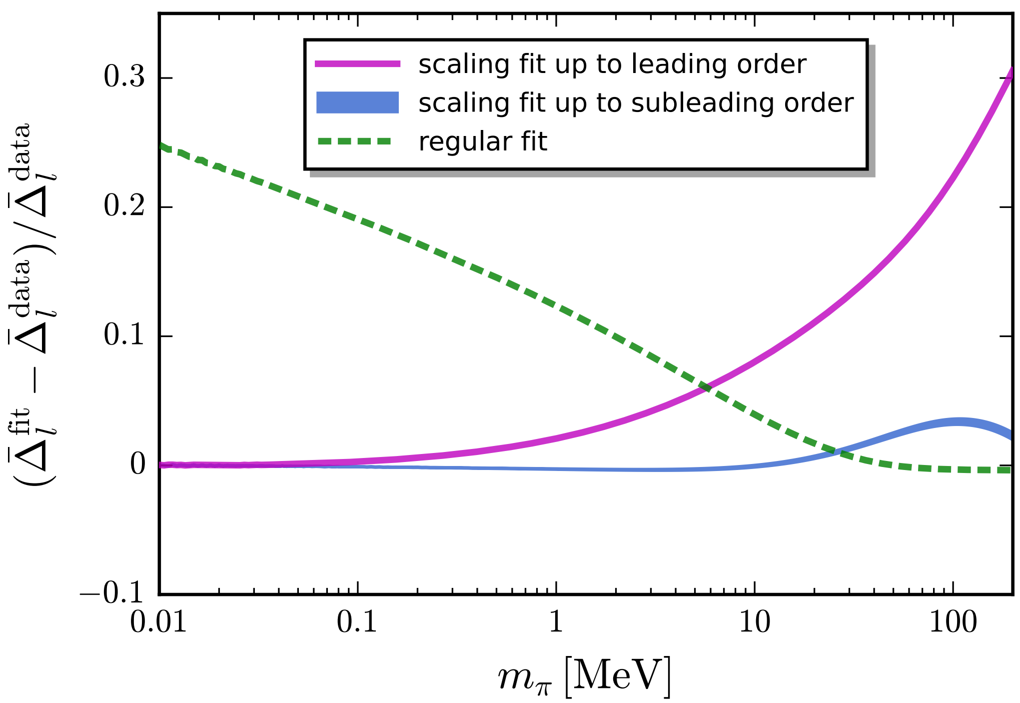
$$T_c^{\rm DSE} \approx 141\,{\rm MeV}\,, \qquad p_{\rm DSE} = 0.9606$$

Gao, Pawlowski, *PRD* 105 (2022) 9, 094020, arXiv: 2112.01395.

- **The almost linear dependence of the pseudo-critical temperature on the pion mass has nothing to do with the criticality.**
- **So what is the size of the critical region in QCD?**

Scaling vs regular fitting

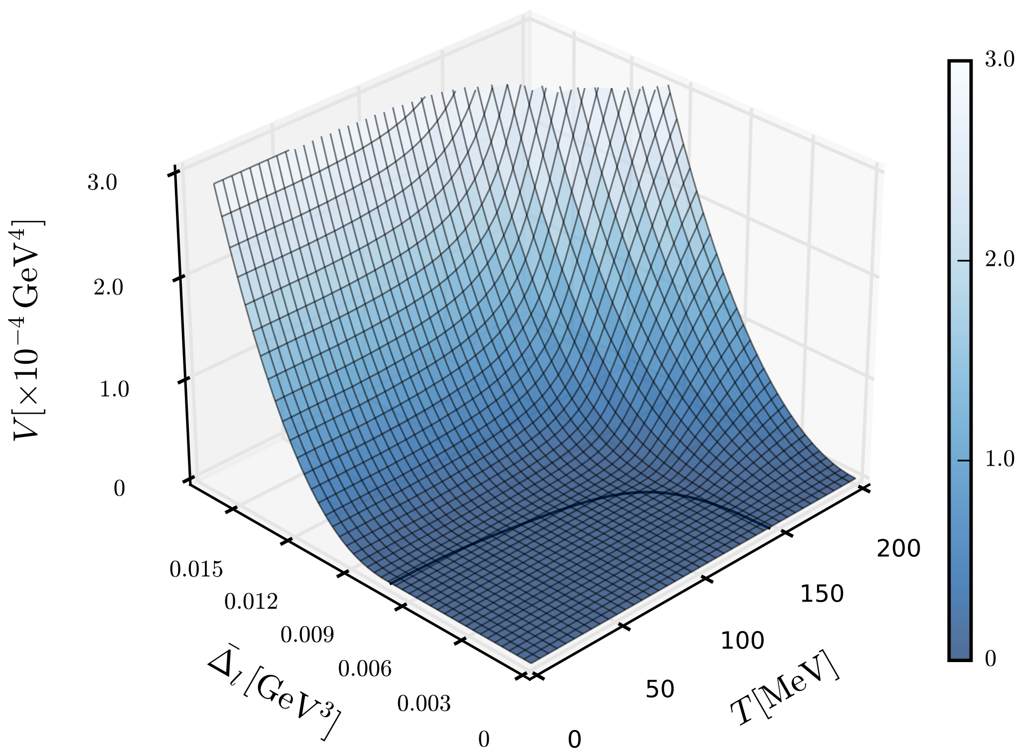
Errors:



$$\bar{\Delta}_l^{(\text{crit})}(m_\pi) = B_c\, m_\pi^{2/\delta}\big[1 + a_m m_\pi^{2\theta_H}\big]$$

$$\Delta_l^{(\text{reg})}(m_\pi) = b_{\frac{1}{5}}\, m_\pi^{2/5} + b_{\frac{3}{5}}\, m_\pi^{6/5} + b_1\, m_\pi^2$$

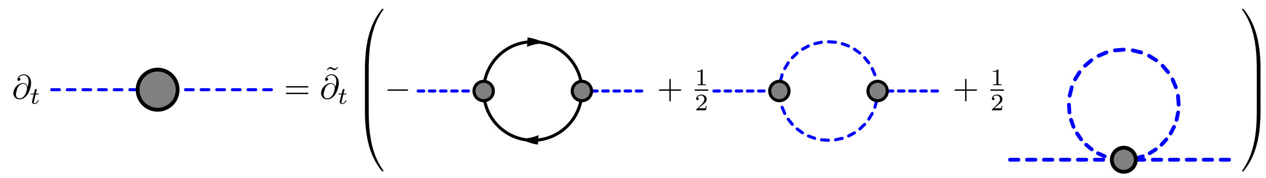
Potential:

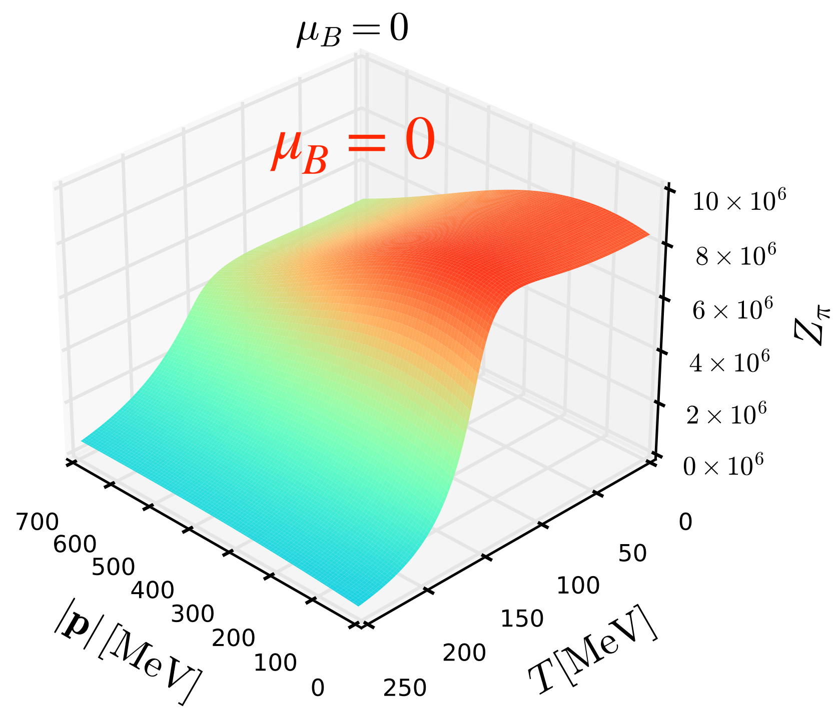


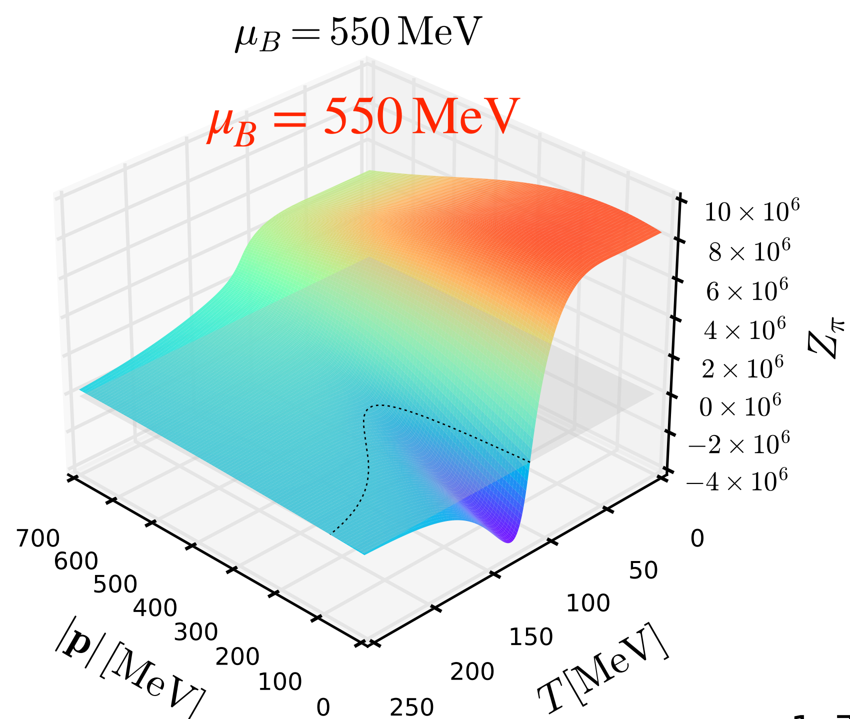
Braun, Chen, WF, Gao, Huang, Ihssen, Pawlowski, Rennecke, Sattler, Tan, Wen, and Yin, arXiv:2310.19853.

Momentum-dependent mesonic wave function

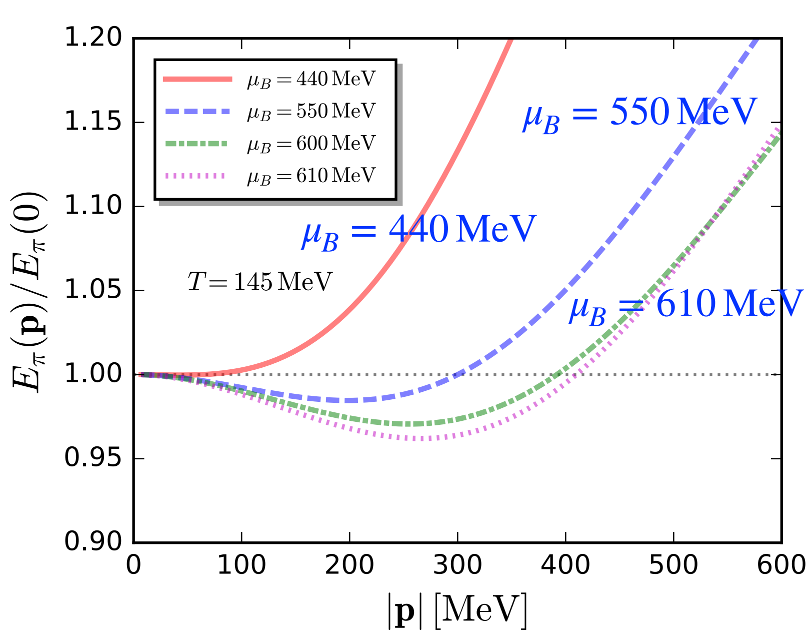
Flow equation for mesonic two-point functions:







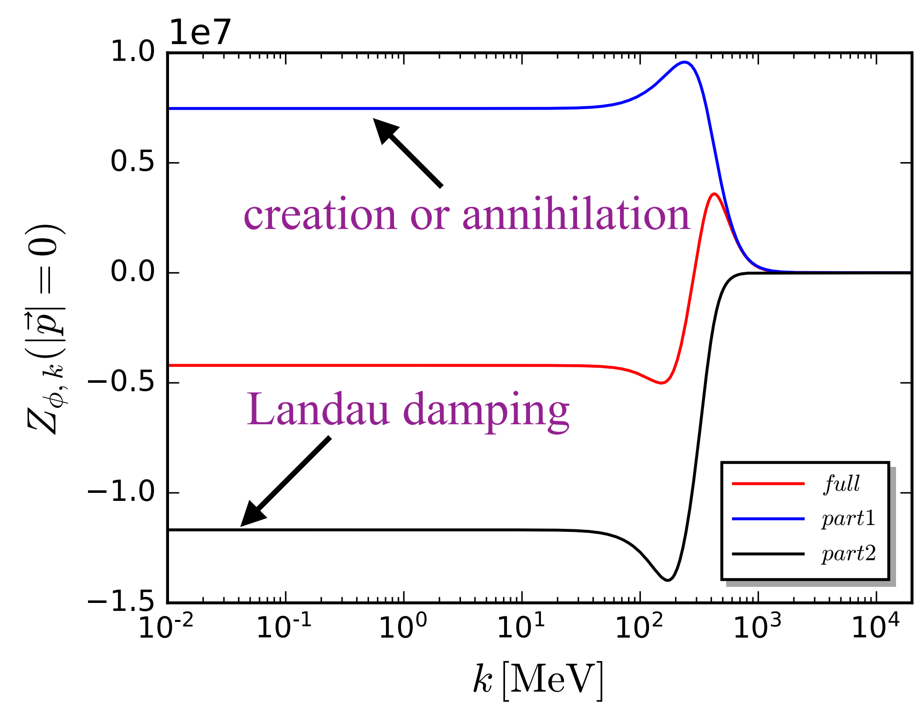
- Inhomogeneous instability is resulted from Landau damping of quarks in thermal bath in the regime of large baryon chemical potential.



Dispersion relation:

$$E_\phi(\mathbf{p}) = \left[Z_\phi^\perp(\mathbf{p})\,\mathbf{p}^2 + m_\phi^2\right]^{1/2}$$

WF, Pawlowski, Pisarski, Rennecke, Wen, Yin, in preparation.

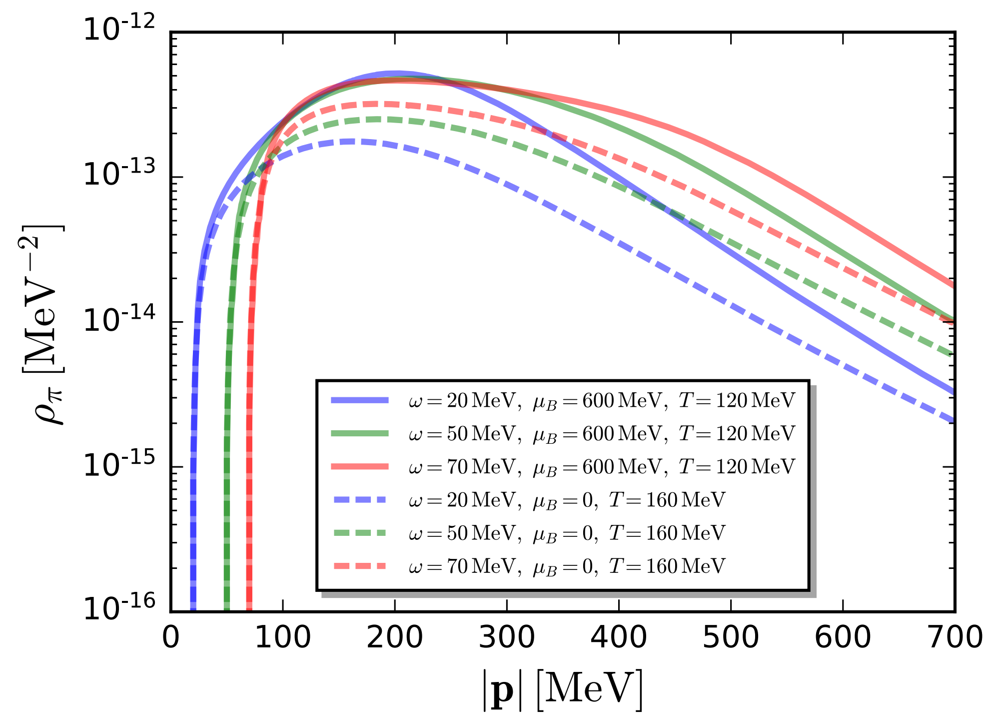


Real-time mesonic two-point functions

Analytic continuation on the flow equation:

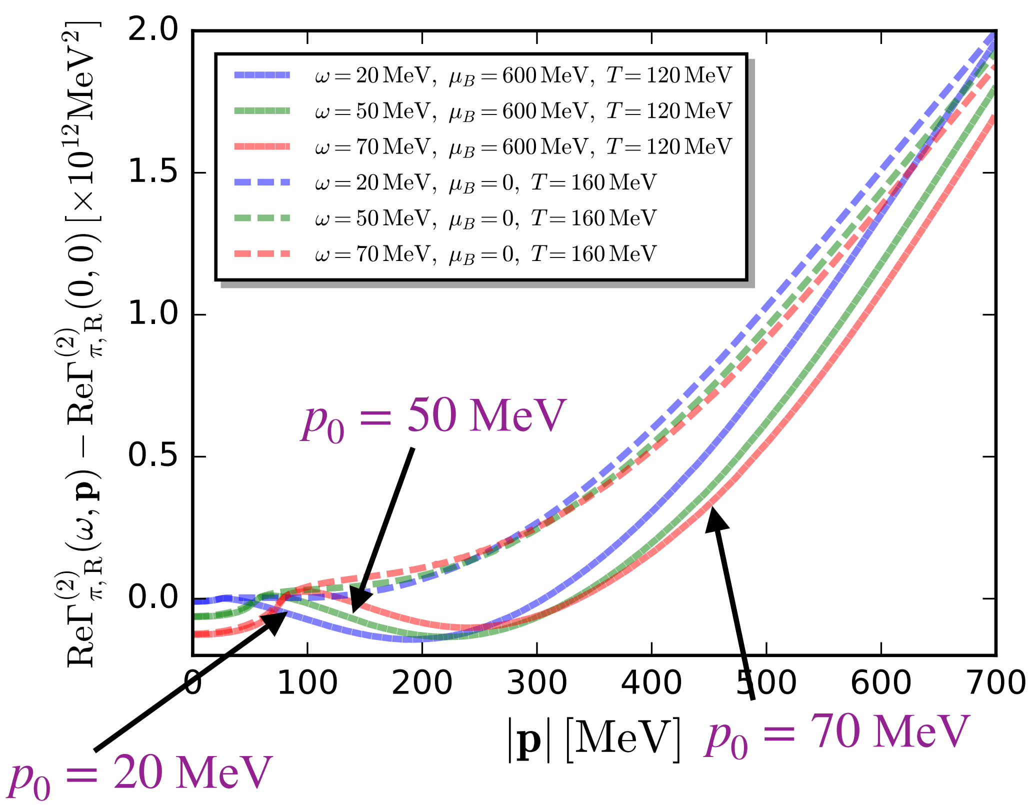
$$\Gamma^{(2)}_{\phi\phi,\mathrm{R}}(\omega,\mathbf{p}) = \lim_{\epsilon\to 0^+}\Gamma^{(2)}_{\phi\phi}\big(-\mathrm{i}(\omega+\mathrm{i}\epsilon),\mathbf{p}\big)$$

Note: not on data!

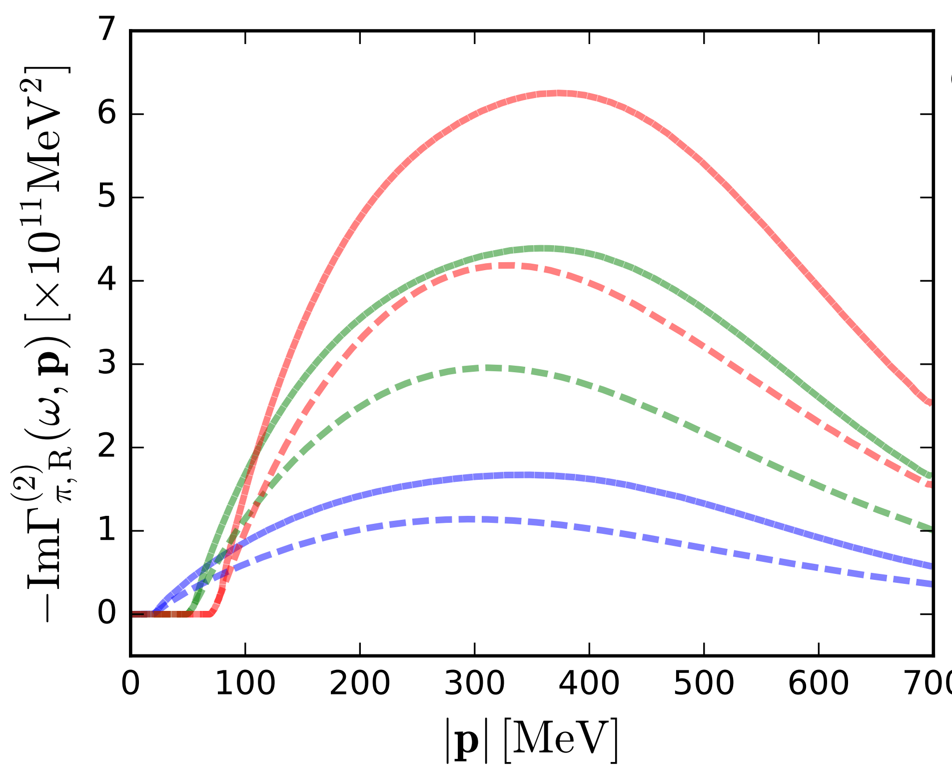


Spectral function

Real part of $\Gamma^{(2)}_{\phi\phi,\mathrm{R}}(p_0,\mathbf{p})$:



Imaginary part of $\Gamma^{(2)}_{\phi\phi,\mathrm{R}}(p_0,\mathbf{p})$:

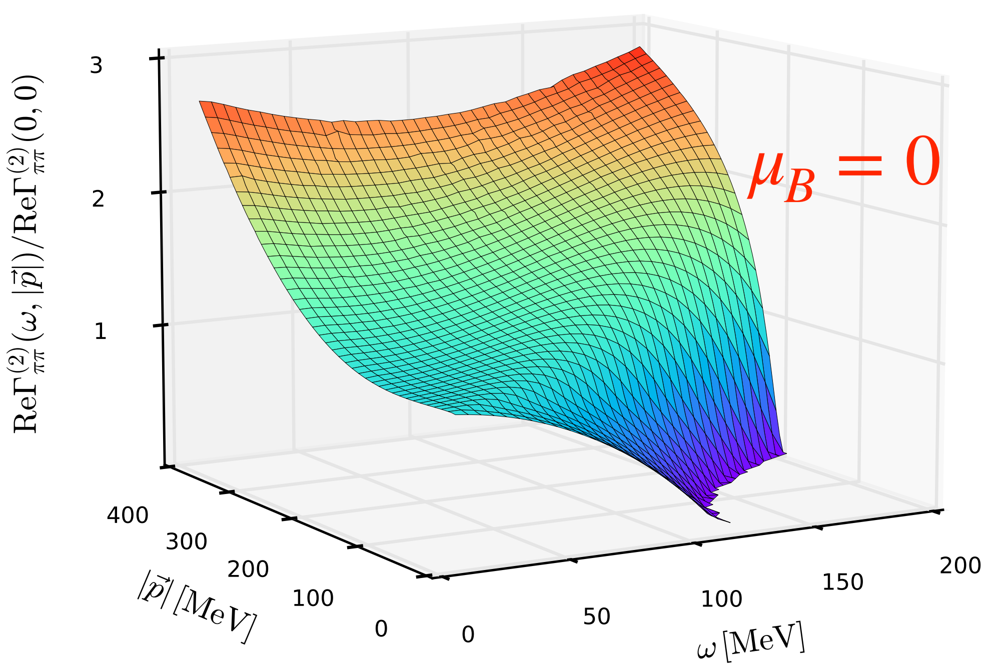


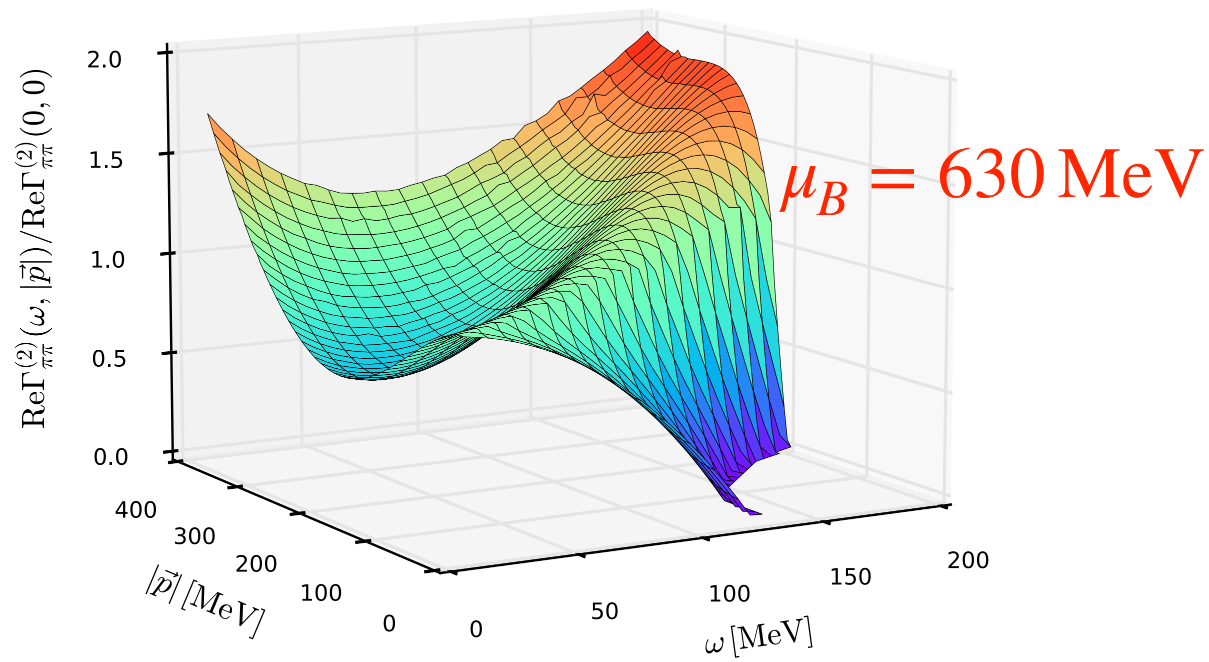
- Imaginary part of the mesonic two-point functions and spectral function are enhanced by the Landau damping effect

WF, Pawlowski, Pisarski, Rennecke, Wen, Yin, in preparation.

Real-time mesonic two-point functions

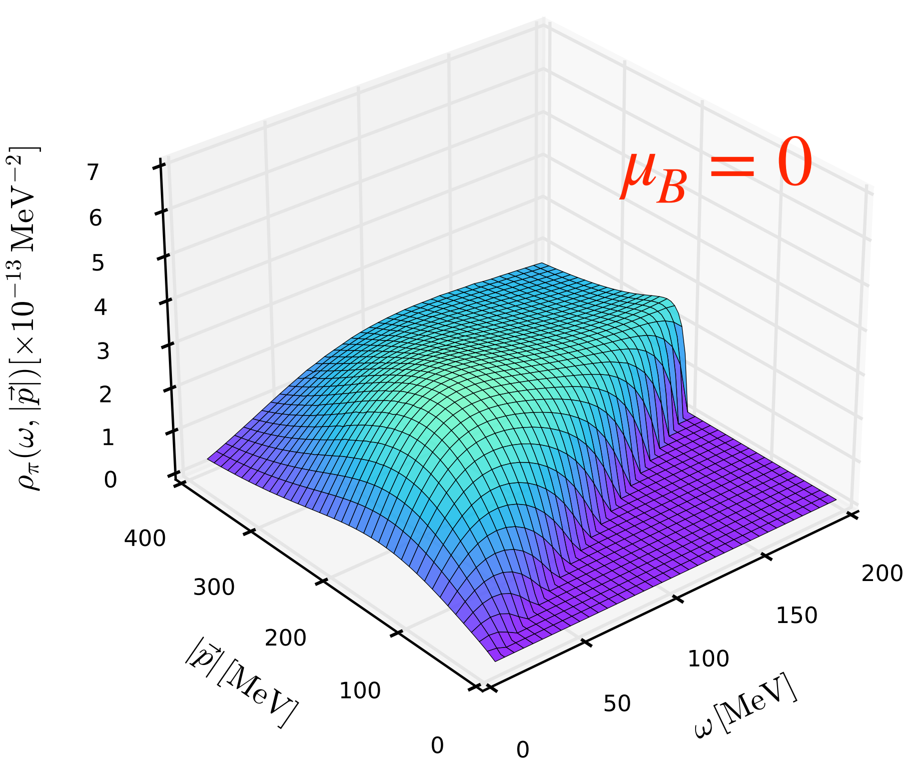
Real part:

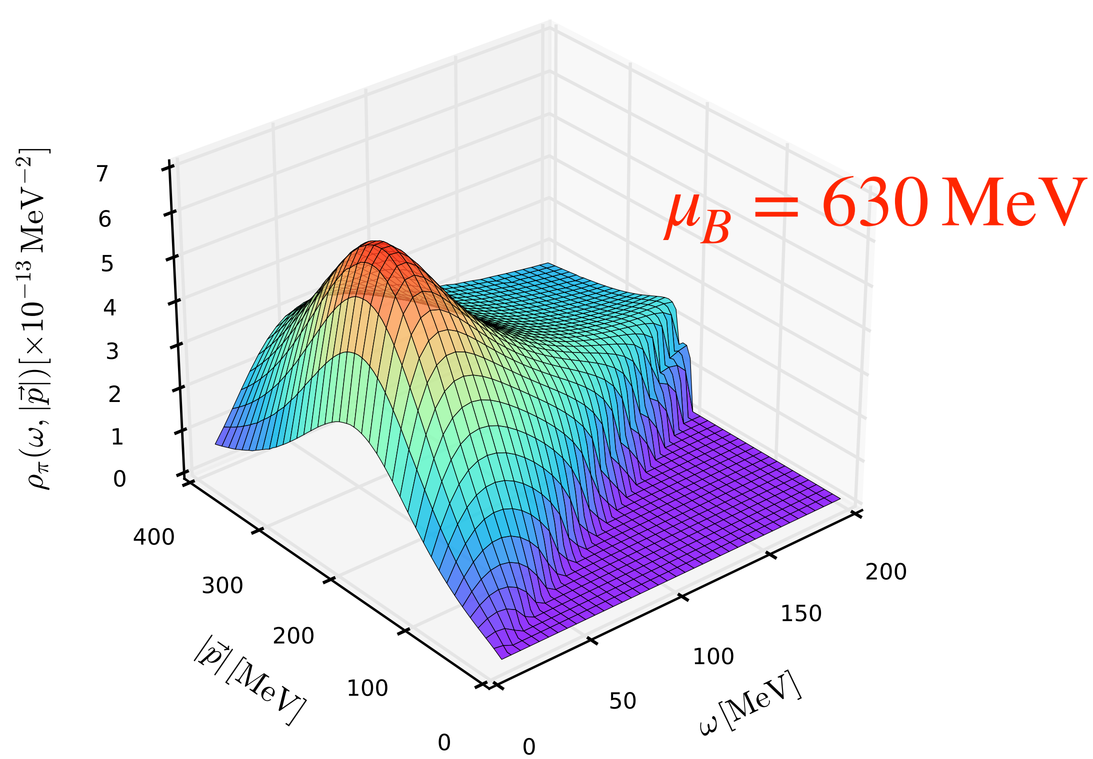




WF, Pawlowski, Pisarski, Rennecke, Wen, Yin, in preparation.

Spectral function:





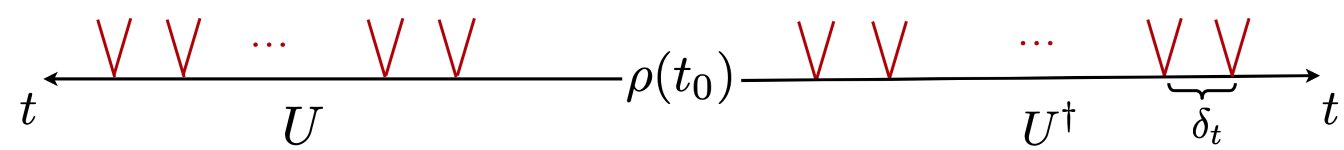
Schwinger-Keldysh path integral

- **Schrödinger equation:**

$U(t, t_0) = e^{-iH(t-t_0)}$

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \quad \longrightarrow \quad |\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle,$$

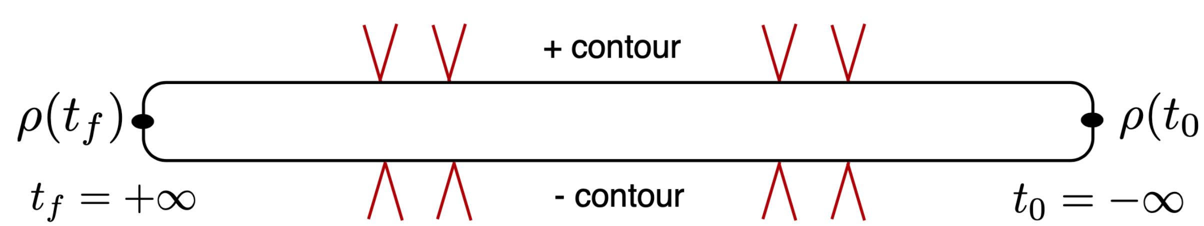
- **von Neumann equation:**



$$\partial_t\rho(t) = -i[H,\rho(t)] \quad \longrightarrow \quad \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0),$$

- **Keldysh partition function:**

$$Z = \mathrm{tr}\,\rho(t),$$



- **two-point closed time-path Green's function:**

$$G(x,y) \equiv -i\mathrm{tr}\{T_p(\phi(x)\phi^\dagger(y)\rho)\}$$

$$\equiv -i\langle T_p(\phi(x)\phi^\dagger(y))\rangle\,,$$

$$G(x,y) = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}$$

$$\equiv \begin{pmatrix} G_F & G_+ \\ G_- & G_{\tilde{F}} \end{pmatrix}\,,$$

$$G_F(x,y) \equiv -i\langle T(\phi(x)\phi^\dagger(y))\rangle\,,$$

$$G_+(x,y) \equiv -i\langle \phi^\dagger(y)\phi(x)\rangle\,,$$

$$G_-(x,y) \equiv -i\langle \phi(x)\phi^\dagger(y)\rangle\,,$$

$$G_{\tilde{F}}(x,y) \equiv -i\langle \tilde{T}(\phi(x)\phi^\dagger(y))\rangle\,,$$

Schwinger, J. Math. Phys. 2, 407 (1961);
Keldysh, Zh. Eksp. Teor. Fiz. 47, 1515 (1964);
Chou, Su, Hao, Yu, Phys. Rept. 118, 1 (1985).

FRG in Keldysh path integral

- **Implement the formalism of fRG in the two time branches:**

$$Z_k[J_c, J_q] = \int (\mathscr{D}\varphi_c \mathscr{D}\varphi_q) \exp\left\{ i\left(S[\varphi] + \Delta S_k[\varphi] + (J_q^i \varphi_{i,c} + J_c^i \varphi_{i,q}) \right) \right\},$$

with

$$\Delta S_k[\varphi] = \frac{1}{2}(\varphi_{i,c}, \varphi_{i,q}) \begin{pmatrix} 0 & R_k^{ij} \\ (R_k^{ij})^* & 0 \end{pmatrix} \begin{pmatrix} \varphi_{j,c} \\ \varphi_{j,q} \end{pmatrix}$$

$$= \frac{1}{2}\left(\varphi_{i,c} R_k^{ij} \varphi_{j,q} + \varphi_{i,q} (R_k^{ij})^* \varphi_{j,c} \right),$$

Keldysh rotation:

$$\begin{cases} \varphi_{i,+} = \frac{1}{\sqrt{2}}(\varphi_{i,c} + \varphi_{i,q}), \\ \varphi_{i,-} = \frac{1}{\sqrt{2}}(\varphi_{i,c} - \varphi_{i,q}), \end{cases}$$

- **Then we derive the flow equation in the closed time path:**

$$\partial_\tau \Gamma_k[\Phi] = \frac{i}{2} \mathrm{STr}\left[\left(\partial_\tau R_k^* \right) G_k \right],$$

$$R_k^{ab} \equiv \begin{pmatrix} 0 & R_k^{ij} \\ (R_k^{ij})^* & 0 \end{pmatrix},$$

$$iG(x,y) = \begin{pmatrix} iG^K(x,y) & iG^R(x,y) \\ iG^A(x,y) & 0 \end{pmatrix},$$

$$\boxed{\begin{aligned} iG^R(x,y) &= \theta(x^0 - y^0)\langle [\phi(x), \phi^*(y)] \rangle, \\ iG^A(x,y) &= \theta(y^0 - x^0)\langle [\phi^*(y), \phi(x)] \rangle, \\ iG^K(x,y) &= \langle \{ \phi(x), \phi^*(y) \} \rangle, \end{aligned}}$$

Tan, Chen, WF, *SciPost Phys*. 12 (2022) 026,
arXiv: 2107.06482

A relaxation critical O(N) model

- **The effective action on the Schwinger-Keldysh contour reads**

Hohenberg and Halperin, *Rev. Mod. Phys*. 49 (1977) 435.

Model A

$$\Gamma[\phi_c,\phi_q]=\int d^4x\left(Z_a^{(t)}\,\phi_{a,q}\,\partial_t\,\phi_{a,c}-Z_a^{(i)}\,\phi_{a,q}\,\partial_i^2\,\phi_{a,c}+V'(\rho_c)\,\phi_{a,q}\,\phi_{a,c}-\underline{2\,Z_a^{(t)}\,T\,\phi_{a,q}^2}-\sqrt{2}c\,\sigma_q\right)$$

$\Gamma=1/Z_a^{(t)}$: relaxation rate

$V'(\rho_c)$: potential $\rho_c\equiv\phi_c^2/4$

$Z_a^{(i)}$: wave function

c: explicit breaking

Gaussian white noise with coefficient determined by fluctuation-dissipation theorem

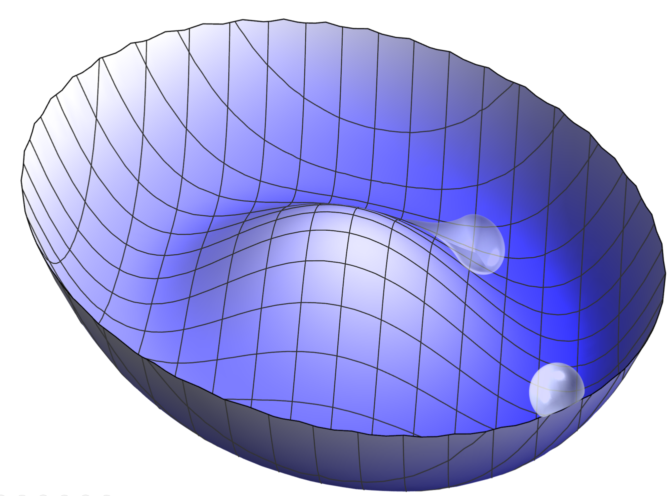
- **Retarded propagator**

$$G_{ab}^R=\left(\frac{\delta^2\Gamma[\phi_c,\phi_q]}{\delta\phi_{a,q}\,\delta\phi_{b,c}}\right)^{-1}$$

Retarded propagator of Goldstone

$$G_{\varphi\varphi}^R(\omega,q)=\frac{1}{-\mathrm{i}Z_\varphi^{(t)}\omega+Z_\varphi^{(i)}\left(q^2+m_\varphi^2\right)}$$

pseudo-Goldstone:



Mass of pseudo-Goldstone

$$m_\varphi^2=\frac{V'(\rho_0)}{Z_\varphi^{(i)}}=\frac{c}{\sigma_0 Z_\varphi^{(i)}}$$

Gell-Mann--Oakes--Renner (GMOR) relation

Universal damping or not?

From the pole of the retarded propagator of Goldstone

$$G^R_{\varphi\varphi}(\omega, q) = \frac{1}{-\mathrm{i}Z^{(t)}_\varphi \omega + Z^{(i)}_\varphi \left(q^2 + m^2_\varphi\right)}$$

One obtains the dispersion relation of a damped mode

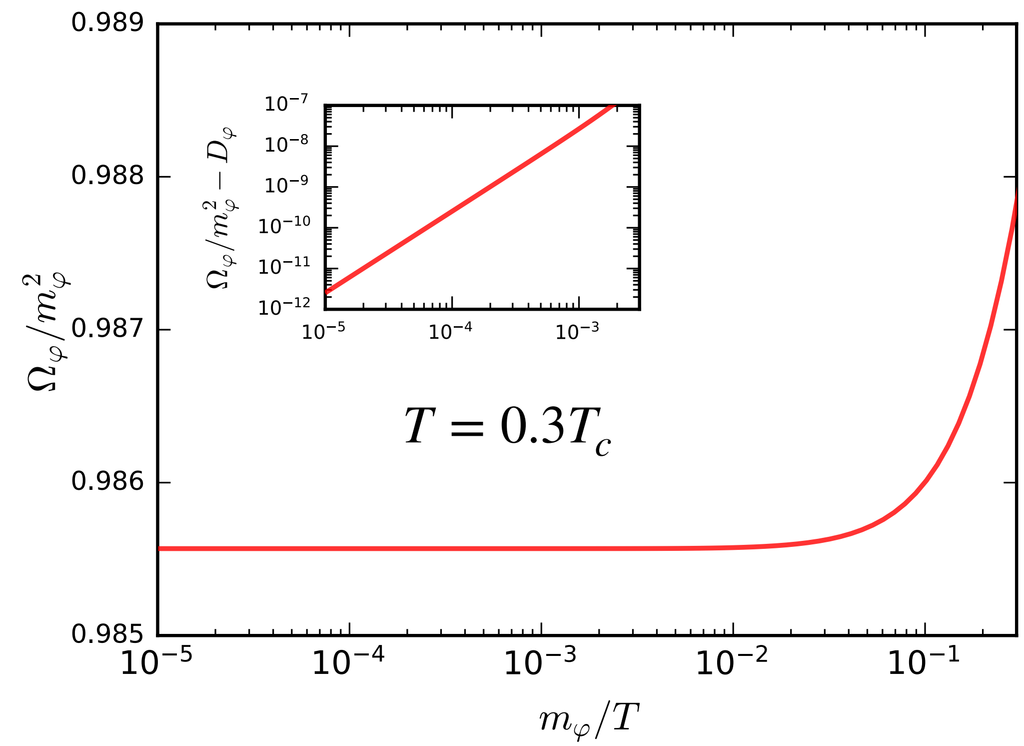
$$\omega(q) = -\,\mathrm{i}\,\frac{Z^{(i)}_\varphi}{Z^{(t)}_\varphi}\left(m^2_\varphi + q^2\right)$$

The relaxation rate at zero momentum reads

$$\Omega_\varphi \equiv -\,\mathrm{Im}\,\omega(q=0) = \frac{Z^{(i)}_\varphi}{Z^{(t)}_\varphi}\, m^2_\varphi$$

- **If** $T \ll T_c$

$$\frac{\Omega_\varphi}{m^2_\varphi} \simeq D_\varphi(T) + \mathcal{O}\left(\frac{m^2_\varphi}{T^2}\right) \quad \textbf{with} \quad D_\varphi(T) \equiv \frac{Z^{(i)}_\varphi(T, c=0)}{Z^{(t)}_\varphi(T, c=0)}$$



Tan, Chen, WF, Li, arXiv: 2403.03503

This seemingly appears as a **universal** relation that was also observed in Holographics, Hydrodynamics, and EFT

Holographics:
Amoretti, Areán, Goutéraux, Musso, *PRL* 123 (2019) 211602;
Amoretti, Areán, Goutéraux, Musso, *JHEP* 10 (2019) 068;
Ammon *et al*., *JHEP* 03 (2022) 015;
Cao, Baggioli, Liu, Li, *JHEP* 12 (2022) 113

Hydrodynamics:
Delacrétaz, Goutéraux, Ziogas, *PRL* 128 (2022) 141601

EFT:
Baggioli, *Phys. Rev. Res.* 2 (2020) 022022;
Baggioli, Landry, *SciPost Phys*. 9 (2020) 062

Breaking down of the universal damping in the critical region

In the critical region, the two wave function renormalizations read

$$Z_\varphi^{(i)} = t^{-\nu\eta} f^{(i)}(z)\,, \qquad Z_\varphi^{(t)} = t^{-\nu\eta_t} f^{(t)}(z)$$

Here $f^{(i)}(z), f^{(t)}(z)$: scaling functions; $z \equiv tc^{-1/(\beta\delta)}$: scaling variable; $t \equiv (T_c - T)/T_c$: reduced temperature.The static and dynamic anomalous dimensions are

$$\eta = -\frac{\partial_\tau Z_\varphi^{(i)}}{Z_\varphi^{(i)}}\,, \qquad \eta_t = -\frac{\partial_\tau Z_\varphi^{(t)}}{Z_\varphi^{(t)}}$$

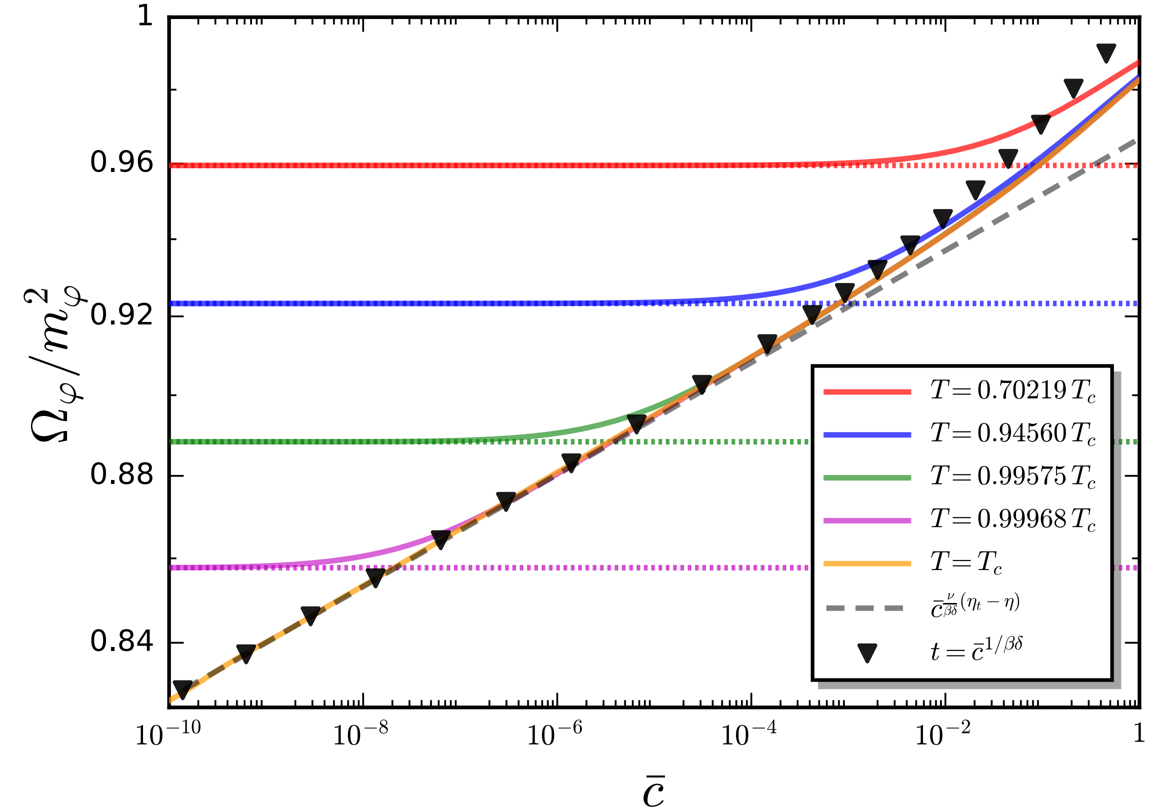
RG time $\tau = \ln(k/\Lambda)$

- In the case of $c \to 0$

$$\frac{Z_\varphi^{(i)}}{Z_\varphi^{(t)}} \propto t^{\nu(\eta_t - \eta)}$$

- In the other case of $t \to 0$

$$\frac{Z_\varphi^{(i)}}{Z_\varphi^{(t)}} \propto c^{\frac{\nu}{\beta\delta}(\eta_t-\eta)} \propto m_\varphi^{(\eta_t-\eta)} \quad \text{with} \quad m_\varphi^2 \propto c^{\frac{2\nu}{\beta\delta}}$$



Tan, Chen, WF, Li, arXiv: 2403.03503

From the fixed-point equation we determine in the O(4) symmetry

$$\eta \approx 0.0374, \quad \eta_t \approx 0.0546$$

Thus

$$\Delta_\eta \equiv \eta_t - \eta \approx 0.0172$$

Estimate of size of the dynamic critical region:

$$m_{\pi 0} \lesssim 0.1 \sim 1\ \text{MeV}$$

Large N limit

In the large N limit, the static and dynamic anomalous dimensions can be solved analytically

$$\eta = \frac{5}{N-1}\frac{(1+\eta)(1-2\eta)^2}{(5-\eta)(2-\eta)^2}$$

and

$$\eta_t = \frac{1}{9(N-1)}\frac{(1-2\eta)^2(13+15\eta-2\eta^3)}{(2-\eta)^2}$$

Tan, Chen, WF, Li, arXiv: 2403.03503

- In the limit $N \to \infty$, the breaking down of the universal damping disappears.
- One should not expect that the anomalous scaling regime can be observed in classical holographic models.