Non-Equilibrium Phase Transitions and Critical Dynamics in QCD

Les Diablerets, 25 September 2024

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JHEP 10 (2023) 065; arXiv:2403.4573; arXiv:2409.14470

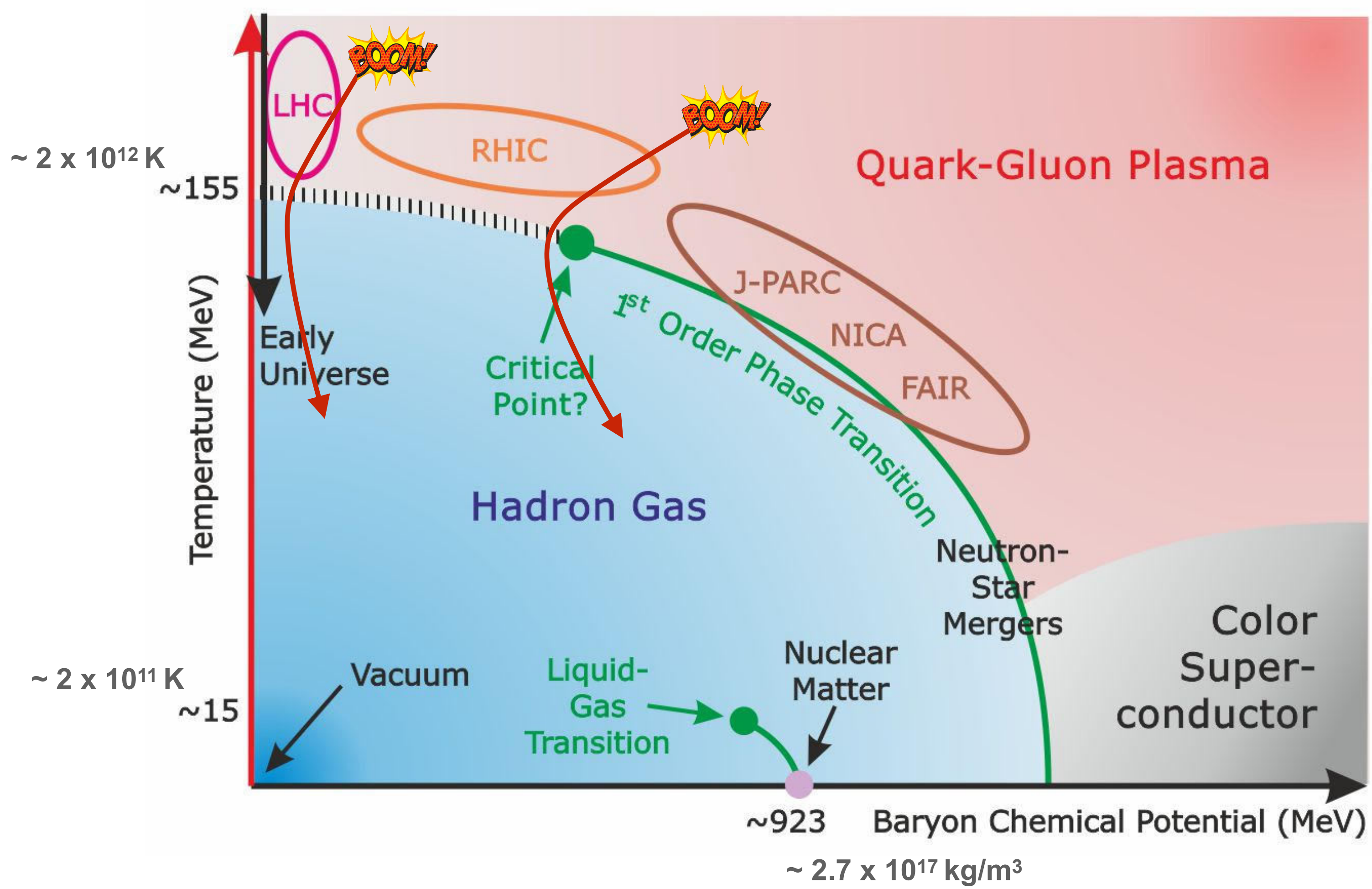
SRS
SwissMAP Research Station in Les Diablerets

12th International Conference on the Exact Renormalization Group 2024 (ERG2024)

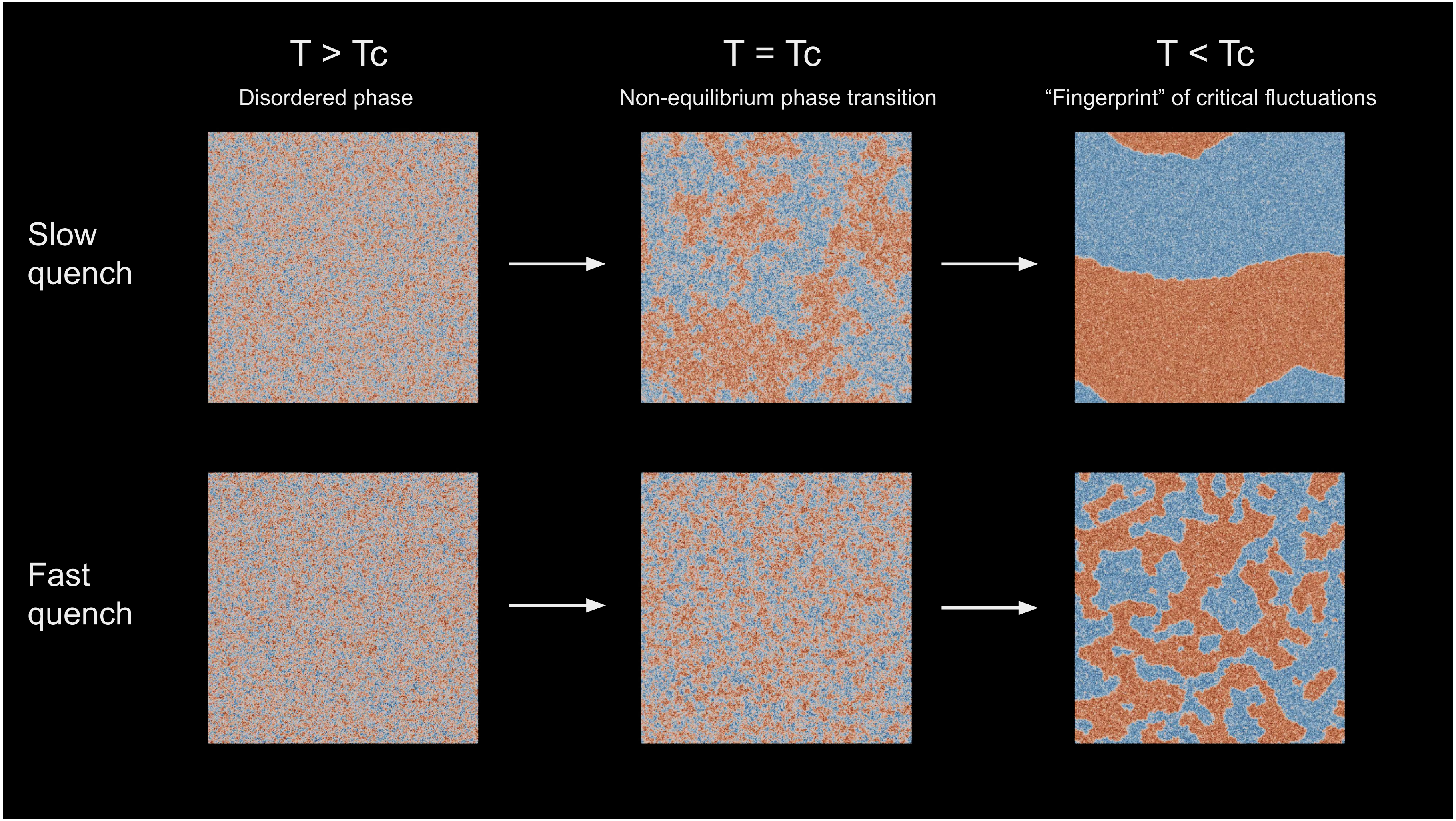
CRC–TR 211
Strong-interaction matter under extreme conditions

Phase Diagram

Strong-Interaction (QCD) Matter



Non-Equilibrium Phase Transitions



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Outline

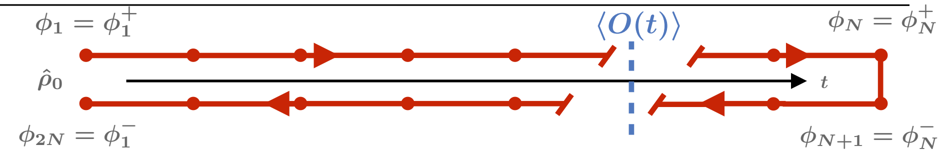
JUSTUS-LIEBIG-UNIVERSITÄT GIESSEN

- Non-Equilibrium, Closed-Time Path, Keldysh
- Open Quantum Systems and Classical Limit
- Non-Equilibrium Phase Transitions
- Dynamic Universality Classes
- Real-Time FRG for Critical Dynamics

U. C. Täuber, *Critical Dynamics: A Field Theory Approach to Equilibrium and Non-Equilibrium Scaling behavior*, Cambridge, 2014

Non-Equilibrium Dynamics

- **path integral on CTP:**



$$Z = \mathrm{tr}\, U_{\mathcal{C}}\, \hat\rho_0$$

$$\langle O \rangle = \int_{\rho_0} \mathcal{D}[\phi^+, \phi^-]\; e^{iS[\phi^+,\phi^-]}\; O(\phi^+, \phi^-)$$

initial state | non-equilibrium dynamics | insert observable

- **Keldysh rotation:**

time ordered, lesser; greater, anti time ordered $\;\to\;$ Keldysh, retarded; advanced

$$\begin{pmatrix} G^T(t,t') & G^<(t,t') \\ G^>(t,t') & G^{\tilde T}(t,t') \end{pmatrix} \longrightarrow \begin{pmatrix} G^K(t,t') & G^R(t,t') \\ G^A(t,t') & 0 \end{pmatrix}$$

- **parametrize:**

$$G^K = G^R \circ F - F \circ G^A$$

distribution function (hermitian): $F(t,t') \overset{\text{equilibrium}}{\longrightarrow} F(t-t')$

- **couple (system) fields ensemble of Gaussian fields (environment E)**

defined by some spectral density $J_{\mathrm{E}}(\omega, \vec{p}\,)$

e.g. ensemble $\rho_{\mathrm{E}}(m^2)$ of Klein-Gordon fields: $= 2\pi\, \mathrm{sgn}(\omega)\theta(p^2)\, \rho_{\mathrm{E}}(p^2)$

- **integrate Gaussian ensemble**

obtain self-energy $\Sigma_{\mathrm{E}}(\omega, \vec{p}\,)$ for system $\begin{pmatrix} 0 & \Sigma_{\mathrm{E}}^{A} \\ \Sigma_{\mathrm{E}}^{R} & \Sigma_{\mathrm{E}}^{K} \end{pmatrix}$

$$\Sigma_{\mathrm{E}}^{R/A}(\omega, \vec{p}\,) = \int_0^\infty \frac{d\omega'}{2\pi}\, \frac{2\omega' J_{\mathrm{E}}(\omega', \vec{p}\,)}{(\omega \pm i\varepsilon)^2 - \omega'^2}$$

E: heat bath at temperature T $\rightsquigarrow$

$$\Sigma_{\mathrm{E}}^{K}(\omega, \vec{p}\,) = \coth\left(\frac{\omega}{2T}\right) \underbrace{\left(\Sigma_{\mathrm{E}}^{R}(\omega, \vec{p}\,) - \Sigma_{E}^{A}(\omega, \vec{p}\,)\right)}$$

$$= 2\mathrm{i}\, \mathrm{Im}\, \Sigma_{\mathrm{E}}^{R}(\omega, \vec{p}\,) = -\mathrm{i}\, J_{\mathrm{E}}(\omega, \vec{p}\,)$$

Keldysh Action

- **open quantum system:**

plus interactions

$$S_0[\Phi] =$$

$$\int \frac{d^4p}{(2\pi)^4}\,\Phi^T(-\omega,\vec{p}) \begin{pmatrix} 0 & \omega^2-\omega_p^2-\Sigma_E^A(\omega,\vec{p}) \\ \omega^2-\omega_p^2-\Sigma_E^R(\omega,\vec{p}) & \mathrm{i}\coth\left(\frac{\omega}{2T}\right) J_E(\omega,\vec{p}) \end{pmatrix} \Phi(\omega,\vec{p})$$

- **(an-)harmonic oscillator in Ohmic bath:**

$$\Phi = \begin{pmatrix} \varphi^c \\ \varphi^q \end{pmatrix}$$

$$J_E(\omega) = 2\gamma\omega\,\theta(\Lambda-|\omega|)$$

for $|\omega| \ll \Lambda$

- **Caldeira-Leggett model:**

$$S_0[\Phi] = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\Phi^T(-\omega) \begin{pmatrix} 0 & \omega^2-\mathrm{i}\gamma\omega-\omega_0^2 \\ \omega^2+\mathrm{i}\gamma\omega-\omega_0^2 & 2\mathrm{i}\gamma\omega\coth\left(\frac{\omega}{2T}\right) \end{pmatrix} \Phi(\omega)$$

Classical Limit

- **on Keldysh contour:**

$$\varphi^{\pm} = \varphi^c \pm \hbar\,\varphi^q$$

- **equilibrium distribution function:**

$$\hbar\,\omega \ll T$$

$$F(\omega) \;=\; \coth\left(\frac{\hbar\,\omega}{2T}\right) \;\longrightarrow\; \frac{2T}{\hbar\,\omega}$$

Rayleigh-Jeans limit

- **Keldysh action:**

with interactions: $\omega_0^2 \varphi^c \to V'(\varphi^c)$, classical force

$$S_0[\Phi] \;\to$$

$$\frac{1}{\hbar}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,(\varphi^c, \hbar\,\varphi^q)\begin{pmatrix} 0 & \omega^2 - i\gamma\omega - \omega_0^2 \\ \omega^2 + i\gamma\omega - \omega_0^2 & 4i\gamma\dfrac{T}{\hbar}\end{pmatrix}\begin{pmatrix}\varphi^c \\ \hbar\,\varphi^q\end{pmatrix}$$

$$= \int dt\,\Big\{2\varphi^q\big(-\ddot{\varphi}^c - \gamma\dot{\varphi}^c - V'(\varphi^c)\big) \;+\; 4i\gamma T\,(\varphi^q)^2\Big\}$$

classical Martin-Siggia-Rose (MSR) action

Langevin Dynamics

- **dissipative equation of motion:**

$$\ddot{\varphi}^c = -\gamma\dot{\varphi}^c - V'(\varphi^c) + \xi(t)$$

friction force, kinetic coefficient γ (drag)

- **stochastic force:**

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\xi(t') \rangle = 2\gamma T\,\delta(t-t')$$

Einstein relation (classic example of FDR)

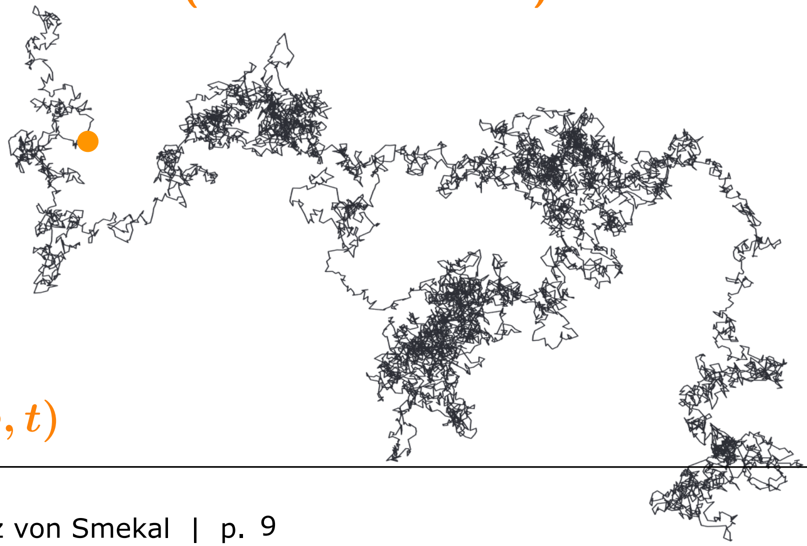
strength of random force (Brownian motion)

- **restricted partition function:**

$$\langle \delta(\varphi^c(t) - \varphi) \rangle =$$

observable $O(\varphi^c)$

$$Z\Big|_{\varphi^c(t)\,=\,\varphi} = \mathcal{P}(\varphi, t)$$



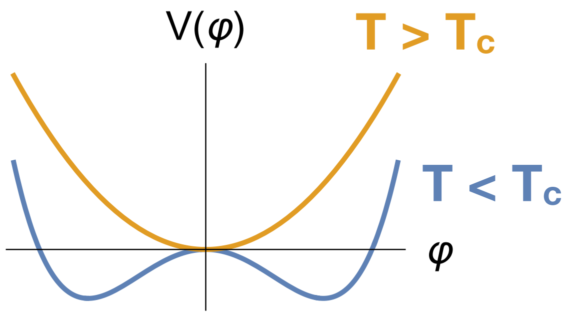
probability distribution of φ at time t

$\rightsquigarrow$ derive Fokker-Planck equation for $\mathcal{P}(\varphi, t)$

Model A

- **replace potential by Landau-Ginzburg-Wilson functional:**

$$F[\varphi] = \int \mathrm{d}^d x \, \left\{ \frac{1}{2} (\vec{\nabla}\varphi)^2 + V(\varphi) \right\}$$



for statics, with Z_2 SSB

- **dissipative equation of motion:**

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi(x)$$

or 1st order form

$$\partial_t \varphi = \pi$$
$$\partial_t \pi = -\gamma \pi - \frac{\delta F}{\delta \varphi} + \xi(x)$$

- **stochastic force:**

$$\langle \xi(x) \xi(x') \rangle = 2\gamma T \, \delta(x - x')$$

- **spectral functions from classical FDR:**

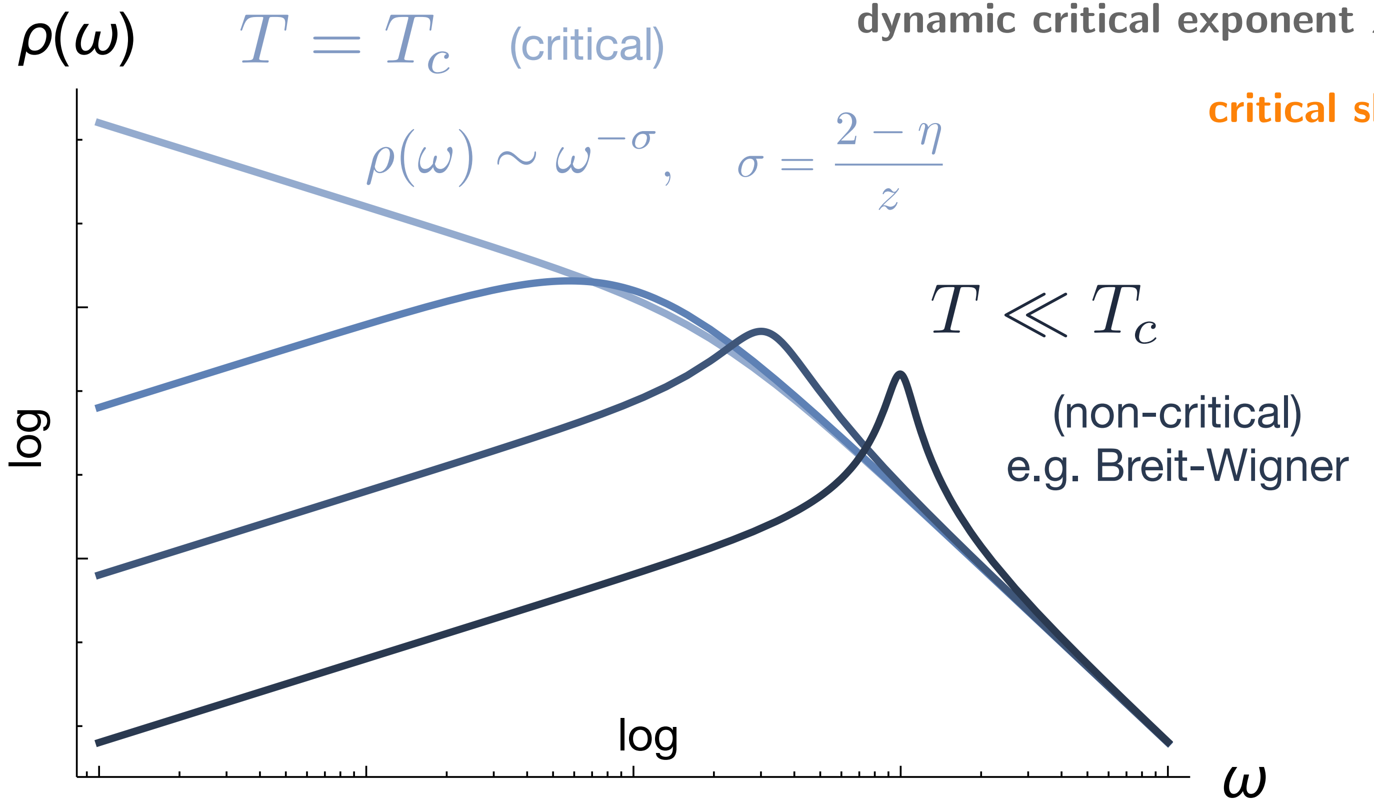
$$\rho(t, \vec{x}) = -\frac{1}{T} \, \partial_t \langle \varphi(t, \vec{x}) \varphi(0,0) \rangle = -\frac{1}{T} \, \langle \pi(t, \vec{x}) \varphi(0,0) \rangle$$

Critical Spectral Functions

- **obtain universal dynamic scaling functions**

dynamic critical exponent z: $\xi_t \sim \xi^z$

critical slowing down

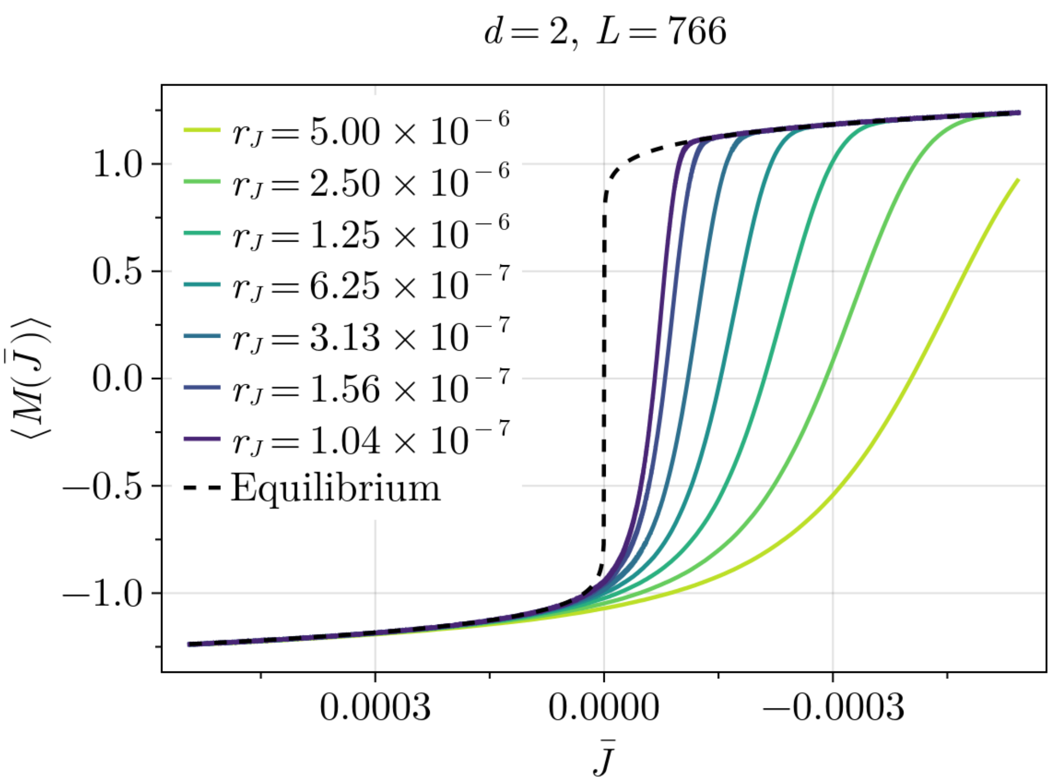


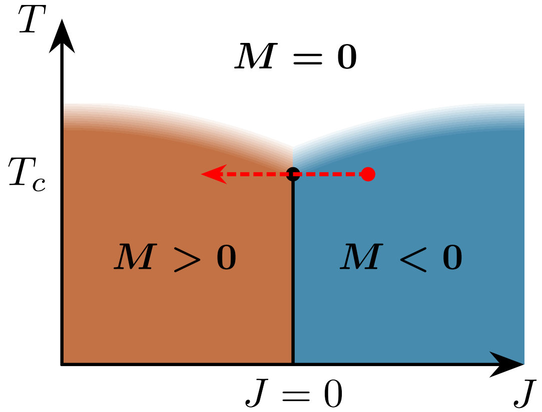
Schlichting, Smith, LvS, NPB 950 (2020) 114868

Schweitzer, Schlichting, LvS, NPB 960 (2020) 115165; NPB 984 (2022) 115944

Non-Equilibrium Phase Transitions

- **trans-critical linear magnetic quench:** $J(t) = -r_J\, t$
- **measure magnetization:**

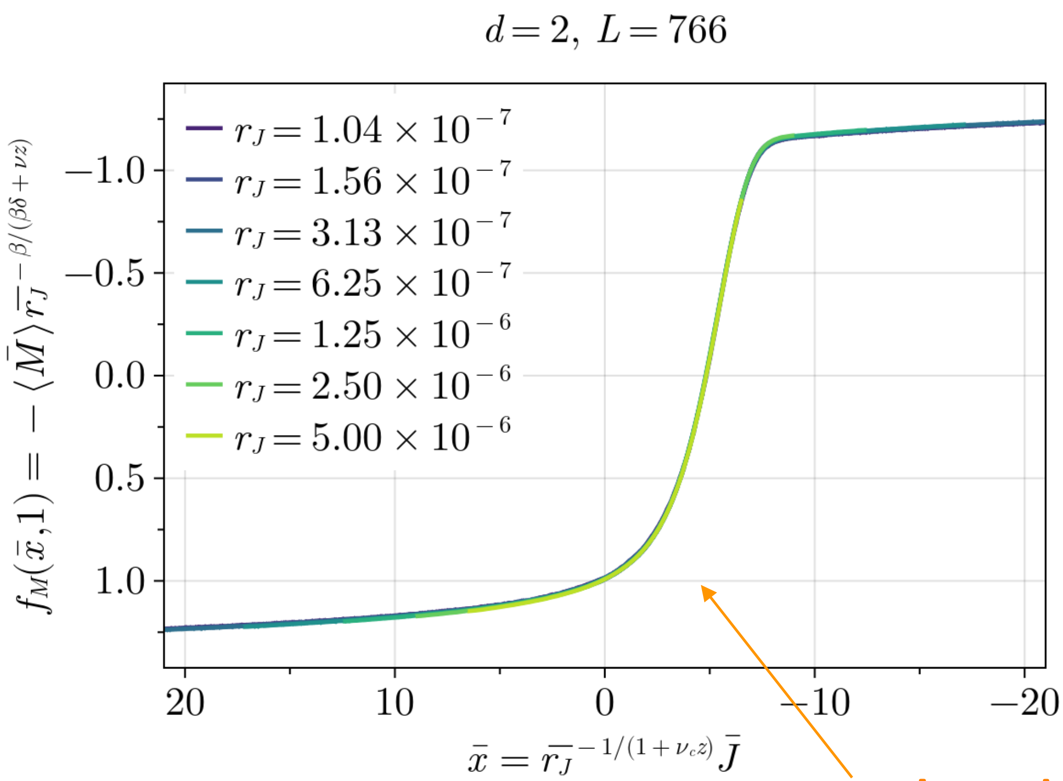


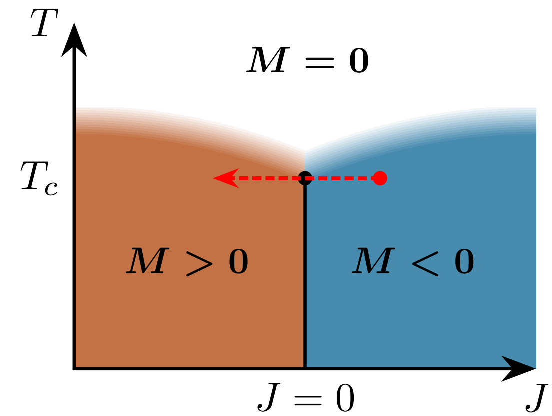


system falls out of equilibrium when $\dot{\xi}_t \approx 1$

adiabatically: $\xi_t \sim J^{-\frac{\nu z}{\beta\delta}}$

- **trans-critical linear magnetic quench:** $J(t) = -r_J\, t$
- **rescale:**





at Kibble-Zurek time:

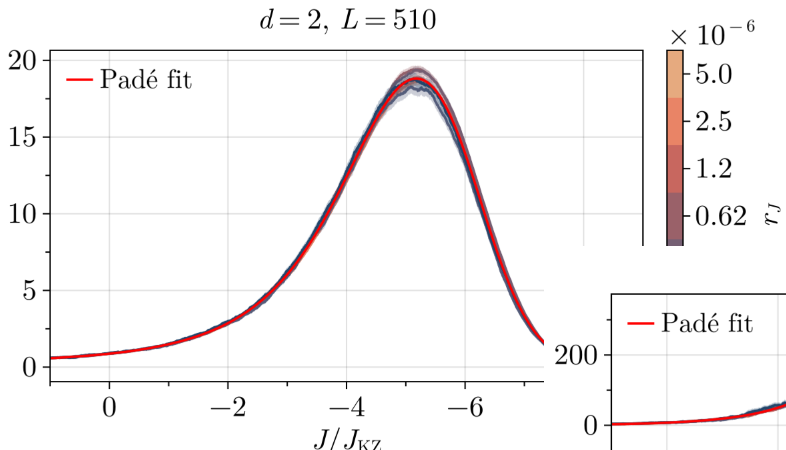
$$J \sim r_J^{\,1/\left(1+\frac{\nu z}{\beta\delta}\right)}$$

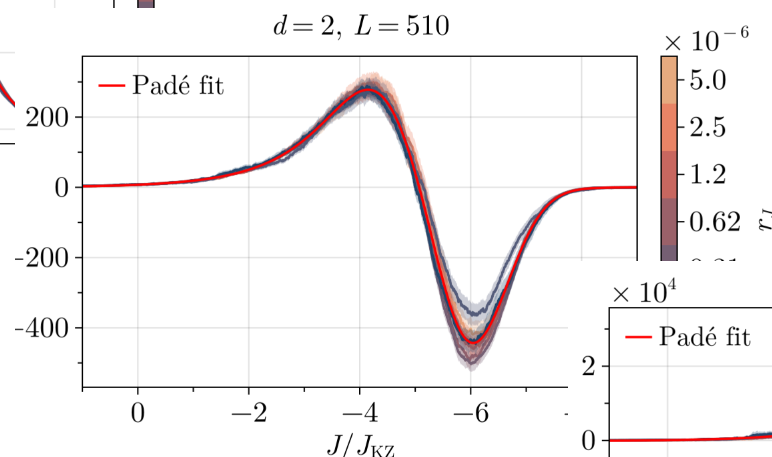
Kibble-Zurek scaling

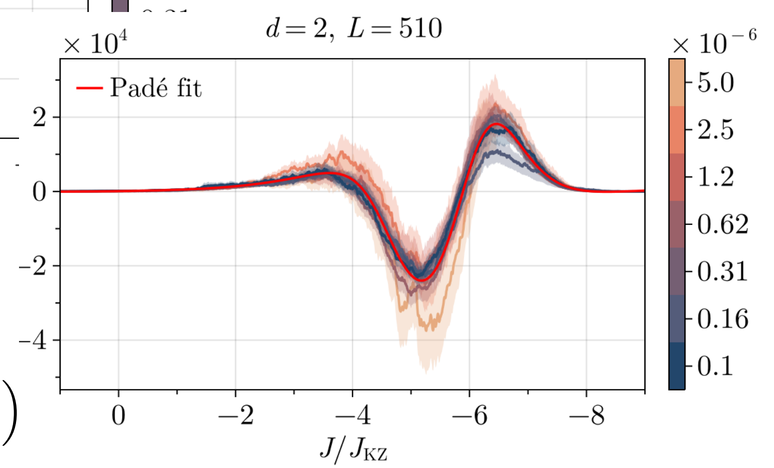
universal non-equilibrium scaling function

Non-Equilibrium Scaling Functions

- **susceptibility, skewness, kurtosis:**







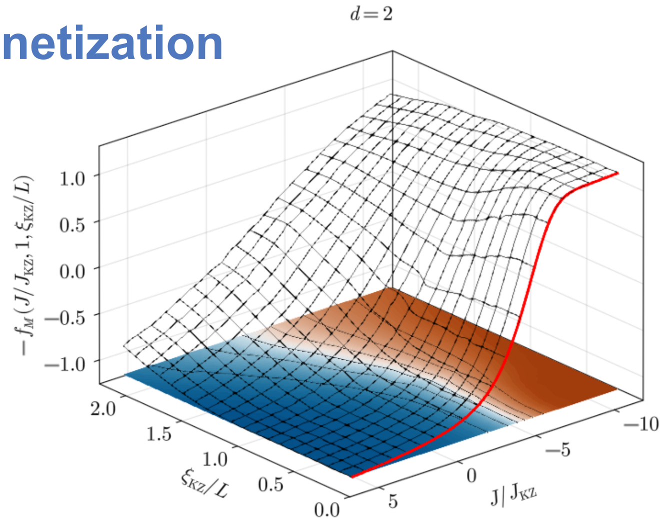
$$\chi = \frac{V}{T}\left(\langle M^2\rangle - \langle M\rangle^2\right)$$

$$\kappa_3 = \left(\frac{V}{T}\right)^2 \left(\langle M^3\rangle - 3\langle M^2\rangle\langle M\rangle + 2\langle M\rangle^3\right)$$

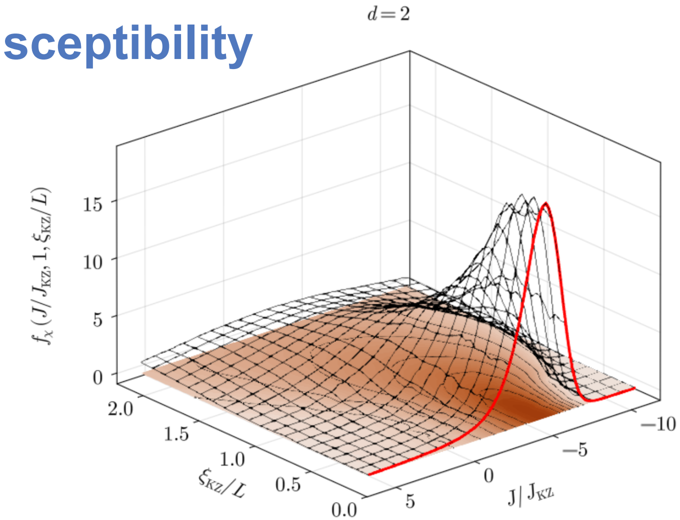
$$\kappa_4 = \left(\frac{V}{T}\right)^3 \left(\langle M^4\rangle - 4\langle M^3\rangle\langle M\rangle - 3\langle M^2\rangle^2 + 12\langle M^2\rangle\langle M\rangle^2 - 6\langle M\rangle^4\right)$$

Non-Equilibrium Finite-Size Scaling

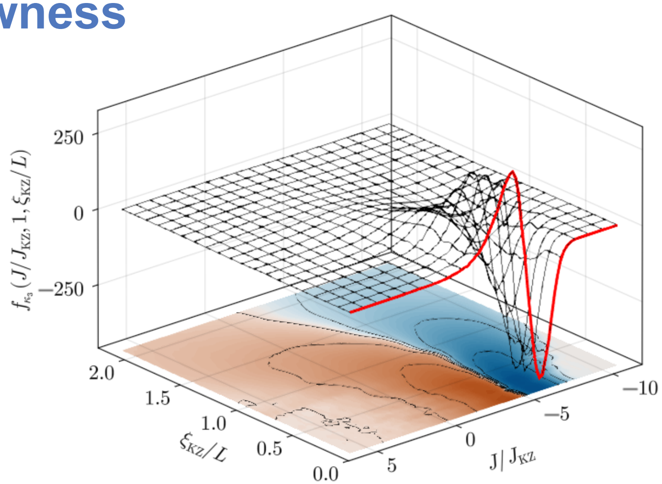
magnetization



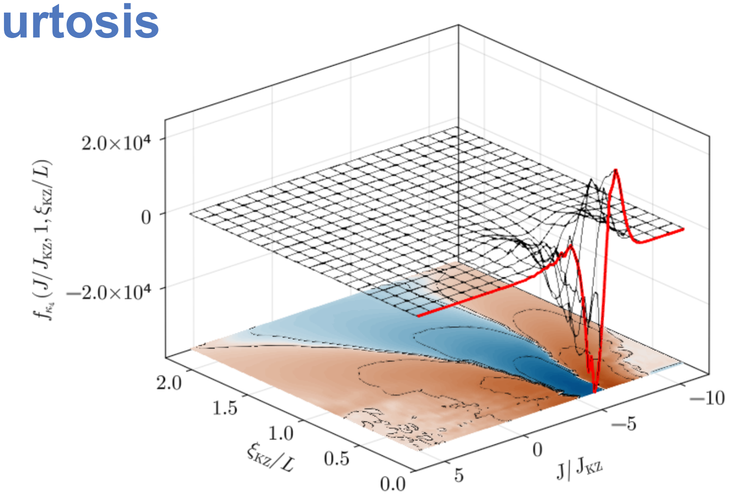
susceptibility



skewness



kurtosis

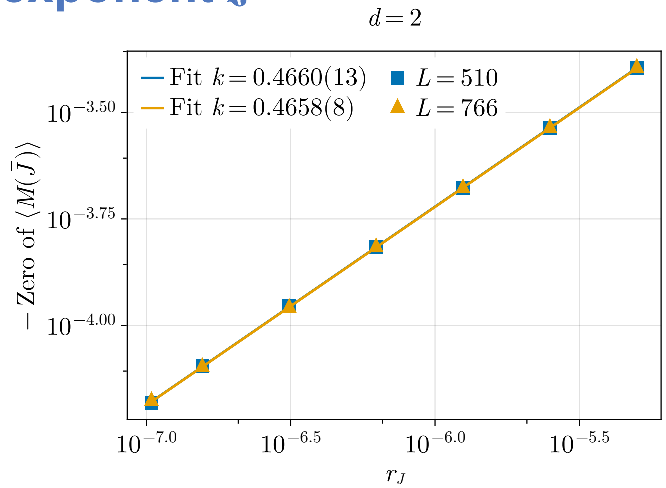


Kibble-Zurek Scaling

- **allows accurately determining dynamic critical exponent z**

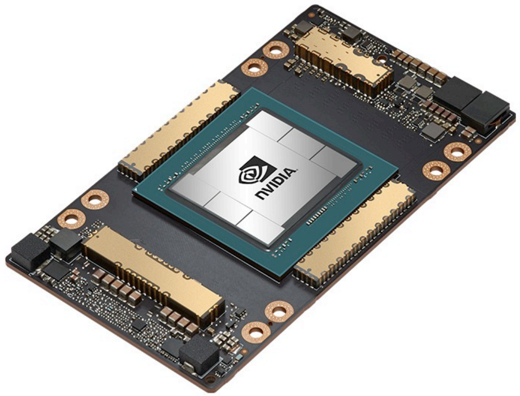
  Sieke, Harhoff, Schlichting, LvS, in preparation

z	$d = 2$	$d = 3$
KZ scaling	2.142(49)	1.949(54)
Crit. SFs	2.10(4)[1]	1.92(11)[1]
Monte Carlo	2.1667(5)[2]	2.0245(15)[3]
ϵ expansion	2.14(2)[4]	2.0236(8)[4]
FRG	2.15[5]	2.024[5]
Experiment	2.09(6) (95% confidence)[6]	1.96(11)[7]



obtain from $J(M=0) \sim r_J^{\,1/\left(1+\frac{\nu z}{\beta\delta}\right)}$

not necessary to know Kibble-Zurek time



[1]Schweitzer, Schlichting, LvS (2020); [2]Nightingale, Blöte (2000); [3]Hasenbusch (2020); [4]Adzhemyan et al. (2022); [5]Duclut, Delamotte (2017); [6]Dunlavy, Venus (2005); [4]Livet et al. (2018)

Dynamic Universality Classes

- **classified as Model A, B, C,… — Model J**

  Hohenberg, Halperin (1977)

- **describe full set of critical/hydrodynamic modes**

  order parameter, Goldstone modes, conserved charges, reversible mode couplings

- **critical dynamics in QCD:**

  - **chiral phase transition: Model G — Rajagopal, Wilczek (1993)**

    classical-statistical: Florio, Grossi, Soloviev, Teaney, PRD **105** (2022) 054512
    Florio, Grossi, Teaney, PRD **109** (2024) 054037

    FRG: Roth, Ye, Schlichting, LvS, arXiv:2403.04573

  - **QCD critical point: Model H — Son, Stephanov (2004)**

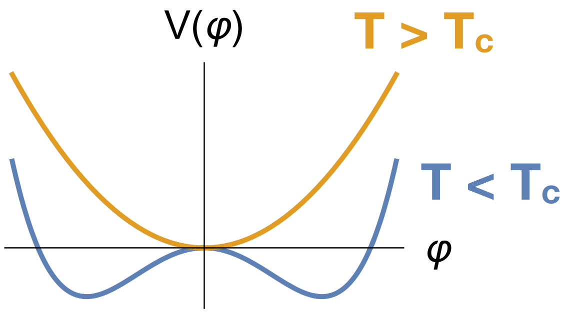
    classical-statistical: Chattopadhyay, Ott, Schaefer, Skokov, PRL **133** (2024) 032301

    FRG: Chen, Tan, Fu, arXiv:2406.00679
    Roth, Ye, Schlichting, LvS, arXiv:2409.14470

Dynamic Universality Classes

- **Landau-Ginzburg-Wilson functional:**

$$F[\varphi] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right\}$$



for statics, with Z_2 SSB

- **Langevin dynamics:**

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi$$

Gaussian white noise

- **no conservation laws**

FRG: Canet, Chate, J. Phys. A **40** (2007) 1937,
Canet, Chate, Delamotte, J. Phys. A **44** (2011) 495001
Duclut, Delamotte, PRE **95** (2017) 012107
Roth, LvS, JHEP 10 (2023) 065
Batini, Grossi, Wink, PRD **108** (2023) 125021

Model A
$z = 2 + c\eta$

Dynamic Universality Classes

Son, Stephanov, PRD 70 (2004) 056001

- **LGW functional:**

$$F[\varphi, n] = \int \mathrm{d}^d x \, \Big\{ \frac{1}{2} (\vec{\nabla}\varphi)^2 + V(\varphi) \; + B\,\varphi n \; + \frac{n^2}{2\chi_n} \Big\}$$

- **equations of motion:**

(chiral) order parameter

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi(x)$$

$$\tau_R \partial_t^2 n + \partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}(x)$$

conserved (baryon) density

with linear coupling B to conserved (baryon) density $n(x)$ (non-critical)

$$\langle \xi(x) \xi(x') \rangle_\beta = 2\gamma T \delta(x - x')$$

$$\langle \zeta^i(x) \zeta^j(x') \rangle_\beta = 2\bar{\lambda} T \delta^{ij} \delta(x - x')$$

- **slow critical mode diffusive**

Model B

$z = 4 - \eta$

FRG: Roth, LvS, JHEP 10 (2023) 065

Dynamic Universality Classes

Berdnikov, Rajagobal, PRD 62 (2000) 105017

- **LGW functional:**

$$F[\varphi, n] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) + \frac{g}{2}\varphi^2 n + \frac{n^2}{2\chi_n} \right\}$$

- **equations of motion:**

(chiral) order parameter

with quadratic coupling g to conserved (energy) density $n(x)$

$$\partial_t^2 \varphi + \gamma \partial_t \varphi = -\frac{\delta F}{\delta \varphi} + \xi(x)$$

$$\langle \xi(x)\xi(x') \rangle_\beta = 2\gamma T \delta(x - x')$$

$$\tau_R \partial_t^2 n + \partial_t n = \bar{\lambda} \vec{\nabla}^2 \frac{\delta F}{\delta n} + \vec{\nabla} \cdot \vec{\zeta}(x)$$

$$\langle \zeta^i(x)\zeta^j(x') \rangle_\beta = 2\bar{\lambda} T \delta^{ij} \delta(x - x')$$

conserved (energy) density

FRG: Mesterházy, Stockemer, Palhares, Berges, PRB 88 (2013) 174301
Roth, LvS, JHEP 10 (2023) 065

Model C
$z = 2 + \alpha/\nu$

Dynamic Universality Classes

- **LGW functional:**

now static O(4) universality

$$F[\varphi,n] = \int \mathrm{d}^d x \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 + \frac{1}{4\chi_n} n_{ab} n_{ab} \right\}$$

- **equations of motion:**

with conserved iso-vector and iso-axialvector charge densities

(chiral) order parameter

$$\partial_t \phi_a = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \xi_a + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}}$$

$$\partial_t n_{ab} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + \vec{\nabla}\cdot\vec{\zeta}_{ab} + g\{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab}, n_{cd}\}\frac{\delta F}{\delta n_{cd}}$$

conserved O(4) densities

aka: SSS Model

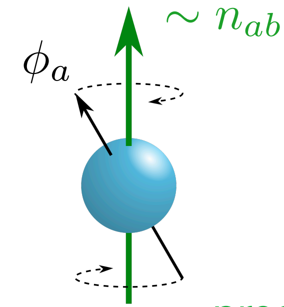
Sasvári, Schwabl, Szépfalusy, Physica A 81 (1975) 108

Model G

$z = d/2$

Dynamic Universality Classes

- **equations of motion:**
  with reversible mode couplings



Larmor precession

$$\partial_t \phi_a = -\Gamma_0 \frac{\delta F}{\delta \phi_a} + \xi_a + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}}$$

$$\partial_t n_{ab} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + \vec{\nabla}\cdot\vec{\zeta}_{ab} + g\{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab}, n_{cd}\}\frac{\delta F}{\delta n_{cd}}$$

reversible (ideal) time evolution

- **Poisson brackets (commutators):**

$$\{\phi_a, n_{bc}\} = \phi_b \delta_{ac} - \phi_c \delta_{ab}$$

$$\{n_{ab}, n_{cd}\} = \delta_{ac} n_{bd} + \delta_{bd} n_{ac} - \delta_{ad} n_{bc} - \delta_{bc} n_{ad}$$

Model G
$z = d/2$

Dynamic Universality Classes

- **equations of motion:**
  with reversible mode couplings

advection

convection

$$\partial_t \phi = \sigma \vec{\nabla}^2 \frac{\delta F}{\delta \phi} + \xi + \frac{g}{2} \{\phi, j_l\} \frac{\delta F}{\delta j_l}$$

$$\partial_t j_l = \mathcal{T}_{lm} \left[\eta \vec{\nabla}^2 \frac{\delta F}{\delta j_m} + \zeta_m + g \{j_m, \phi\} \frac{\delta F}{\delta \phi} + \frac{g}{2} \{j_m, j_n\} \frac{\delta F}{\delta j_n} \right]$$

reversibility

$$\langle \xi(x)\xi(x')\rangle_\beta = -2\sigma T \vec{\nabla}^2 \delta(x-x')$$

$$\langle \zeta_l(x)\zeta_m(x')\rangle_\beta = -2\eta T \delta_{lm} \vec{\nabla}^2 \delta(x-x')$$

- **Poisson brackets:**

$$\{\phi(\vec{x}), j_l(\vec{x}')\} = \phi(\vec{x}') \frac{\partial}{\partial x'_l} \delta(\vec{x}-\vec{x}')$$

$$\{j_l(\vec{x}), j_m(\vec{x}')\} = \left[j_l(\vec{x}') \frac{\partial}{\partial x'_m} - j_m(\vec{x}) \frac{\partial}{\partial x_l} \right] \delta(\vec{x}-\vec{x}')$$

Model H

$z = 4 - \eta - x_\sigma$

FRG: Chen, Tan, Fu, arXiv:2406.00679
Roth, Ye, Schlichting, LvS, arXiv:2409.14470

Real-Time FRG

- **causal regulators:**

$$\Phi = (\phi^c, \phi^q)^T$$

$$\Delta S_k[\Phi] = \frac{1}{2}\int_{xy} \Phi(x)^T R_k(x-y)\Phi(y)$$

$$R_k(\omega, \boldsymbol{p}) = \begin{pmatrix} 0 & R_k^R(\omega, \boldsymbol{p}) \\ R_k^A(\omega, \boldsymbol{p}) & R_k^K(\omega, \boldsymbol{p}) \end{pmatrix}$$

- **introduce fictitious heat-bath *J*:**

$$R^{R/A}(\omega, \boldsymbol{p}) = R^{R/A}(0, \boldsymbol{p}) - \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\omega^2 J(\omega', \boldsymbol{p})}{\omega'((\omega \pm i\epsilon)^2 - \omega'^2)}$$

frequency-independent regulator

subtracted spectral representation (from Kramers-Kronig relations)

with FRG scale k dependent

$$J_k(\omega, \boldsymbol{p}) = \pm 2\, \mathrm{Im}\, R_k^{R/A}(\omega, \boldsymbol{p})$$

and $R_k^K(\omega, \boldsymbol{p})$ from FDR

- **maintain causality, Lorentz invariance, UV and IR finiteness — except positivity**

Braun et al., SciPost Phys.Core 6 (2023) 061
Roth, LvS, JHEP 10 (2023) 065

Truncation Schemes

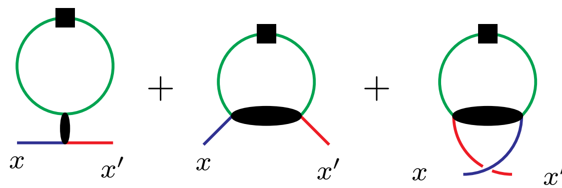
- **Models A & C:**

expand about infrared minimum $\phi^c = \phi^c_0$

combined vertex and loop expansion

- **2-point function:** two-loop exact

$$\partial_k \Gamma_k^{cq}(x,x') = -\frac{i}{2}\Big\{$$

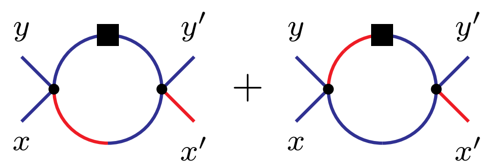


$$\Big\}$$

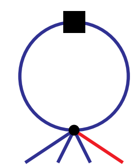
reduce number of loops

- **4-point function:** one-loop exact

$$\partial_k V_k^{cl,A}(x,x') = -i \int_{\substack{x-y,\\ x'-y'}} \Big\{$$

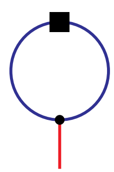


$$\Big\} - \frac{i}{6}\int_{\substack{x-y,\\ x'-y'}}$$



- **higher *n*-point functions:** local vertices

$$\partial_k V_k'(\varphi) = -\frac{i}{\sqrt{8}}$$



for quantum mechanical applications, see:

S. Huelsmann, S. Schlichting, P. Scior, Phys. Rev. D **102**, 096004 (2020)

J. V. Roth, D. Schweitzer, L. J. Sieke, L.v.S., Phys. Rev. D **105**, 116017 (2022)

Truncation Schemes

- **Models A & B:** expand around scale-dependent minimum $\phi^c_{0,k}$

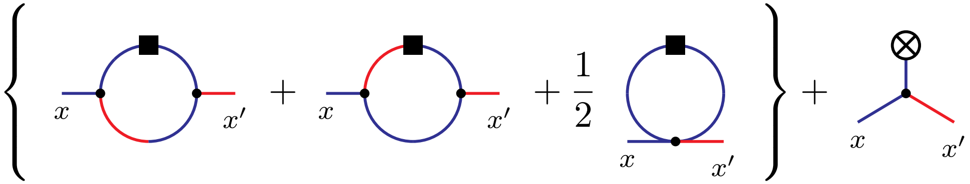
  - **effective average action:**

$$\Gamma_k = \frac{1}{2}\int_{xx'} \left(\phi^c(x) - \phi^c_{0,k}, \phi^q(x)\right) \begin{pmatrix} 0 & \Gamma^{cq}_k(x-x') \\ \Gamma^{qc}_k(x-x') & \Gamma^{qq}_k(x-x') \end{pmatrix} \begin{pmatrix} \phi^c(x') - \phi^c_{0,k} \\ \phi^q(x') \end{pmatrix}$$
$$- \frac{\kappa_k}{\sqrt{8}}\int_x \left(\phi^c(x) - \phi^c_{0,k}\right)^2 \phi^q(x) - \frac{\lambda_k}{12}\int_x \left(\phi^c(x) - \phi^c_{0,k}\right)^3 \phi^q(x)$$

one order less in combined expansion

  - **2-point function:** one-loop exact

$$\partial_k \Gamma^{cq}_k(x,x') = -i \left\{ \dots + \dots + \frac{1}{2} \dots \right\} + \dots$$



  - **4-point and higher:** local vertices

Critical Spectral Functions

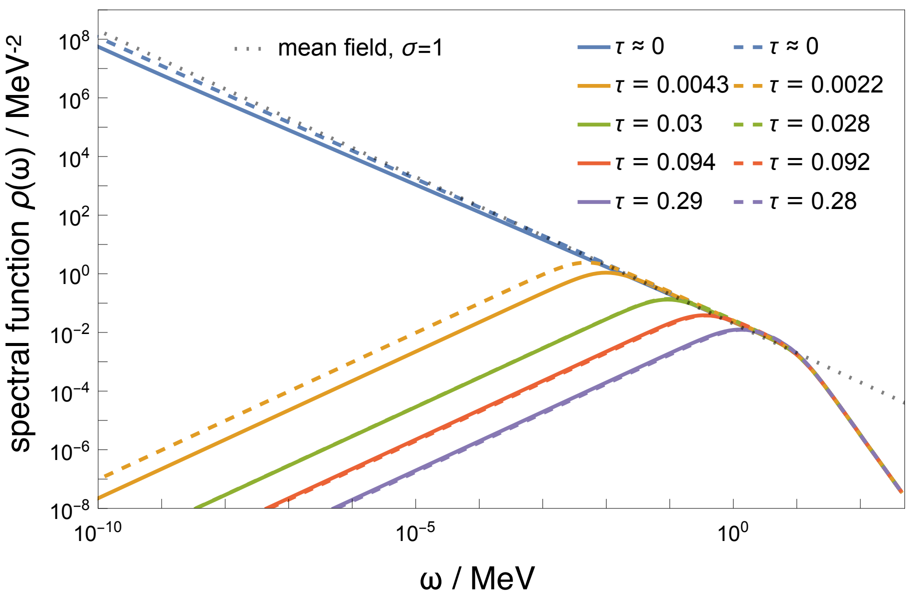
Model A

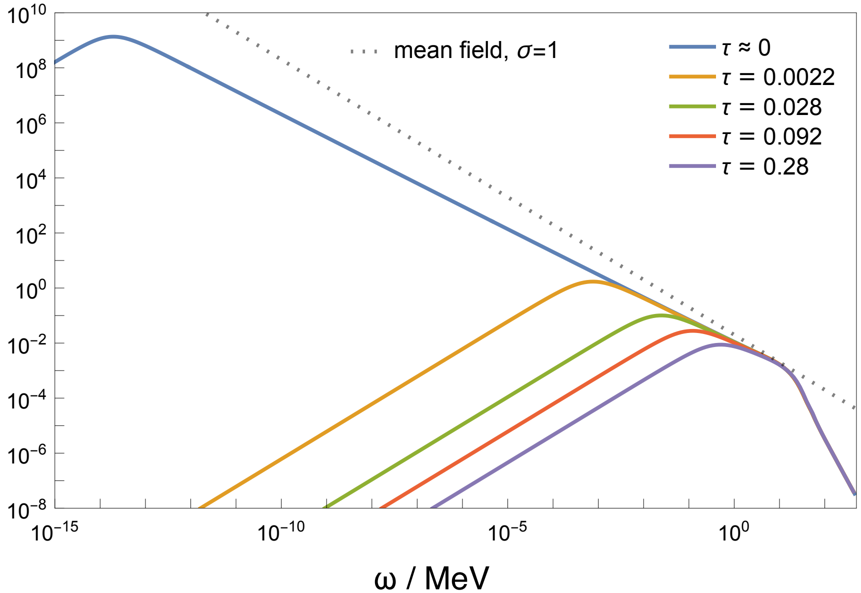
$z = 2 + c\eta$

$$\rho(\omega) \sim \omega^{-\sigma} \quad \text{with} \quad \sigma = \frac{2-\eta}{z}$$

Model C

$z = 2 + \alpha/\nu$





$z \approx 2.042$ (dashed)

$z \approx 2.035$ (solid)

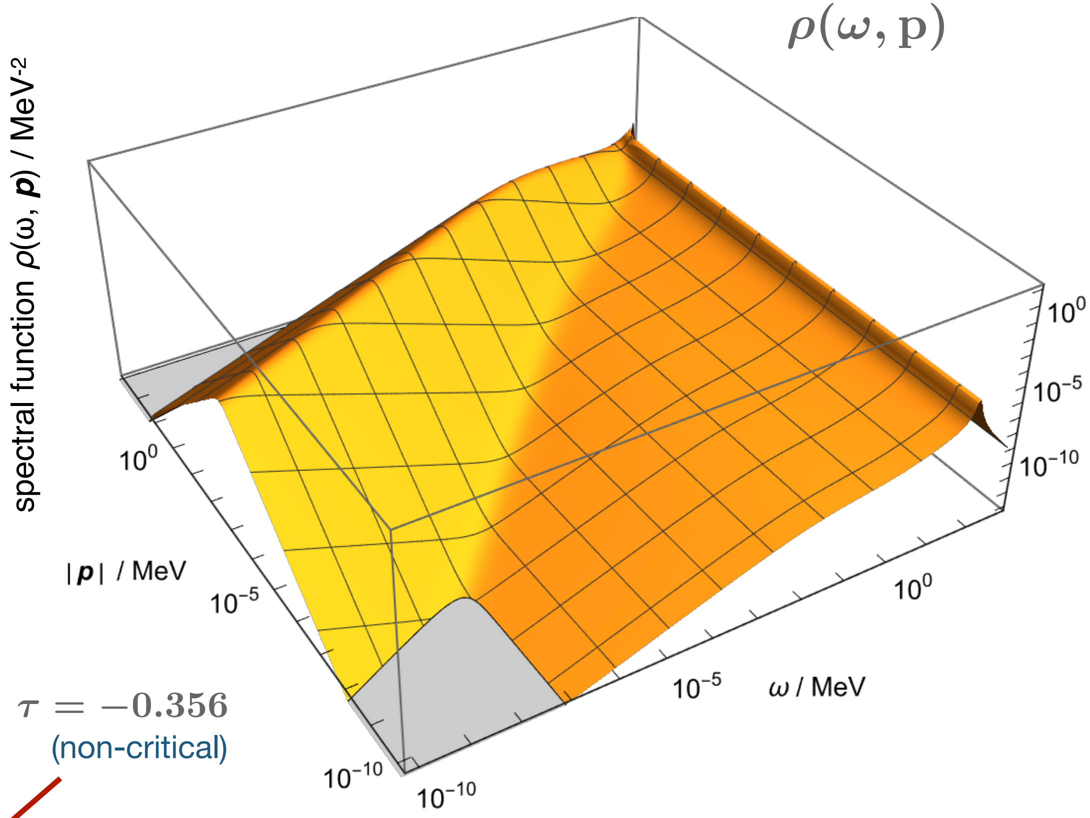
$z \approx 2.31$

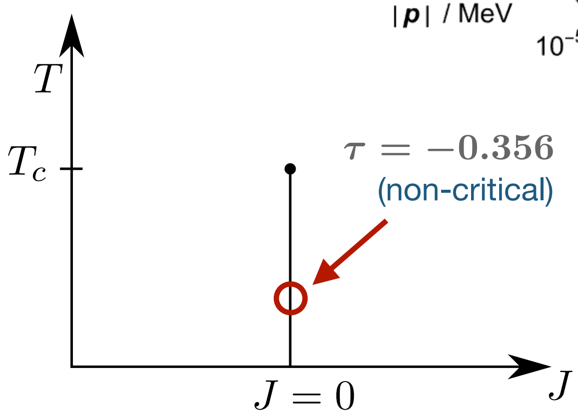
J. V. Roth, L.v.S., JHEP **10**, 065 (2023)

Critical Spectral Functions

- **spectral function:**

Model B
$z = 4 - \eta$





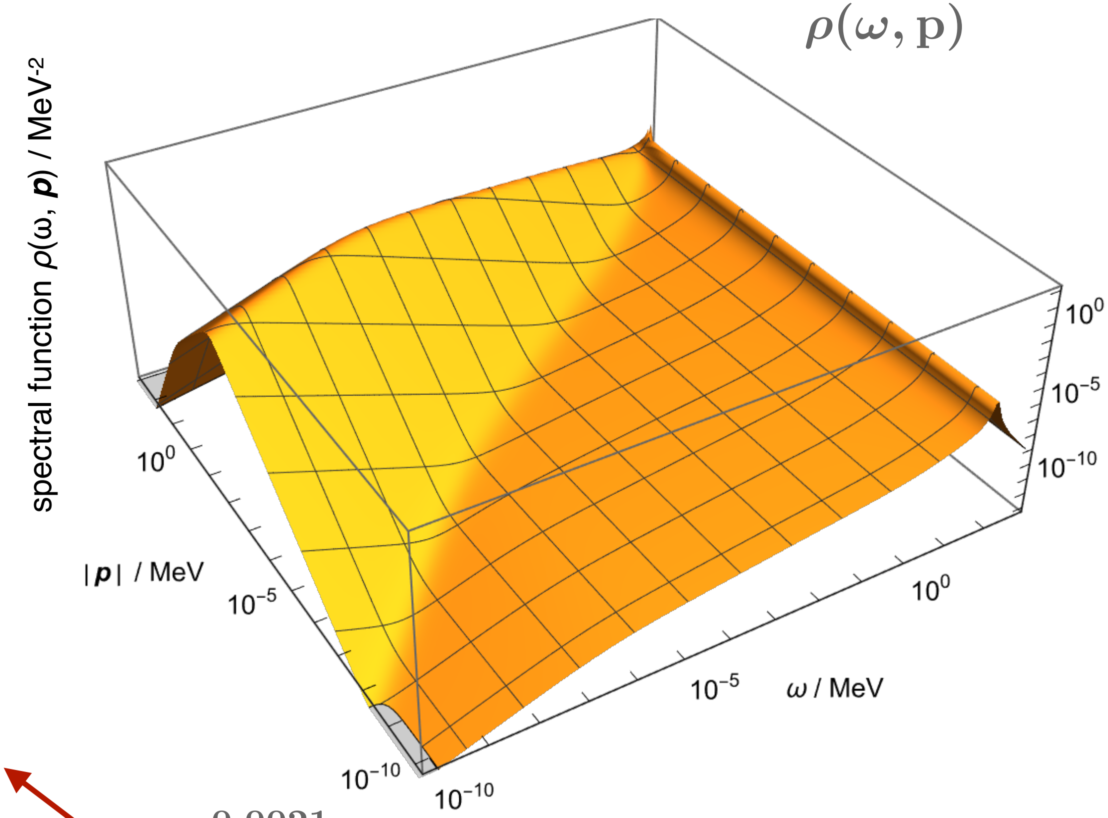
$\tau = (T - T_c)/T_c$
reduced temperature

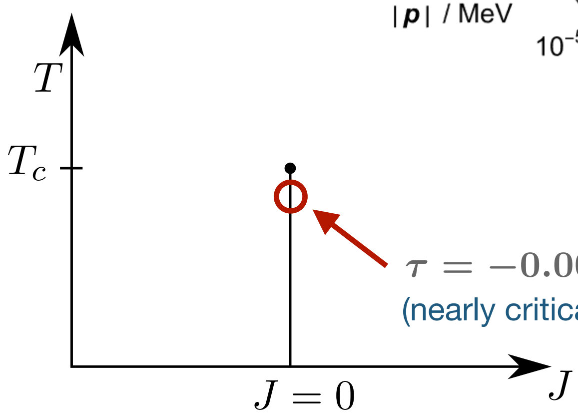
J. V. Roth, L.v.S., JHEP **10**, 065 (2023)

Critical Spectral Functions

- **spectral function:**

Model B
$z = 4 - \eta$





$\tau = -0.0021$
(nearly critical)

$\tau = (T - T_c)/T_c$
reduced temperature

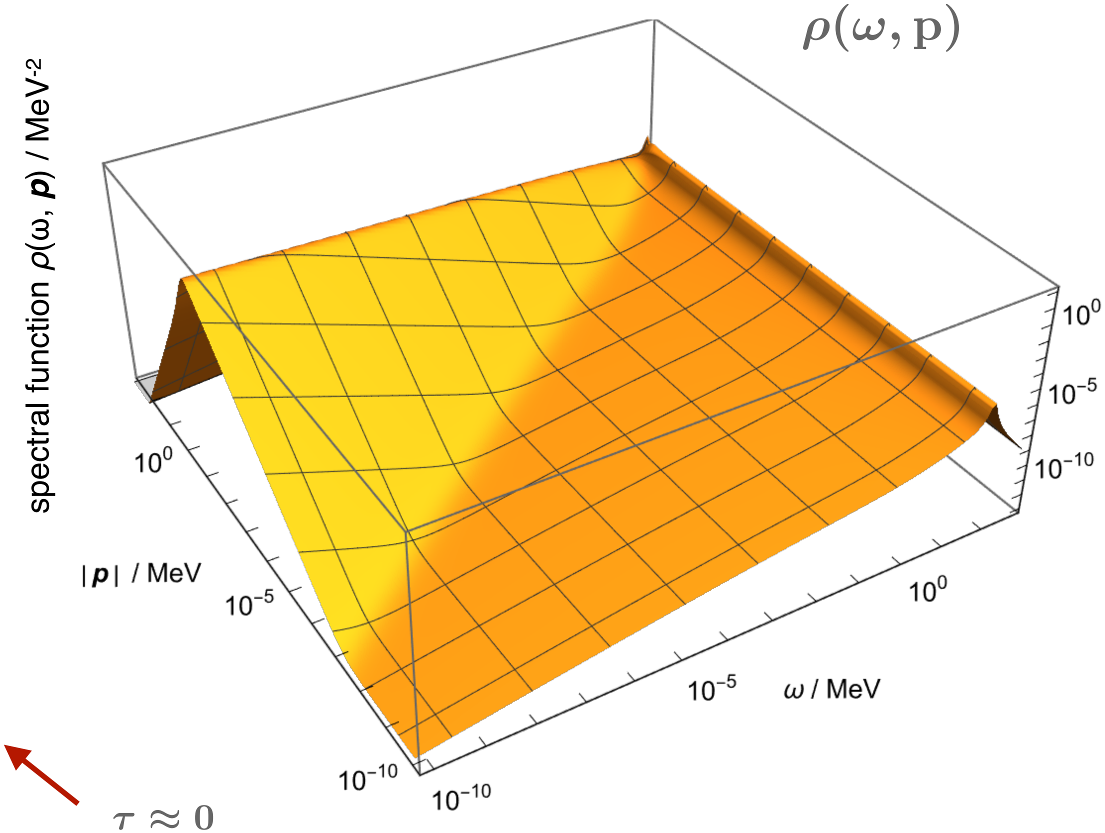
J. V. Roth, L.v.S., JHEP **10**, 065 (2023)

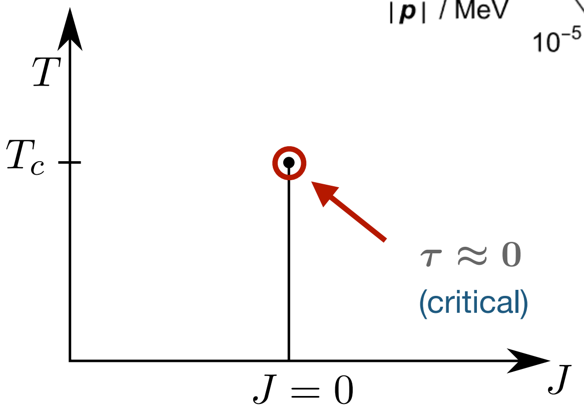
Critical Spectral Functions

- **spectral function:**

Model B

$z = 4 - \eta$





$\tau = (T - T_c)/T_c$

reduced temperature

J. V. Roth, L.v.S., JHEP **10**, 065 (2023)

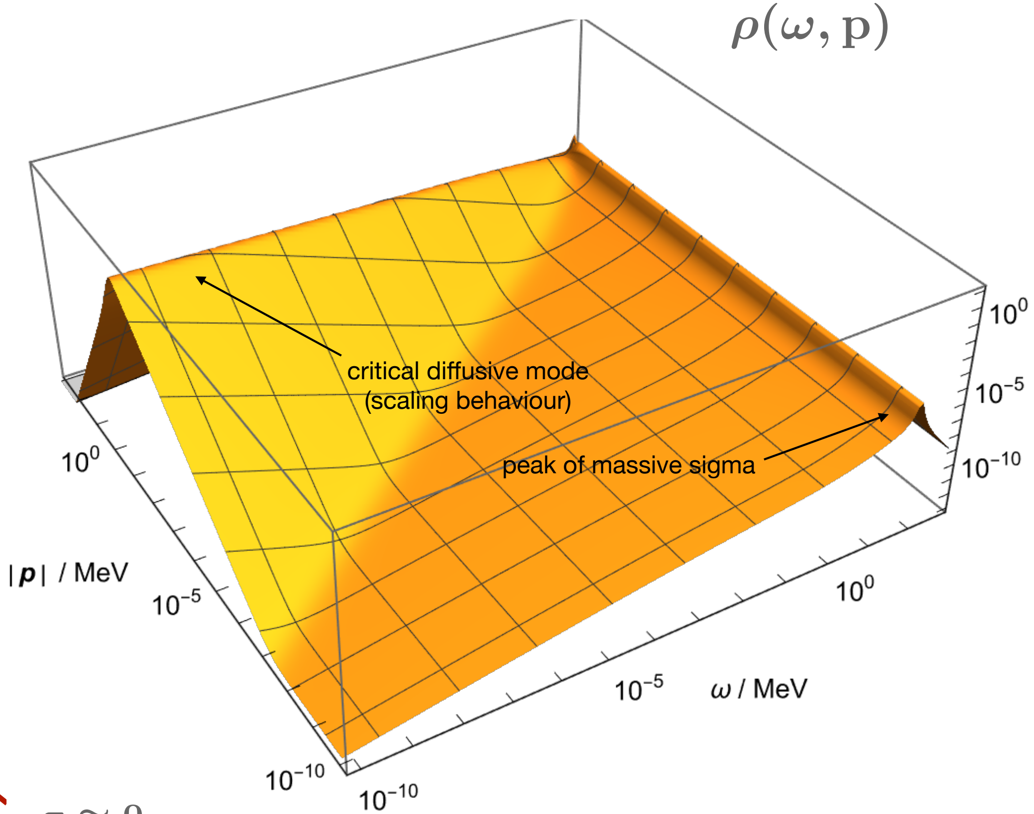
Critical Spectral Functions

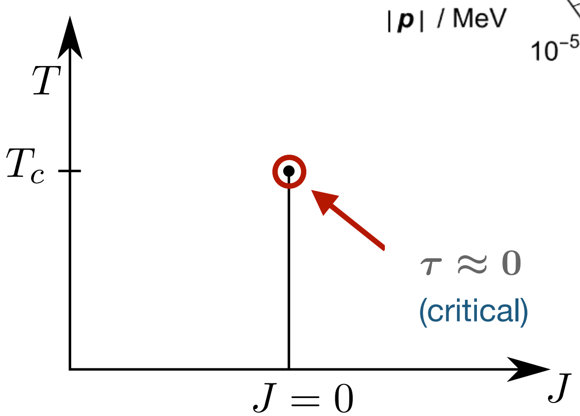
- **spectral function:**

$\rho(\omega, \mathrm{p})$

Model B
$z = 4 - \eta$

$\omega_p \sim p^z,$
$p \sim \xi^{-1}$





$\tau \approx 0$
(critical)

$\tau = (T - T_c)/T_c$
reduced temperature

J. V. Roth, L.v.S., JHEP **10**, 065 (2023)

Critical Dynamics

- **strong-scaling hypothesis:**
  in d spatial dimensions
  (SSS Model)

$$z_\phi = z_n = \frac{d}{2}$$

Model G
$z = d/2$

Sásvari, Schwabl, Szépfalusy, Physica A **81** (1975) 108
Rajagopal, Wilczek, Nucl. Phys. B **399** (1993) 395

- **MSR action:**

$$S = \int_x \Big[-\tilde{\phi}_a \left(\frac{\partial \phi_a}{\partial t} + \Gamma_0 \frac{\delta F}{\delta \phi_a} - \frac{g}{2}\{\phi_a, n_{bc}\} \frac{\delta F}{\delta n_{bc}} \right) - \frac{1}{2}\tilde{n}_{ab} \left(\frac{\partial n_{ab}}{\partial t} - \gamma \boldsymbol{\nabla}^2 \frac{\delta F}{\delta n_{ab}} - g\{n_{ab}, \phi_c\} \frac{\delta F}{\delta \phi_c} - \frac{g}{2}\{n_{ab}, n_{cd}\} \frac{\delta F}{\delta n_{cd}} \right) + iT\tilde{\phi}_a \Gamma_0 \tilde{\phi}_a - \frac{1}{2} iT \tilde{n}_{ab} \gamma \boldsymbol{\nabla}^2 \tilde{n}_{ab} \Big]$$

- **symmetries:**
  – charge conservation
  – thermal equilibrium symmetry
  – temporal (non-Abelian) gauge symmetry
  – BRST symmetry

Canet, Delamotte, Wschebor, PRE **93** (2016) 6, 063101
Crossley, Glorioso, Liu, JHEP 09 (2017) 095

Critical Dynamics

Model G

$z = d/2$

- **add regulators to LGW functional:**

$$F \to F + \frac{1}{2}\int_{\boldsymbol{x}\boldsymbol{y}} \Big(\phi_a(\boldsymbol{x}) R_k^{\phi}(\boldsymbol{x},\boldsymbol{y}) \phi_a(\boldsymbol{y}) + \frac{1}{2} n_{ab}(\boldsymbol{x}) R_k^{n}(\boldsymbol{x},\boldsymbol{y}) n_{ab}(\boldsymbol{y}) \Big)$$

$\rightsquigarrow$ regulators necessarily cubic in fields

- **Ansatz for effective average action:**

$$\Gamma_k = \int_x \Big[-\tilde{\phi}_{a,k} \Big(Z^{\omega}_{\phi,k} \frac{\partial \phi_a}{\partial t} + \gamma_{\phi,k}(\boldsymbol{\nabla}) \frac{\delta F_k}{\delta \phi_a} - \frac{g_k^{\phi n}}{2} \{\phi_a, n_{bc}\} \frac{\delta F_k}{\delta n_{bc}} \Big)$$

$$- \frac{1}{2} \tilde{n}_{ab,k} \Big(Z^{\omega}_{n,k} \frac{\partial n_{ab}}{\partial t} + \gamma_{n,k}(\boldsymbol{\nabla}) \frac{\delta F_k}{\delta n_{ab}} - g_k^{n\phi} \{n_{ab}, \phi_c\} \frac{\delta F_k}{\delta \phi_c} - \frac{g_k^{nn}}{2} \{n_{ab}, n_{cd}\} \frac{\delta F_k}{\delta n_{cd}} \Big)$$

$$+ Z^{\omega}_{\phi,k} iT \tilde{\phi}_{a,k} \gamma_{\phi,k}(\boldsymbol{\nabla}) \tilde{\phi}_{a,k} + \frac{1}{2} Z^{\omega}_{n,k} iT \tilde{n}_{ab,k} \gamma_{n,k}(\boldsymbol{\nabla}) \tilde{n}_{ab,k} \Big]$$

Ward identity:

$$g_k^{\phi n} = g_k^{n\phi} = g_k^{nn} = g$$

kinetic coefficients:

$$\gamma_{\phi,k}(\boldsymbol{p},\tau) = \Gamma_k^{\phi}(\tau) + \mathcal{O}(\boldsymbol{p}^2)$$

$$\gamma_{n,k}(\boldsymbol{p},\tau) = \boldsymbol{p}^2 D_k^{n}(\boldsymbol{p},\tau)$$

charge diffusion coefficient

Critical Dynamics

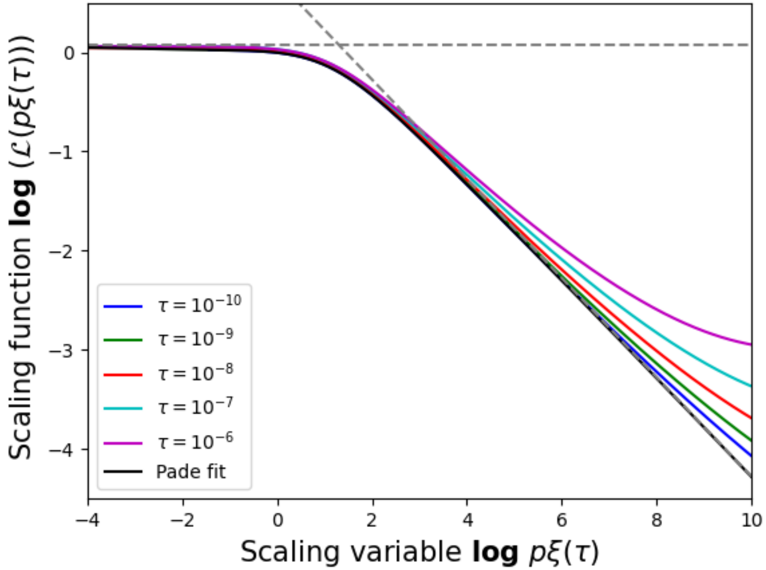
- **scaling of charge diffusion coefficient:**

Model G

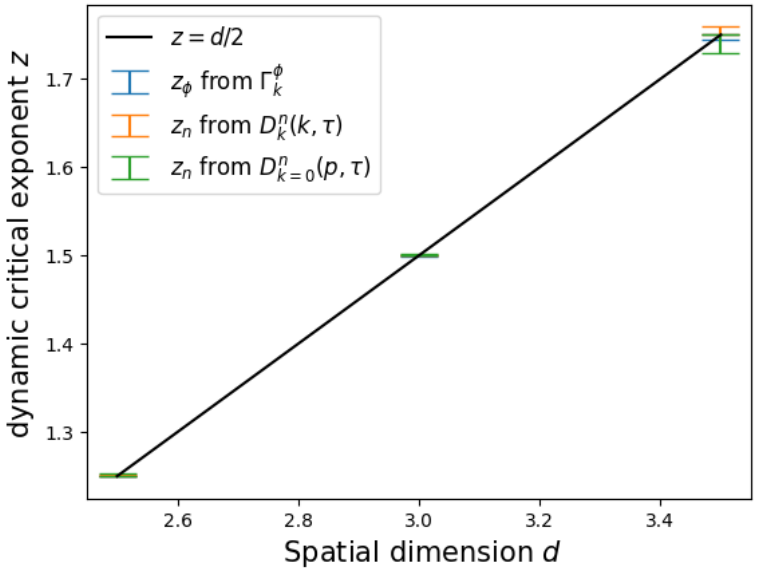
$z = d/2$

$$D_n(p,\tau) = s^{2-z} D_n(sp, s^{1/\nu}\tau)$$

$$\rightsquigarrow \quad D_n(p,\tau) \sim \tau^{-\nu(2-z)} \mathcal{L}(\tau^{-\nu}\bar{p}) \quad , \quad \bar{p} = f^+ p$$



universal dynamic scaling function

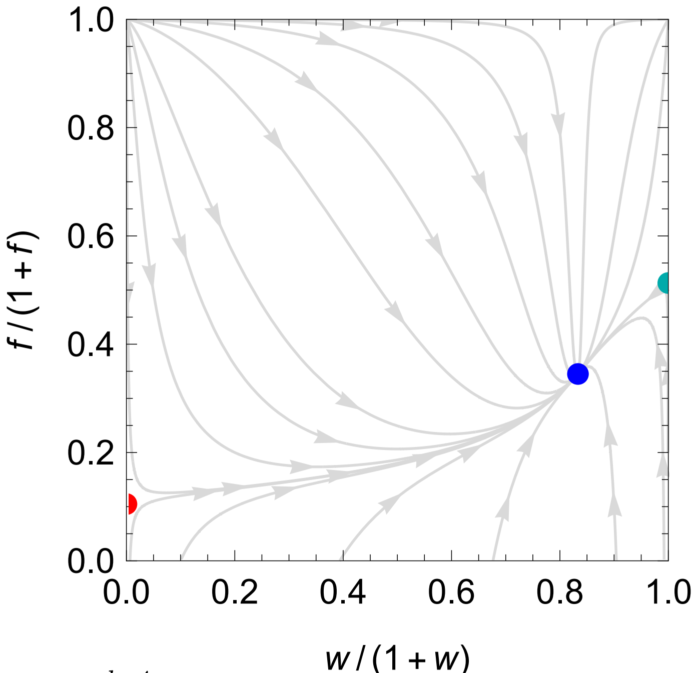


strong scaling

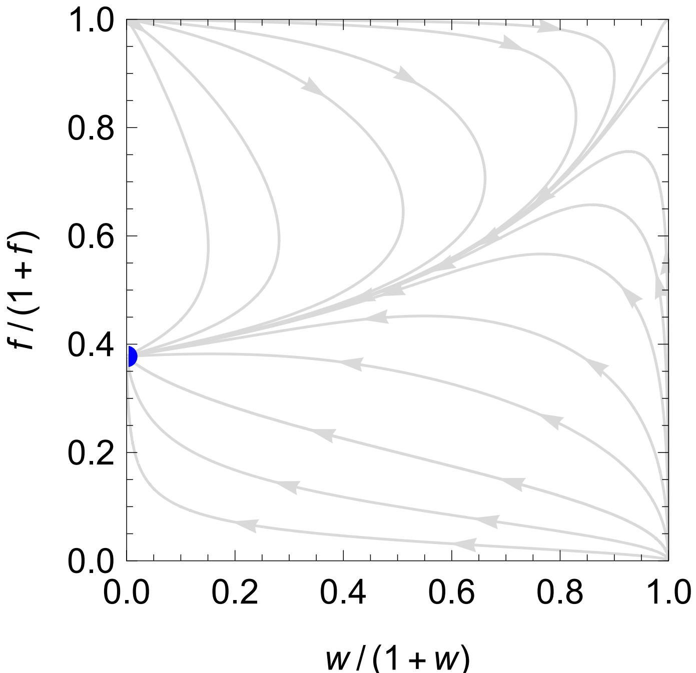
Roth, Ye, Schlichting, LvS, arXiv:2403.04573

Strong and Weak Scaling FPs

Model G



Model H



$$f \propto \frac{Tk^{d-4}}{\Gamma_k^{\phi}\gamma_k} \qquad w = \chi\,\frac{\Gamma_k^{\phi}}{\gamma_k}$$

$$f \propto \frac{Tk^{d-4}}{\sigma_k\eta_k} \qquad w = \rho\,\frac{\sigma_k k^2}{\eta_k}$$

$$\partial_t f = (d - 4 - \eta_\perp + x_{\Gamma^\phi} + x_\gamma)\, f$$

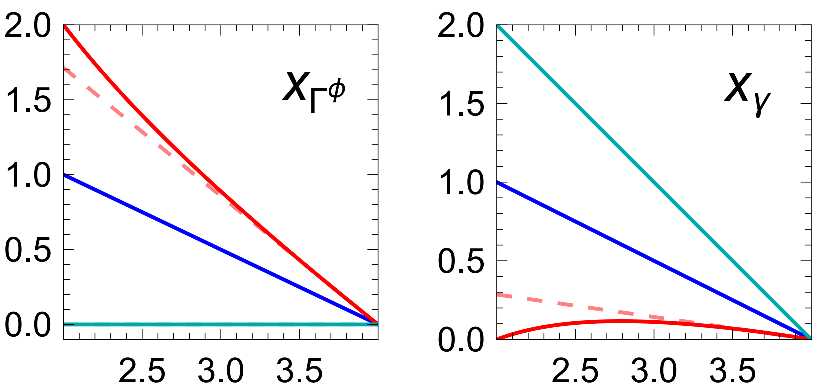
$$\partial_t w = (x_\gamma - x_{\Gamma^\phi} - \eta_\perp)\, w$$

$$\partial_t f = (d - 4 - \eta_\perp + x_\sigma + x_\eta) f$$

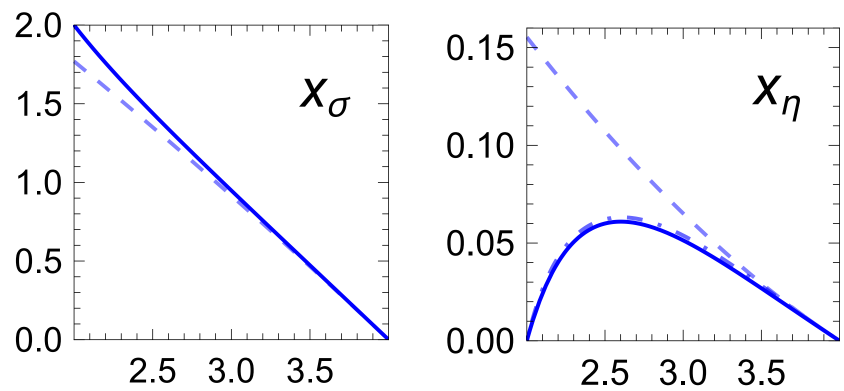
Roth, Ye, Schlichting, LvS, arXiv:2409.14470

Dynamic Scaling Relations

Model G



Model H



$1/\Gamma^\phi$: $\rightsquigarrow$ order-parameter damping

γ: charge mobility

σ: order-parameter diffusion

η: shear voscosity

Model H (d = 3): $x_\sigma \approx 0.949$

$x_\eta \approx 0.051$

$z_\phi \approx 3.051$

- **weak-scaling relations:** $x_{\Gamma^\phi} + x_\gamma = x_\sigma + x_\eta = 4 - d - \eta_\perp$
- **strong-scaling relation:** $x_{\Gamma^\phi} = x_\gamma - \eta_\perp$

$\Rightarrow$ **only Model G:** $z_\phi = z_n = d/2$

Summary & Conclusion

- **real-time methods for non-equilibrium phase transitions**
  - compute universal non-equilibrium scaling functions
  - determine non-equilibrium scaling regions

- **real-time FRG for critical dynamics**
  - quantify universal aspects of QCD chiral dynamics and critical point, Model G and Model H
  - determine universal dynamic scaling functions and dynamic scaling regions

Thank you for your attention!