

Bosonization for fermionic fRG at weak and strong couplings

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In collaboration with:

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M. Patricolo, **M. Krämer**, **F. Krien**, **A. Toschi**, **S. Andergassen** (TU Wien), **P. M. Bonetti** (Harvard),
D. Vilardi, **M. Meixner**, **T. Schäfer** (MPI Stuttgart), **N. Wentzell** (CCQ New York)

- ① Introduction to SBE fRG
- ② Investigations at weak couplings
- ③ Investigations at strong couplings

Introduction to SBE fRG

Consider a fermionic theory with a quartic interaction:

$$S[\bar{\psi}, \psi] = - \sum_{\alpha_1, \alpha_2} \bar{\psi}_{\alpha_1} G_{0, \alpha_1 | \alpha_2}^{-1} \psi_{\alpha_2} + \frac{1}{4} \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} U_{\alpha_1 \alpha_2 | \alpha_3 \alpha_4} \bar{\psi}_{\alpha_1} \bar{\psi}_{\alpha_2} \psi_{\alpha_3} \psi_{\alpha_4}$$

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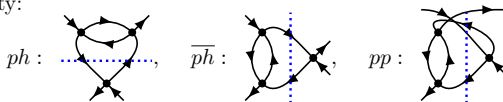
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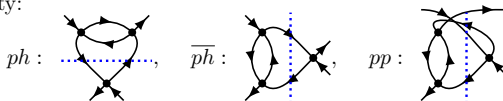


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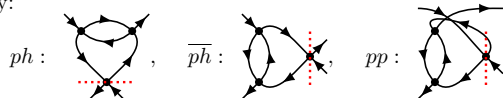
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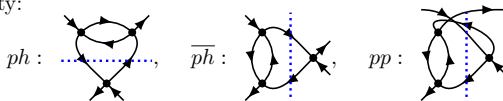


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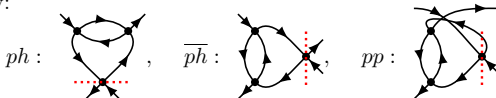
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⚠ U-reducibility \Rightarrow 2P-reducibility

⚠ 2P-reducibility $\not\Rightarrow$ U-reducibility

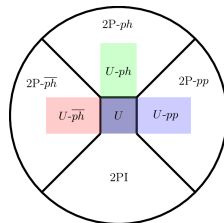
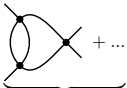


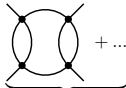
Fig. adapted from Krien, Valli, Capone, PRB 100, 155149 (2019)

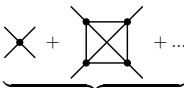
Single-boson exchange (SBE) decomposition:

[Krien, Valli, Capone, PRB 100, 155149 (2019)]

$$\Gamma^{(4)} = \frac{\delta^4 \Gamma[\bar{\psi}, \psi]}{\delta \bar{\psi} \delta \bar{\psi} \delta \psi \delta \psi} \Bigg|_{\psi = \bar{\psi} = 0} = \sum_{X=\{pp, ph, p\bar{h}\}} (\nabla^X + M^X - U) + \mathcal{I}^{2PI}$$

with $\nabla^X =$  + ...
 only U -reducible diagrams

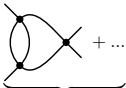
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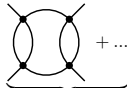
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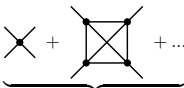
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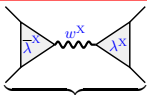
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Heart of the SBE decomposition:

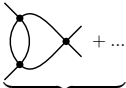
$\nabla^X =$ 
 1 boson exchanged

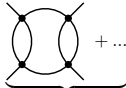
w^X : bosonic propagator
 λ^X : Yukawa coupling
 M^X : rest function

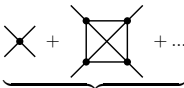
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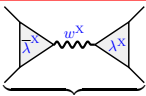
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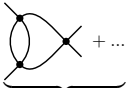
⇒ Example for the Hubbard model:

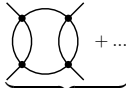
$$\nabla_{kk'}^X(Q) = \bar{\lambda}_k^X(Q) w^X(Q) \lambda_{k'}^X(Q)$$

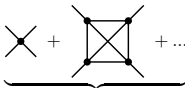
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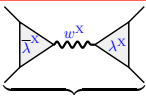
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⇒ SBE decomposition introduces **bosonic dofs** and treats **all channels equitably**

Start from 1ℓ flow equations obtained from the vertex expansion:

[Morris, IJMP A09, 2411 (1994)]

[Metzner, Salmhofer, Honerkamp, Meden, Schönhammer, RMP 84, 299 (2012)]

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$$\partial_\Lambda \Gamma_\Lambda^{(4)} = \text{Diagram 2} + \text{Diagram 3}$$

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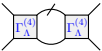
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⚠ SBE formalism and Hubbard-Stratonovich transformations can yield same flow equations at 1ℓ

Multiloop fRG:

[Kugler, von Delft, PRL 120, 057403 (2018), PRB 97, 035162 (2018), NJP 20, 123029 (2018)]

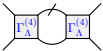
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Idea: substitute $\partial_\Lambda \Gamma_\Lambda^{(4)}$ =  with flow equations derived from the **Bethe-Salpeter** equation:

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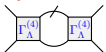
$$\phi^X = \text{Diagram}(\Gamma^{(4)} - \phi^X, \Gamma^{(4)}) , \quad \Gamma^{(4)} = \sum_X \phi^X + \text{Diagram}(\Gamma^{(4)}, U)$$

The diagram on the left shows a four-point vertex with two internal lines forming a loop, labeled with $\Gamma^{(4)} - \phi^X$ and $\Gamma^{(4)}$. The diagram on the right shows a four-point vertex with two internal lines forming a loop, labeled with $\Gamma^{(4)}$ and U .

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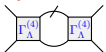
$$\Downarrow G_0 \rightarrow G_{0,\Lambda}$$

$$\partial_\Lambda \phi_\Lambda^X = \partial_\Lambda \left(\text{Diagram}(\Gamma_\Lambda^{(4)} - \phi_\Lambda^X, \Gamma_\Lambda^{(4)}) \right)$$

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$$\partial_\Lambda \phi_\Lambda^X = \partial_\Lambda \left(\text{Diagram: } \Gamma_\Lambda^{(4)} - \phi_\Lambda^X \text{ and } \Gamma_\Lambda^{(4)} \text{ in a loop} \right)$$

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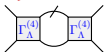
$$\partial_\Lambda \phi_\Lambda^{X(1)} = \text{Diagram: } \Gamma_\Lambda^{(4)} \text{ with a loop and } \partial_\Lambda G_\Lambda^{-1} \text{ on the loop}$$

$$\partial_\Lambda \phi_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{Diagram: } \partial_\Lambda \phi_\Lambda^{X'(n-1)} \text{ and } \Gamma_\Lambda^{(4)} \text{ in a loop} \right) + \left(\text{Diagram: } \Gamma_\Lambda^{(4)} \text{ and } \partial_\Lambda \phi_\Lambda^{X'(n-2)} \text{ in a loop} \right) + \left(\text{Diagram: } \Gamma_\Lambda^{(4)} \text{ and } \partial_\Lambda \phi_\Lambda^{X'(n-1)} \text{ in a loop} \right)$$

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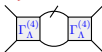
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What about the self-energy flow?

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What about the self-energy flow?

[Hille, Rohe, Honerkamp, Andergassen, PRR 2, 033068 (2020)]

[Patricolo, Gievers, Fraboulet, Heinzlmann, Al-Eryani, Bonetti, Toschi, Vilardi, Andergassen, in prep.]

[Talk by M. Patricolo]

Advantageous to use **Schwinger-Dyson equation** within the truncated unity approach: $\partial_\Lambda \Sigma_\Lambda = \partial_\Lambda \left(\text{Diagram}(\text{loop}, \Gamma_\Lambda^{(4)}) \right)$

Insert SBE decomposition into multiloop fRG equations:

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[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]

$$\partial_\Lambda w_\Lambda^{X(1)} = \text{diagram}, \quad \partial_\Lambda \lambda_\Lambda^{X(1)} = \text{diagram}, \quad \partial_\Lambda M_\Lambda^{X(1)} = \text{diagram}$$

The diagrams represent the first-order derivatives of the self-energy, vertex, and mass with respect to the coupling Λ . Each diagram features a central loop with a cross on top, representing a self-energy insertion. The external lines are wavy for w and straight for λ and M . The labels λ_Λ^X and $T_{U_{\text{irr},\Lambda}}^X$ indicate the coupling and the irreducible vertex, respectively.

$$\partial_\Lambda w_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{diagram} \right), \quad \partial_\Lambda \lambda_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{diagram} + \text{diagram} \right)$$

The diagrams for $n > 1$ show higher-order corrections. The first diagram for w has two self-energy insertions. The second diagram for λ has a self-energy insertion and a vertex correction. The shaded regions represent the SBE decomposition of the self-energy and vertex.

$$\partial_\Lambda M_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{diagram} + \text{diagram} + \text{diagram} \right)$$

The diagrams for M show three types of corrections: a self-energy insertion, a vertex correction, and a mass correction. The shaded regions represent the SBE decomposition of the self-energy and vertex.

$$\partial_\Lambda w_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda w_\Lambda^{X(\ell)}, \quad \partial_\Lambda \lambda_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda \lambda_\Lambda^{X(\ell)}, \quad \partial_\Lambda M_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda M_\Lambda^{X(\ell)}$$

The summation over ℓ indicates the resummation of all multiloop diagrams into a single series.

Insert SBE decomposition into multiloop fRG equations:

[Gievers, Walter, Ge, von Delft, Kugler, EPJB 95, 108 (2022)]

$$\partial_\Lambda w_\Lambda^{X(1)} = \text{diagram}, \quad \partial_\Lambda \lambda_\Lambda^{X(1)} = \text{diagram}, \quad \partial_\Lambda M_\Lambda^{X(1)} = \text{diagram}$$

$$\partial_\Lambda w_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{diagram} \right), \quad \partial_\Lambda \lambda_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{diagram} + \text{diagram} \right)$$

$$\partial_\Lambda M_\Lambda^{X(n)} = \sum_{X' \neq X} \left(\text{diagram} + \text{diagram} + \text{diagram} \right)$$

$$\partial_\Lambda w_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda w_\Lambda^{X(\ell)}, \quad \partial_\Lambda \lambda_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda \lambda_\Lambda^{X(\ell)}, \quad \partial_\Lambda M_\Lambda^X = \sum_{\ell=1}^{\infty} \partial_\Lambda M_\Lambda^{X(\ell)}$$

⇒ Multiloop SBE fRG enables us to go **beyond conventional 1ℓ fermionic fRG**
 by means of flowing **bosonic** vertices

Investigations at weak couplings

Hubbard model Hamiltonian:

[Hubbard, PRSL A 276, 238 (1963)]

[Gutzwiller, PRL 10, 159 (1963)]

[Kanamori, PTP 30, 275 (1963)]

[Qin, Schäfer, Andergassen, Corboz, Gull, ARCOMP 13, 275 (2022)]

$$\mathcal{H} = \sum_{i \neq j, \sigma} t_{ij} c_{i\sigma}^\dagger c_{j\sigma} + U \sum_i c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow}$$

- t_{ij} : **hopping** between nearest-neighbor sites (t) or between next-nearest-neighbor sites (t')
- U : on-site Coulomb **interaction**

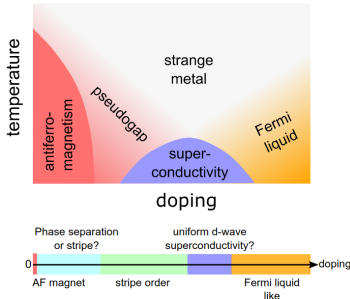
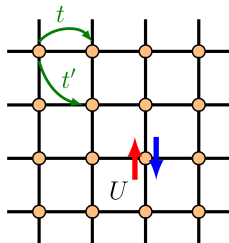


Fig. adapted from Qin, Schäfer, Andergassen, Corboz, Gull, ARCOMP 13, 275 (2022)

Conclusions from 1 ℓ SBE fRG study:

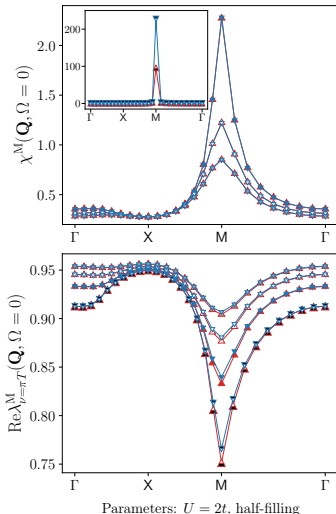
[Fraboulet, Heinzlmann, Bonetti, Al-Eryani, Vilardi, Toschi, Andergassen, EPJB 95, 202 (2022)]

Conclusions from 1 ℓ SBE fRG study:

[Fraboulet, Heinzlmann, Bonetti, Al-Eryani, Vilardi, Toschi, Andergassen, EPJB 95, 202 (2022)]

- Rest functions M^X negligible in most studied regimes (above pseudo-critical transition)
- Divergence when approaching pseudo-critical transition fully encompassed in bosonic propagator w^M ($w^M = U + U^2\chi^M$)

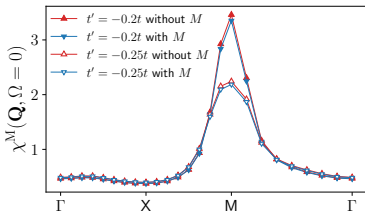
- | | |
|---|--|
| \blacktriangle $T = 0.1t$ without M | \blacktriangledown $T = 0.1t$ with M |
| \blacktriangle $T = 0.2t$ without M | \blacktriangledown $T = 0.2t$ with M |
| \blacktriangle $T = 0.3t$ without M | \blacktriangledown $T = 0.3t$ with M |
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Conclusions from 1 ℓ SBE fRG study:

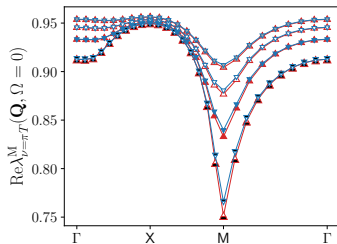
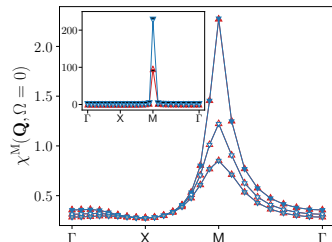
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Parameters: $U = 2t$, $T = 0.2t$, $\mu = 4t'$

- $T = 0.1t$ without M (red triangles)
- $T = 0.1t$ with M (blue inverted triangles)
- $T = 0.2t$ without M (red triangles)
- $T = 0.2t$ with M (blue inverted triangles)
- $T = 0.3t$ without M (red open triangles)
- $T = 0.3t$ with M (blue open inverted triangles)
- $T = 0.4t$ without M (red triangles)
- $T = 0.4t$ with M (blue inverted triangles)



Parameters: $U = 2t$, half-filling

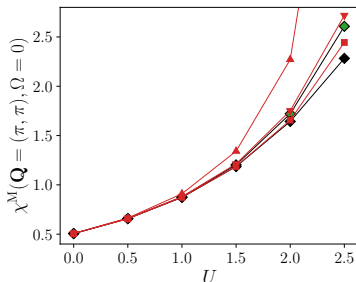
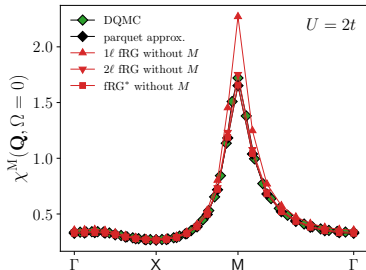
Conclusions from multiloop SBE fRG study:

[Fraboulet, Al-Eryani, Heinzlmann, Andergassen, in prep.]

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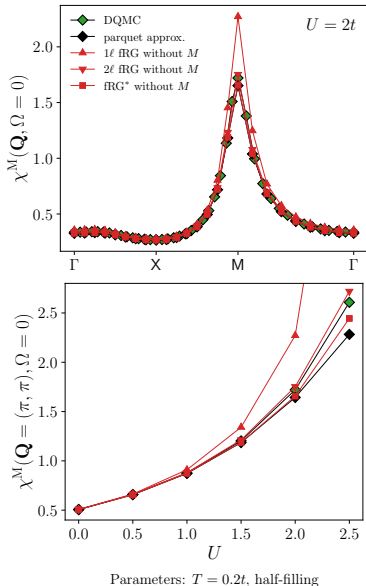
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How can we extend multiloop SBE fRG to access **strong couplings**?



Investigations at strong couplings

Dynamical mean-field theory:

[Metzner, Vollhardt, PRL 62, 324 (1989)]

[Georges, Kotliar, Krauth, Rozenberg, RMP 68, 13 (1996)]

Idea: treat a **lattice** problem by studying an effective **impurity** problem with a bath adjusted *self-consistently*

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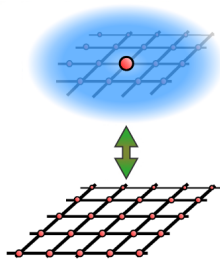
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$$\mathcal{S}_{\text{imp}} = - \int_0^\beta d\tau d\tau' \sum_\sigma c_\sigma^\dagger(\tau) G_{0,\text{imp}}^{-1}(\tau - \tau') c_\sigma(\tau') + \mathcal{S}_{\text{int}}$$

$$G_{0,\text{imp}}^{-1}(\nu) = i\nu + \mu - \Delta(\nu)$$



$$\mathcal{S}_{\text{latt}} = - \int_0^\beta d\tau d\tau' \sum_{\mathbf{k}, \sigma} c_\sigma^\dagger(\mathbf{k}, \tau) G_{0,\text{latt}}^{-1}(\mathbf{k}, \tau - \tau') c_\sigma(\mathbf{k}, \tau') + \mathcal{S}_{\text{int}}$$

$$G_{0,\text{latt}}^{-1}(\mathbf{k}, \nu) = i\nu + \mu - \epsilon_{\mathbf{k}}$$

Fig. adapted from Bonetti, Toschi, Hille, Andergassen, Vilardi, PRR 4, 013034 (2022)

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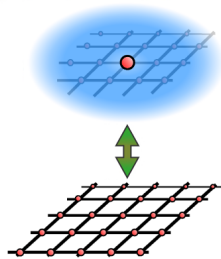
$$G_{0,\text{imp}}^{-1}(\nu) = i\nu + \mu - \Delta(\nu)$$

Determination of the dynamical mean-field $\Delta(\nu)$:

Make **local** approximation $\Sigma_{\text{imp}}(\nu) = \Sigma_{\text{latt},ii}(\nu)$

and solve Dyson equation with it

$$G_{\text{imp}}^{-1}(\nu) = G_{0,\text{imp}}^{-1}(\nu) - \Sigma_{\text{latt},ii}(\nu)$$



$$\mathcal{S}_{\text{latt}} = - \int_0^\beta d\tau d\tau' \sum_{\mathbf{k},\sigma} c_\sigma^\dagger(\mathbf{k}, \tau) G_{0,\text{latt}}^{-1}(\mathbf{k}, \tau - \tau') c_\sigma(\mathbf{k}, \tau') + \mathcal{S}_{\text{int}}$$

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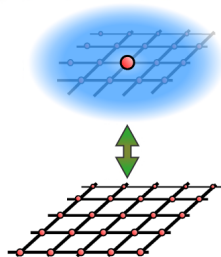
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⚠ **Local** correlations fully captured

⚠ DMFT mapping exact in infinite dimensions



$$\mathcal{S}_{\text{latt}} = - \int_0^\beta d\tau d\tau' \sum_{\mathbf{k}, \sigma} c_\sigma^\dagger(\mathbf{k}, \tau) G_{0,\text{latt}}^{-1}(\mathbf{k}, \tau - \tau') c_\sigma(\mathbf{k}, \tau') + \mathcal{S}_{\text{int}}$$

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DMF²RG (DMFT+fRG):

[Taranto, Andergassen, Bauer, Held, Katanin, Metzner, Rohringer, Toschi, PRL 112, 196402 (2014)]

[Vilardi, Taranto, Metzner, PRB 99, 104501 (2019)]

[Bonetti, Toschi, Hille, Andergassen, Vilardi, PRR 4, 013034 (2022)]

[Fraboulet, Al-Eryani, Andergassen, in prep.]

Idea: use DMFT results as starting point for fRG flows

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Idea: use DMFT results as starting point for fRG flows

$$G_{0,\Lambda}(\mathbf{k}, \nu) = (1 - \Lambda) G_{0,\text{imp}}(\nu) + \Lambda G_{0,\text{latt}}(\mathbf{k}, \nu)$$

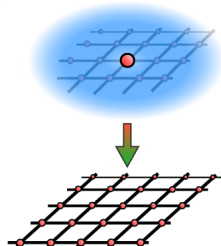
$$\text{At } \Lambda = \Lambda_{\text{ini}} = 0 : G_{0,\Lambda_{\text{ini}}}(\mathbf{k}, \nu) = G_{0,\text{imp}}(\nu)$$

$$\text{At } \Lambda = \Lambda_{\text{fin}} = 1 : G_{0,\Lambda_{\text{fin}}}(\mathbf{k}, \nu) = G_{0,\text{latt}}(\mathbf{k}, \nu)$$

$$\Lambda = \Lambda_{\text{ini}} = 0$$

$$S_{\text{imp}} = - \int_0^\beta d\tau d\tau' \sum_{\mathbf{k}, \sigma} c_\sigma^\dagger(\mathbf{k}, \tau) G_{0,\Lambda_{\text{ini}}}^{-1}(\mathbf{k}, \tau - \tau') c_\sigma(\mathbf{k}, \tau') + S_{\text{int}}$$

$$G_{0,\Lambda_{\text{ini}}}(\mathbf{k}, \nu) = G_{0,\text{imp}}(\nu)$$



$$\Lambda = \Lambda_{\text{fin}} = 1$$

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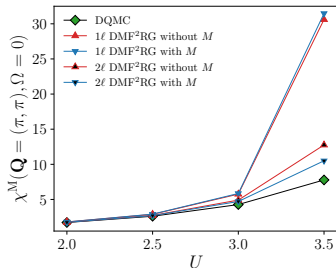
[Fraboulet, Al-Eryani, Andergassen, in prep.]

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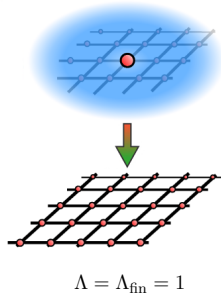


Parameters: $T = 0.2t$, half-filling

$$\Lambda = \Lambda_{\text{ini}} = 0$$

$$S_{\text{imp}} = - \int_0^\beta d\tau d\tau' \sum_{\mathbf{k}, \sigma} c_\sigma^\dagger(\mathbf{k}, \tau) G_{0,\Lambda_{\text{ini}}}^{-1}(\mathbf{k}, \tau - \tau') c_\sigma(\mathbf{k}, \tau') + S_{\text{int}}$$

$$G_{0,\Lambda_{\text{ini}}}(\mathbf{k}, \nu) = G_{0,\text{imp}}(\nu)$$



$$\Lambda = \Lambda_{\text{fin}} = 1$$

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Conclusion

Main conclusions:

- Efficient bosonization of fermionic fRG provided by the **SBE formalism**
- Possibility to combine SBE formalism with **multiloop fRG** and/or **DMF²RG** to achieve better (quantitative) accuracy and/or to tackle strong couplings

Main conclusions:

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Further remarks:

- SBE fRG already used to investigate other models (extended Hubbard model, Hubbard-Holstein model, X-ray-edge singularity)

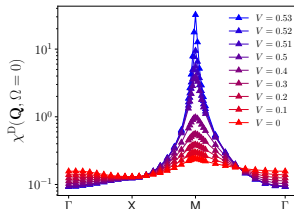
[Heinzlmann, Al-Eryani, Fraboulet, Krien, Andergassen, in prep.]

[Posters by A. Al-Eryani and M. Gievers]

- Ongoing work to improve starting point of DMF²RG with cellular DMFT

[Krämer, Meixner, Fraboulet, Bonetti, Vilardi, Schäfer, Toschi, Andergassen, in prep.]

[Poster by M. Krämer]



Parameters: $U = 2t$, $T = 0.1t$, half-filling