Universal critical dynamics in QCD

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Based on
JR, Yunxin Ye, Sören Schlichting, and Lorenz von Smekal
arXiv:2403.04573 & 2409.14470

ERG 2024, Les Diablerets, 25 September 2024

Motivation

Long-term goal: detect **critical point** in the **QCD phase diagram** by identifying signatures of **criticality** in **heavy-ion collisions**

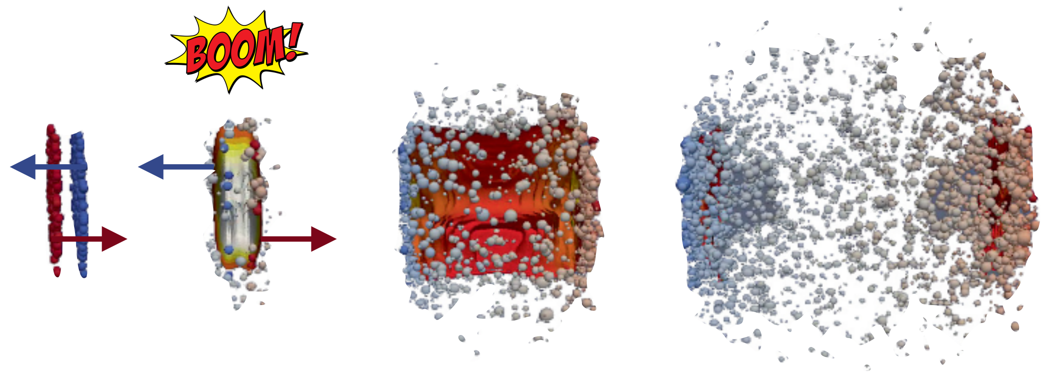


Figure from
MADAI collaboration

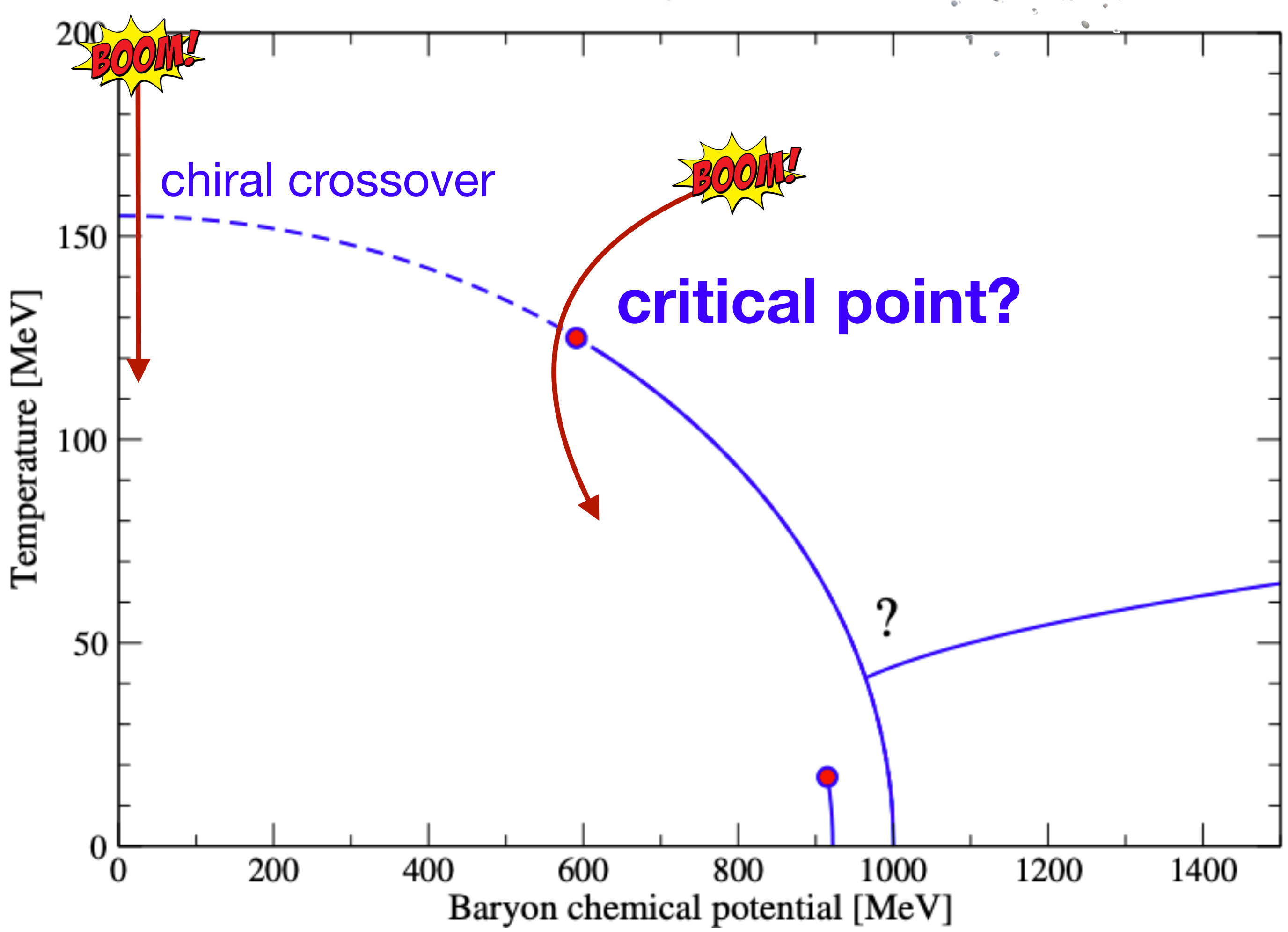


Figure adapted from C. S. Fischer, Prog. Part. Nucl. Phys. **105**, 1 (2019)

- Fireball is rapidly evolving ⇝ Need to understand **dynamics** of **critical fluctuations**
- **Idea:** dynamics of critical fluctuations **universal** ⇝ study simpler system from same **dynamic universality class**

Outline

1. Dynamic universality classes of chiral phase transition and QCD's critical point
2. FRG flow for systems with reversible mode couplings
3. Results for fixed points & critical exponents

Universal dynamics

dynamic critical exponent

critical slowing down: $\xi_t \sim \xi^z$

correlation time — correlation length

- Universality extends to critical **dynamics**
- z determined by **dynamic** universality class

Hohenberg & Halperin,
Rev. Mod. Phys. **49**, 435 (1977)

Halperin & Hohenberg: **Dynamic** universality class depends on

- order parameter conserved/not conserved
- other slow modes in the system, how they interact with order parameter

Dynamic universality classes

Static universality classes split up into **dynamic** universality classes:

classified into Models 'A, B, C, H, E, F, G & J'
[see Hohenberg & Halperin, Rev. Mod. Phys. **49**, 435 (1977)]

Dynamic universality class of chiral phase transition

Rajagopal and Wilczek, Nucl. Phys. B **399** (1993) 395-425

Z_2

Model A
Model B
Model C
Model H

$O(4)$

Model A
Model G

Dynamic universality class of QCD's critical point

Son and Stephanov, Phys. Rev. D **70**, 056001 (2004)

FRG:
JR, Ye, Schlichting, von Smekal, arXiv:2409.14470

Classical-statistical simulations:
Chattopadhyay, Ott, Schaefer, Skokov, Phys. Rev. Lett. **133**, 032301 (2024)

FRG:
JR, Ye, Schlichting, von Smekal, arXiv:2403.04573

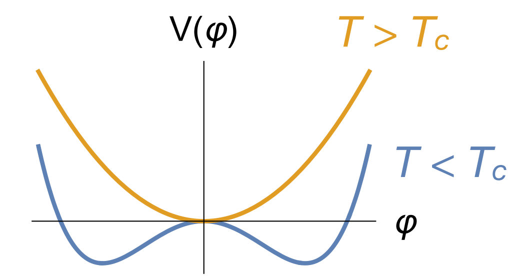
Classical-statistical simulations:
Florio, Grossi, Soloviev, Teaney, Phys. Rev. D **105**, 054512 (2022)
Florio, Grossi, Teaney, Phys. Rev. D **109**, 054037 (2024)

Dynamic universality class of chiral phase transition

CRC-TR 211

Chiral order parameter: $\phi = (\sigma, \vec{\pi})$

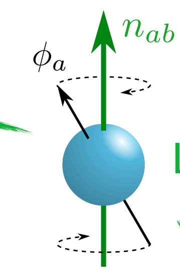
Statics: $O(4)$ Landau-Ginzburg-Wilson (LGW) free energy



$$F = \int d^d x \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{4\chi} n_{ab} n_{ab} \right\}$$

conserved iso-vector & iso-axial-vector charges

Equations of motion: **dissipation & noise** → relaxation towards thermal equilibrium $e^{-F/T}$ **ideal time evolution**



$$\partial_t \phi_a = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}}$$

$$\partial_t n_{ab} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + \vec{\nabla}\cdot\vec{\zeta}_{ab} + g\{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab}, n_{cd}\}\frac{\delta F}{\delta n_{cd}}$$

reversibility

Poisson brackets:

$$\{\phi_a, n_{bc}\} = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

(non-linear) reversible mode couplings

$$\{n_{ab}, n_{cd}\} = \delta_{ac} n_{bd} + \delta_{bd} n_{ac} - \delta_{ad} n_{bc} - \delta_{bc} n_{ad}$$

Model G

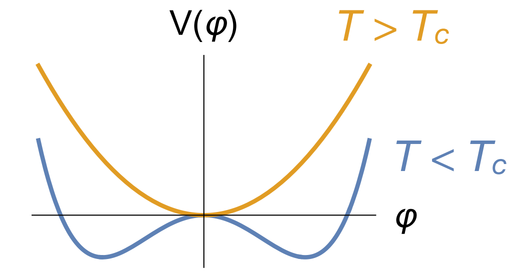
$z = d/2$

$O(4)$ Heisenberg **antiferro**magnet

Dynamic universality class of QCD's critical point

Order parameter: $\phi \sim \delta(s/n)$ (entropy per baryon)

Statics: Z_2 Landau-Ginzburg-Wilson (LGW) free energy



$$F = \int d^d x \left\{ \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{24}\phi^4 + \frac{\vec{j}^2}{2\rho} \right\}$$

(transverse) momentum density

Equations of motion:

dissipation & noise ⇝ relaxation towards thermal equilibrium $e^{-F/T}$

ideal time evolution

$$\frac{\partial \phi}{\partial t} = \sigma \vec{\nabla}^2 \frac{\delta F}{\delta \phi} + \theta + g\{\phi, \vec{j}\} \cdot \frac{\delta F}{\delta \vec{j}}$$

advection (replaces Larmor precession)

$$\frac{\partial j_l}{\partial t} = \mathcal{T}_{lm}[\eta \vec{\nabla}^2 \frac{\delta F}{\delta j_m} + \xi_m + g\{j_m, \phi\} \frac{\delta F}{\delta \phi} + g\{j_m, j_n\} \frac{\delta F}{\delta j_n}]$$

Poisson brackets:

$$\{\phi(\vec{x}), j_l(\vec{x}')\} = \phi(\vec{x}') \frac{\partial}{\partial x'_l} \delta(\vec{x} - \vec{x}')$$

$$\{j_l(\vec{x}), j_m(\vec{x}')\} = \left[j_l(\vec{x}') \frac{\partial}{\partial x'_m} - j_m(\vec{x}) \frac{\partial}{\partial x_l} \right] \delta(\vec{x} - \vec{x}')$$

(non-linear) reversible mode couplings

Model H

$$z = 4 - \eta - x_\sigma$$

Liquid-gas transition in pure fluid

Structure very similar to **Model G** ⇝ apply **same** real-time FRG method!

Temporal gauge symmetry (here for Model G)

- Apply external 'magnetic' field $\mathcal{H}_{ab}$: $F \to F - \frac{1}{2}\int d^d x\, \mathcal{H}_{ab} n_{ab}$

'magnetization'

- Equations of motion change according to:

$$\partial_t \phi_a = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2}\{\phi_a, n_{bc}\}\left[\frac{\delta F}{\delta n_{bc}} - \mathcal{H}_{bc}\right]$$

$$\partial_t n_{ab} = \gamma \vec{\nabla}^2\left[\frac{\delta F}{\delta n_{ab}} - \mathcal{H}_{ab}\right] + \vec{\nabla}\cdot\vec{\zeta}_{ab} + g\,\{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab}, n_{cd}\}\left[\frac{\delta F}{\delta n_{cd}} - \mathcal{H}_{cd}\right]$$

Temporal gauge symmetry (here for Model G)

CRC-TR 211

- Apply external 'magnetic' field $\mathcal{H}_{ab}$: $\quad F \to F - \frac{1}{2}\int d^dx\, \mathcal{H}_{ab} n_{ab}$

'magnetization'

- Equations of motion change according to:

$$\partial_t \phi_a - g(\mathcal{H}\phi)_a = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}}$$

$$\partial_t n_{ab} - g[\mathcal{H}, n]_{ab} = \gamma \vec{\nabla}^2\left[\frac{\delta F}{\delta n_{ab}} - \mathcal{H}_{ab}\right] + \vec{\nabla}\cdot\vec{\zeta}_{ab} + g\{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab}, n_{cd}\}\frac{\delta F}{\delta n_{cd}}$$

Covariant time-derivatives,
in which $\mathcal{H}_{ab}$ = zero-component
of **external** $O(4)$ **gauge field**

Inhomogeneous
(time dependent) term
drops out here

- Equations of motion **invariant** under **time-gauged** $O(4)$ transformations

$$\phi(t,\vec{x}) \to O(t)\phi(t,\vec{x})$$

$$n(t,\vec{x}) \to O(t)n(t,\vec{x})O^T(t)$$

$$\mathcal{H}(t,\vec{x}) \to O(t)\mathcal{H}(t,\vec{x})O^T(t) + \frac{1}{g}O(t)\partial_t O^T(t)$$

- **Goal:** preserve during FRG flow (next)

2. FRG for systems with reversible mode couplings

FRG flow for systems with reversible mode couplings

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- **Problem:** preserve temporal gauge symmetry during FRG flow
- **Path-integral formulation:** Martin-Siggia-Rose (MSR) technique
- for every classical field ψ, introduce 'response' field $\tilde{\psi}$ (schematically)

$$Z = \int \mathcal{D}\psi \mathcal{D}\tilde{\psi}\, e^{iS} = 1 \qquad S = \int \Big[-\tilde{\psi}\Big(\partial_t \psi + (\gamma - g\{\psi,\psi\})\frac{\delta F}{\delta \psi}\Big) + i\gamma T \tilde{\psi}^2 \Big]$$

- Effective MSR action Γ: introduce sources J for $\tilde{\psi}$ and $\tilde{J}$ for ψ

one step earlier: $F \to F - \int J\psi$

$\Downarrow$

instead of:

$$S \to S + \int [\tilde{J}\psi + J\tilde{\psi}] \qquad \text{vs.} \qquad S \to S + \int \big[\tilde{J}\psi + J\underbrace{(\gamma + g\{\psi,\psi\})\tilde{\psi}}\big]$$

⇝ physical sources J couple to **'composite'** response fields $\tilde{\Psi}$

✓ **Problem solved:** temporal gauge symmetry becomes **extended** symmetry of effective MSR action Γ

see also Canet, Delamotte, Wschebor, Phys. Rev. E **93** (2016) 6, 063101
Floerchinger, Grossi, Phys. Rev. D **105** (2022) 8, 085015

FRG flow for systems with reversible mode couplings

CRC-TR 211

- Similarly, add regulators also on level of LGW free energy:

$$F \to F + \frac{1}{2}\int \psi R_k \psi \qquad \Longrightarrow \qquad S \to S - \frac{1}{2}\int \tilde{\Psi} R_k \psi$$

+ symmetries intact (temporal gauge, thermal equilibrium, BRST, …)
– regulators couple to composite response fields $\tilde{\Psi}$

- **Exact results:**
  - Ward identities: flow of g vanishes
  - Flow of statics (i.e., F_k) **independent** of dynamics

- Truncation:

$$m^2 \to m_k^2 \qquad \sigma \to \sigma_k \qquad \Gamma^\phi \to \Gamma^\phi_k$$
$$\lambda \to \lambda_k \qquad \eta \to \eta_k \qquad \gamma \to \gamma_k$$

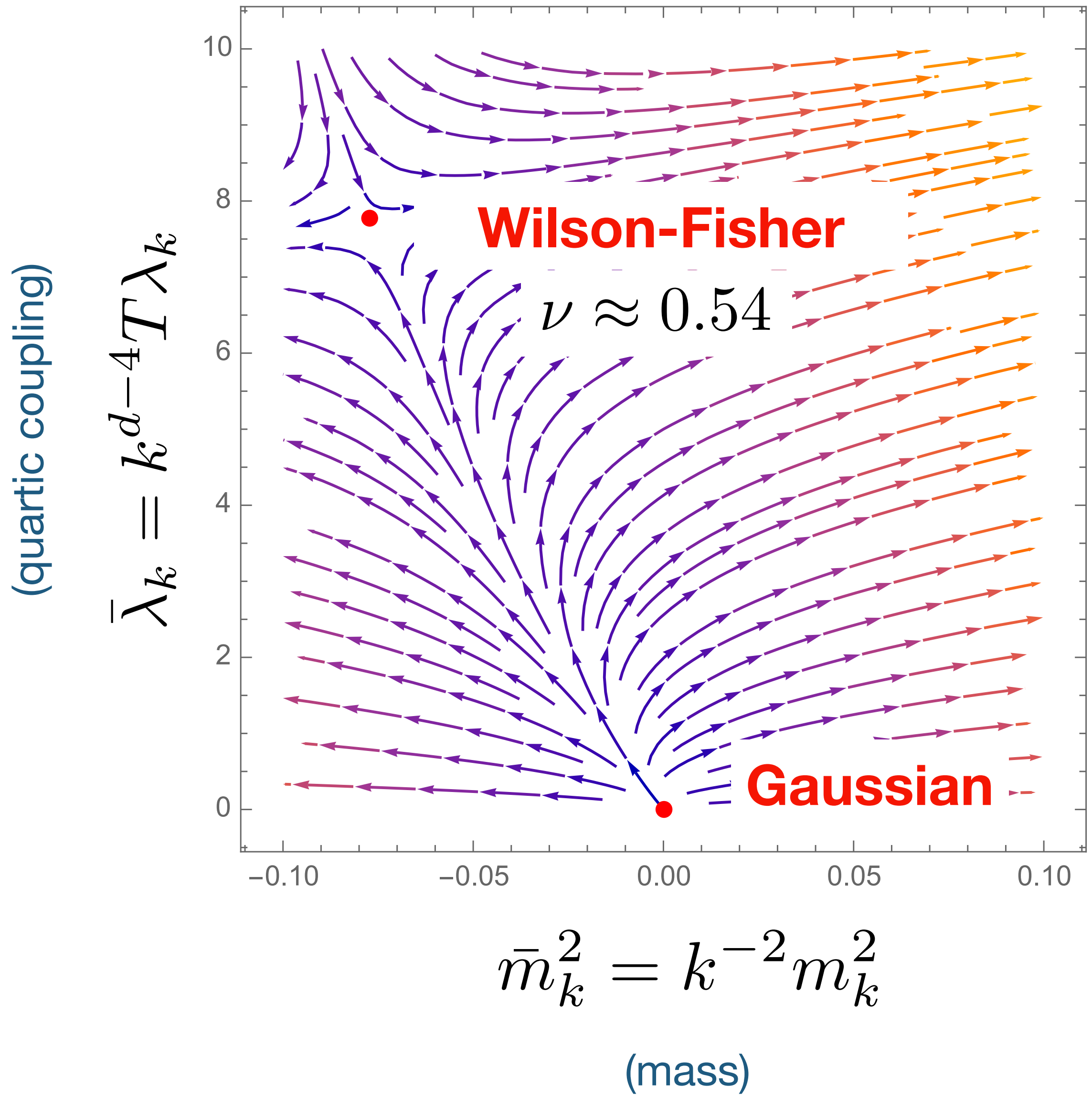
($g,\ \chi,\ \rho$ protected from renormalization)

3. Results for fixed points & critical exponents

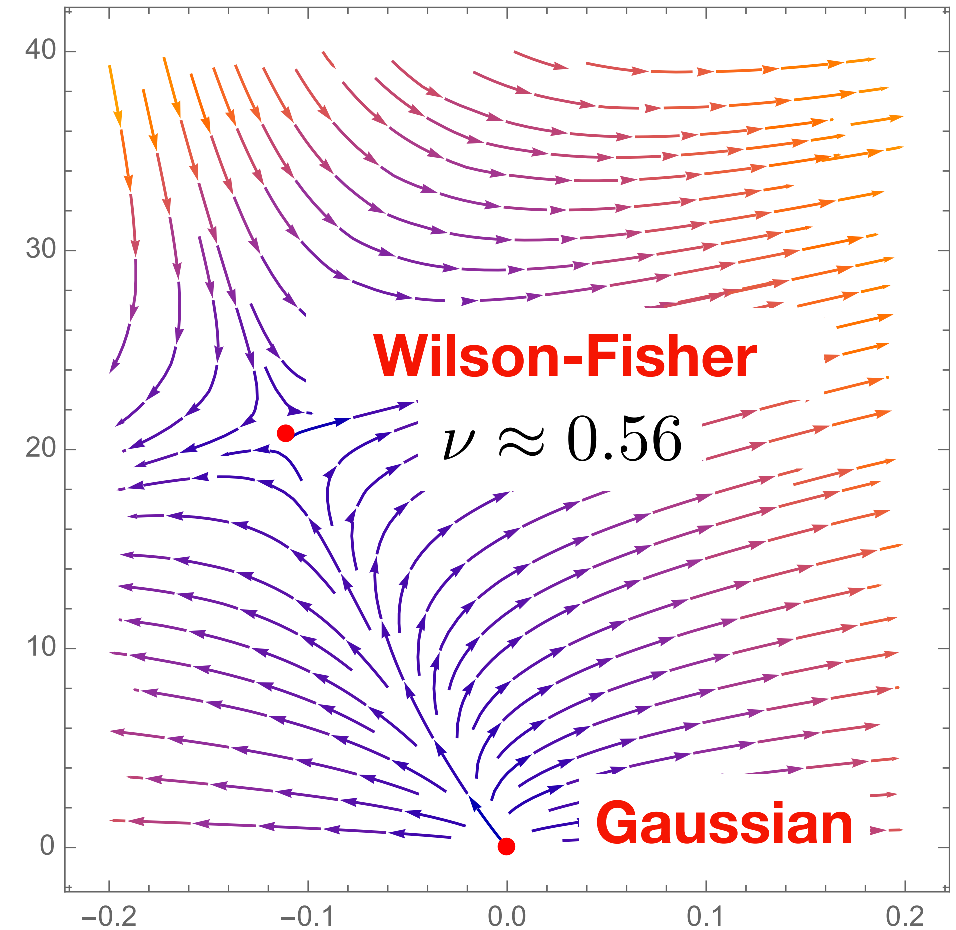
Fixed points of the FRG flow (statics)

First look at flow of LGW functional ⤳ flow of (static) couplings $m_k^2,\ \lambda_k$

Z₂



O(4)



Fixed points of the FRG flow (dynamics)

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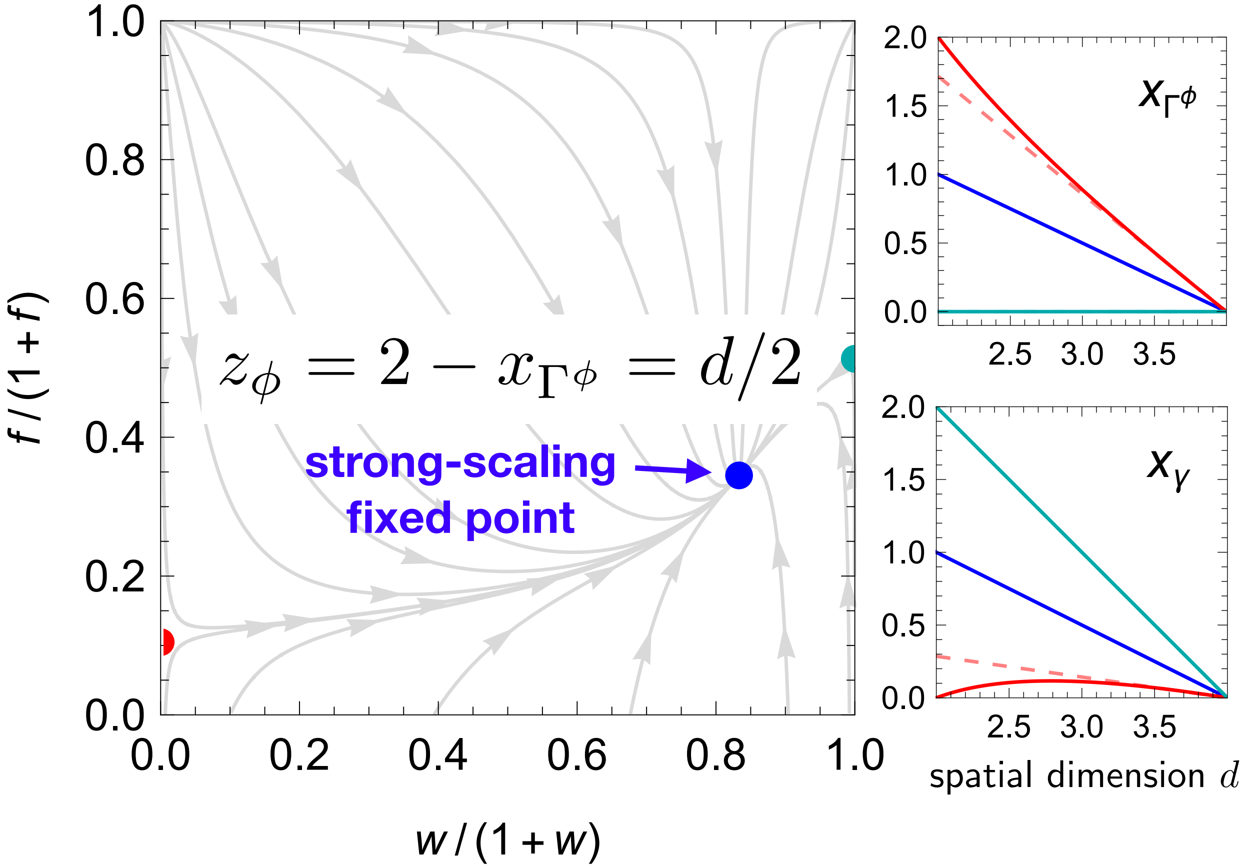
flow of (dimensionless) dynamic couplings:

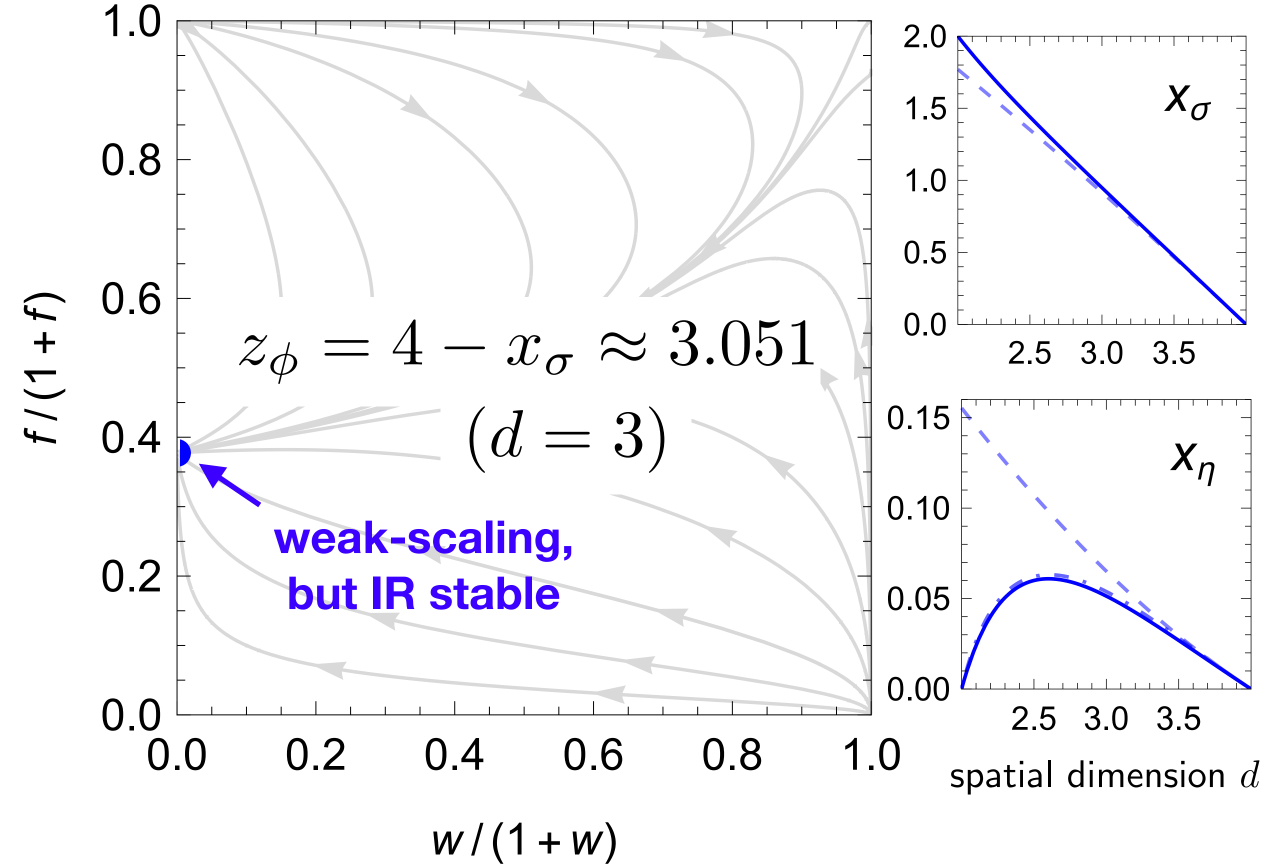
Model G

$$w_G \equiv \chi \frac{\Gamma_k^\phi}{\gamma_k}\,, \qquad f_G \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\Gamma_k^\phi \gamma_k}$$



Model H

$$w_H \equiv \rho\, \frac{\sigma_k k^2}{\eta_k}\,, \qquad f_H \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\sigma_k \eta_k}$$



at fixed points:

$$\Gamma^\phi \sim k^{-x_{\Gamma^\phi}} \qquad \gamma \sim k^{-x_\gamma} \qquad \sigma \sim k^{-x_\sigma} \qquad \eta \sim k^{-x_\eta}$$

see also Täuber, *Critical Dynamics* (Cambridge University Press, 2014)

Summary & Outlook

Summary:

- FRG flow for systems with reversible mode couplings **(Model G & H)**

  JR, Ye, Schlichting, von Smekal, arXiv:2403.04573 & 2409.14470

Outlook:

- dynamics of **Model G** for non-vanishing external fields (quark masses)
- dynamic universal scaling functions of **Model H**
- real-time dynamics of the Quark-Meson model

  JR, Ye, Schlichting, von Smekal, in prep.

Thank you for your attention!

Backup

Detailed formulas (Model H)

- Generating functional

$$Z[H,\tilde{H},\vec{A},\tilde{\vec{A}}] = \int \mathcal{D}\phi\,\mathcal{D}\tilde{\phi}\,\mathcal{D}n\,\mathcal{D}\tilde{n}\,\exp\Big\{iS + i\int_x \big(\tilde{H}\phi + \tilde{A}_l j_l\big) +$$
$$i\int_x H\big(-\sigma_k\vec{\nabla}^2\tilde{\phi} + g\{\phi, j_m\}\mathcal{T}_{mo}\tilde{j}_o\big) + i\int_x A_l\big(-\eta_k\vec{\nabla}^2\mathcal{T}_{lo}\tilde{j}_o + g\mathcal{T}_{lm}\{j_m,\phi\}\tilde{\phi} + g\mathcal{T}_{lm}\{j_m,j_n\}\mathcal{T}_{no}\tilde{j}_o\big)\Big\}$$

- MSR action

$$S = \int_x \Big\{ -\tilde{\phi}\left(\frac{\partial\phi}{\partial t} - \sigma\vec{\nabla}^2\frac{\delta F}{\delta\phi} - g\{\phi,\vec{j}\}\cdot\frac{\delta F}{\delta\vec{j}}\right)$$
$$-\tilde{j}_l\left(\frac{\partial j_l}{\partial t} - \mathcal{T}_{lm}\left[\eta\vec{\nabla}^2\frac{\delta F}{\delta j_m} + g\{j_m,\phi\}\frac{\delta F}{\delta\phi} + g\{j_m,j_n\}\frac{\delta F}{\delta j_n}\right]\right)$$
$$-iT\tilde{\phi}\sigma\vec{\nabla}^2\tilde{\phi} - iT\tilde{j}_l\eta\mathcal{T}_{lm}\vec{\nabla}^2\tilde{j}_m\Big\}$$

- Composite operators

$$\tilde{\Phi} \equiv -\sigma_k\vec{\nabla}^2\tilde{\phi} + g\{\phi, j_m\}\mathcal{T}_{mo}\tilde{j}_o$$
$$\tilde{J}_l \equiv -\eta_k\vec{\nabla}^2\mathcal{T}_{lo}\tilde{j}_o + g\mathcal{T}_{lm}\{j_m,\phi\}\tilde{\phi} + g\mathcal{T}_{lm}\{j_m,j_n\}\mathcal{T}_{no}\tilde{j}_o$$

Detailed FRG setup (Model H)

- effective average MSR action

$$\Gamma_k[\phi, \tilde{\Phi}, \vec{j}, \tilde{\vec{J}}] \equiv \sup_{H,\tilde{H},\vec{A},\tilde{\vec{A}}} \Big\{ - i \log Z_k[H, \tilde{H}, \vec{A}, \tilde{A}] - \int_x \big(\tilde{H}\phi + H\tilde{\Phi} + \tilde{A}_l j_l + A_l \tilde{J}_l\big) \Big\}$$

- full (truncated) propagators

$$G^{R/A}_{\phi,k}(\omega, \vec{p}) = -\frac{\sigma_k \vec{p}^2}{\pm i\omega - \sigma_k \vec{p}^2 (m_k^2 + \vec{p}^2 + R^\phi_k(\vec{p}))}\,, \qquad G^{R/A}_{j,k}(\omega, \vec{p}) = -\frac{\eta_k \vec{p}^2}{\pm i\omega - \eta_k \vec{p}^2 (1/\rho + R^j_k(\vec{p}))}$$

$$iF_{\phi/j,k}(\omega, \vec{p}) = \frac{T}{\omega}\Big(G^R_{\phi/j,k}(\omega,\vec{p}) - G^A_{\phi/j,k}(\omega,\vec{p})\Big)$$

- (truncated) 1PI vertex functions

$$\Gamma_k^{\tilde{\Phi}\phi j_l}(p,q,r) = -g\,\frac{r^0\,(\mathcal{T}_{\vec{r}}\vec{p})_l}{\eta_k \vec{r}^2\,\sigma_k \vec{p}^2}$$

$$\Gamma_k^{\tilde{J}_l\phi\phi}(p,q,r) = g\,\frac{q^0\,(\mathcal{T}_{\vec{p}}\vec{q})_l}{\eta_k \vec{p}^2\,\sigma_k \vec{q}^2} + g\,\frac{r^0\,(\mathcal{T}_{\vec{p}}\vec{r})_l}{\eta_k \vec{p}^2\,\sigma_k \vec{r}^2}$$

$$\Gamma_k^{\tilde{\Phi}\tilde{\Phi}\phi\phi}(p,q,r,s) = \frac{2ig^2T}{\sigma_k\vec{p}^2\,\sigma_k\vec{q}^2}\left(\frac{\mathcal{T}_{lm}(\vec{p}+\vec{r})}{\eta_k(\vec{p}+\vec{r})^2} + \frac{\mathcal{T}_{lm}(\vec{q}+\vec{r})}{\eta_k(\vec{q}+\vec{r})^2}\right) r_l s_m$$

$$\Gamma_k^{\tilde{J}_l\tilde{J}_m\phi\phi}(p,q,r,s) = 2ig^2T\,\frac{(\mathcal{T}_{\vec{p}}(\vec{p}+\vec{r}))_l\,(\mathcal{T}_{\vec{q}}(\vec{q}+\vec{s}))_m}{\eta_k\vec{p}^2\,\eta_k\vec{q}^2\,\sigma_k(\vec{p}+\vec{r})^2} + 2ig^2T\,\frac{(\mathcal{T}_{\vec{p}}(\vec{p}+\vec{s}))_l\,(\mathcal{T}_{\vec{q}}(\vec{q}+\vec{r}))_m}{\eta_k\vec{p}^2\,\eta_k\vec{q}^2\,\sigma_k(\vec{q}+\vec{r})^2}$$

Detailed FRG flow equations

- projection onto kinetic coefficients (Model H)

$$\partial_k \sigma_k = -\frac{\sigma_k^2}{2iT} \lim_{\vec{p}\to 0} \vec{p}^2 \lim_{\omega\to 0} \frac{\delta^2 \partial_k \Gamma_k}{\delta\tilde{\Phi}(-p)\delta\tilde{\Phi}(p)}\bigg|_0$$

$$\partial_k \eta_k = -\frac{\eta_k^2}{2iT} \lim_{\vec{p}\to 0} \vec{p}^2 \lim_{\omega\to 0} \frac{\mathcal{T}_{lm}(\vec{p})}{d-1} \frac{\delta^2 \partial_k \Gamma_k}{\delta\tilde{J}_l(-p)\delta\tilde{J}_m(p)}\bigg|_0$$

- analytical result:

$$\partial_k \sigma_k = \frac{2g^2 \Omega_d k^{d-1} T}{(2\pi)^d} \frac{d-1}{d-2} \frac{1}{\eta_k} \left(\frac{\sigma_k^2}{(\eta_k/\rho + \sigma_k(k^2+m_k^2))^2} - \frac{1}{(k^2+m_k^2)^2} \right)$$

$$\partial_k \eta_k = -\frac{g^2 \Omega_d k^{d+1} T}{(2\pi)^d \, (2+d) \sigma_k (k^2+m_k^2)^3}$$

- analytical result (Model G):

$$\partial_k \Gamma_k^\phi = \frac{g^2 \,(N-1)\, d\, \Omega_d k^{d-1} T}{(2\pi)^d \,\left(k^2+m_k^2\right)\, \gamma_k} \left\{ \frac{\Gamma_k^\phi}{k^2 \gamma_k/\chi + \Gamma_k^\phi (k^2+m_k^2)} - \frac{2+(d-4)\; {}_2F_1\left(1, \frac{d-2}{2}; \frac{d}{2}; -\frac{k^2\gamma_k/\chi}{\Gamma_k^\phi (k^2+m_k^2)}\right)}{(d-2)\, \left(k^2+m_k^2\right)} \right\}$$

$$\partial_k \gamma_k = -\frac{2g^2 \Omega_d k^{d+1} T}{(2\pi)^d \, \Gamma_k^\phi \, \left(k^2+m_k^2\right)^3}$$

Beta functions

- dynamic couplings (Model G):

$$w_G \equiv \chi \frac{\Gamma_k^\phi}{\gamma_k}\,, \qquad f_G \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\Gamma_k^\phi \gamma_k}$$

$$k\partial_k f_G = f_G(d-4) + f_G^2\left(\frac{2}{d(1+\bar{m}_k^2)^3} - (N-1) I_d(\bar{m}^2, w_G)\right)$$

$$k\partial_k w_G = w_G f_G\left[\frac{2}{d(1+\bar{m}_k^2)^3} + (N-1) I_d(\bar{m}^2, w_G)\right]$$

with $I_d(\bar{m}^2, w_G) \equiv -\frac{1}{(1+\bar{m}^2)^2}\left\{\frac{1}{1+(1+\bar{m}^2)w_G} + \frac{4-d}{d-2}\left[1 - {}_2F_1\left(1, \frac{d-2}{2}; \frac{d}{2}; -\frac{1}{(1+\bar{m}^2)w_G}\right)\right]\right\}$

- dynamic couplings (Model H):

$$w_H \equiv \rho \frac{\sigma_k k^2}{\eta_k}\,, \qquad f_H \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\sigma_k \eta_k}$$

$$\begin{aligned} k\partial_k f_H = {} & f_H(d-4) \\ & - f_H^2 \frac{2}{d-2}\left(\frac{(d-1)}{d(1/w_H + (1+\bar{m}^2))^2} - \frac{d-1}{d(1+\bar{m}^2)^2}\right) \\ & + f_H^2 \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3} \end{aligned}$$

$$\begin{aligned} k\partial_k w_H = {} & 2w_H + w_H f_H\left[\frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3} \right. \\ & \left. + \frac{2}{d-2}\left(\frac{(d-1)}{d(1/w_H + (1+\bar{m}^2))^2} - \frac{d-1}{d(1+\bar{m}^2)^2}\right)\right]. \end{aligned}$$

Fixed points & critical exponents (dynamic sector)

CRC-TR 211

- Model G (strong-scaling FP):

critical exponents:

$$f_G^* = \frac{(4-d)d(1+\bar{m}^2)^3}{4}$$

$$I_d(\bar{m}^{*2}, w_G^*) = -\frac{2}{(N-1)d(1+\bar{m}^{*2})^3}$$

numerical inversion: $\rightsquigarrow w_G^*$

$$x_{\Gamma\phi} = x_\gamma = 2 - \frac{d}{2}$$

- Model G (weak-scaling FP 1):

$$f_G^* = \frac{(4-d)(d-2)d(1+\bar{m}^{2*})^3}{2d(N(1+\bar{m}^{2*}) - \bar{m}^{2*}) - 4}$$

$$w_G^* = 0$$

$$x_{\Gamma\phi} = \frac{(N-1)(4-d)d(1+\bar{m}^{2*})}{d(N(1+\bar{m}^{2*}) - \bar{m}^{2*}) - 2}$$

$$x_\gamma = \frac{(4-d)(d-2)}{d(N(1+\bar{m}^{2*}) - \bar{m}^{2*}) - 2}$$

- Model G (weak-scaling FP 2):

$$f_G^* = \frac{1}{2}(4-d)d(1+\bar{m}^{2*})^3$$

$$w_G^* = \infty$$

$$x_{\Gamma\phi} = 0 \qquad x_\gamma = d$$

- Model H:

$$f_H^* = \frac{4-d}{\frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} + \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}}$$

$$w_H^* = 0$$

$$x_\sigma = \frac{4-d}{\frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} + \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}} \frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} \quad (26)$$

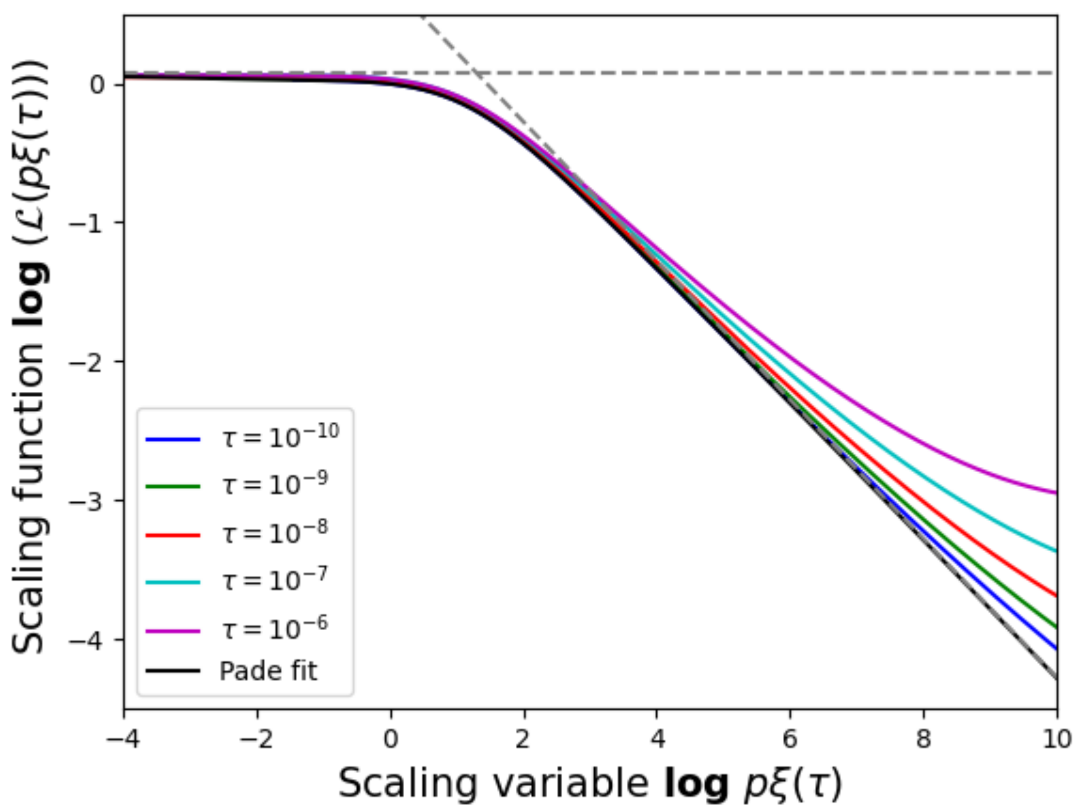
$$x_\eta = \frac{4-d}{\frac{2}{d-2}\frac{d-1}{d(1+\bar{m}^2)^2} + \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3}} \frac{1}{d(d+2)}\frac{1}{(1+\bar{m}^2)^3} \quad (27)$$

Include momentum dependence in Model G

- **Strong-scaling** of charge diffusion coefficient in Model G

$$D_n(\boldsymbol{p},\tau) = s^{2-z} D_n(s\boldsymbol{p}, s^{1/\nu}\tau)$$

$$\rightsquigarrow \quad D_n(p,\tau) \sim \tau^{-\nu(2-z)} \mathcal{L}(\tau^{-\nu}\bar{p}) \qquad \bar{p} = f^+ p$$



dynamic universal scaling function

JR, Ye, Schlichting, von Smekal, arXiv:2403.04573

Dynamic (hyper-)scaling relations

- What can we say about the scaling exponents? Investigate fixed-point equation of f:

$\Gamma^\phi \sim k^{-x_{\Gamma^\phi}}$ $\quad \sigma \sim k^{-x_\sigma}$

$\gamma \sim k^{-x_\gamma}$ $\quad \eta \sim k^{-x_\eta}$

$$f_G \equiv \frac{d\,\Omega_d\,g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\Gamma^\phi_k \gamma_k} \implies \partial_t f_G = (d-4+x_{\Gamma^\phi}+x_\gamma) f_G \overset{f_G^* \neq 0}{\implies} x_{\Gamma^\phi}+x_\gamma = 4-d$$

$$f_H \equiv \frac{d\,\Omega_d\,g^2 T}{(2\pi)^d}\,\frac{k^{d-4}}{\sigma_k \eta_k} \implies \partial_t f_H = (d-4+x_\sigma+x_\eta) f_H \overset{f_H^* \neq 0}{\implies} x_\sigma+x_\eta = 4-d$$

in both Model G and H: **'weak-scaling'** relation

- Try same argument with fixed-point eq. of w:

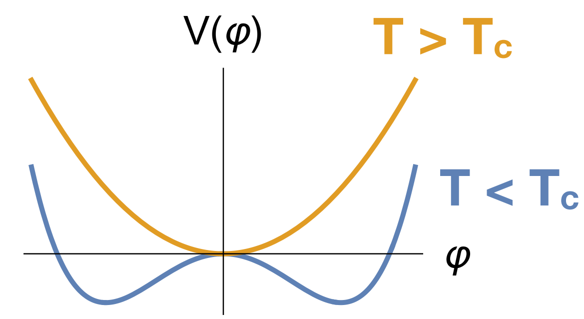
$$w_G \equiv \chi\,\frac{\Gamma^\phi_k}{\gamma_k} \implies \partial_t w_G = (x_\gamma - x_{\Gamma^\phi}) w_G \overset{w_G^* \neq 0}{\implies} x_{\Gamma^\phi} = x_\gamma$$

$$w_H \equiv \rho\,\frac{\lambda_k k^2}{\eta_k} \implies \partial_t w_H = (x_\eta - x_\sigma - 2) w_H \;\not\Longrightarrow$$

doesn't lead to new scaling relation, since we have $w_H^* = 0$ in Model H

only at **'strong-scaling' fixed point** of Model G: also **'strong-scaling'** relation

Statics: $O(4)$ Landau-Ginzburg-Wilson (LGW) functional



$$F = \int d^dx \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{4\chi} n_{ab}n_{ab} \right\}$$

six conserved iso-vector and iso-axialvector charge densities

(higher-order terms irrelevant)

Dynamics: need equations of motion which drive system towards $e^{-F/T}$

[see Landau & Lifshitz, *Statistical Physics, Part 1* (Butterworth-Heinemann, Oxford, 1980)]

fixed by fluctuation-dissipation (Einstein) relations, e.g.:

$$\langle \theta_a(x)\theta_b(x')\rangle = 2\Gamma_0 T \delta_{ab}\delta(x-x')$$

damping/diffusion

stochastic forces

$$\frac{\partial \phi_a}{\partial t} = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \theta_a$$

$$\frac{\partial n_{ab}}{\partial t} = +\gamma \nabla^2 \frac{\delta F}{\delta n_{ab}} + \nabla \cdot \zeta_{ab}$$

Langevin dynamics

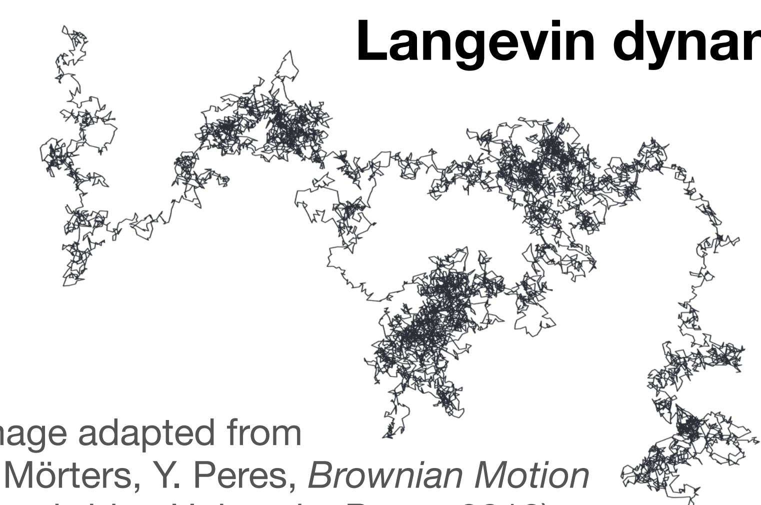


Image adapted from P. Mörters, Y. Peres, *Brownian Motion* (Cambridge University Press, 2010)

Reversible mode couplings

CRC-TR 211

- Besides **dissipative** part, equations of motion also contain **ideal** part

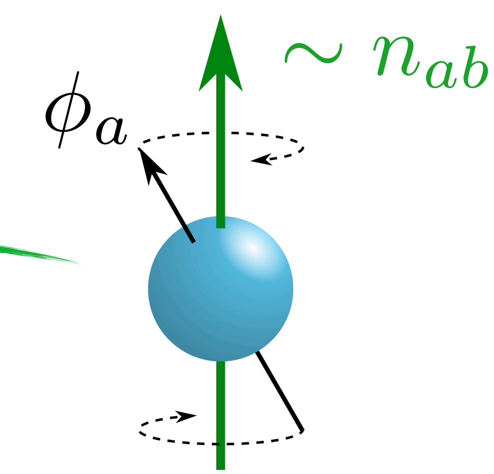
- $O(4)$ Lie algebra:

$$[n_{ab}, n_{cd}] = i\,(\delta_{ac} n_{bd} + \delta_{bd} n_{ac} - \delta_{ad} n_{bc} - \delta_{bc} n_{ad})$$

$$[\phi_a, n_{bc}] = i\,(\delta_{ac}\phi_b - \delta_{ab}\phi_c)$$

(n's generate $O(4)$ transformations)

- **Reversible (ideal) part:** Poisson bracket technique



$$\frac{\partial \phi_a}{\partial t} = \{\phi_a, F\} = \frac{1}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}}$$

$$\frac{\partial n_{ab}}{\partial t} = \{n_{ab}, F\} = \{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{1}{2}\{n_{ab}, n_{cd}\}\frac{\delta F}{\delta n_{cd}}$$

Larmor precession

conserve F exactly

$\{\cdot,\cdot\} \to -i[\cdot,\cdot]$

reversibility

non-Abelian nature of O(4)

Summary of Schwinger-Keldysh formalism

Goal: compute **non-equilibrium** correlation functions

⇝ Path integral requires **doubling number of fields:** L.V. Keldysh, Sov. Phys. JETP **20** (1965) 1018

$$\langle O(t)\rangle = \mathrm{tr}\left(O(t)\rho_0\right)$$ (Heisenberg picture)

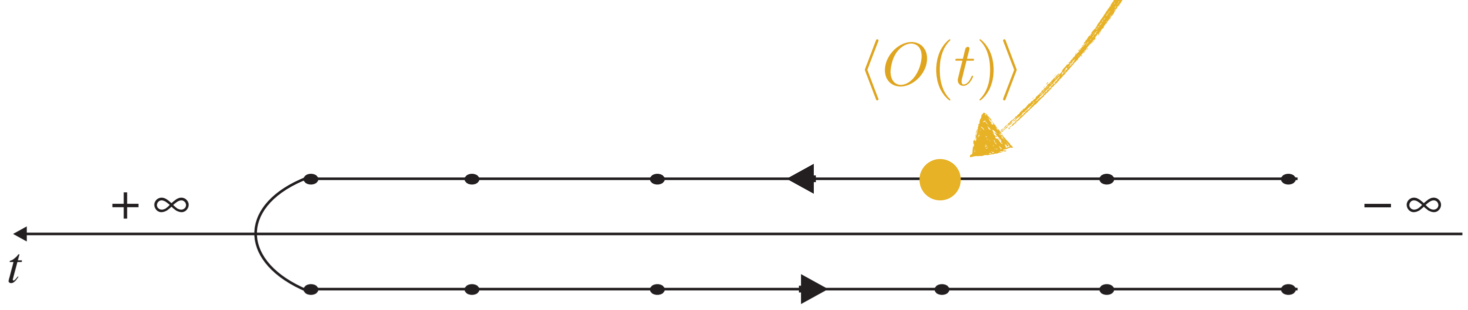
$$= \mathrm{tr}\left(U(-\infty,t)OU(t,-\infty)\rho_0\right)$$

(extend evolution to t = +∞)

$$= \int_{\rho_0} \mathcal{D}\phi^+\,\mathcal{D}\phi^-\, e^{i(S[\phi^+]-S[\phi^-])}\, O\left(\phi^+(t),\phi^-(t)\right)$$

⇝ in particular: **direct access** to **real-time** Green functions

$$G^K(t,t') = i\langle\{\phi(t),\phi(t')\}\rangle$$
$$G^R(t,t') = i\theta(t-t')\langle[\phi(t),\phi(t')]\rangle$$
$$G^A(t,t') = i\theta(t'-t)\langle[\phi(t'),\phi(t)]\rangle$$
$$G^{\tilde{K}}(t,t') = 0$$



⇝ **Causal structure** built into the formalism!

Figure adapted from Kamenev, *Field Theory of Non-Equilibrium Systems* (Cambridge University Press, 2011)

closed-time path

Summary of Martin-Siggia-Rose (MSR) formalism

in classical simulations:

solve Langevin equation

$$\partial_t^2\varphi + \gamma\partial_t\varphi = -\frac{\delta F}{\delta\varphi} + \xi(x)$$

$$\langle\xi(x)\rangle = 0$$
$$\langle\xi(x)\xi(x')\rangle = 2\gamma T\delta(x-x')$$

Introduce (Hubbard) response field $\tilde{\varphi}$

integrate $\tilde{\varphi}$

However, we need:

$$Z = \int \mathcal{D}\tilde{\varphi}\,\mathcal{D}\varphi\, e^{iS[\tilde{\varphi},\varphi]} \qquad S[\tilde{\varphi},\varphi] = \int_x \left[-\tilde{\varphi}\left(\partial_t^2\varphi + \gamma\partial_t\varphi + \frac{\delta F}{\delta\varphi}\right) + i\gamma T\tilde{\varphi}^2\right]$$

fluctuations

deterministic part of eom's

Path-integral formulation

for (real-time) FRG

Dynamic universality classes in more detail

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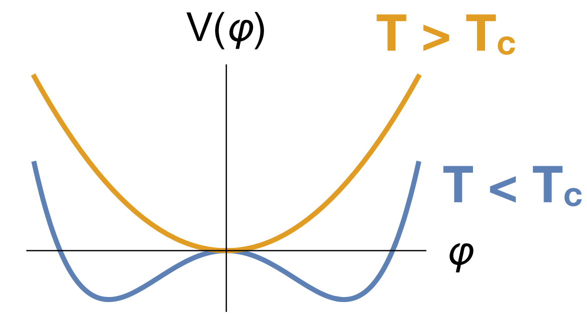
Model A
$z = 2 + c\eta$

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right\}$$

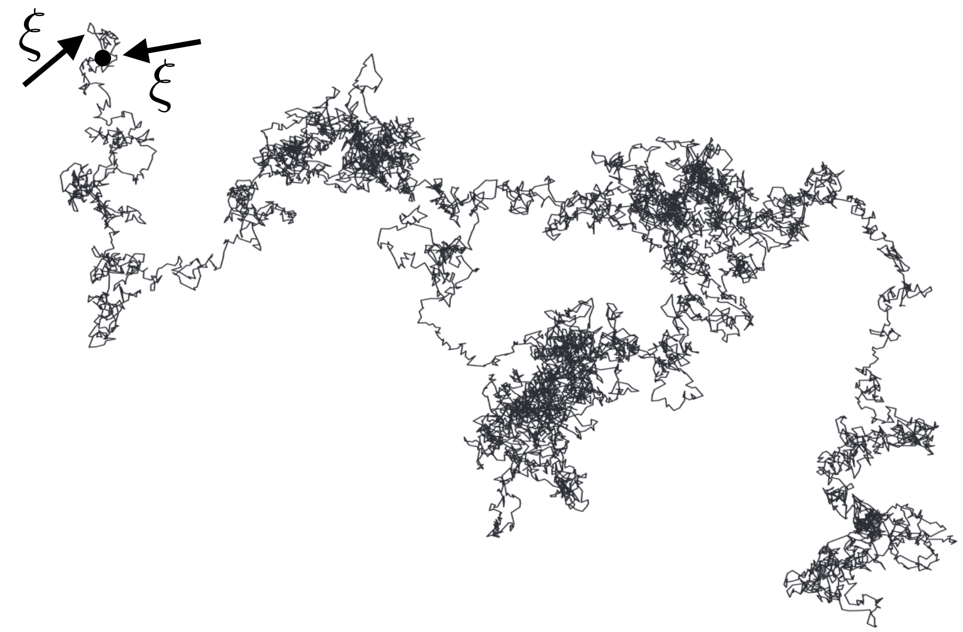


- **Dynamics:** Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noise

describes particle submerged in heat bath:



- No conservation laws here! ⇝ **Model A**
- **Slow modes** determine critical dynamics

(e.g. densities of conserved quantities) (generally true!)

Image adapted from P. Mörters, Y. Peres, *Brownian Motion* (Cambridge University Press, 2010)

Dynamic universality classes in more detail

Model B
$z = 4 - \eta$

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + B\varphi n + \frac{n^2}{2\chi_0} \right\}$$

- **Dynamics:** Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Critical dynamics dominated by diffusion ⇝ **Model B**
- Include hydrodynamic shear modes of energy-momentum tensor ⇝ **Model H**

Dynamic universality classes in more detail

CRC-TR 211

Model C
$z = 2 + \alpha/\nu$

Statics: Landau-Ginzburg-Wilson functional

$$F = \int d^d x \, \left\{ \frac{1}{2} (\vec{\nabla}\varphi)^2 + V(\varphi) + \frac{g}{2}\varphi^2 n + \frac{n^2}{2\chi_0} \right\}$$

- **Dynamics:** Langevin equations of motion

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noises

$$\partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}$$

diffusive!

- Order parameter not conserved but interacts non-linearly with conserved (energy) density ⇝ **Model C**

Wilson’s RG in Minkowski spacetime?

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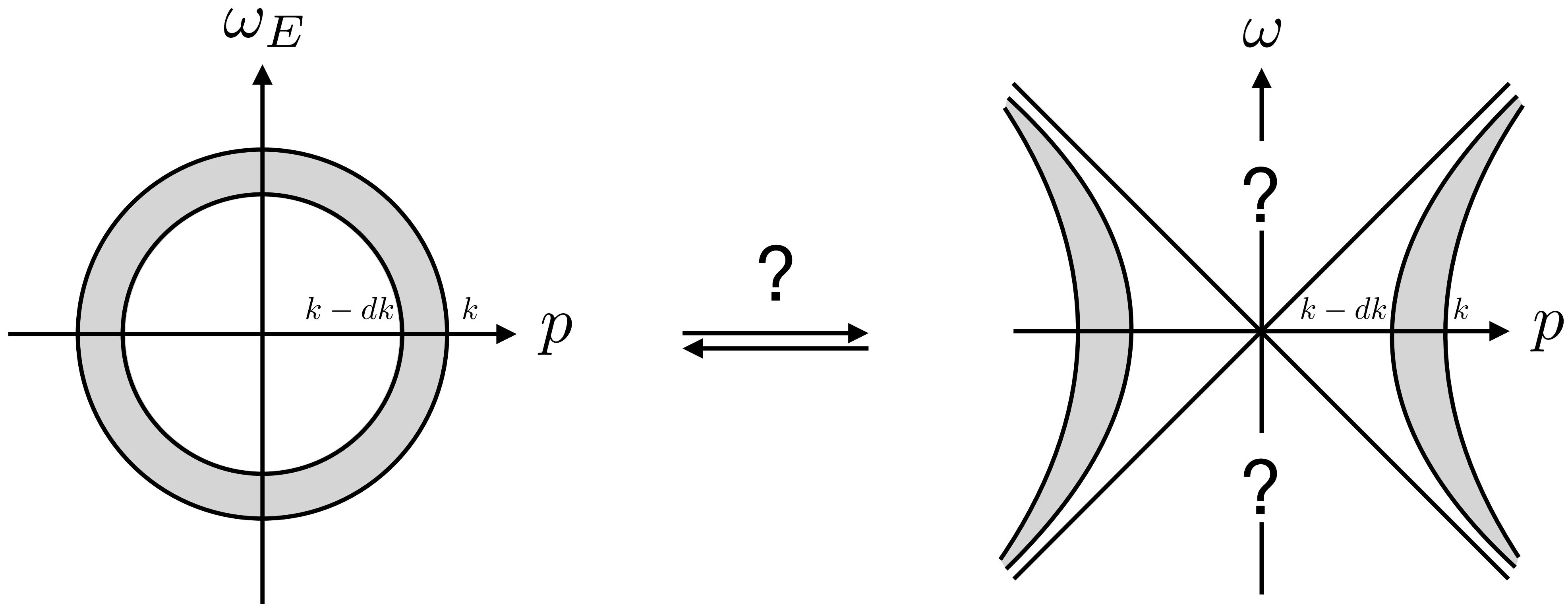

Wilsonian renormalization in Euclidean spacetime

vs.

Wilsonian renormalization in Minkowski spacetime

Conceptually straightforward:
integrate out (hyper-)spheres
no need to worry about causality (at least naively)

Conceptually intricate:
integrate hyperboloids?
timelike momenta?
causal structure of propagators?
...

Problem: Frequency-dependent regulators usually violate **causal structure**

Solution: Interpret regulator as fictitious scale-dependent heat bath
$\Rightarrow$ **Spectral representation**

JR, L. von Smekal, JHEP **10**, 065 (2023)

Causal regulators

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Solution: Observe that regulator is a self-energy

- Self-energies generally inherit **causal structure**

  ⤳ **Spectral representation** from (subtracted) Kramers-Kronig relations

mass-like part (trivially causal)

'spectral density'

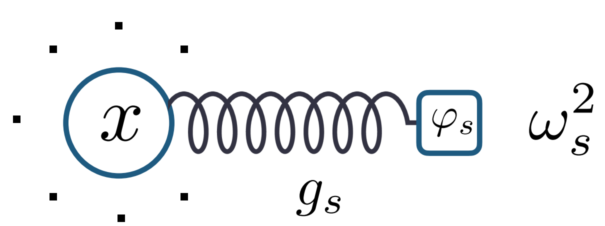
$$R_k^{R/A}(\omega, \boldsymbol{p}) = R_k^{R/A}(0, \boldsymbol{p}) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega', \boldsymbol{p})}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)}$$

$$J_k(\omega, \boldsymbol{p}) = 2\,\mathrm{Im}\, R_k^R(\omega, \boldsymbol{p})$$

JR, von Smekal, JHEP **10**, 065 (2023)
JR, Schweitzer, Sieke, von Smekal, Phys. Rev. D **105**, 116017 (2022)

⇳

- Interpret as coupling to **fictitious heat bath:**
  (Hubbard-Stratonovich transformation)



here: $J_k(\omega) = \pi \sum_s \frac{g_s^2(k)}{\omega_s(k)} \left(\delta(\omega - \omega_s(k)) - \delta(\omega + \omega_s(k))\right)$

⤳ **Physical** only for **positive-semidefinite** spectral densities $J_k(\omega, \boldsymbol{p}) \geq 0 \quad (\omega > 0)$

⤳ Spectral density encodes **spectrum of bath oscillators**

QM example for causal regulator

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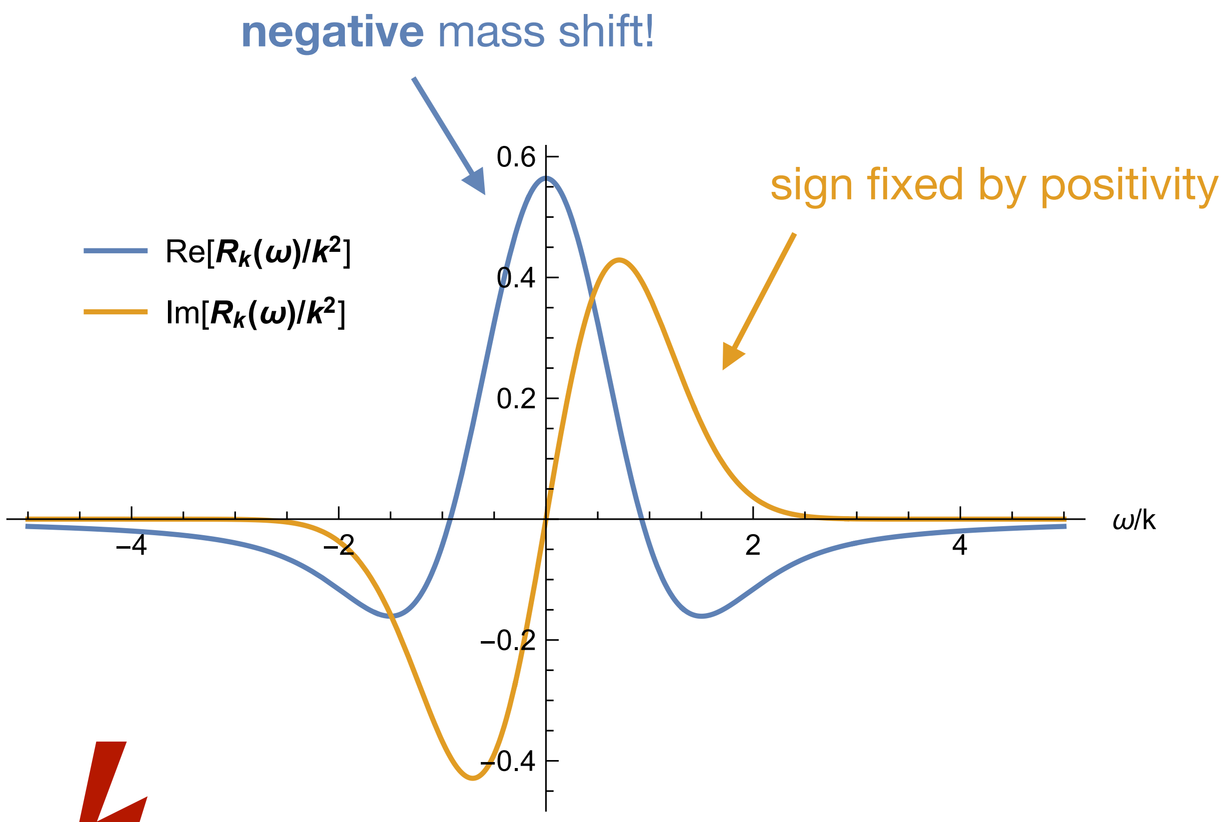
$$R_k^{R/A}(\omega) = R_k^{R/A}(0) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J_k(\omega')}{\omega'((\omega \pm i\varepsilon)^2 - \omega'^2)} \quad \text{in} \quad {\Gamma_k^{(2)}}^R(\omega) = (\omega + i\varepsilon)^2 - m^2 + R_k^R(\omega)$$

- spectral density: $\rightsquigarrow$ **Regulator** (retarded part):

$$J_k(\omega) = 2k\omega\, e^{-\omega^2/k^2} = 2\,\mathrm{Im}\, R_k^R(\omega)$$



- assume UV finiteness:

$$\Delta M_{UV}^2(k) = -R_k^{R/A}(0) + \underbrace{\int_0^\infty \frac{d\omega'}{\pi} \frac{J_k(\omega')}{\omega'}}_{\geq 0} \stackrel{!}{=} 0$$

(positivity)

⇒ IR mass shift:

$$\Delta M_{IR}^2(k) = -R_k^{R/A}(0) < 0$$

is negative!

Solution: choose IR mass shift $\Delta M_{IR}^2(k) > 0$ positive (at cost of **UV finiteness**)