Analytic solutions for scaling dimensions of highly irrelevant operators in LPA and f(R) approximations

ERG2024
26/09/24

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Single component real scalar field in LPA:

$$\Gamma_\Lambda = \int d^d x \left(\frac{1}{2} (\partial_\mu \varphi)^2 + V_\Lambda(\varphi) \right)$$

The LPA approximation amounts to setting the field φ in the Hessian to a spacetime constant, thus dropping from a derivative expansion all terms that do not take the form of a correction to the potential. The flow equation for $V_\Lambda(\varphi)$ then takes the form:

$$\left(\partial_t + d_\varphi \varphi \frac{\partial}{\partial \varphi} - d \right) V_\Lambda(\varphi) = -\frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \frac{\dot{\Delta}}{\Delta} \frac{1}{1 + \Delta V''_\Lambda(\varphi)} , \tag{2.1}$$

where $\partial_t = -\Lambda \partial_\Lambda$, t being the renormalization group 'time' which, following [7], we have chosen to flow towards the IR. Here the momentum, potential and field are already scaled by the appropriate power of Λ to make them dimensionless. Then $\Delta = C(q^2)/q^2$ no longer depends on Λ. The same is true of $\partial_t \Delta_\Lambda$, which after scaling we write as $\dot{\Delta}$, where

$$\dot{\Delta} = 2\, C'(q^2) \,. \tag{2.2}$$

Since $C(q^2)$ is monotonically increasing, we have that $\dot{\Delta} > 0$.

The scaling dimension of the field is $d_\varphi = \frac{1}{2}(d - 2 + \eta)$, where η is the anomalous dimension.

Asymptotic behaviour

Fixed point potential:

$$V(\varphi) = A|\varphi|^{d/d_\varphi} + \cdots \qquad \text{as} \qquad \varphi \to \pm\infty$$

=> discrete set of fixed points

Eigenoperator equation:

$$V_\Lambda(\varphi) = V(\varphi) + \varepsilon\, v(\varphi)\, \mathrm{e}^{\lambda t}$$

ε being infinitesimal. Here λ is the RG eigenvalue. It is the scaling dimension of the corresponding coupling, and is positive (negative) for relevant (irrelevant) operators. The scaling dimension of the operator $v(\varphi)$ itself is then $d - \lambda$. We write the eigenoperator equation in the same form as refs. [34, 35, 41]:

$$-\, a_2(\varphi) v''(\varphi) + a_1(\varphi) v'(\varphi) + a_0(\varphi) v(\varphi) = (d - \lambda) v(\varphi)\,, \tag{2.4}$$

where the φ-dependent coefficients multiplying the eigenoperators are given by:

$$a_0(\varphi) = 0\,, \tag{2.5}$$

$$a_1(\varphi) = d_\varphi \varphi\,, \tag{2.6}$$

$$a_2(\varphi) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d} \frac{\dot{\Delta}}{(1+\Delta V'')^2} > 0\,, \tag{2.7}$$

Asymptotic behaviour

Eigenoperator equation:

$$-a_2(\varphi)v''(\varphi) + a_1(\varphi)v'(\varphi) + a_0(\varphi)v(\varphi) = (d-\lambda)v(\varphi)$$

$$a_0(\varphi) = 0\,,$$
$$a_1(\varphi) = d_\varphi \varphi\,,$$
$$a_2(\varphi) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d}\frac{\dot{\Delta}}{(1+\Delta V'')^2} \propto F|\varphi|^{2(2-d/d_\varphi)} + \cdots$$

Cutoff-dependent

+ RG => $v(\varphi) \propto |\varphi|^{\frac{d-\lambda}{d_\varphi}} + \cdots$

=> Sturm-Liouville type

Liouville Normal Form:

Thus from SL analysis [32], we know that the eigenvalues λ_n are real, discrete, with a most positive (relevant) eigenvalue and an infinite tower of ever more negative (more irrelevant) eigenvalues, $\lambda_n \to -\infty$ as $n \to \infty$ [30]. Let us define a 'coordinate' x:

$$x = \int_0^\varphi \frac{1}{\sqrt{a_2(\varphi')}} \, d\varphi' \tag{2.17}$$

(always taking the positive root in fractional powers). Defining the wave-function as

$$\psi(x) = a_2^{1/4}(\varphi) w^{1/2}(\varphi) v(\varphi) \, , \tag{2.18}$$

enables us to recast (2.15) as:

$$w(\varphi) = \frac{1}{a_2(\varphi)} \exp\left\{ - \int_0^\varphi d\varphi' \frac{a_1(\varphi')}{a_2(\varphi')} d\varphi' \right\} \, ,$$

$$- \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = (d - \lambda)\psi(x) \, . \tag{2.19}$$

This is a one-dimensional time-independent Schrödinger equation for a particle of mass $m = 1/2$, with energy $E = d - \lambda$ *i.e.* just the eigenoperator scaling dimension, and with potential [34,35,41]:

$$U(x) = \frac{a_1^2}{4a_2} - \frac{a_1'}{2} + a_2' \left(\frac{a_1}{2a_2} + \frac{3a_2'}{16a_2} \right) - \frac{a_2''}{4} \, , \tag{2.20}$$

where the terms on the right hand side are functions of φ.

Asymptotic behaviour

$$-\frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = (d-\lambda)\psi(x)$$

$$U(x) = \frac{1}{4}(d-d_\varphi)^2 x^2 + \cdots$$

SHM. Universal!

WKB:

$$\int_{-x_n}^{x_n} dx\,\sqrt{E_n - U(x)} = \left(n + \frac{1}{2}\right)\pi$$

$$E_n = d - \lambda_n = U(x_n) = n(d-d_\varphi) + \cdots$$

$$\lambda_n = -n(d - d_\varphi) + O\left(n^{\frac{d/d_\varphi}{2(d/d_\varphi - 1)}}\right) \quad \text{as} \quad n \to \infty$$

Find double this for O(N) scalar field theory,
for all fixed N $\geqslant$0 and N=-2,-4,...

f(R) approximation

Split into background + fluctuation:

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

IR cutoff: $\mathcal{R} \sim \mathcal{R}(-\nabla^2/k^2)$

Single metric, or background field, ansatz:

Replace $\dfrac{\delta^2 \Gamma_k}{\delta h_{\mu\nu}(x)\, \delta h_{\alpha\beta}(y)}$ with $\dfrac{\delta^2 \Gamma_k}{\delta g_{\mu\nu}(x)\, \delta g_{\alpha\beta}(y)}$

This allows: $\Gamma_k[g] = \displaystyle\int d^4x \sqrt{g}\, f_k(R)$

by specialising to a maximally symmetric background manifold

D. Benedetti Europhys. Lett. (2013) 102:20007

$$\Delta_0 = -\nabla^2 - \frac{R}{3}, \quad \Delta_1 = -\nabla^2 - \frac{R}{4}, \quad \Delta_2 = -\nabla^2 + \frac{R}{6}.$$

Non-adaptive cutoff: $\mathcal{R}_k^\phi = k^{m_\phi} c_\phi r(\Delta_s + \alpha_s R)$

$$V\big(\partial_t f_k(R) + 2E_k(R)\big) = \mathcal{T}_2 + \mathcal{T}_0^{\bar{h}} + \mathcal{T}_1^{Jac} + \mathcal{T}_0^{Jac}, \tag{18}$$

where $V = \int d^4x\sqrt{g}$ is the volume of the manifold, and the $\mathcal{T}$ objects are the following spacetime traces:

$$\mathcal{T}_2 = \mathrm{Tr}\Big[\frac{d_t\mathcal{R}_k^T}{-f_k'(R)\Delta_2 - E_k(R)/2 + 2\mathcal{R}_k^T}\Big], \tag{19}$$

$$\mathcal{T}_0^{\bar{h}} = \mathrm{Tr}\Big[\frac{8\, d_t\mathcal{R}_k^{\bar{h}}}{9f_k''(R)\Delta_0^2 + 3f_k'(R)\Delta_0 + E_k(R) + 16\mathcal{R}_k^{\bar{h}}}\Big], \tag{20}$$

$$\mathcal{T}_1^{Jac} = -\frac{1}{2}\mathrm{Tr}\Big[\frac{d_t\mathcal{R}_k^V}{\Delta_1 + \mathcal{R}_k^V}\Big], \tag{21}$$

$$\mathcal{T}_0^{Jac} = \frac{1}{2}\mathrm{Tr}\Big[\frac{d_t\mathcal{R}_{S_1}^V}{\Delta_0 + R/3 + \mathcal{R}_k^{S_1}}\Big] - \mathrm{Tr}\Big[\frac{2\, d_t\mathcal{R}_{S_2}^V}{(3\Delta_0 + R)\Delta_0 + 4\mathcal{R}_k^{S_2}}\Big]. \tag{22}$$

$$r(z) = \frac{z}{\exp(az^b) - 1}, \qquad a > 0, b \geq 1.$$

Asymptotic analysis: fixed points

$$r(z) = \frac{z}{\exp(az^b) - 1}, \qquad a > 0, b \geq 1 .$$

4-Sphere:

$$f(R) = \frac{5a\alpha_0^b}{768\pi^2} R^{2+b} + \frac{5}{768\pi^2} R^2 \ln R + AR^2 + \frac{16 c_{\bar{h}}}{5ab(1+b)\alpha_0^b} \left(\alpha_0 - \frac{1}{3}\right) \mathrm{e}^{-a\left(\alpha_0 - \frac{1}{3}\right)^b R^b} + \cdots$$

A_S

4-Hyperboloid:

$$f(R) = AR^2 + \frac{c_{S1}}{96\sqrt{3\pi a^3 b^3}} \left(\frac{25}{48} - \alpha_0\right)^{\frac{5-3b}{2}} (-R)^{2-\frac{3b}{2}} \left\{1 + O\left(|R|^{-\frac{1}{2}}\right)\right\} \mathrm{e}^{-a\left[\left(\alpha_0 - \frac{25}{48}\right)R\right]^b} + \cdots$$

A_H

To get discrete set of fixed points, match sphere and hyperboloid solutions smoothly through flat space ($R \to 0$).

Asymptotic analysis: eigenoperators

Substitute $f_k(R) = f(R) + \epsilon\, v_k(R) \qquad \Longrightarrow \qquad v_k(R) = v(R)\,\mathrm{e}^{-\theta t}$

$$-a_2(R)\, v''(R) + a_1(R)\, v'(R) + a_0(R)\, v(R) = (4-\theta) v(R)$$

$a_2(R) \to 0 \qquad a_1(R) \to 2R \qquad a_0(R) \to 0$

$$a_2 = \frac{144 c_{\bar{h}}}{V} \mathrm{Tr}\left[\frac{\Delta_0^2(2r(\Delta_0+\alpha_0 R) - (\Delta_0+\alpha_0 R) r'(\Delta_0+\alpha_0 R))}{(9f''(R)\Delta_0^2 + 3f'(R)\Delta_0 + E(R) + 16 c_h r(\Delta_0+\alpha_0 R))^2}\right] > 0$$

+ RG => $\qquad v(R) \propto |R|^{2-\frac{\theta}{2}} + \cdots$

=> Sturm-Liouville type

$$r(z) = \frac{z}{\exp(az^b) - 1},$$

Liouville Normal Form

$$U(x) = (bx \ln|x|)^2 \left\{ 1 + O\left(\frac{\ln \ln |x|}{\ln |x|} \right) \right\} \qquad x \to \pm\infty$$

WKB:

$$\int_{-x_n}^{x_n} dx \sqrt{E_n - U(x)} = \left(n + \frac{1}{2} \right) \pi$$

$$\theta_n = -b\,(n \ln n) \left\{ 1 + O\left(\frac{\ln \ln n}{\ln n} \right) \right\} \quad \text{as} \quad n \to \infty\,.$$

All non-universal parameters have dropped out except for b (and this is caused by the single metric approximation).

Why single metric ansatz is at fault:

$$a_2(\varphi) = \frac{1}{2}\int \frac{d^d q}{(2\pi)^d} \frac{\dot{\Delta}}{(1+\Delta V'')^2} \propto F|\varphi|^{2(2-d/d_\varphi)} + \cdots$$

$$a_2 = \frac{144 c_h}{V} \mathrm{Tr}\left[\frac{\Delta_0^2 (2r(\Delta_0+\alpha_0 R) - (\Delta_0+\alpha_0 R) r'(\Delta_0+\alpha_0 R))}{(9 f''(R)\Delta_0^2 + 3 f'(R)\Delta_0 + E(R) + 16 c_h r(\Delta_0+\alpha_0 R))^2} \right].$$

This `ought' to be ~

$$a_2 = \frac{144 c_h}{V} \mathrm{Tr}\left[\frac{\Delta_0^2 (2r(\Delta_0+\alpha_0 R) - (\Delta_0+\alpha_0 R) r'(\Delta_0+\alpha_0 R))}{(9 f''(\hat{R})\Delta_0^2 + 3 f'(\hat{R})\Delta_0 + E(\hat{R}) + 16 c_h r(\Delta_0+\alpha_0 R))^2} \right].$$

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and f(R) approximations

$$\lambda_n = -n(d-d_\varphi) + O\left(n^{\frac{d/d_\varphi}{2(d/d_\varphi-1)}}\right) \quad \text{as} \quad n \to \infty$$

Double this for O(N) scalar field theory,
for all fixed N $\geqslant$0 and N=-2,-4,...

$$\theta_n = -b\,(n\ln n)\left\{1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right\} \quad \text{as} \quad n \to \infty\,.$$