When the RG must be functional and nonperturbative… Lessons from disordered systems

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Disordered systems in physics: Interplay of interactions and quenched disorder anu quenche in large dimension *N*

• Multiple low-energy metastable states $(energy minima) \Rightarrow Complex, rugged,$ energy or free-energy landscape.

 $N \gg 1$

- Due to quenched disorder, long-distance physics influenced by nonuniform configurations, singular collective events (avalanches), and/or rare excitations (droplets) on all scales.
- Often: Disorder-induced fluctuations dominate over thermal (quantum) fluctuations. Disorder grows under coarse-graining.

Disordered systems in physics: Interplay of interactions and quenched disorder

- How to describe these phenomena? Standard perturbative RG fails!
- Random field, random anisotropy systems, elastic manifolds in a disordered environment, Bose glass, etc.: nontrivial **zero-temperature fixed points** and **functional RG**.
- Spin-glass phases and alike: nontrivial zero-temperature fixed points but so far no satisfactory RG description.
- Random transverse field Ising model, quantum Griffiths phases, MBL, etc.: **Real space functional RG** (strong disorder RG and infinite randomness fixed points).

Outline

- **• Why a nonperturbative functional RG (from a physical/phenomenological perspective) and how?**
- **• Oddities of the resulting fixed-point theories.**

Why and how a functional RG?

• Illustration for the random-field Ising model (RFIM)

Lattice version:
$$
\mathcal{H}_h[S] = -J \sum_{\langle i,j \rangle} S_i S_j - \sum_i (h_i + H) S_i
$$

Field theory:

$$
S_h[\varphi] = \int d^d x \{ \frac{1}{2} (\nabla \varphi(x))^2 + \frac{\tau}{2} \varphi(x)^2 + \frac{g}{4!} \varphi(x)^4 - h(x) \varphi(x) \}
$$

+ quenched random field: $\overline{h(x)} = 0$, $\overline{h(x)h(y)} = \Delta \delta^{(d)}(x - y)$

• Main puzzling feature: **dimensional reduction** (DR) to the pure model in *d-*2 associated with supersymmetry (**Parisi-Sourlas SUSY**) and its breakdown as a function of space dimension *d*.

Avalanches in the *T*=0 equilibrium RFIM $\overline{}$ σ algorithm to algorithm to calculate the equilibrium M-H curve. Calculate the energies σ $s_{\rm eff} = {1 \over 2}$ and $C_{22} = {1 \over 2}$, respectively, as a function of H. According to P and $C_{22} = {1 \over 2}$

• Computer simulation: Discontinuous changes of the ground state (avalanches or shocks) under applied source H. $\mathcal{L}_{\mathcal{L}}$ and the commutation of $\mathcal{L}_{\mathcal{L}}$ is continuous changes of the ground state omputer simulation: Discontinuous changes of the ground state equilibrium M-H curve is shown in Fig. 2. The curve is shown in Fig. 2.

Sequence of avalanches in the GS of a *3d*-RFIM sample [Liu-Dahmen, '08] 0.5

Cross-section of a spanning avalanche at criticality in 3*d* D II iticality to J

 \bullet At criticality, avalanches on all scales; in a finite system (*L*) they have a size (*S*) distribution ucanty, avaianci
LCC distribution

$$
\rho_L(S) = \rho_{0,L} \frac{e^{-S/S_{\text{max}}(L)}}{S^{\tau}}
$$

$$
S_{\text{max}}(L) \sim L^{d_f}, \quad 1 < \tau < 2
$$

Direct signature of avalanches (a)

0 "avalanche"

 $\setminus \bigwedge^{\bullet} \widetilde{G}(0; -\delta J, \delta J)$

!G(0; −δJ, δJ)

'*GS*

h

 $\overline{0}$

 ϕ_G

 φ

 J

J + *h*

 $U(\varphi) - (J + h)\varphi$

- The ground state (GS) jumps discontinuously (possibly large avalanches) at a sample-depdt $J_h \approx 0$ at criticality)
- Then, the second cumulant of the magnetization and the associated pair correlation function for **slightly different sources** $\pm \delta J$, has a nonanalytic behavior (a "cusp" in $\sqrt{\delta}J^2$) when $\delta J \rightarrow 0$.

$$
L^{d} \overline{\phi_{\mathrm{GS,h}}(-\delta J)\phi_{\mathrm{GS,h}}(\delta J)} = L^{d} \overline{\phi_{\mathrm{GS,h}}(0)^{2}} + O(\delta J^{2})
$$

$$
- |\delta J| \frac{1}{2L^{d}} \int_{S_{\mathrm{min}}}^{\infty} dS S^{2} \rho_{L}(S) \equiv \widetilde{G}_{L}(q=0; -\delta J, \delta J)
$$

Cusp amplitude = Second moment of the avalanche distribution which diverges with the system size *L* at criticality. lation function in the toy model of the *d* = 0 RFIM stud- $SIZE$ L at Chucancy.

Why functional (in the fields)?

e.g., for the free-energy functional $\mathcal{W}_h[J]=\ln \mathcal{Z}_h[J]$: • Effect of avalanches (and low-energy droplet excitations at *T>0*) can be captured in a disorder-averaged description, provided one keeps the **functional dependence** of the cumulants on the sources/fields,

 $W_1[J_1] = W_h[J_1], W_2[J_1, J_2] = W_h[J_1]W_h[J_2]$ cum *, ···*

- For instance, the "cusp" shows up in the second derivative of second cumulant, $\partial_{J_1} \partial_{J_2} W_2 [J_1,J_2]$, when $J_1 \rightarrow J_2$.
- Equivalently, study the 1-PI cumulants associated with the effective action: $\Gamma_1[\phi_1], \Gamma_2[\phi_1, \phi_2], \Gamma_3[\phi_1, \phi_2, \phi_3], \cdots$
- Importantly, effect is already captured in low-order cumulants and for uniform source/field configurations. Not true for Griffiths phase and exponentially rare large-scale excitations.

Exact Functional RG for the 1-PI cumulants

• Evolution of the scale-dependent effective action Γ*k*, with decreasing *k* is described by an **exact (functional) RG equation** [Wetterich, 1993]:

$$
\partial_k \Gamma_k[\{\phi_a\}] = \frac{1}{2} \text{Trace}\{(\Gamma_k^{(2)}[\{\phi_a\}] + \mathcal{R}_k)^{-1} \partial_k \mathcal{R}_k\}
$$

with $\Gamma_k[\{\phi_a\}]$ obtained by coupling copies/replicas of the disordered system to distinct sources J_a and then Legendre transform.

• From it, hierarchy of exact functional RG equations for the **flow of the 1-PI cumulants**:

$$
\partial_k \Gamma_{k1}[\phi_1] = \mathcal{F}_{k1}[\Gamma_{k1}^{(2)}, \Gamma_{k2}^{(2)}],
$$

\n
$$
\partial_k \Gamma_{k2}[\phi_1, \phi_2] = \mathcal{F}_{k2}[\Gamma_{k1}^{(2)}, \Gamma_{k2}^{(2)}, \Gamma_{k3}^{(2)}],
$$

\n
$$
\partial_k \Gamma_{k3}[\phi_1, \phi_2, \phi_3] = \mathcal{F}_{k3}[\Gamma_{k1}^{(2)}, \Gamma_{k2}^{(2)}, \Gamma_{k3}^{(2)}, \Gamma_{k4}^{(2)}],
$$
 etc.

Nonperturbative FRG

- Guided by considerations on avalanches, nonperturbative approximation scheme = combined truncation of
	- ✴ Expansion in the **number of derivatives of the fields** (around uniform configurations)
	- ✴ Expansion in **cumulant order**.
- Example of truncation (referred to as DE2):

$$
\Gamma_{k1}[\phi] = \int_x [U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x)) (\partial_x \phi(x))^2 + O(\partial^4)],
$$

\n
$$
\Gamma_{k2}[\phi_1, \phi_2] = \int_x [V_k(\phi_1(x), \phi_2(x)) + O(\partial^2)],
$$

\n
$$
\Gamma_{k,p \ge 3} = 0.
$$

- Crucial not to explicitly break underlying Parisi-Sourlas SUSY.
- Key quantity: $\Delta_k(\phi_1, \phi_2) = \partial_{\phi_1} \partial_{\phi_2} V_k(\phi_1, \phi_2) = V_k^{(11)}(\phi_1, \phi_2)$

Cusp in the functional dependence and SUSY/DR breakdown

• In dimensionless form (appropriate for a **zero-temperature fixed point**):

 $\partial_t u'_k(\varphi) = \beta_{u'}(\varphi), \quad t = \ln(k/\Lambda_\mathrm{UV}),$ $\partial_t z_k(\varphi) = \beta_z(\varphi),$ $\partial_t \delta_k(\varphi_1, \varphi_2) = \beta_\delta(\varphi_1, \varphi_2)$

- Second cumulant of the RF δ_k with $\varphi = (\varphi_1 + \varphi_2)/2$, $\delta \varphi = \varphi_1 \varphi_2$ $\text{No cusp}: \; \delta(\varphi,\delta\varphi) = \delta_0(\varphi) + \frac{1}{2}$ 2 $\delta_2(\varphi)\delta\varphi^2 + \cdots$ $\text{Cusp}: \delta(\varphi, \delta\varphi) = \delta_0(\varphi) + a_{\text{cusp}}(\varphi)|\delta\varphi| +$ 1 2 $\widetilde{\delta}_2(\varphi)\delta\varphi^2 + \cdots$
- Cusp breaks SUSY and dimensional reduction: Takes place below some d_{DR} .
- A simple signature in $\partial^2 \delta(\varphi, \delta \varphi) / \partial \delta \varphi^2 |_{\delta \varphi = 0}$

Solution of the NP-FRG flaw equations: cusp versus ₁₅₀₀ cusp Signature in the flow of the second cumulant $\delta_k(\varphi_1,\varphi_2)$; Cusp in $y \equiv \delta \varphi = (\varphi_1 - \varphi_2)$ below $d_{DR} \approx 5.1$ versus no cusp above. By DUCTION OF THE IN $\overline{\textbf{0}}$ pendence on the chosen view value dDR ⇒ 5.1 obtained here is consistent with the construction of the construc \mathcal{C}^* and the RFO(N)M when \mathcal{C}^* $t₁$ bighature in the flow of 0 0.002 0.004 0.006 0.008 0.01 \int cumulant $\delta_k(\varphi_1,\varphi_2)$!2000 -1500 -1000 -500 $\frac{1}{16}$ \overline{u}

 $\partial^2 \delta_k / \partial^2 y (y=0)$ blows up in a finite RG time *t* for $d < d_{DR}$ (red curve), not for $d > d_{DR}$ (blue curve) for definition of \mathbf{z} and \mathbf{z} and \mathbf{z} and \mathbf{z} and \mathbf{z} **b** BC time t for $d < d_{DR}$ (red curve),

 $\mathbf{r} = \mathbf{r} - \mathbf$

Flow of the dimensionless second $cumulant \delta_k$ in $d=4 < d_{DR}$ Cumulant σ_K in σ \rightarrow \sim σ _{DK} Culturalit σ_k in σ \rightarrow \sim σ _{DR}

 Γ FIG, THD, ZOTT, ZOTZ Γ [M. Tissier, G.T., PRL, PRB, 2011, 2012] \mathbf{p} ivitial conditions at \mathbf{p} (i.e., the \mathbf{p} ′ ^k(ρ) **RI** PRR

Oddities of the resulting fixed-point theories

Transition between different types of critical behavior at a nontrivial critical dimension d_{DR}

- Below d_{DR} , nonanalytic (zero-temperature) fixed-point effective action
- Unconventional pattern of disappearance/appearance of fixed points around d_{DR} : requires a functional description!

Oddities and need for a functional description

• In d_{DR} , coalescence of two cuspless fixed points and emergence of a cuspy fixed point below d_{DR} thru a boundary-layer mechanism:

$$
d \to d_{\operatorname{DR}}^- : \delta(\varphi, \delta \varphi) = \delta_0(\varphi) + (d_{\operatorname{DR}} - d)^{\mu} f(\varphi, \frac{|\delta \varphi|}{(d_{\operatorname{DR}} - d)^{\mu/2}}) + \cdots
$$

• Cannot be pictured in a simple diagram with few coupling constants. Some quantities and exponents are continuous, some are discontinuous as a function of *d*.

Anomalous field dimensions are continuous Eigenvalue of the cuspy perturbation is discontinuous

Anomalous increase of the correction-toscaling exponent below $d_{DR} \approx 5.1$ $\overline{}$ Ivan Balog[∗] Institute of Physics, P.O.Box 304, Bigger and Physics, P.O.Box 304, Bigger and P

 Δ s a noout of the bo As a result of the boundary-layer mechanism, anomalous square-root behavior:

> NP-FRG result for the lowest irrelevant eigenvalue (correction to scaling) λ near d_{DR}

[[]Balog, G.T., Tissier, PRE, 2020] FIG. 1: Lowest irrelevant eigenvalue of the stability matrix of the stability

The need to be nonperturbative...

Slava Rychkov and his collaborators solved a 40-year-old problem
 Slava Rychkov and his collaborators solved a 40-year-old problem posed by the Parisi-Sourlas conjecture (*F*6)*^L* = (12 ³")class ⁷ "² + *...,* (3.53)

Press release – 8 September 2022

• Perturbative RG description of destabilization of the SUSY/DR fixed point by irrelevant operators in $d=6$ that become less so as ϵ =6-d increases. [Rychkov, Trevisani et al., 2021-2024] conformal bootstrap. *articles that answer a question which has puzzled physicists for more than four decades. They are thus finally shedding light on the role played by disorder in some fundamental models in statistical physics, such as the Random Field Ising Model and the branched point by irrelevant operators* ds E–0-0 IIICICdsCS. [KyClikOV, IICVISdIII Ct dl., 2021-2024] *articles that answer a question which has puzzled physicists for more than four decades. They are thus finally shedding light on the role played by disorder in some fundamental models in statistical physics, such as the* see that both these scaling dimensions crossed marginality line = *d* for *d* between 4 and 5: (*F*6)*^L* = *d* at *d* = *dc*² ⇡ 4*.*2*.* (3.54)

of the singlets *F*⁴ and *F*6, see Section 3.7. Their scaling dimensions are given by:

where we show separately the classical dimension and the anomalous one, arising at α

- $\mathsf{mod}_{\mathcal{V}}$ in EDC decarintion: To determine the role of the extra term, physicists set out to study how solutions to the RFIM might $(1/\Omega)$ Ω (0.8) Ω (0.2) Γ
	- $-1/(4104)\psi$ \rightarrow \rightarrow \rightarrow \sim certain materials might be affected by their inherent disorder. W_{max} started started started studying the 1970s they could be 1970s they could be extracted that the extra the extracted started + *···* \rightarrow F_4 \rightarrow F_6

The need to be nonperturbative…

• Perturbative RG is intrinsically unable to capture the disappearance of the SUSY/DR cuspless fixed point at d_{DR} . Focus on eigenvalue A_2 (associated with δ_2 and the most dangerous operator F_4) above d_{DR}

of NID ERC route by study of succ point in the results by study of succ *Fare formation* in *Pers* = $>$ *Ranid annarent convergent* μ us \rightarrow respublique apparent convergent $\zeta(A): d_{\text{max}} \sim 5.11 + 0.00$ ICT Tiesig μ perturbative approximation scheme of the FRG (CPA), the FRG (FRG μ fixed points. Note that the SUCCESSIVE fixed point and of the spectrum of \mathbf{r} onvergence (from LPA' to convergence α $N₁$ and \sim 0.000 μ balog, 2024] $\;$ converge to the same value with the same slope when *d* ! 6. the value of *d*DR. We find *d*DR ⇡ 5*.*2005 for the lowest order approximation LPA', *d*DR ⇡ 5*.*0180 for the LPA", **IG.T. Tissier, Balog. 20** $\sum_{i=1}^n$ $\sum_{i=1}^n$ $\sum_{i=1}^n$ $\sum_{i=1}^n$ $\sum_{i=1}^n$ $\sum_{i=1}^n$ \mathbf{A} A' to scale-free collective phenomena collective ph approximation orders => Rapid apparent convergence (from LPA' to criticality. All of this is well supported by state-of-the- \mathbf{a} • Check robustness of NP-FRG results by study of successive LPA'' to DE2 to DE4): $d_{\text{DR}} \approx 5.11 \pm 0.09$ [G.T., Tissier, Balog, 2024]

Conclusion

- Many collective phenomena in disordered systems require a **functional RG** (functional in the order-parameter fields).
	- ➡ For random-field, random-manifold and alike models, solution can be obtained through a combined truncation of the cumulants series and the derivative expansion.
- Resulting zero-temperature fixed-point theories are unusual (e.g., nonanalyticities in the functional dependence of the 1-PI cumulants).
	- ➡ In random-field models disappearance of the SUSY/dim.-red. fixed point and emergence of a "cuspy" fixed point is highly unconventional and can only be described via a **functional and nonperturbative** RG.
- Exponentially rare events and Griffiths phenomena: Functional but whole distributions?