Vacuum energy density and the Gradient Flow ERG

Hidenori SONODA

Visiting Research Associate, Department of Physics & Astronomy, The University of Iowa, USA

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Abstract

In the first part I overview the recent work with Carlo Pagani (Uni. Mainz) on the vacuum energy density in ERG. In the second part I overview the work with Hiroshi Suzuki (Kyushu Univ.) on Gradient Flow ERG that keeps BRST invariance manifestly.

Plan of the talk

1. A brief overview of ERG for a real scalar field (with Carlo Pagani)
   - ERG transformation as a total differential
   - The field independent part of the Wilson action gives the vacuum energy.

2. Gradient Flow ERG for QED (with Hiroshi Suzuki)
   - BRST invariant diffusion (gradient flow) of Lüscher & Weisz
   - GFERG is ERG that reproduces BRST invariant diffusion of field variables

ERG transformation as a total differential

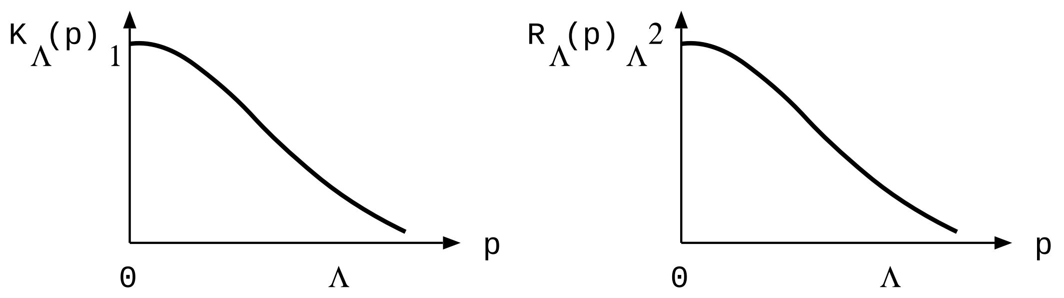
1. "Master formula" for the Wilson action $S_\Lambda[\phi]$

$$e^{S_\Lambda[\phi]}$$

$$= \int [d\phi'] \exp \left[S_{\mathrm{bare}}[\phi'] - \frac{1}{2} \int_p \left(\phi'(p) - \frac{\phi(p)}{K_\Lambda(p)} \right) R_\Lambda(p) \left(\phi'(-p) - \frac{\phi(-p)}{K_\Lambda(p)} \right) \right]$$

$$\Big/ \int [d\phi''] \exp \left(-\frac{1}{2} \int_p \phi''(p) \phi''(-p) \frac{R_\Lambda(p)}{K_\Lambda(p)^2} \right)$$

where



The partition function is preserved. [Wegner&Houghton, K. Wilson, ...]

$$\int [d\phi] e^{S_\Lambda[\phi]} = \int [d\phi] e^{S_{\mathrm{bare}}[\phi]}$$

Hence, $\partial_\Lambda e^{S_\Lambda[\phi]}$ is a total differential w.r.t. ϕ.

2. S_{bare} may not exist, but the master formula gives the correct ERG equation.

$$\begin{aligned} &- \Lambda \partial_\Lambda e^{S_\Lambda[\phi]} \\ &= \int_p \frac{\delta}{\delta \phi(p)} \left[\left\{ \Lambda \frac{\partial \ln K_\Lambda(p)}{\partial \Lambda} \phi(p) + \frac{1}{2} \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} \cdot \frac{K_\Lambda(p)^2}{R_\Lambda(p)} \frac{\delta}{\delta \phi(p)} \right\} e^{S_\Lambda[\phi]} \right] \end{aligned}$$

Total differential as expected.[Pagani & H.S. 2024]

3. The cutoff independent correlation functions are given by

$$\langle \phi(p_1)\cdots\phi(p_n)\rangle_{S_{\mathrm{bare}}} = \langle\!\langle \phi(p_1)\cdots\phi(p_n)\rangle\!\rangle_{S_\Lambda}$$
$$\equiv \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)} \left\langle \exp\left(-\frac{1}{2}\int_p \frac{K_\Lambda(p)^2}{R_\Lambda(p)} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1)\cdots\phi(p_n) \right\rangle_{S_\Lambda}$$

4. Using Wilson's spin variables $\sigma(p) \equiv \frac{\sqrt{R_\Lambda(p)}}{K_\Lambda(p)}\phi(p)$, the master formula is simple:

$$e^{S_\Lambda[\sigma]} = \int [d\phi'] e^{S_{\mathrm{bare}}[\phi']}$$
$$\times \exp\left[-\frac{1}{2}\int_p \left(\sigma(p) - \sqrt{R_\Lambda(p)}\phi'(p)\right) \left(\sigma(-p) - \sqrt{R_\Lambda(p)}\phi'(-p)\right) \right]$$

ERG equation is simple:

$$-\Lambda\partial_\Lambda e^{S_\Lambda[\sigma]} = \int_p \Lambda \frac{\partial \ln \sqrt{R_\Lambda(p)}}{\partial \Lambda} \frac{\delta}{\delta\sigma(p)} \left\{ \left(\sigma(p) + \frac{\delta}{\delta\sigma(-p)} \right) e^{S_\Lambda[\sigma]} \right\}$$

Asymptotic behavior is simple:

$$\lim_{\Lambda\to 0+} S_\Lambda[\sigma] = -\frac{1}{2}\int_p \sigma(p)\sigma(-p) - \varepsilon_{\mathrm{vac}} \underbrace{\int d^D x}_{=\delta(0)}$$

where

$$e^{-\varepsilon_{\mathrm{vac}} \int d^D x} = \int [d\sigma] e^{S_\Lambda[\sigma]}$$

5. Expand

$$S_\Lambda[\sigma] = c_\Lambda \int d^D x + \frac{1}{2}\int_p c_{2\Lambda}(p)\sigma(p)\sigma(-p) + \cdots$$

so that

$$\boxed{-\Lambda \partial_\Lambda c_\Lambda = \int_p \Lambda \frac{\partial \ln \sqrt{R_\Lambda(p)}}{\partial \Lambda} \cdot (1 + c_{2\Lambda}(p))}$$

Example: free massive theory [Pagani& H.S. 2024]

$$c_{2\Lambda}(p) = -\frac{p^2 + m^2}{p^2 + m^2 + R_\Lambda(p)}$$

For $R_\Lambda(p) = \Lambda^2 R(p/\Lambda)$,

$$-\Lambda \partial_\Lambda c_\Lambda = \Lambda^D \int_p \frac{\left(1 - \frac{1}{2} p \cdot \partial_p\right) R(p)}{p^2 + m^2/\Lambda^2 + R(p)}$$

This gives, for $2 < D < 4$,

$$c_\Lambda = c - \Lambda^D \frac{1}{D} \int_p \frac{(1 - 1/2 p \cdot \partial_p) R(p)}{p^2 + R(p)} + \Lambda^{D-2} m^2 \frac{1}{D-2} \int_p \frac{(1 - 1/2 p \cdot \partial_p) R(p)}{(p^2 + R(p))^2} - \frac{1}{2} \int_p \left[\ln \frac{p^2 + m^2 + R_\Lambda(p)}{p^2 + R_\Lambda(p)} - \frac{m^2}{p^2 + R_\Lambda(p)} \right]$$

In the limit $\Lambda \to 0+$,

$$\varepsilon_{\mathrm{vac}}(m^2) = -c + \frac{1}{2} \int_p \left[\ln \frac{p^2 + m^2}{p^2} - \frac{m^2}{p^2} \right] = -c - \frac{1}{2(4\pi)^{\frac{D}{2}}} \Gamma\left(-\frac{D}{2}\right) (m^2)^{\frac{D}{2}}$$

$$= -c + \frac{1}{2} \int \frac{d^{D-1} p}{(2\pi)^{D-1}} \sqrt{p^2 + m^2} \quad (\text{dim reg})$$

For $D = 4$ we obtain

$$c_\Lambda = c + c'_\mu m^4 - \frac{\Lambda^4}{4} \int_p \frac{\left(1 - \frac{1}{2} p \cdot \partial_p\right) R(p)}{p^2 + R(p)}$$
$$+ \frac{m^2 \Lambda^2}{2} \int_p \frac{\left(1 - \frac{1}{2} p \cdot \partial_p\right) R(p)}{\left(p^2 + R(p)\right)^2} - m^4 \ln \frac{\Lambda}{\mu} \int_p \frac{\left(1 - \frac{1}{2} p \cdot \partial_p\right) R(p)}{\left(p^2 + R(p)\right)^3}$$
$$+ \Lambda^4 F\left(\frac{m^2}{\Lambda^2}\right)$$

where c'_μ is to cancel the μ-dependence, and

$$\Lambda^4 F\left(\frac{m^2}{\Lambda^2}\right) \equiv \int_\Lambda^\infty \frac{d\Lambda'}{\Lambda'} \Lambda'^{\,4} \int_p \left(1 - \frac{1}{2} p \cdot \partial_p\right) R(p)$$
$$\times \left[\frac{1}{p^2 + \frac{m^2}{\Lambda'^{\,2}} + R(p)} - \frac{1}{p^2 + R(p)} + \frac{m^2}{\Lambda'^{\,2}} \frac{1}{\left(p^2 + R(p)\right)^2} - \frac{m^4}{\Lambda'^{\,4}} \frac{1}{\left(p^2 + R(p)\right)^3} \right]$$

This gives

$$\begin{aligned} \varepsilon_{\rm vac} &= - \lim_{\Lambda \to 0+} c_\Lambda \\ &= -c - c''_\mu m^4 + m^4 \frac{1}{2} \ln \frac{m^2}{\mu^2} \cdot \underbrace{\int_p \frac{\left(1 - \frac{1}{2} p \cdot \partial_p\right) R(p)}{\left(p^2 + R(p)\right)^3}}_{\frac{1}{2(4\pi)^2}} \end{aligned}$$

where $\mu \partial_\mu c''_\mu = - \frac{1}{2(4\pi)^2}$. This is to be compared with

$$\lim_{\epsilon \to 0} \left(\mu^\epsilon \int \frac{d^{D-1} p}{(2\pi)^{D-1}} \frac{1}{2} \sqrt{p^2 + m^2} + \frac{m^4}{(4\pi)^2} \frac{1}{2\epsilon} \right) = m^4 \frac{1}{4(4\pi)^2} \ln \frac{m^2}{\mu^2 e^{\gamma - \frac{3}{2}}}$$

where $D = 4 - \epsilon$.

6. For a constant field $\sigma(p) = \sigma\,\delta(p)$, $S_\Lambda[\sigma] = s_\Lambda(\sigma)\,\delta(0)$.

$$-\Lambda\partial_\Lambda s_\Lambda(\sigma) = \int_p \Lambda\partial_\Lambda \ln\sqrt{R_\Lambda(p)} \cdot (1 + s_{2\Lambda}(p,\sigma)) + \left(\sigma + \frac{\partial s_\Lambda(\sigma)}{\partial\sigma}\right)\frac{\partial s_\Lambda(\sigma)}{\partial\sigma}$$

where $\frac{\delta^2 S_\Lambda[\sigma]}{\delta\sigma(p)\delta\sigma(-q)}\Big|_{\text{constant}\,\sigma} = \delta(p-q)\cdot s_{2\Lambda}(p,\sigma)$

7. $W_\Lambda[J] \equiv S_\Lambda[\sigma] + \frac{1}{2}\int_p \sigma(p)\sigma(-p)$, where $J(p) \equiv \sqrt{R_\Lambda(p)}\,\sigma(p)$

   (a) Master formula

$$e^{W_\Lambda[J]} = \int [d\phi'] \exp\left[S_{\text{bare}}[\phi'] - \frac{1}{2}\int_p R_\Lambda(p)\phi'(p)\phi'(-p) + \int_p J(p)\phi'(-p)\right]$$

   (b) ERG

$$-\Lambda\partial_\Lambda e^{W_\Lambda[J]} = \int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{1}{2}\frac{\delta^2}{\delta J(p)\delta J(-p)}e^{W_\Lambda[J]}$$

(c) the generating functional $\mathcal{W}[J] = \lim_{\Lambda \to 0+} W_\Lambda[J]$
(d) the vacuum energy density $-\varepsilon_{\rm vac}\,\delta(0) = \mathcal{W}[J=0]$

8. $\Gamma_\Lambda[\Phi] - \frac{1}{2}\int_p R_\Lambda(p)\Phi(p)\Phi(-p) = W_\Lambda[J] - \int_p J(p)\Phi(-p)$
where $\Phi(p) \equiv \frac{\delta W_\Lambda[J]}{\delta J(-p)}$.

(a) ERG

$$-\Lambda\partial_\Lambda \Gamma_\Lambda[\Phi] = \int_p \Lambda\partial_\Lambda R_\Lambda(p) \frac{1}{2} \frac{\delta^2 W_\Lambda[J]}{\delta J(p)\delta J(-p)}$$

(b) the 1PI generating functional $\Gamma_{\rm eff}[\Phi] = \lim_{\Lambda \to 0+} \Gamma_\Lambda[\Phi]$
(c) the vacuum energy density $-\varepsilon_{\rm vac}\,\delta(0) = \Gamma_{\rm eff}[\Phi]\Big|_{\Phi(p)=v\,\delta(p)}$ where $J=0$ corresponds to $\Phi(p) = v\,\delta(p)$.

9. An anomalous dimension γ_Λ can be introduced.

Cutoff function $\sqrt{R_\Lambda}(x,y)$ and diffusion

1. The master formula in coordinate space:

$$e^{S_\Lambda[\sigma]}$$

$$= \int [d\phi'] \exp \left[S_{\rm bare}[\phi'] - \frac{1}{2} \int d^D x \left(\sigma(x) - \int d^D y \sqrt{R_\Lambda}(x,y) \phi'(y) \right)^2 \right]$$

2. For a real scalar field, we can choose $\sqrt{R_\Lambda}(p) = \Lambda e^{-\frac{p^2}{\Lambda^2}}$.[Wilson&Kogut 1974]

   $\sqrt{R_\Lambda}(x,y) = \sqrt{R_\Lambda}(y,x) = \int_p e^{-\frac{p^2}{\Lambda^2} + ip(x-y)}$ satisfies the diffusion equation

$$-\Lambda \partial_\Lambda \sqrt{R_\Lambda}(x,y) = \left(-1 + \frac{2}{\Lambda^2} \partial_x^2 \right) \sqrt{R_\Lambda}(x,y)$$

The correlation function

$$\mathcal{C}_{n\Lambda}(x_1, \cdots, x_n) \equiv \left\langle \exp\left(-\frac{1}{2} \int d^D x \frac{\delta^2}{\delta\sigma(x)\delta\sigma(x)} \right) \sigma(x_1) \cdots \sigma(x_n) \right\rangle_{S_\Lambda[\sigma]}$$

satisfies the diffusion equation:

$$-\Lambda \frac{\partial}{\partial \Lambda} \mathcal{C}_{n\Lambda}(x_1, \cdots, x_n) = \sum_{i=1}^{n} \left(-1 + \frac{2}{\Lambda^2} \partial_{x_i}^2 \right) \mathcal{C}_{n\Lambda}(x_1, \cdots, x_n) \,.$$

3. For a complex scalar field under the background U(1) gauge field $\bar{A}_\mu(x)$, $\sqrt{R_\Lambda}(x, y) = \sqrt{R_\Lambda}(y, x)^*$ satisfies

$$-\Lambda \partial_\Lambda \sqrt{R_\Lambda}(x, y) = \left(-1 + \frac{2}{\Lambda^2} \left(\partial_\mu - i \bar{A}_\mu(x) \right)^2 \right) \sqrt{R_\Lambda}(x, y)$$

The master formula is background gauge invariant.

$$e^{S_\Lambda[\sigma,\sigma^*;\bar{A}_\mu]}$$
$$= \int [d\phi'] \exp\left[S_{\rm bare}[\phi', \phi'^{\,*}; \bar{A}_\mu] - \frac{1}{2}\int d^D x \left| \sigma(x) - \int d^D y \sqrt{R_\Lambda}(x,y)\phi'(y)\right|^2 \right]$$

The correlation function

$$\mathcal{C}_{n\Lambda}(x_1, \cdots, x_n; y_1, \cdots, y_n)$$
$$\equiv \left\langle \exp\left(- \int d^D x \frac{\delta^2}{\delta\sigma(x)\delta\sigma^*(x)}\right) \sigma(x_1)\cdots\sigma(x_n)\sigma^*(y_1)\cdots\sigma^*(y_n) \right\rangle_{S_\Lambda}$$

satisfies the covariant diffusion equation:

$$
\begin{aligned}
&-\Lambda\partial_\Lambda \mathcal{C}_{n\Lambda}(x_1,\cdots,x_n;y_1,\cdots,y_n) = \sum_{i=1}^n \Bigg\{ \left(-1+\frac{2}{\Lambda^2}\left(\frac{\partial}{\partial x_{i\mu}} - i\bar{A}_\mu(x_i)\right)^2\right) \\
&+ \left(-1+\frac{2}{\Lambda^2}\left(\frac{\partial}{\partial y_{i\mu}} + i\bar{A}_\mu(y_i)\right)^2\right)\Bigg\} \mathcal{C}_{n\Lambda}(x_1,\cdots,x_n;y_1,\cdots,y_n)
\end{aligned}
$$

4. In GFERG, we replace $\bar{A}_\mu$ by a dynamical gauge field. [Hiroshi Suzuki 2018, H.S.& Hiroshi Suzuki, 2019, 2020]

GFERG for QED

[H.S.& Hiroshi Suzuki, 2111.15529, 2201.04448]

1. Master formula

$$e^{S_\Lambda[\sigma_\mu,\sigma_c,\bar\sigma_c,\sigma_F,\bar\sigma_F]} \equiv \int [dA'_\mu dc' d\bar c' d\psi' d\bar\psi'] e^{S_{\rm bare}[A'_\mu,c',\bar c',\psi',\bar\psi']}$$
$$\times \exp\Bigg[-\frac{1}{2}\int d^4x \left(\sigma_\mu - z_\Lambda \Lambda A'_{\Lambda\mu}\right)^2 - \int d^4x \left(\bar\sigma_c - \frac{1}{z_\Lambda}\Lambda \bar c'_\Lambda\right)\left(\sigma_c - z_\Lambda \Lambda c'_\Lambda\right)$$
$$+ i \int d^4x \, \left(\bar\sigma_F - z_{F\Lambda}\sqrt{\Lambda}\bar\psi'_\Lambda\right)\left(\sigma_F - z_{F\Lambda}\sqrt{\Lambda}\psi'_\Lambda\right)\Bigg]$$

where $S_{\rm bare}$ is a BRST invariant bare action, and $A'_{\Lambda\mu}, c'_\Lambda, \bar c'_\Lambda, \psi'_\Lambda, \bar\psi'_\Lambda$ are diffused fields.

(a) The bare action

$$S_{\rm bare}[A_\mu, c, \bar{c}, \psi, \bar{\psi}] = - \int d^4x \left[\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2\xi_0} (\partial \cdot A)^2 + \partial_\mu \bar{c} \partial_\mu c \right.$$
$$\left. + \bar{\psi} \left\{ \gamma_\mu \left(-i\partial_\mu - e_0 A_\mu \right) + i m_0 \right\} \psi \right]$$

is invariant under the BRST transformation

$$\delta_\eta A_\mu = \eta \partial_\mu c$$

$$\delta_\eta c = 0, \qquad \delta_\eta \bar{c} = \eta \frac{1}{\xi_0} \partial_\mu A_\mu$$

$$\delta_\eta \psi = i e_0 \eta c \psi, \qquad \delta_\eta \bar{\psi} = -i e_0 \eta c \bar{\psi}$$

(b) Diffused fields satisfy the diffusion equations [Lüscher & Weisz, 2011]:

$$-\Lambda\partial_\Lambda A'_{\Lambda\mu} = \frac{2}{\Lambda^2}\left(\partial_\nu F'_{\Lambda\nu\mu} + \alpha_0\partial_\mu\partial\cdot A'_\Lambda\right)$$

$$-\Lambda\partial_\Lambda c'_\Lambda = \frac{2}{\Lambda^2}\alpha_0\partial^2 c'_\Lambda, \quad -\Lambda\partial_\Lambda \bar{c}'_\Lambda = \frac{2}{\Lambda^2}\alpha_0\partial^2 \bar{c}'_\Lambda$$

$$-\Lambda\partial_\Lambda \psi'_\Lambda = \frac{2}{\Lambda^2}\left\{\left(\partial_\mu - ie_0 A'_{\Lambda\mu}\right)^2 + \alpha_0 ie_0\partial\cdot A'_\Lambda\right\}\psi'_\Lambda,$$

$$-\Lambda\partial_\Lambda \bar{\psi}'_\Lambda = \frac{2}{\Lambda^2}\left\{\left(\partial_\mu + ie_0 A'_{\Lambda\mu}\right)^2 - \alpha_0 ie_0\partial\cdot A'_\Lambda\right\}\bar{\psi}'_\Lambda$$

(c) The diffusion commutes with BRST:

$$\delta_\eta A'_{\Lambda\mu} = \eta\partial_\mu c'_\Lambda$$

$$\delta_\eta c'_\Lambda = 0, \qquad \delta\bar{c}'_\Lambda = \eta\frac{1}{\xi_0}\partial_\mu A'_{\Lambda\mu}$$

$$\delta_\eta\psi'_\Lambda = ie_0\eta c'_\Lambda\psi'_\Lambda, \qquad \delta_\eta\bar{\psi}'_\Lambda = -ie_0\eta c'_\Lambda\bar{\psi}'_\Lambda$$

2. GFERG equation (simpler by choosing $\alpha_0 = 1$)

$$
\begin{aligned}
&- \Lambda\partial_\Lambda e^{S_\Lambda[\sigma_\mu,\sigma_c,\bar\sigma_c,\sigma_F,\bar\sigma_F]}\\
&= \int d^4x \Bigg[\frac{\delta}{\delta\sigma_\mu(x)} \left\{ \left(-\gamma_\Lambda + 1 - \frac{2}{\Lambda^2}\partial^2 \right) \left(\sigma_\mu(x) + \frac{\delta}{\delta\sigma_\mu(x)} \right) e^{S_\Lambda} \right\}\\
&\quad + \left\{ \left(\gamma_\Lambda - 1 + \frac{2}{\Lambda^2}\partial^2 \right) \left(\sigma_c + \frac{\overrightarrow{\delta}}{\delta\bar\sigma_c} \right) e^{S_\Lambda} \right\} \frac{\overleftarrow{\delta}}{\delta\sigma_c} + \frac{\overrightarrow{\delta}}{\delta\bar\sigma_c} \left\{ e^{S_\Lambda} \cdots \right\}\\
&\quad + \mathrm{Tr} \left\{ \left(\gamma_F - \frac{1}{2} + \frac{2}{\Lambda^2}\partial^2 - i\frac{4e_\Lambda}{\Lambda} \left(\sigma_\mu + \frac{\delta}{\delta\sigma_\mu} \right) \partial_\mu - \frac{2e_\Lambda^2}{\Lambda^2} \left(\sigma_\mu + \frac{\delta}{\delta\sigma_\mu} \right)^2 \right) \right.\\
&\qquad \left. \times \left(\sigma_F + i\frac{\overrightarrow{\delta}}{\delta\bar\sigma_F} \right) e^{S_\Lambda} \right\} \frac{\overleftarrow{\delta}}{\delta\sigma_F} + \mathrm{Tr}\, \frac{\overrightarrow{\delta}}{\delta\bar\sigma_F} \left\{ e^{S_\Lambda} \cdots \right\} \Bigg]
\end{aligned}
$$

where $e_\Lambda = \frac{e_0}{z_\Lambda}$, $-\Lambda\partial_\Lambda z_\Lambda = \gamma_\Lambda z_\Lambda$, $-\Lambda\partial_\Lambda z_{F\Lambda} = \gamma_F z_{F\Lambda}$

3. GFERG for the modified correlation functions

$$
\begin{aligned}
& -\Lambda\partial_\Lambda \left\langle \exp\left(-\frac{1}{2}\int d^4x \frac{\delta^2}{\delta\sigma_\mu(x)\delta\sigma_\mu(x)}\right) \sigma_{\mu_1}(x_1)\cdots\sigma_{\mu_k}(x_k) \right. \\
& \left. \times \sigma_F(y_1)\cdots\sigma_F(y_l) \exp\left(-i\int d^4x \frac{\overleftarrow{\delta}}{\delta\sigma_F(x)}\frac{\overrightarrow{\delta}}{\delta\bar\sigma_F(x)}\right) \bar\sigma_F(z_1)\cdots\bar\sigma_F(z_l) \right\rangle_{S_\Lambda} \\
& = \left\langle \exp\left(-\frac{1}{2}\int d^4x \frac{\delta^2}{\delta\sigma_\mu(x)\delta\sigma_\mu(x)}\right) \left(\gamma_\Lambda - 1 + \frac{2}{\Lambda^2}\partial_{x_1}^2\right) \sigma_{\mu_1}(x_1)\cdots \right\rangle_{S_\Lambda} + \cdots \\
& \quad + \left\langle \cdots \left(\gamma_F - \frac{1}{2} + \frac{2}{\Lambda^2}\partial_{y_1}^2 - i\frac{4e_\Lambda}{\Lambda}\sigma_\mu(y_1) - \frac{2e_\Lambda^2}{\Lambda^2}\sigma_\mu(y_1)^2\right) \sigma_F(y_1)\cdots \right. \\
& \quad\quad \left. \times \exp\left(-\int d^4x \frac{\overleftarrow{\delta}}{\delta\sigma_F}\frac{\overrightarrow{\delta}}{\delta\bar\sigma_F}\right)\cdots \right\rangle_{S_\Lambda} + \cdots
\end{aligned}
$$

This is BRST invariant diffusion.

4. The BRST invariance

$$\int d^4x \Bigg[-\partial_\mu \left(\sigma_c + \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_c} S_\Lambda \right) \frac{\delta S_\Lambda}{\delta \sigma_\mu} - \frac{1}{\xi_\Lambda} \partial_\mu \left(\sigma_\mu + \frac{\delta S_\Lambda}{\delta \sigma_\mu} \right) \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_c} S_\Lambda$$
$$- i e_\Lambda \frac{1}{\Lambda} \left(\sigma_c + \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_c} S_\Lambda \right) \left(-\bar{\sigma}_F \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_F} S_\Lambda + S_\Lambda \frac{\overleftarrow{\delta}}{\delta \sigma_F} \sigma_F \right) \Bigg] = 0$$

where $\xi_\Lambda = z_\Lambda^2 \xi_0$.

The ghost dependence

$$S_{\text{ghost}} = - \int_k \bar{\sigma}_c(-k) \frac{k^2}{\Lambda^2 e^{-2\frac{k^2}{\Lambda^2}} + k^2} \sigma_c(k)$$

gives the BRST invariance as an almost standard WT identity:

$$\frac{\xi_\Lambda \Lambda^2 e^{-2\frac{k^2}{\Lambda^2}} + k^2}{\xi_\Lambda \Lambda^2 e^{-2\frac{k^2}{\Lambda^2}}} k_\mu \frac{\delta}{\delta \sigma_\mu(k)} \left[S_\Lambda + \frac{1}{2} \int_k \sigma_\mu(k) \sigma_\nu(-k) \frac{k_\mu k_\nu}{\xi_\Lambda \Lambda^2 e^{-2\frac{k^2}{\Lambda^2}} + k^2} \right]$$

$$= \frac{e_\Lambda}{\Lambda} \int_p \left[\bar\sigma_F(-p-k) \frac{\overrightarrow{\delta}}{\delta \bar\sigma_F(-p)} S_\Lambda - S_\Lambda \frac{\overleftarrow{\delta}}{\delta \sigma_F(p+k)} \sigma_F(p) \right]$$

This is to be contrasted with the "usual" non-linear WT identity:

$$\frac{1}{\xi_\Lambda} k^2 k_\mu \mathcal{A}_\mu(k) = -k_\mu \sqrt{R_\Lambda(k)} \frac{\delta S_\Lambda}{\delta \sigma_\mu(-k)}$$

$$- e_\Lambda e^{-S_\Lambda} \int_p \sqrt{R_\Lambda(p)} \mathrm{Tr} \left(\Psi(p+k) e^{S_\Lambda} \right) \frac{\overleftarrow{\delta}}{\delta \sigma_F(p)}$$

$$+ e_\Lambda \int_p \sqrt{R_\Lambda(p)} \mathrm{Tr} \frac{\overrightarrow{\delta}}{\delta \bar\sigma_F(-p)} \left(e^{S_\Lambda} \bar\Psi(-p+k) \right)$$

where

$$\mathcal{A}_\mu(k) \equiv \frac{1}{\sqrt{R_\Lambda(k)}} \left(\sigma_\mu(k) + \frac{\delta S_\Lambda}{\delta \sigma_\mu(-k)} \right)$$

$$\Psi(p) \equiv \frac{1}{\sqrt{R_\Lambda(p)}} \left(\sigma_F(p) + i \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_F(-p)} S_\Lambda \right)$$

$$\bar{\Psi}(-p) \equiv \frac{1}{\sqrt{R_\Lambda(p)}} \left(\bar{\sigma}_F(-p) + i S_\Lambda \frac{\overleftarrow{\delta}}{\delta \sigma_F(p)} \right)$$

5. 1-loop calculations give ξ-independent results [H.S. & Hiroshi Suzuki, 2111.1529]:

$$\gamma_\Lambda \simeq \frac{4}{3} \frac{e^2}{(4\pi)^2}, \quad \beta_m \simeq 6 \frac{e^2}{(4\pi)^2}, \quad \gamma_{F\Lambda} \simeq 4 \frac{e^2}{(4\pi)^2}$$

The result for $\gamma_{F\Lambda}$ agrees with Lüscher& Weisz.

1PI for GFERG

1. New variables

$$\mathcal{A}_\mu(x) \equiv \frac{1}{\Lambda}\left(\sigma_\mu(x) + \frac{\delta S_\Lambda}{\delta\sigma_\mu(x)}\right)$$

$$C(x) \equiv \frac{1}{\Lambda}\left(\sigma_c(x) + \frac{\overrightarrow{\delta}}{\delta\bar{\sigma}_c(x)} S_\Lambda\right), \qquad \bar{C}(x) \equiv \frac{1}{\Lambda}\left(\bar{\sigma}_c(x) + S_\Lambda \frac{\overleftarrow{\delta}}{\delta\sigma_c(x)}\right)$$

$$\Psi(x) \equiv \frac{1}{\sqrt{\Lambda}}\left(\sigma_F(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\sigma}_F(x)} S_\Lambda\right), \quad \bar{\Psi}(x) \equiv \frac{1}{\sqrt{\Lambda}}\left(\bar{\sigma}_F(x) + S_\Lambda i\frac{\overleftarrow{\delta}}{\delta\sigma_F(x)}\right)$$

2. Legendre transformation

$$\Gamma_\Lambda + \int d^4x \left(-\frac{\Lambda^2}{2}\mathcal{A}_\mu^2 - \Lambda^2 \bar{C} C + i\Lambda \bar{\Psi}\Psi \right)$$
$$= S_\Lambda + \int d^4x \left(\frac{1}{2}\sigma_\mu^2 + \bar{\sigma}_c \sigma_c - i\bar{\sigma}_F \sigma_F \right)$$
$$- \int d^4x \left(\sigma_\mu \Lambda \mathcal{A}_\mu + \Lambda \bar{C} \sigma_c + \bar{\sigma}_c \Lambda C - i\sqrt{\Lambda}\bar{\Psi}\sigma_F - i\bar{\sigma}_F \sqrt{\Lambda}\Psi \right)$$

This gives

$$\frac{\delta \Gamma_\Lambda}{\delta \mathcal{A}_\mu} = \Lambda \frac{\delta S_\Lambda}{\delta \sigma_\mu}, \quad \Gamma_\Lambda \frac{\overleftarrow{\delta}}{\delta \Psi} = S_\Lambda \sqrt{\Lambda} \frac{\overleftarrow{\delta}}{\delta \sigma_F}, \quad \frac{\overrightarrow{\delta}}{\delta \bar{\Psi}} \Gamma_\Lambda = \sqrt{\Lambda} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_F} S_\Lambda$$

3. BRST invariance rewritten as

$$k_\mu \frac{\delta}{\delta \mathcal{A}_\mu(k)} \left(\Gamma_\Lambda + \frac{1}{2\xi_\Lambda} \int_l l_\mu \mathcal{A}_\mu(k) l_\nu \mathcal{A}_\nu(-l) e^{2\frac{l^2}{\Lambda^2}} \right)$$
$$= e_\Lambda \int_p \left(\bar{\Psi}(-p-k) \frac{\overrightarrow{\delta}}{\delta \bar{\Psi}(-p)} \Gamma_\Lambda - \Gamma_\Lambda \frac{\overleftarrow{\delta}}{\delta \Psi(p+k)} \Psi(p) \right)$$

Hence,

$$\Gamma_\Lambda[\mathcal{A}, C, \bar{C}, \Psi, \bar{\Psi}] = -\int_k e^{2\frac{k^2}{\Lambda^2}} k^2 \bar{C}(-k) C(k) - \frac{1}{2\xi_\Lambda} \int_k e^{2\frac{k^2}{\Lambda^2}} k \cdot \mathcal{A}(k) k \cdot \mathcal{A}(-k)$$
$$+ \Gamma_{\mathrm{inv},\Lambda}[\mathcal{A}_\mu, \Psi, \bar{\Psi}]$$

where $\Gamma_{\mathrm{inv},\Lambda}$ is gauge invariant in the classical sense.
[H.S.& Hiroshi Suzuki, 2201.04448] GFERG equation given in Appendix 3.

Concluding remarks

1. ERG for the vacuum energy density
   [Pagani&H.S., Phys. Rev. **D109** (2024) 12, 125007 [2404.12881]]

2. GFERG for gauge theories [H.S.& Hiroshi Suzuki]
   (a) gauge invariant truncation possible now — potential applications to QED with 4-Fermi interactions in 4D (equivalent to gauged Yukawa model?) & ...
   (b) difficulty with YM — diffusion of $\bar{c}^a$ not compatible with BRST; the introduction of the auxiliary field B^a complicates life
   (c) axial invariance not manifest, since the master formula spoils axial invariance
   (d) GFERG has been applied to 2d nonlinear sigma model [Haruna et al.] and scalar QED [Haruna&Yamada]

Appendix 1: anomalous dimension

1. ERG for $S_\Lambda[\sigma]$

$$-\Lambda \partial_\Lambda e^{S_\Lambda[\sigma]} = \int_p \left(\Lambda \frac{\partial \ln \sqrt{R_\Lambda(p)}}{\partial \Lambda} - \gamma_\Lambda \right) \frac{\delta}{\delta \sigma(p)} \left\{ \left(\sigma(p) + \frac{\delta}{\delta \sigma(-p)} \right) e^{S_\Lambda[\sigma]} \right\}$$

2. ERG for $W_\Lambda[J] = S_\Lambda[\sigma] + \frac{1}{2} \int_p \sigma(p)\sigma(-p)$ where $J(p) \equiv \sqrt{R_\Lambda(p)}\,\sigma(p)$

$$\begin{aligned} &- \Lambda \partial_\Lambda W_\Lambda[J] = \gamma_\Lambda \int_p J(p) \frac{\delta}{\delta J(p)} W_\Lambda[J] \\ &\quad + \int_p \left(\Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} - 2\gamma_\Lambda R_\Lambda(p) \right) \frac{1}{2} \left\{ \frac{\delta W_\Lambda[J]}{\delta J(p)} \frac{\delta W_\Lambda[J]}{\delta J(-p)} + \frac{\delta^2 W_\Lambda[J]}{\delta J(p) \delta J(-p)} \right\} \end{aligned}$$

Generating functional $\mathcal{W}[J] = \lim_{\Lambda \to 0+} W_\Lambda[J/z_\Lambda]$

3. ERG for $\Gamma_\Lambda[\Phi] - \frac{1}{2}\int_p R_\Lambda(p)\Phi(p)\Phi(-p) = W_\Lambda[J] - \int_p J(p)\Phi(-p)$ where $\Phi(p) = \frac{\delta W_\Lambda[J]}{\delta J(-p)}$

$$-\Lambda\partial_\Lambda \Gamma_\Lambda[\Phi] = -\gamma_\Lambda \int_p \Phi(p)\frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(p)} + \int_p \left(\Lambda\partial_\Lambda R_\Lambda(p) - 2\gamma_\Lambda R_\Lambda(p)\right)\frac{1}{2}\frac{\delta^2 W_\Lambda[J]}{\delta J(p)\delta J(-p)}$$

where

$$\int_q \frac{\delta^2 W_\Lambda[J]}{\delta J(p)\delta J(-q)}\left(R(q)\delta(q-r) - \frac{\delta^2 \Gamma_\Lambda[\Phi]}{\delta\Phi(q)\delta\Phi(-r)}\right) = \delta(p-r)$$

Generating functional $\Gamma_{\mathrm{eff}}[\Phi] = \lim_{\Lambda\to 0+}\Gamma_\Lambda[z_\Lambda\Phi]$

Appendix 2: Dimensionless convention

1. Dimensionless parameters and fields

$$\Lambda = \mu e^{-t}$$

$$e_\Lambda = \mu^{-\frac{\epsilon}{2}} e_t$$

$$\mathcal{A}_\mu(k) = \Lambda^{-\frac{D+2}{2}} \tilde{\mathcal{A}}_\mu(k/\Lambda)$$

$$C(k) = \Lambda^{-\frac{D+2}{2}} \tilde{C}(k/\Lambda)$$

$$\bar{C}(-k) = \Lambda^{-\frac{D+2}{2}} \tilde{\bar{C}}(-k/\Lambda)$$

$$\Psi(p) = \Lambda^{-\frac{D+1}{2}} \tilde{\Psi}(p/\Lambda)$$

$$\bar{\Psi}(-p) = \Lambda^{-\frac{D+1}{2}} \tilde{\bar{\Psi}}(-p/\Lambda)$$

$$\Gamma_\Lambda[\mathcal{A}, C, \bar{C}, \Psi, \bar{\Psi}] = \Gamma_t[\tilde{\mathcal{A}}, \tilde{C}, \tilde{\bar{C}}, \tilde{\Psi}, \tilde{\bar{\Psi}}]$$

2.

$$\partial_t \tilde{\Gamma}_t = -\Lambda \partial_\Lambda \Gamma_\Lambda$$
$$+ \int_k \left(\frac{D+2}{2} + k \cdot \partial_k \right) \tilde{\mathcal{A}}_\mu(k) \cdot \frac{\delta \tilde{\Gamma}_t}{\delta \mathcal{A}_\mu(k)}$$
$$+ \int_k \left(\frac{D+2}{2} + k \cdot \partial_k \right) \tilde{\bar{\tilde{C}}}(-k) \frac{\overrightarrow{\delta}}{\delta \tilde{\bar{\tilde{C}}}(-k)} \tilde{\Gamma}_t + \int_k \tilde{\Gamma}_t \frac{\overleftarrow{\delta}}{\delta \tilde{C}(k)} \left(\frac{D+2}{2} + k \cdot \partial_k \right) \tilde{C}(k)$$
$$+ \int_p \left[\tilde{\Gamma}_t \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(p)} \left(\frac{D+1}{2} + p \cdot \partial_p \right) \tilde{\Psi}(p) + \left(\frac{D+1}{2} + p \cdot \partial_p \right) \tilde{\bar{\tilde{\Psi}}}(-p) \frac{\overrightarrow{\delta}}{\delta \tilde{\bar{\tilde{\Psi}}}(-p)} \tilde{\Gamma}_t \right]$$

3.

$$\partial_t = \left(\frac{\epsilon}{2} + \frac{\beta(e^2)}{2e^2} \right) e \partial_e + \left(1 + \beta_m(e^2) \right) m \partial_m$$

Appendix 3: GFERG equation for Γ of QED

We give the GFERG equation for the 1PI Γ in the dimensionless convention, where all the quantities are rendered dimensionless by using appropriate powers of Λ. Though BRST invariance is simple, the differential equation for Γ is complicated.

$$\partial_t \Gamma + \int d^D x \Bigg[\left\{ \left(\frac{D-2}{2} + \gamma + x \cdot \partial \right) \mathcal{A}_\mu(x) + 2\partial^2 \mathcal{A}_\mu(x) \right\} \frac{\delta \Gamma}{\delta \mathcal{A}_\mu(x)}$$

$$+ \left\{ \left(\frac{D-1}{2} + \gamma_F + x \cdot \partial \right) \bar{\Psi} + 2 \left(\partial^2 \bar{\Psi}(x) + 2ie \partial_\mu \bar{\Psi} \cdot \mathcal{A}_\mu - e^2 \bar{\Psi} \mathcal{A}_\mu \mathcal{A}_\mu \right) \right\} \frac{\overrightarrow{\delta}}{\delta \bar{\Psi}(x)} \Gamma$$

$$+ \Gamma \frac{\overleftarrow{\delta}}{\delta \Psi(x)} \left\{ \left(\frac{D-1}{2} + \gamma_F + x \cdot \partial \right) \Psi + 2 \left(\partial^2 \Psi - 2ie \mathcal{A}_\mu \partial_\mu \Psi - e^2 \mathcal{A}_\mu \mathcal{A}_\mu \right) \right\} \Bigg]$$

$$= \int d^D x \Bigg[- 2 \left\{ 2ie \left(\left[\partial_\mu \bar{\Psi} \cdot \mathcal{A}_\mu \right] - \partial_\mu \bar{\Psi} \cdot \mathcal{A}_\mu \right) - e^2 \left(\left[\bar{\Psi} \mathcal{A}_\mu \mathcal{A}_\mu \right] - \bar{\Psi} \mathcal{A}_\mu \mathcal{A}_\mu \right) \right\} \cdot \frac{\overrightarrow{\delta}}{\delta \bar{\Psi}} \Gamma$$

$$- 2 \Gamma \frac{\overleftarrow{\delta}}{\delta \Psi} \left\{ -2ie \left([\mathcal{A}_\mu \partial_\mu \Psi] - \mathcal{A}_\mu \partial_\mu \Psi \right) - e^2 \left([\mathcal{A}_\mu \mathcal{A}_\mu \Psi] - \mathcal{A}_\mu \mathcal{A}_\mu \Psi \right) \right\}$$

$$- \frac{\delta}{\delta \sigma_\mu(x)} \left\{ (-1 + \gamma) \mathcal{A}_\mu(x) + 2\partial^2 \mathcal{A}_\mu(x) \right\}$$

$$+ \mathrm{Tr} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_F(x)} \left\{ \left(-\frac{1}{2} + \gamma_F \right) \bar{\Psi}(x) \right\} + \mathrm{Tr} \left\{ \left(-\frac{1}{2} + \gamma_F \right) \Psi(x) \right\} \frac{\overleftarrow{\delta}}{\delta \sigma_F(x)}$$

$$+ 2\mathrm{Tr} \frac{\overrightarrow{\delta}}{\delta \bar{\sigma}_F(x)} \left\{ \partial^2 \bar{\Psi}(x) + 2ie \left[\mathcal{A}_\mu \partial_\mu \bar{\Psi} \right] - e^2 \left[\mathcal{A}_\mu^2 \bar{\Psi} \right] \right\}$$

$$+ 2\mathrm{Tr} \left\{ \partial^2 \Psi(x) - 2ie \left[\mathcal{A}_\mu \partial_\mu \Psi \right] - e^2 \left[\mathcal{A}_\mu^2 \Psi \right] \right\} \frac{\overleftarrow{\delta}}{\delta \sigma_F(x)} \Bigg]$$

where

$$\left[\bar{\Psi}(x)\mathcal{A}_\mu(x')\right] \equiv \bar{\Psi}(x)\mathcal{A}_\mu(x') + \frac{\delta}{\delta\sigma_\mu(x')}\bar{\Psi}(x)$$

$$\begin{aligned}\left[\bar{\Psi}(x)\mathcal{A}_\mu(x')\mathcal{A}_\nu(x'')\right] \equiv\ & \bar{\Psi}(x)\mathcal{A}_\mu(x')\mathcal{A}_\nu(x'') + \bar{\Psi}(x)\frac{\delta\mathcal{A}_\mu(x')}{\delta\sigma_\nu(x'')} \\ & + \mathcal{A}_\nu(x'')\frac{\delta}{\delta\sigma_\mu(x')}\bar{\Psi}(x) + \mathcal{A}_\mu(x')\frac{\delta}{\delta\sigma_\nu(x'')}\bar{\Psi}(x) \\ & + \frac{\delta^2}{\delta\sigma_\mu(x')\delta\sigma_\nu(x'')}\bar{\Psi}(x)\end{aligned}$$

Appendix 4: Models

NJL model(2) $$\mathcal{L} = \bar{\psi}\gamma_\mu\partial_\mu\psi - g_V\left(\bar{\psi}\gamma_\mu\psi\right)^2 - g_A\left(\bar{\psi}\gamma_\mu\gamma_5\psi\right)^2$$

Yukawa(3) $$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^*\partial_\mu\phi + \frac{\lambda}{2}\left(\phi^*\phi - v^2\right)^2 \\ &\quad + \bar{\psi}\gamma_\mu\partial_\mu\psi - g\left(\phi^*\bar{\psi}_R\psi_L + \phi\bar{\psi}_L\psi_R\right)\end{aligned}$$

gauged NJL model(3) $$\begin{aligned}\mathcal{L} &= \frac{1}{4}F^2 + \bar{\psi}\gamma_\mu\left(\partial_\mu - ieA_\mu\right)\psi \\ &\quad - g_V\left(\bar{\psi}\gamma_\mu\psi\right)^2 - g_A\left(\bar{\psi}\gamma_\mu\gamma_5\psi\right)^2\end{aligned}$$

gauged Yukawa(4) $$\begin{aligned}\mathcal{L} &= \frac{1}{4}F^2 + \bar{\psi}\gamma_\mu\left(\partial_\mu - ieA_\mu\right)\psi \\ &\quad + \partial_\mu\phi^*\partial_\mu\phi + \frac{\lambda}{2}\left(\phi^*\phi - v^2\right)^2 \\ &\quad - g\left(\phi^*\bar{\psi}_R\psi_L + \phi\bar{\psi}_L\psi_R\right)\end{aligned}$$