Essential Renormalization Group Equation for Gravity coupled to Scalar Matter

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Partly based on
K. Falls, N.O. and R. Percacci, "Towards the determination of the dimension of the critical surface in asymptotically safe gravity," Phys. Lett. B 810 (2020) 135773 [arXiv:2004.04126 [hep-th]].

H. Kawai and N.O., "Wave function renormalization and flow of couplings in asymptotically safe quantum gravity," Phys. Rev. D 107 (2023) no.12, 126025 [arXiv:2305.10591 [hep-th]].

N.O. and M. Yamada (in preparation)

1 Introduction

I would like to explain an approach to Quantum gravity (QG) using RG.

The fundamental problem is that the Einstein theory is non-renormalizable perturbatively.

Superstring is not yet at such a stage to study quantum gravity
$\Rightarrow$ We need nonperturbative technique $\Rightarrow$ renormalization group

$\Rightarrow$ Quantum gravity within the framework of local field theory.

Known facts

- **Higher-derivative (curvature) terms always appear in QG,**
  e.g. quantized Einstein theory
  and (low-energy effective theory of) superstring theories!
- **In 4D, quadratic (higher derivative) theory is renormalizable!**

[K. S. Stelle, Phys. Rev. D16 (1977) 953.]

$\Rightarrow$ Possible UV completion? But it is non-unitary! (on flat backgrounds)

It is natural to consider the higher derivative theory in the formulation.

HDG

$$S_{HDG} = \int d^4x \sqrt{-g} \Big[\mathcal{V} - Z_N R + \frac{1}{2\lambda} {C_{\mu\nu\rho\lambda}}^2 + \frac{1}{\xi} R^2 - \frac{1}{\rho} E \Big],$$

$$
\begin{aligned}
&{C_{\mu\nu\rho\lambda}}^2 = R_{\mu\nu\alpha\beta}^2 - 2R_{\mu\nu}^2 + \frac{1}{3}R^2, \qquad E = R_{\mu\nu\alpha\beta}^2 - 4R_{\mu\nu}^2 + R^2, \\
&Z_N = \frac{1}{16\pi G_N}\ , \ \mathcal{V} = 2\Lambda Z_N\ ,
\end{aligned}
$$

To fully understand the theory, we need nonperturbative method because there is this non-unitarity problem and other difficulties (strong couplings).

The hope is that the nonperturbative effects might cure the ghost problem.

$\Rightarrow$ (Functional or Exact) Renormalization Group!

Here comes the Asymptotic Safety.

2 Asymptotic Safety in a nutshell

We consider effective "average" action obtained by integrating out all fluctuations of the fields with momenta larger than k.

$$\Gamma_k = \sum_i g_i(k)\mathcal{O}_i$$

where $\mathcal{O}_i$ are the operator basis representing interactions.

We apply functional RG equations (FRGE) to gravity system.

FRGE gives flow of the effective action in the theory (coupling) space defined by suitable bases $\mathcal{O}_i$.

$$\frac{d\Gamma_k}{dt} = \sum_i \beta_i \mathcal{O}_i, \quad \beta_i = \frac{dg_i}{dt}, \qquad t \equiv \ln k$$

We set initial conditions at some point and then flow to $k \to \infty$.

The flows may stop at FPs where $\beta = 0$.

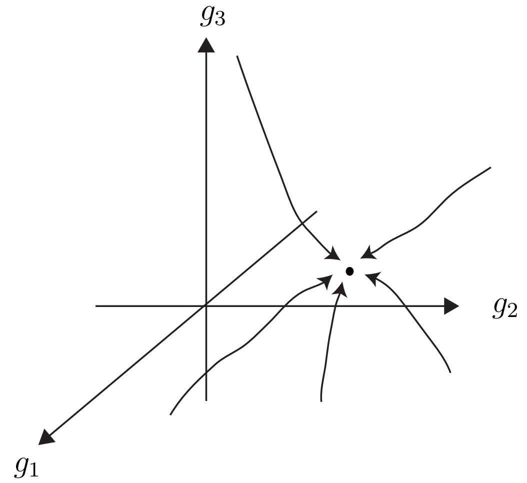


Figure 1: RG flow

Asymptotic safety

All couplings go to finite FPs at UV, giving the UV finite theory
+ There are finite number of the couplings $\Rightarrow$ Predictability

When integrated to $k = 0$, we get the standard effective action $\Gamma_{k=0}[\phi]$.

Newton coupling goes to finite value in the high energy unless there are too many matters (spin 0 and 1/2) (gravity and gauge field make it).

Those operators whose couplings go to FPs in the infinite energy are called relevant operators, and repell irrelevant operators and others marginal.

Irrelevant operators are not included in our fundamental action; they are just like nonrenormalizable interactions in perturbation theory.

Scale invariance is realized in the large energy limit!
$\Rightarrow$ Possible connection to string theory!

The Important problem

How many relevant operators we need? $\cdots$ "Nonperturbatively Renormalizable theory" or Predictability

3 Beta functions

To formulate the theory, we need truncation (keep finite no. of operators). (We cannot deal with infinite no. of couplings.)

Consider up to quadratic curvature terms.

Gauss-Bonnet term is topological, and its coupling does not contribute.

Other beta functions from dim. reg.

J. Julve, M. Tonin, Nuovo Cim. 46B, 137 (1978).

E.S. Fradkin, A.A. Tseytlin, Phys. Lett. 104 B, 377 (1981).

I.G. Avramidi, A.O. Barvinski, Phys. Lett. 159 B, 269 (1985).

Theory is AF $(\lambda \to 0\ ,\ \omega(\equiv -\frac{3\lambda}{\xi}) \to -0.0228\ ,\ \theta \to 0.327)$
only if $\xi > 0$ **(scalar tachyon) and** $\rho > 0$**.**

No nontrivial coupling for λ and ξ was found.

A. Codello and R. Percacci, Phys. Rev. Lett. 97 (2006) 22.

M. Niedermaier, Nucl. Phys. B 833 (2010) 226.

N. O. and R. Percacci, Class. Quant. Grav. 31 (2014) 015024 [arXiv:1308.3398]

K. Groh, S. Rechenberger, F. Saueressig and O. Zanusso, arXiv:1111.1743 [hep-th].

All calculations found Gaussian FPs for dimensionless couplings.

When the contribution from Newton coupling is included, nontrivial FPs are found with 3 relevant operators.

K. Falls, D. Litim, K. Nikolakopulos and C. Rahmede, "A bootstrap towards asymptotic safety," Phys. Rev. D 93 (2016) 104022 [arXiv:1410.4815 [hep-th]].
D. Benedetti, P. F. Machado and F. Saueressig, Nucl. Phys. B 824 (2010) 168 [arXiv:0902.4630 [hep-th]]
K. Falls, N. Ohta and R. Percacci, Phys. Lett. B 810 (2020) 135773 [arXiv:2004.04126 [hep-th]].

However these studies are made either on the sphere, Einstein space or to finite order in $Z_N = \frac{1}{G_N}$, and not sufficient to conclude the result.

(In order to study the behavior down to low-energy, we have to get beta functions to all order in Z_N.)

We tried to find nontrivial fixed point including all order terms in Z_N, and find only trivial Gaussian FP, and then the flow to low energy. But the flow was very unstable.

Kawai pointed out that in this analysis, I should take the wave function renormalization.

4 Wave function renormalization

H. Kawai and N.O., Phys. Rev. D 107 (2023) 126025 [arXiv:2305.10591 [hep-th]].

The wave function renormalization "constant" is redundant (inessential) parameter which does not affect any physical quantities.

In order compare two theories with quantum corrections, we have to normalize kinetic term, otherwise there is no standard to compare the two!

N. Christiansen, B. Knorr, J.M. Pawlowski and A. Rogigast, PRD93 (2016) 044036 [arXiv:1403.1232]. (Numerically)

4.1 Einstein gravity

Consider the Einstein theory with the vacuum energy:

$$S = \int d^4x \sqrt{g} \left(\rho - \frac{1}{16\pi G_N} R \right).$$

Under the wave function renormalization

$$g_{\mu\nu} = Z g'_{\mu\nu} \quad \Rightarrow \quad \sqrt{g} \to Z^2 \sqrt{g'}, \qquad \sqrt{g} R \to Z \sqrt{g} R'$$

The vacuum energy ρ and the Newton coupling changes as

$$\rho \to Z^2 \rho', \qquad G_N \to Z^{-1} G'_N \quad \Rightarrow \quad \Lambda G_N^2 \text{ is invariant}$$

$$\Delta \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu \Rightarrow Z^{-1} \Delta$$

The usual optimized cutoff

$$R_k = (k^2 - \Delta)\theta(k^2 - \Delta)$$

breaks the invariance under the wave function renormalization unless we transform momentum cutoff k as

$$k'^2 = k^2 Z.$$

It is convenient to define invariants:

$$\ell^2 = \frac{k^2}{\sqrt{\rho}}, \qquad \eta = 16\pi G_N \sqrt{\rho}$$

We can derive the FRGE as

$$\frac{d\ell^2}{dt} = (2-g)\ell^2, \qquad \frac{d\eta}{dt} = g\eta + f\eta^2 \cdots \text{Everything invariant}$$

$$\begin{aligned} f &= -\frac{4\ell^2(11\ell^4 - 9\ell^2\eta + 7\eta^2)}{192\pi^2(\ell^2-\eta)^2 - \ell^4\eta(\ell^2+5\eta)}, \\ g &= \frac{\ell^4[\ell^4(107\ell^2 - 10\eta)\eta + 576\pi^2(\ell^4 + 3\ell^2\eta - 4\eta^2)]}{96\pi^2[192\pi^2(\ell^2-\eta)^2 - \ell^4\eta(\ell^2+5\eta)]}. \end{aligned}$$

The real fixed points:

$$(\eta_*, \ell_*^2) = \mathbf{FP}_1 : (3.7065, 9.5923), \ \ \mathbf{FP}_2 : (0, 8\pi).$$

$$\Rightarrow \mathbf{FP}_1 : (\tilde{\rho}_*, \tilde{G}_*) = \left(\ell_*^{-4}, \frac{\eta_* \ell_*^2}{16\pi}\right) = (0.01087, 0.7073),$$

The eigenvalues of stability matrix:

$$\mathbf{FP}_1 : -1.4753 \pm 3.0432i,$$
$$\mathbf{FP}_2 : -4, 2.$$

FP_1: attractive in the UV, and the flow is converging to this point with spiral curve.

FP_2: a separatorix; there is a trajectory into the point $\rho = 0$ in the IR under the very fine tuning, but there remains arbitrary constant value, positive or negative, if the initial condition is slightly away from it.

When the quadratic curvature terms are included, which are unchanged under w.f.r., $\Rightarrow$ the flow trajectory becomes more stable.

Unfortunately there is no nontrivial fixed point in the dimensionless couplings (though it is not 100% excluded.)

How are the Gaussian fixed point and its properties?

4.2 Gaussian fixed points in the dimensionless couplings

For small λ, the beta functions to the second order are

$$\beta_\lambda = -\frac{133}{160\pi^2}\lambda^2 + O(\lambda^3), \qquad \beta_\xi = -\frac{5(72\lambda^2 - 36\lambda\xi + \xi^2)}{576\pi^2} + O(\lambda^3, \xi^3).$$

Usual study examines only the stability matrix (first order gradient) and $\lambda = \xi = 0$ would be marginal FP $\Rightarrow$ This is not enough

Setting $\xi = x\lambda$,

$$\frac{d\xi}{d\lambda} \propto (x - 131.2)(x - 0.5987)$$

Flow is drawn on the right:
If we start in the regions (1) and (2), these couplings flow to the origin.

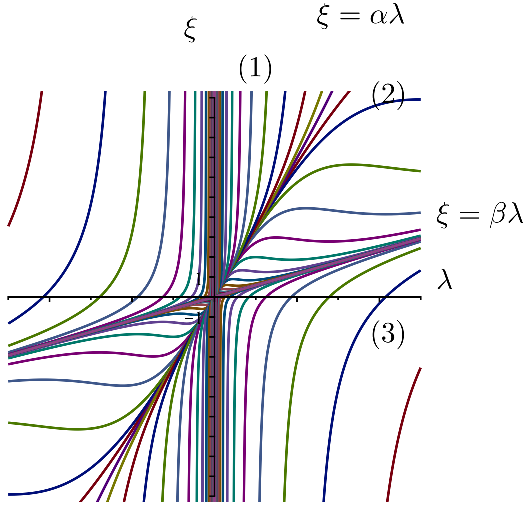


Figure 2: Renormalization group flow. The flow direction is from right to left. $\alpha = 131.2, \beta = 0.598$.

The UV FP is $\lambda_* = 0$ and $\omega_* = -\frac{3\lambda_*}{\xi_*} \simeq -0.0229\,(1/x = 131.2)$, corresponding to relevant direction, giving 4 relevant directions.

Surprize: the number of relevant operators at this AF FP is 4, in agreement with perturbation theory.

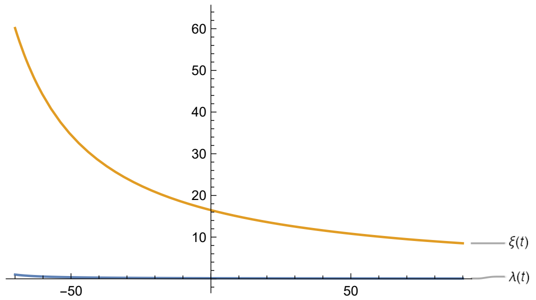
Summary:

1. **In the high energy limit, the Newton coupling g is gradually approaching the FP value 2.38.**

2. **The higher derivative couplings λ is becoming smaller and smaller with the ratio $\omega = -\frac{3\lambda}{\xi}$ fixed around its fixed point value -0.0229. Both λ and ξ go to zero, the asymptotic freedom.**

   This means that in the high energy, the curvature squared terms become dominant, compared to Einstein term.

3. **In the low-energy limit, we expect that $g(= G_N k^2)$ goes to zero (or small), whereas λ and ξ becomes large.**



Flows of λ and ξ.

However, if there is only Gaussian fixed point, perturbative approximation is good, and no way to evade the ghosts with negative metric!

5 REG and essential couplings

S. Weinberg,Ultraviolet divergences in quantum theories of gravitation, 1980

Wave function renormalization is one of the inessential couplings.

More generally, if any operator could be removed by field redefinition, such an operators do not affect physical quantities and are redundant.

$\Rightarrow$ We should formulate the ERG by considering this.

Change a coupling γ_0 by small ϵ, the Lagrangian changes $\mathcal{L} \to \epsilon \frac{\partial \mathcal{L}}{\partial \gamma_0}$
Let's try to produce this change by field redefinition

$$\psi_n(x) \to \psi_n(x) + \epsilon F_n(\psi(x), \partial_\mu \psi(x), \cdots)$$

The change in $\mathcal{L}$ is

$$\begin{aligned} \delta\mathcal{L} &= \epsilon \sum_n \left[\frac{\partial \mathcal{L}}{\partial \psi_n} F_n + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} \partial_\mu F_n + \cdots \right] \\ &= \epsilon \sum_n \left[\frac{\partial \mathcal{L}}{\partial \psi_n} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} \right) + \cdots \right] F_n + \text{(total derivative)} \end{aligned}$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \gamma_0} = \sum_n \left[\frac{\partial \mathcal{L}}{\partial \psi_n} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)} \right) + \cdots \right] F_n + \text{(total derivative)}$$

The coupling γ_0 is inessential if the term is proportional to the field equation!

In the Einstein gravity, the field equation is

$$R_{\mu\nu} = 0, \qquad R = 0$$

So any term proportional to these may be eliminated, thus

R^2 and $R_{\mu\nu}^2$ terms are inessential!

In perturbation, it has been known that the theory is renormalizable "on shell"

⇒ G. 't Hooft and M. J. G. Veltman, "One loop divergencies in the theory of gravitation," Ann. Inst. H. Poincare A Phys. Theor. 20 (1974), 69

What about the next term, cubic in Riemann tensor $C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\alpha\beta}C_{\alpha\beta}{}^{\mu\nu}$?
— so-called Goroff-Sagnotti term

A. Baldazzi, K. Falls, Y. Kluth and B. Knorr, [arXiv:2312.03831 [hep-th]].

H. Gies, B. Knorr, S. Lippoldt and F. Saueressig, Phys. Rev. Lett. 116 (2016) 211302 [arXiv:1601.01800 [hep-th]].

It turns out that this is irrelevant!

So only the Einstein term (and CC) could be considered as UV completion in pure gravity in this approach!

The question still remains if this still makes sense when matter is involved (work in progress with Yamada).

't Hooft and Veltman showed that when the scalar matter is included,

$$\mathcal{L} = \sqrt{g}\left(-R - \frac{1}{2}\partial_\mu\phi g^{\mu\nu}\partial_\nu\phi\right)$$

there remains divergence that cannot be removed by field equation.

How about in essential FRGE with additional interactions, possibly with $\xi = \frac{1}{6}$?

$$S = \int dx^4 x\sqrt{g}\left[\rho - \frac{1}{16\pi G_N}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{\xi}{2}R\phi^2\right]$$

Preliminary results

We find that there are divergences of the form

$$R^2_{\alpha\beta\gamma\delta}, \qquad R^2_{\alpha\beta}, \qquad R^2, \qquad (g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi)^2, \qquad Rg^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

Redundant operator can be expressed in terms of the basis

$$\left\{g_{\mu\nu}, \quad Rg_{\mu\nu}, \quad S_{\mu\nu}, \quad R^2 g_{\mu\nu}, \quad RS_{\mu\nu}, \quad S_\mu{}^\sigma S_{\sigma\nu}, \quad \phi g_{\mu\nu}, \quad g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right\}, \quad \{\phi\},$$

where $S_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R$ is the traceless Ricci tensor.

We have

$$
\begin{aligned}
&-2\gamma_g \varrho + \left(\frac{\gamma_g}{16\pi G_N} - 2\varrho\gamma_R\right) R + \left(\frac{\gamma_R}{16\pi G_N} - 2\gamma_{R^2}\varrho\right) R^2 \\
&-\left(\frac{\gamma_S}{16\pi G_N} + 2\gamma_{RSq}\varrho + \frac{\gamma_{S^2}}{2}\rho\right) S^{\mu\nu} S_{\mu\nu} - 2\gamma_{g\phi}\varrho\phi^2 + \frac{\gamma_g}{2}(\partial_\mu\phi)^2 - \frac{\gamma_S}{2} S^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \\
&+\left(\frac{\gamma_{g\phi}}{16\pi G_N} - \frac{\gamma_g \xi}{2} + \gamma_\phi \xi\right)\phi^2 R - \frac{\gamma_R}{2}(\partial_\mu\phi)^2 R + \gamma_R(\partial_\mu\phi\partial^\mu\phi)^2 - \gamma_\phi \phi\Delta\phi + \ldots
\end{aligned}
$$

These divergences may be removed by the field redefinitions.

The important difference from 't Hooft and Veltman is the presence of the vacuum energy, which enables to eliminate divergence.

As long as the Riemann tensor terms are not generated, this procedure can be done.

Such operators appear at higher order with higher dimensions, which would be more irrelevant, like Goroff-Sagnotti term.

If true, this gives strong evidence that the theory of Einstein and cosmological terms is UV complete in the framework of FRG.

6 Summary

- **The beta functions including all order in Z_N indicates that the non-trivial FPs (asymptotically safe points) might be a fake.**
- **We have found that there are 4 relevant operators in the asymptotically free fixed point, in agreement with perturbation theory.**
- **It appears that it is smoothly connected to the perturbative gravity regime in the low-energy limit.**
- **These are nice properties, but if this is true, the ghost problem is more serious!**
- **More promising approach seems to be essential ERG, throwing away redundant operators.**

  $\Rightarrow$ Only the Einstein theory with cosmological term should be considered for pure gravity.
- **When matter is present, it appears that the divergences may be removed by field redefinition (to be confirmed).**
- **Possible connection with strings?**