Trace anomaly and renormalization group

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Weyl symmetry and the trace anomaly

Classical Weyl symmetry

Local Weyl rescalings

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad \Phi \to \Phi' = \mathrm{e}^{w_\Phi \sigma} \Phi$$

Metric is the **source** of the energy-momentum tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

Nöther identities of Diff and Weyl symmetries on-shell

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T^\mu{}_\mu = 0$$

Quantum Weyl symmetry

From the path-integral

$$\mathrm{e}^{-\Gamma} = \int [\mathrm{d}\Phi]\, \mathrm{e}^{-S}$$

The renormalized EMT

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}}$$

Trace is dimension d operator and the anomaly **Duff, Deser-Schwimmer, Jack-Osborn ...**

$$\langle T^{\mu}{}_{\mu} \rangle = \text{geometry} + \text{renormalization group} + \text{flavor and EOMs}$$

Plan

What I would like to understand is this structure

$$\langle T^{\mu}{}_{\mu} \rangle = \text{geometry} + \text{renormalization group} + \text{flavor and EOMs}$$

Plan: discussion on symmetries and then tools for the job:

- renormalization group $\Longrightarrow$ Local RG
- geometry $\Longrightarrow$ Ambient space and nonlocal actions
- flavor and EOMs $\Longrightarrow$ Flavor current and B-functions

Weyl vs conformal symmetry

Weyl symmetry $\Longrightarrow$ Conformal symmetry

Conformal (isometries) group

$$\delta_\sigma^W g_{\mu\nu} + \delta_\xi^E g_{\mu\nu} = 2\sigma g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

Solution $\sigma = -\frac{1}{d}\nabla_\mu \xi^\mu$

$$\delta_\xi^C = \delta_\sigma^W + \delta_\xi^E$$

Flat space limit $g_{\mu\nu} \to \eta_{\mu\nu}$ has $\frac{(d+1)(d+2)}{2} = \frac{d(d+1)}{2} + d + 1$ generators $(d \neq 2)$

$$P_\mu\,,\;\; J_{\mu\nu}\,,\;\; D\,,\;\; K_\mu$$

Conformal symmetry $\Longrightarrow$ Weyl symmetry?

Conformal invariance in flat space $d \neq 2$ implies **Polchinski**

$$T^{\mu}{}_{\mu} = \partial_\mu \partial_\nu X^{\mu\nu}$$

There exists a **new EMT**

$$T'_{\mu\nu} = T_{\mu\nu} + \frac{1}{d-2}\Big(2\partial_\alpha \partial_{(\mu} X^{\alpha}{}_{\nu)} - \partial^2 X_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha \partial_\beta X^{\alpha\beta}\Big) + \frac{1}{(d-1)(d-2)}\Big(\eta_{\mu\nu}\partial^2 X - \partial_\mu \partial_\nu X\Big)$$

New action "improved" with Schouten $K_{\mu\nu} = \frac{1}{d-2}\big(R_{\mu\nu} - \frac{1}{2(d-1)} R\, g_{\mu\nu}\big)$

$$S[\Phi, g] \longrightarrow S'[\Phi, g] = S[\Phi, g] + \int \mathrm{d}^d x\, K_{\mu\nu} X^{\mu\nu}$$

A side note: primary EMT

Representation theory $\hat{P}_\mu \sim a^\dagger$ and $\hat{K}_\mu \sim a$ similar to that of harmonic oscillator

$$\hat{K}_\mu T_{\alpha\beta}\big|_{x=0} = 0$$

In $d \neq 2$ and 4 can be made primary including $Z_{\mu\nu}$ **Stergiou-Osborn, Stergiou et al.**

$$T''_{\mu\nu} = T'_{\mu\nu} + \frac{1}{d-2}\Big(2\partial_\alpha\partial^2\partial_{(\mu}Z^\alpha{}_{\nu)} + \ldots\Big) - \frac{1}{d-1}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\partial_\alpha\partial_\beta Z^{\alpha\beta}$$

Geometrically Weyl invariance fixes $Z_{\mu\nu}$ with Bach $B_{\mu\nu} = \nabla^2 R_{\mu\nu} + \ldots$

$$S'[\Phi, g] \longrightarrow S''[\Phi, g] = S'[\Phi, g] + \frac{1}{d-4}\int \mathrm{d}^d x\, B_{\mu\nu} Z^{\mu\nu} + \ldots$$

Weyl/conformal vs scale invariance

Scale vs conformal symmetry

For a rigid scale transformation $\sigma = \mathrm{const.}$

$$\int \mathrm{d}^d x \sqrt{g}\, T^{\mu}{}_{\mu} = 0$$

Implies the existence of a *virial* current D_μ

$$T^{\mu}{}_{\mu} = \nabla_\mu D^\mu$$

The current *must not* have anomalous dimension, and ideally

$$\langle T^{\mu}{}_{\mu} - \nabla_\mu D^\mu \rangle = \mathrm{geometry} + \mathrm{beta\ terms}$$

A source for D_μ: gauging the Weyl group

Introduce an Abelian gauge potential **Iorio et al., Percacci et al., Sauro et al.**

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad S_\mu \to S'_\mu = S_\mu - \partial_\mu \sigma \qquad \Phi \to \Phi' = \mathrm{e}^{w_\Phi \sigma} \Phi$$

The unique gauged covariant *compatible* derivative

$$\hat{\nabla}_\mu \Phi = \nabla_\mu \Phi + L_\mu \cdot \Phi + w_\Phi S_\mu \Phi \qquad \to \qquad \hat{\nabla}'_\mu \Phi' = \mathrm{e}^{w_\Phi \sigma} \hat{\nabla}_\mu \Phi$$

It contains “disformation” because dilatations do not commute with Poincaré

$$(L_\mu)^\alpha{}_\beta = \frac{1}{2}(S_\beta \delta^\alpha_\mu + S_\mu \delta^\alpha_\beta - S^\alpha g_{\beta\mu})$$

Also, $S_\mu \sim T_\mu$ can be interpreted as torsion vector **Karananas-Shaposhnikov, Sauro et al.**

Consequences of gauging Weyl

There is a new dilation current

Sauro et al.

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \qquad\qquad D^{\mu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta S_{\mu}}$$

Classically gauged Weyl and Diff symmetries with $W = \mathrm{d}S$ imply

$$T^{\mu}{}_{\mu} = \nabla^{\mu} D_{\mu} \qquad\qquad \hat{\nabla}_{\mu} T^{\mu\nu} + D_{\mu} W^{\mu\nu} = 0$$

In the limit $S_{\mu} \to 0$ we have scale invariance and D_{μ} is virial

$$T^{\mu}{}_{\mu} = \nabla^{\mu} D_{\mu} \qquad\qquad \nabla_{\mu} T^{\mu\nu} = 0$$

Example of scale-but-not-conformal model

Dipolar ferromagnet (Aharony-Fisher)

$$S[\varphi] = \int \mathrm{d}^3 x \Big\{ \frac{1}{2}(\partial_i \phi^j)^2 + \frac{m^2}{2}\phi^i\phi^i + \frac{\lambda}{4!}(\phi^i\phi^i)^2 + \int \mathrm{d}^3 x \int \mathrm{d}^3 y\, \phi^i(x) U_{ij}(x-y)\phi^j(y) \Big\}$$

Notice that ϕ^i "remembers" that it is a vector. In momentum space (localized with B)

$$U(q) \sim v \frac{q_i q_j}{q^2}$$

Renormalization of virial current protected by hidden shift symmetry **Gimenez-Grau et al.**
Later we see elasticity/membrane briefly

Local RG analysis of the anomaly

$$\langle T^{\mu}{}_{\mu}\rangle = \text{geometry} + \boxed{\text{renormalization group}} + \text{flavor and EOMs}$$

Renormalization with local couplings

Use local couplings to source observables **Shore, Osborn, Jack-Osborn**

$$S \supset -\int \mathrm{d}^d x \sqrt{g}\lambda^i(x)\mathcal{O}_i$$

Currents source the expectation values

$$\langle T^{\mu\nu}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}} \qquad \langle D^{\mu}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta S_{\mu}} \qquad \langle \mathcal{O}_i\rangle = -\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta \lambda^i}$$

We expect the path-integral to give the anomaly

$$\langle T^{\mu}{}_{\mu} - \nabla^{\mu}D_{\mu}\rangle = \text{geometry} + \text{beta terms}$$

Local rg interpretation

Local scale transformation on the geometrical sources

$$\Delta_\sigma^W = \int \left\{ 2\sigma g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \partial_\mu \sigma \frac{\delta}{\delta S_\mu} \right\}$$

Local scale transformation caused by the rg beta functions

$$\Delta_\sigma^\beta = - \int \sigma \beta^i \frac{\delta}{\delta \lambda^i}$$

The anomaly of Γ is expressed

$$\Delta_\sigma^W \Gamma = \Delta_\sigma^\beta \Gamma + A_\sigma \qquad A_\sigma \supset \{\partial_\mu \lambda^i, R, S_\mu, W_{\mu\nu} \cdots \}$$

Wess-Zumino consistency

Rewrite

$$\Delta_\sigma \Gamma = (\Delta_\sigma^W - \Delta_\sigma^\beta)\Gamma = A_\sigma$$

For Wess-Zumino's consistency and Abelian transf.

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = 0$$

Consistency condition for the anomaly

$$(\Delta_\sigma^W - \Delta_\sigma^\beta)A_{\sigma'} - (\sigma \leftrightarrow \sigma') = 0$$

Example: two dimensions

Most general parametrization of A_σ using $\hat{R} = R - 2\nabla^\mu S_\mu$

$$A_\sigma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\Big\{\sigma\frac{\beta_\Phi}{2}\hat{R} - \sigma\frac{\chi_{ij}}{2}\partial_\mu\lambda^i\partial^\mu\lambda^j - \partial_\mu\sigma w_i\partial^\mu\lambda^i$$
$$+ \sigma\beta_\Psi\nabla_\mu S^\mu + \sigma\frac{\beta_2^S}{2}S_\mu S^\mu + \sigma z_i\partial_\mu\lambda^i S^\mu\Big\}$$

Apply Wess-Zumino's

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\big(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma\big)\mathcal{Z}^\mu = 0$$

Condition $\mathcal{Z}_\mu = \partial_\mu\lambda^i\mathcal{Y}_i + S_\mu\mathcal{X} = 0$ among tensors becomes (here $\partial_i = \partial/\partial g_i$)

$$\mathcal{Y}_i = -\partial_i\beta_\Psi + \chi_{ij}\beta^j - \beta^j\partial_j w_i - w^j\partial_i\beta_j + z_i$$
$$\mathcal{X} = \beta_2^S - \beta^i\partial_i\beta_3^S - z_i\beta^i$$

A special scalar charge

Define a new charge

$$\tilde{\beta}_\Psi = \beta_\Psi + w_i \beta^i$$

Using $\Theta = \beta^i O_i$ and $\mathcal{T} = T - \partial \cdot D$

$$\langle \mathcal{T}(x) \mathcal{T}(0) \rangle - \langle \Theta(x) \Theta(0) \rangle \sim \tilde{\beta}_\Psi \partial^2 \delta^{(2)}(x)$$

Using WZ consistency

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \tilde{\beta}_\Psi = \beta^i \partial_i \tilde{\beta}_\Psi = \chi_{ij} \beta^i \beta^j + \beta_2^S$$

(Ir)reversibility and gradient structure

For flows between **unitary CFTs** there is $\chi_{ij} \to G_{ij} = \frac{1}{8}\left|x\right|^4 \langle \mathcal{O}_i(x)\mathcal{O}_j(0)\rangle > 0$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu}\tilde{\beta}_\Psi \geq 0 \qquad \longrightarrow \qquad \text{Zamolodchikov's theorem}$$

For flows between **nonunitary CFTs** we only have A such that

$$\beta^i = \gamma^{ij}\partial_j A \qquad \text{identifying} \qquad \chi_{ij} \leftrightarrow \gamma_{(ij)}\,, \quad A \leftrightarrow \tilde{\beta}_\Psi$$

For flows between **scale-but-not-conformal-invariant theories**

$$\beta_2^S \quad \text{is an obstruction to both properties}$$

Simple application: theory of elasticity

Elastic $2d$ membrane with strain $u_{\mu\nu} = \partial_{(\mu} u_{\nu)}$ considered by Cardy-Riva

$$S[u] = \frac{1}{2} \int \mathrm{d}^2 x \Big\{ 2\, g\, u_{\mu\nu} u^{\mu\nu} + \, k\, u_\mu{}^\mu u_\nu{}^\nu \Big\}$$

Gauging $u_{\mu\nu} \to \hat{\nabla}_{(\mu} u_{\nu)}$ we find

Benfatto et al. in prep.

$$A_\sigma = \frac{1}{2\pi} \int \mathrm{d}^2 x \sqrt{g} \sigma \Big\{ \frac{13g + 5k}{6(2g+k)} R - \frac{3g+k}{2g+k} \nabla^\mu S_\mu - \frac{(3g+k)^2}{4g(2g+k)} S_\mu S^\mu \Big\} + \cdots$$

Charges $\beta_\Phi = \beta_\Psi = \frac{2}{3}$ and $\beta_2^S = 0$ in the global conformal limit $3g + k = 0$

$$\beta_\Phi = \frac{5}{3} + \frac{g}{(2g+k)} \qquad \beta_\Psi = \frac{2}{3} \qquad \beta_2^S = -\frac{(3g+k)^2}{4g(2g+k)}$$

Extension to $4d$

In $d = 4$ (schematically, case $S_\mu = 0$) **Osborn, Jack-Osborn**

$$A_\sigma \supset \int \mathrm{d}^4x\sqrt{g}\,\sigma\Big\{\beta_a E_4 + \cdots + \chi^a_{ij}\Box\lambda^i\Box\lambda^j + \chi^g_{ij}R^{\mu\nu}\partial_\mu\lambda^i\partial^\mu\lambda^j + \cdots\Big\}$$

The "positive" metric is $\chi^a_{ij} \sim \langle\mathcal{O}_i\mathcal{O}_j\rangle$, but

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\beta_a \sim \chi^g_{ij}\beta^i\beta^j$$

Not obvious to establish positivity of χ^g_{ij} because it is a 3pf (only perturbatively)

$$\chi^g_{ij} \sim \langle\mathcal{O}_i\mathcal{O}_j T\rangle$$

Conformal geometry and ambient space

$$\langle T^{\mu}{}_{\mu} \rangle = \boxed{\text{geometry}} + \text{renormalization group} + \text{flavor and EOMs}$$

Lightcone embedding in flat space

Move from $\mathbb{R}^d$ to $\mathbb{R}^{d+2}$ on the lightcone

$$Y^A = (Y^\mu, Y^+, Y^-) \qquad \eta_{AB} Y^A Y^B = 0 \qquad Y^A \sim \lambda Y^A$$

Spacetime embedding in the lightcone

$$x^\mu \to Y^A = (Y^\mu, Y^+, Y^-) = Y^+(x^\mu, 1, -x^2)$$
$$Y^A \to x^\mu = \frac{Y^\mu}{Y^+}$$

Embedding Lorentz generates conformal on spacetime

$$(Y'^+)^2 \eta_{\mu\nu} dx'^\mu dx'^\nu = (Y^+)^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

Fefferman-Graham ambient space

Use Cartesian coordinates, $X^2 = 2t^2\rho$, $t = X^+$

$$Y^A \to X^A = (X^\mu, X^{d+1}, X^{d+2}) \stackrel{*}{=} t\left(x^\mu, \frac{1+2\rho-x^2}{2}, \frac{1-2\rho+x^2}{2}\right)$$

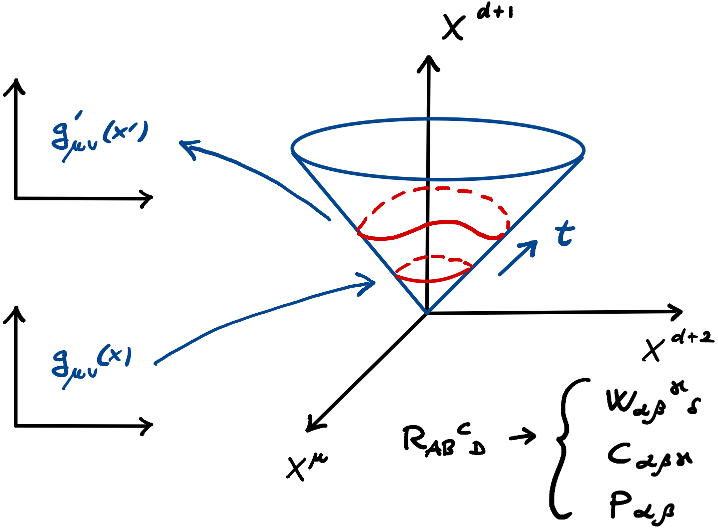
The flat embedding metric

$$\tilde{\eta} = \eta_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

In curved space: FG metric with $R_{AB} = 0$, $\mathcal{L}_{t\partial_t}\tilde{g} = 2\tilde{g}$ and $h_{\mu\nu}(x, \rho = 0) = g_{\mu\nu}$

$$\tilde{g} = \tilde{g}_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 h_{\mu\nu}(x,\rho) dx^\mu dx^\nu$$

Ambient Space in a nutshell



Relation with AdS/CFT

Coordinates $\rho = -\frac{r^2}{2}$ and $t = \frac{s}{r}$

$$\tilde{g} = -ds^2 + s^2\Big(\frac{dr^2 + h_{\mu\nu}(x,r)dx^\mu dx^\nu}{r^2}\Big)$$

Asymptotically (in r) local (in s) AdS space **Parisini-Skenderis-Withers**

Fixed s: approaching lightcone with hyperobolas. Geometrical fundation of AdS/CFT

Note: If you are familiar with AdS/CFT you can replace $\rho \leftrightarrow r$ otherwise don't worry

PBH diffeomorphisms

A diffeomorphism of the ambient **Imbimbo et al.**

$$\delta_\zeta \tilde{g}_{AB} = \mathcal{L}_\zeta \tilde{g}_{AB} = \zeta^C \partial_C \tilde{g}_{AB} + \tilde{g}_{AC} \partial_B \zeta^C + \tilde{g}_{BC} \partial_A \zeta^C$$

If it preserves the form of the ambient metric **Penrose-Rindler, Brown-Henneaux**

$$\zeta^t = t\,\sigma(x) \qquad \zeta^\rho = -2\rho\,\sigma(x) \qquad \zeta^\mu = \xi^\mu(x) + \cdots$$

It generates Diff $\ltimes$ Weyl on spacetime

$$\delta_\zeta h_{\mu\nu}|_{\rho=0} = \delta_\zeta g_{\mu\nu} = \delta_{\sigma,\xi} g_{\mu\nu} = 2\sigma g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu$$

Ricci-flatness determines $h_{\mu\nu}$

Expand in ρ to solve $\tilde{R}_{AB} = 0$ iteratively

$$h_{\mu\nu}(x,\rho) = g_{\mu\nu}(x) + \rho h^{(1)}{}_{\mu\nu} + \frac{1}{2}\rho^2 h^{(2)}{}_{\mu\nu} + \cdots$$

The coefficients find obstructions in even d **Fefferman-Graham**

$$h^{(1)}{}_{\mu\nu} = +\frac{2}{d-2}\left(R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu}\right) = 2K_{\mu\nu}$$

$$h^{(2)}{}_{\mu\nu} = -\frac{2}{d-4} B_{\mu\nu} + 2K_{\mu\sigma}K^{\sigma}{}_{\nu}$$

$$h^{(3)}{}_{\mu\nu} = +\frac{2}{d-6} B'_{\mu\nu} + \cdots$$

Ambient Laplacian(s)

Scalar Laplacian of the embedding

$$-\Box_{\tilde{g}}\Phi = -\frac{1}{t^2}\Box_h\Phi - \frac{2}{t}\partial_t\partial_\rho\Phi - \frac{1}{2t}\partial_t\Phi - \frac{d-2}{t^2}\partial_\rho\Phi + \frac{\rho}{t^2}{h'}_\mu{}^\mu\partial_\rho\Phi$$

Consider a scaling scalar field $\Phi = t^{\Delta_\varphi}\varphi(x)$ and project to Yamabe

$$-\Box_{\tilde{g}}\Phi|_{\rho=0} = t^{\Delta_\varphi-2}\left(-\Box_g - \frac{d-2}{4(d-1)}R\right)\varphi$$

We can construct a family of powers of conformal GJMS Laplacians **Graham et al.**

$$P_{2n}\varphi(x) \equiv t^{-\frac{2n+d}{2}}(-\Box_{\tilde{g}})^n(t^{\frac{2n-d}{2}}\varphi)|_{\rho=0}$$

Conformal Laplacians and conformal invariants

There are derivative $\Delta_{2n} \sim \partial^{2n}$ and constant parts (exist in $d \geq 2n$)

$$P_{2n}\varphi(x) = \Delta_{2n} + \frac{d-2n}{2} Q_{2n}$$

Constant part transforms nicely: Q-curvatures in $d = 2n$ **Branson et al.**

$$\sqrt{g}\, Q_d = \sqrt{\bar{g}}(\bar{Q}_d + \bar{\Delta}_d \sigma)$$

Conformal invariants are also easy to obtain

$$\tilde{\mathcal{R}}_i^n \longrightarrow \mathcal{W}_i \qquad \text{e.g.} \qquad \tilde{R}^2_{ABCD} \longrightarrow W^2_{\mu\nu\alpha\beta}$$

On Cardy's conjecture

Cardy's conjecture

$$\langle T^{\mu}{}_{\mu} \rangle = \sum_i b_i \mathcal{W}_i + a E_d$$

However, we have all "integrable" geometrical terms such that

$$[\delta_\sigma, \delta_{\sigma'}]\, \mathcal{W}_i = 0 \qquad\qquad [\delta_\sigma, \delta_{\sigma'}]\, Q_d = 0$$

Notice: $E_d \sim Q_d$ for conformally equivalent! The natural ansatz:

$$\langle T^{\mu}{}_{\mu} \rangle = \sum_i b_i \mathcal{W}_i + a Q_d$$

Ambiguities and integration

Integration is possible (schematically)

$$\Gamma = \Gamma_c[g] + \int d^d x \sqrt{g} \left(b_i \mathcal{W}_i + a Q_d \right) \frac{1}{\Delta_d} Q_d$$

In $d = 2$ works wonderfully

$$\Gamma \supset \frac{c}{96\pi} \int d^2 x \sqrt{g} R \frac{1}{\Box} R \qquad \longrightarrow \qquad \langle T(x) T(0) \rangle \sim c / |x|^4$$

But

- In $d = 4$ disagreement with momentum space CFT from $\langle \mathcal{O}\mathcal{O}\mathcal{O} \rangle$ **Corianò's group**
- From $d = 6$ ambiguities: $\tilde{Riem} \neq 0$ param. families of (Δ_6, Q_6) **Paci et al. in prep.**

An eye towards Wilsonian RG

FRG-based relations for central charge in $2d$ LPA **Codello-D'Odorico-Pagani**

$$k\partial_k c_k = -24\pi \; k\partial_k \Gamma_{\mathrm{e}^{-\sigma}k}[\mathrm{e}^{w\sigma}\varphi, \mathrm{e}^{2\sigma} g]\Big|_{\int(\partial\sigma)^2}$$

Integrable form of Γ_k in $2d$ LPA **Codello-D'Odorico-Pagani**

$$\Gamma_k \supset \int \sqrt{g} V_k(\varphi) - \frac{1}{2}\int \sqrt{g}\, k\partial_k V_k(\varphi)\frac{1}{\Delta_2}R - \frac{c_k}{96\pi}\int \sqrt{g}\, R\frac{1}{\Delta_2}R$$

Furthermore:

- Local RG analysis of Polchinski eqn. in presence of a UV cutoff **Ellwanger, ...**
- Irrelevant operators require beta function for $g_{\mu\nu}$!!! **Schwimmer-Theisen**

And one more thing...

$$\langle T^{\mu}{}_{\mu} \rangle = \text{geometry} + \text{renormalization group} + \boxed{\text{flavor and EOMs}}$$

The troubles with going on-shell and many flavors

Introducing a "fundamental" field Φ

$$S[\Phi] = S_{\mathrm{kin}}[\Phi] + \int \mathrm{d}^d x \Big(\lambda^i \mathcal{O}_i + \mathcal{J} \cdot \Phi \Big)$$

Global flavor symmetry $\omega \in G_F$ of S_{kin} promoted to local by introducing gauge A_μ

$$\Gamma[\Phi, g_{\mu\nu}, \lambda^i, A_\mu] \qquad\qquad J_F^\mu = \frac{1}{\sqrt{g}} \frac{\delta\Gamma}{\delta A_\mu}$$

Trace-anomaly in flat-space constant couplings limit **Jack-Osborn, Herren-Thomsen**

$$T^\mu{}_\mu = \beta^i \mathcal{O}_i + \upsilon \cdot \partial_\mu J_F^\mu \quad \longrightarrow \quad T^\mu{}_\mu = B^i \mathcal{O}_i$$

Implications of flavor-improved RG

Conformal invariance

$$T^{\mu}{}_{\mu} = 0 \qquad \Longleftrightarrow \qquad B^i = \beta^i - \upsilon \cdot \lambda^i = 0$$

In practice, there is an antisymmetric part to wavefunction of the kinetic term fixed by

$$\Delta^F_\omega \Gamma = 0$$

Example #1: β-functions of SM are ill-defined, but B-functions are not **Herren-Thomsen**

Example #2: ϕ^4 RG has gradient structure only in terms of B-functions **Pannell-Stergiou**

Conclusions

- Protected dimension d and $d-1$ operators: $\langle T \rangle$ and $\langle D^\mu \rangle$
- Rich geometrical structure of conformal anomaly linked to scale
- Implications for Quantum Gravity?
- Largely unexplored relations with cutoff-based schemes **Ellwanger**

Thank you