When the RG must be functional and nonperturbative...

Lessons from disordered systems

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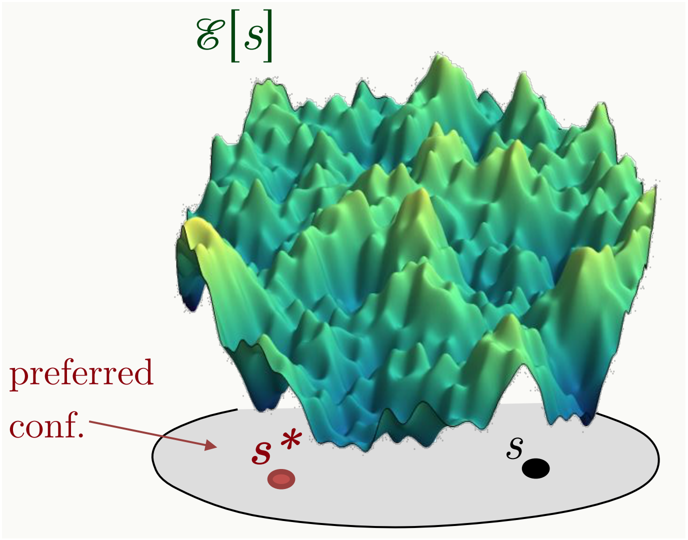
Disordered systems in physics: Interplay of interactions and quenched disorder

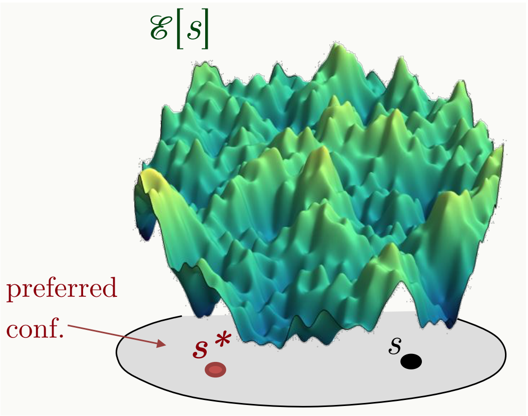
- Multiple low-energy metastable states (energy minima) => Complex, rugged, energy or free-energy landscape.



- Due to quenched disorder, long-distance physics influenced by nonuniform configurations, singular collective events (avalanches), and/or rare excitations (droplets) on all scales.

- Often: Disorder-induced fluctuations dominate over thermal (quantum) fluctuations. Disorder grows under coarse-graining.

Disordered systems in physics: Interplay of interactions and quenched disorder



- How to describe these phenomena? Standard perturbative RG fails!
- Random field, random anisotropy systems, elastic manifolds in a disordered environment, Bose glass, etc.: nontrivial **zero-temperature fixed points** and **functional RG**.
- Spin-glass phases and alike: nontrivial zero-temperature fixed points but so far no satisfactory RG description.
- Random transverse field Ising model, quantum Griffiths phases, MBL, etc.: **Real space functional RG** (strong disorder RG and infinite randomness fixed points).

Outline

- **Why a nonperturbative functional RG (from a physical/phenomenological perspective) and how?**

- **Oddities of the resulting fixed-point theories.**

Why and how a functional RG?

- Illustration for the random-field Ising model (RFIM)

Lattice version: $\mathcal{H}_h[S] = -J \sum_{<i,j>} S_i S_j - \sum_i (h_i + H) S_i$

Field theory:

$$S_h[\varphi] = \int d^d x \{ \frac{1}{2}(\nabla\varphi(x))^2 + \frac{\tau}{2}\varphi(x)^2 + \frac{g}{4!}\varphi(x)^4 - h(x)\varphi(x) \}$$

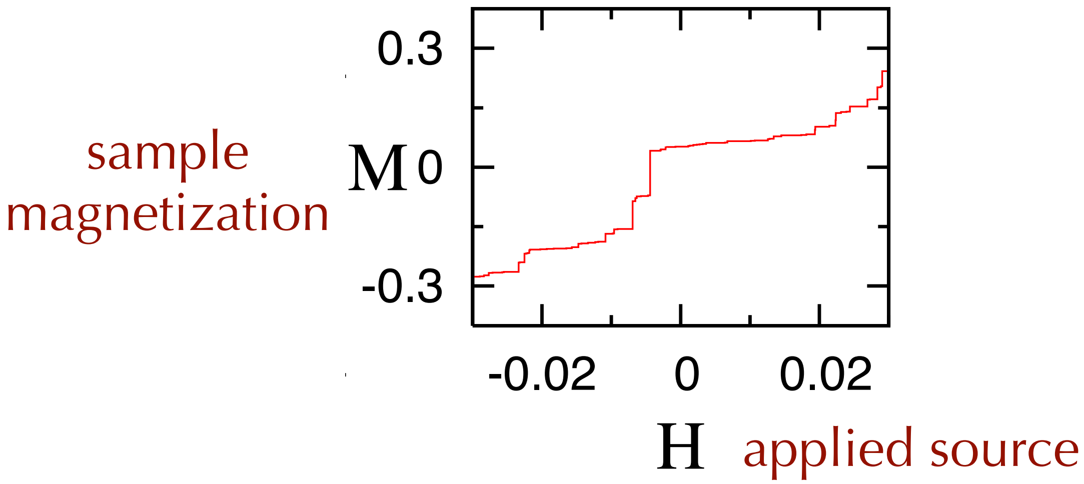
+ quenched random field: $\overline{h(x)} = 0,\ \overline{h(x)h(y)} = \Delta\delta^{(d)}(x-y)$

- Main puzzling feature: **dimensional reduction** (DR) to the pure model in *d*-2 associated with supersymmetry (**Parisi-Sourlas SUSY**) and its breakdown as a function of space dimension *d*.

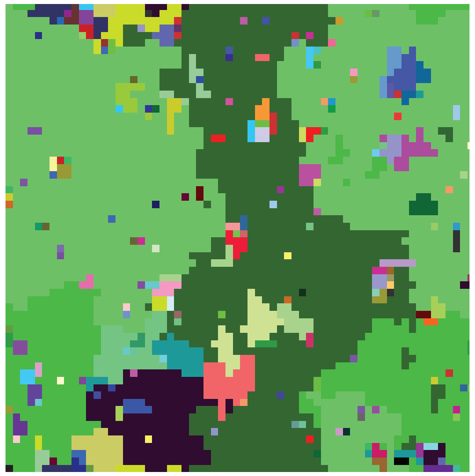
Avalanches in the *T=0* equilibrium RFIM

- Computer simulation: Discontinuous changes of the ground state (avalanches or shocks) under applied source *H*.

Sequence of avalanches in the GS of a *3d*-RFIM sample [Liu-Dahmen, '08]



Cross-section of a spanning avalanche at criticality in 3*d*



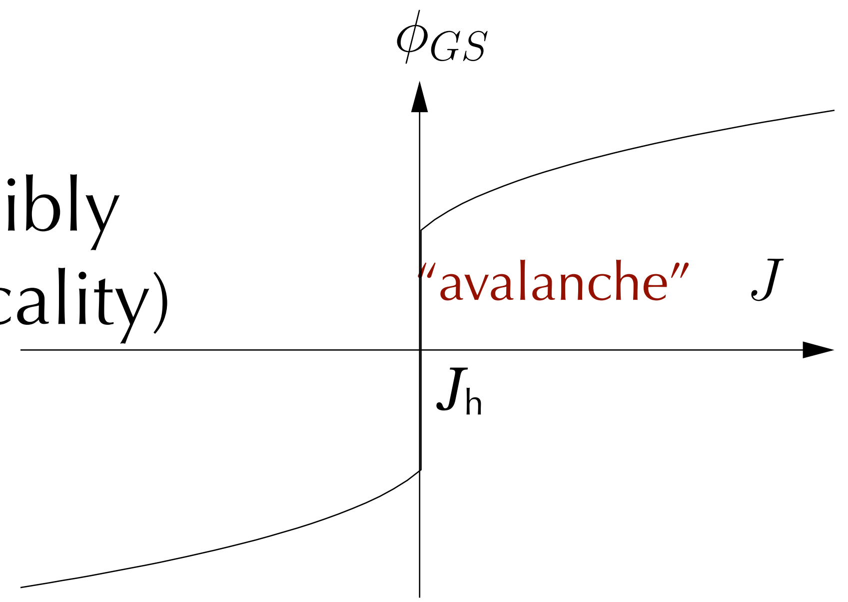
- At criticality, avalanches on all scales; in a finite system (*L*) they have a size (*S*) distribution

$$\rho_L(S) = \rho_{0,L} \frac{e^{-S/S_{\max}(L)}}{S^\tau}$$

$$S_{\max}(L) \sim L^{d_f}, \quad 1 < \tau < 2$$

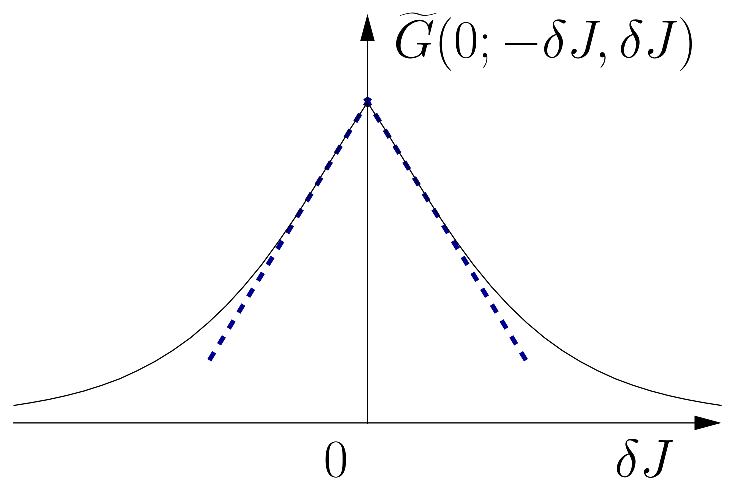
Direct signature of avalanches

- The ground state (GS) jumps discontinuously (possibly large avalanches) at a sample-depdt J_h (≈ 0 at criticality)



- Then, the second cumulant of the magnetization and the associated pair correlation function for **slightly different sources** $\pm\delta J$, has a nonanalytic behavior (a "cusp" in $\sqrt{\delta J^2}$) when $\delta J \to 0$.

$$L^d\,\overline{\phi_{\mathrm{GS,h}}(-\delta J)\phi_{\mathrm{GS,h}}(\delta J)} = L^d\,\overline{\phi_{\mathrm{GS,h}}(0)^2} + \mathrm{O}(\delta J^2)$$
$$-\,|\delta J|\frac{1}{2L^d}\int_{S_{\min}}^{\infty} dS\,S^2\rho_L(S) \equiv \widetilde{G}_L(q=0;-\delta J,\delta J)$$



Cusp amplitude = Second moment of the avalanche distribution which diverges with the system size L at criticality.

Why functional (in the fields)?

- Effect of avalanches (and low-energy droplet excitations at *T>0*) can be captured in a disorder-averaged description, provided one keeps the **functional dependence** of the cumulants on the sources/fields, e.g., for the free-energy functional $\mathcal{W}_h[J] = \ln \mathcal{Z}_h[J]$:

$$W_1[J_1] = \overline{\mathcal{W}_h[J_1]},\ W_2[J_1, J_2] = \overline{\mathcal{W}_h[J_1] W_h[J_2]}^{\text{cum}}, \cdots$$

- For instance, the "cusp" shows up in the second derivative of second cumulant, $\partial_{J_1} \partial_{J_2} W_2[J_1, J_2]$, when $J_1 \to J_2$.

- Equivalently, study the 1-PI cumulants associated with the effective action: $\Gamma_1[\phi_1],\ \Gamma_2[\phi_1, \phi_2],\ \Gamma_3[\phi_1, \phi_2, \phi_3], \cdots$

- Importantly, effect is already captured in low-order cumulants and for uniform source/field configurations. Not true for Griffiths phase and exponentially rare large-scale excitations.

Exact Functional RG for the 1-PI cumulants

- Evolution of the scale-dependent effective action Γ_k, with decreasing k is described by an **exact (functional) RG equation** [Wetterich, 1993]:

$$\partial_k \Gamma_k[\{\phi_a\}] = \frac{1}{2} \mathrm{Trace}\{(\Gamma_k^{(2)}[\{\phi_a\}] + \mathcal{R}_k)^{-1} \, \partial_k \mathcal{R}_k\}$$

with $\Gamma_k[\{\phi_a\}]$ obtained by coupling copies/replicas of the disordered system to distinct sources J_a and then Legendre transform.

- From it, hierarchy of exact functional RG equations for the **flow of the 1-PI cumulants**:

$$\partial_k \Gamma_{k1}[\phi_1] = \mathcal{F}_{k1}[\Gamma_{k1}^{(2)}, \Gamma_{k2}^{(2)}],$$

$$\partial_k \Gamma_{k2}[\phi_1, \phi_2] = \mathcal{F}_{k2}[\Gamma_{k1}^{(2)}, \Gamma_{k2}^{(2)}, \Gamma_{k3}^{(2)}],$$

$$\partial_k \Gamma_{k3}[\phi_1, \phi_2, \phi_3] = \mathcal{F}_{k3}[\Gamma_{k1}^{(2)}, \Gamma_{k2}^{(2)}, \Gamma_{k3}^{(2)}, \Gamma_{k4}^{(2)}], \ \text{etc.}$$

Nonperturbative FRG

- Guided by considerations on avalanches, nonperturbative approximation scheme = combined truncation of
  - Expansion in the **number of derivatives of the fields** (around uniform configurations)
  - Expansion in **cumulant order**.
- Example of truncation (referred to as DE2):

$$\Gamma_{k1}[\phi] = \int_x [U_k(\phi(x)) + \frac{1}{2} Z_k(\phi(x))(\partial_x \phi(x))^2 + \mathrm{O}(\partial^4)],$$

$$\Gamma_{k2}[\phi_1, \phi_2] = \int_x [V_k(\phi_1(x), \phi_2(x)) + \mathrm{O}(\partial^2)],$$

$$\Gamma_{k,p\geq 3} = 0.$$

- Crucial not to explicitly break underlying Parisi-Sourlas SUSY.
- Key quantity: $\Delta_k(\phi_1, \phi_2) = \partial_{\phi_1}\partial_{\phi_2} V_k(\phi_1, \phi_2) = V_k^{(11)}(\phi_1, \phi_2)$

Cusp in the functional dependence and SUSY/DR breakdown

- In dimensionless form (appropriate for a **zero-temperature fixed point**):

$$\partial_t u'_k(\varphi) = \beta_{u'}(\varphi), \quad t = \ln(k/\Lambda_{\rm UV}),$$
$$\partial_t z_k(\varphi) = \beta_z(\varphi),$$
$$\partial_t \delta_k(\varphi_1, \varphi_2) = \beta_\delta(\varphi_1, \varphi_2)$$

- Second cumulant of the RF δ_k with $\varphi = (\varphi_1 + \varphi_2)/2, \; \delta\varphi = \varphi_1 - \varphi_2$

$$\text{No cusp}: \; \delta(\varphi, \delta\varphi) = \delta_0(\varphi) + \frac{1}{2}\delta_2(\varphi)\delta\varphi^2 + \cdots$$

$$\text{Cusp}: \; \delta(\varphi, \delta\varphi) = \delta_0(\varphi) + a_{\rm cusp}(\varphi)|\delta\varphi| + \frac{1}{2}\tilde{\delta}_2(\varphi)\delta\varphi^2 + \cdots$$

- **Cusp breaks SUSY and dimensional reduction:** Takes place below some $d_{\rm DR}$.

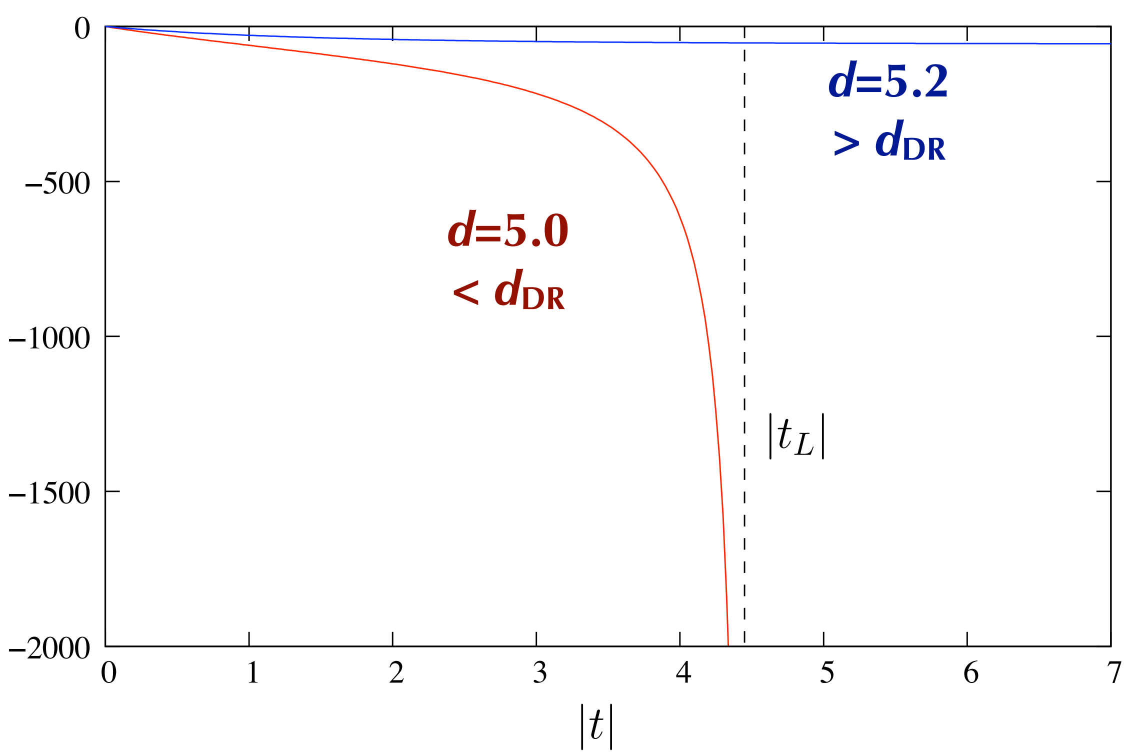
- A simple signature in $\partial^2\delta(\varphi, \delta\varphi)/\partial\delta\varphi^2|_{\delta\varphi=0}$

Solution of the NP-FRG flow equations: cusp versus no cusp

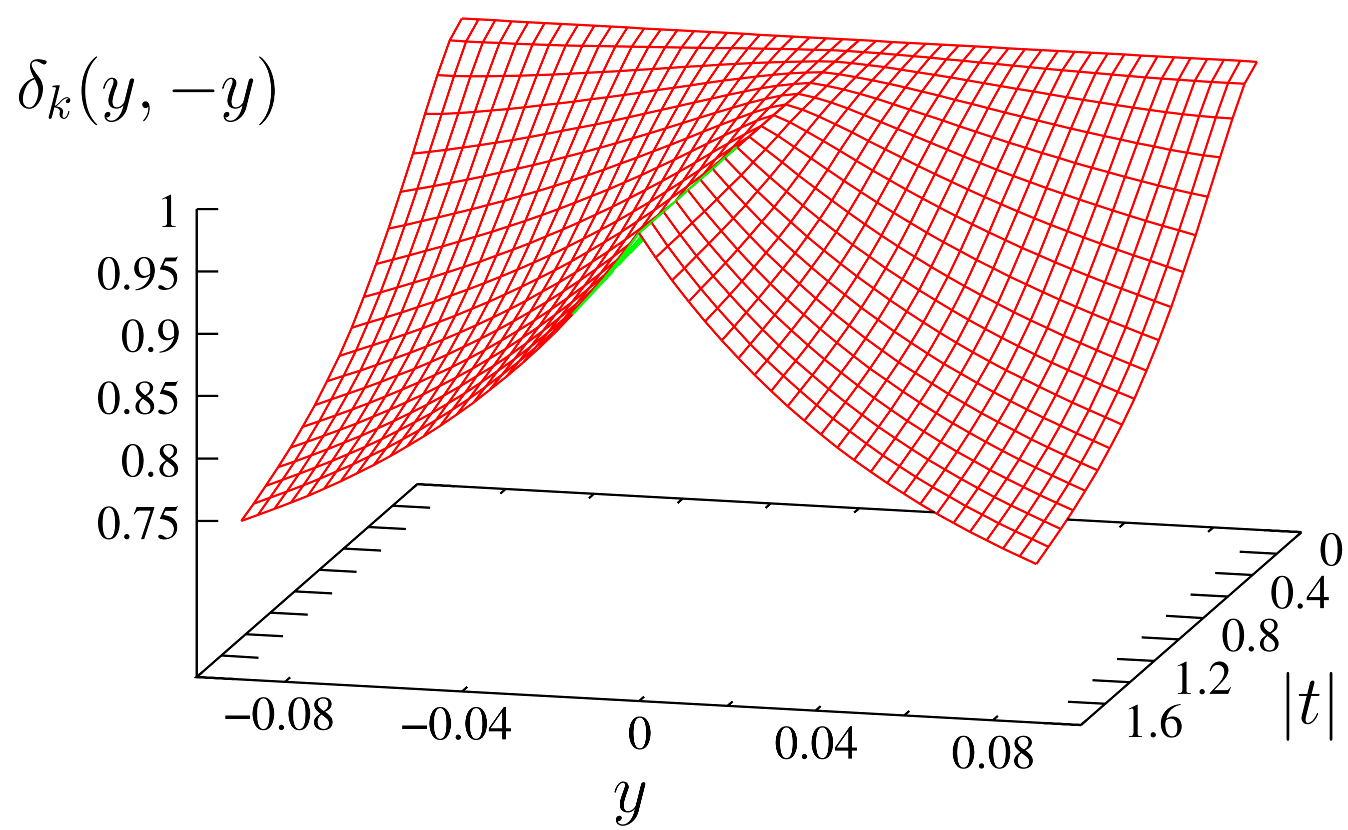
Signature in the flow of the second cumulant $\delta_k(\varphi_1,\varphi_2)$:

Cusp in $y \equiv \delta\varphi = (\varphi_1 - \varphi_2)$ below $\boldsymbol{d_{DR}} \approx \mathbf{5.1}$ versus no cusp above.

$\partial^2\delta_k/\partial^2 y(y=0)$ blows up in a finite RG time t for $d < d_{DR}$ (red curve), not for $d > d_{DR}$ (blue curve)



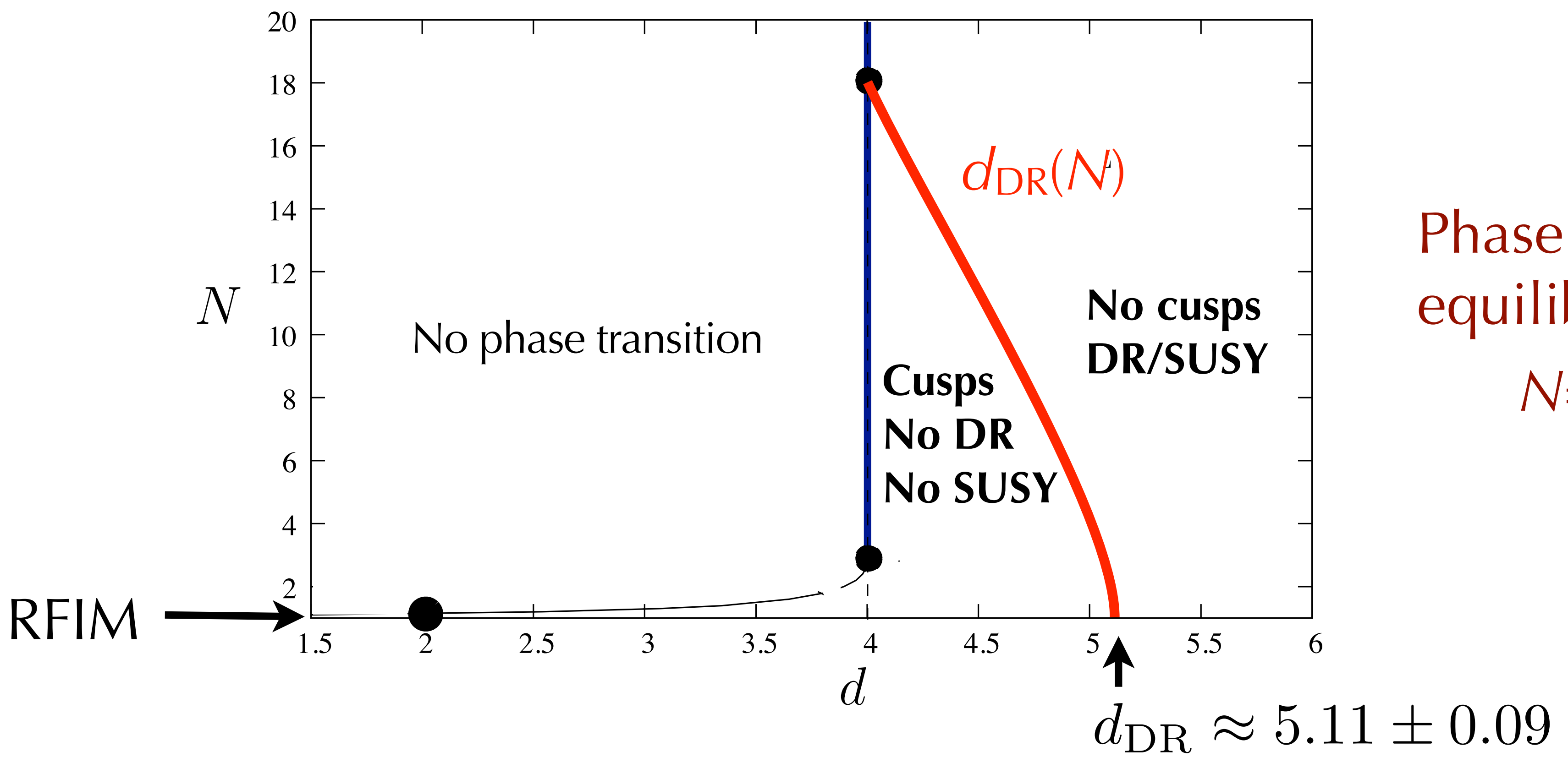
Flow of the dimensionless second cumulant δ_k in $d=4 < d_{DR}$



[M. Tissier, G.T., PRL, PRB, 2011, 2012]

Oddities of the resulting fixed-point theories

Transition between different types of critical behavior at a nontrivial critical dimension d_{DR}



Phase diagram of the equilibrium RFO(N)M

N=1 → RFIM

- Below d_{DR}, nonanalytic (zero-temperature) fixed-point effective action
- Unconventional pattern of disappearance/appearance of fixed points around d_{DR}: requires a functional description!

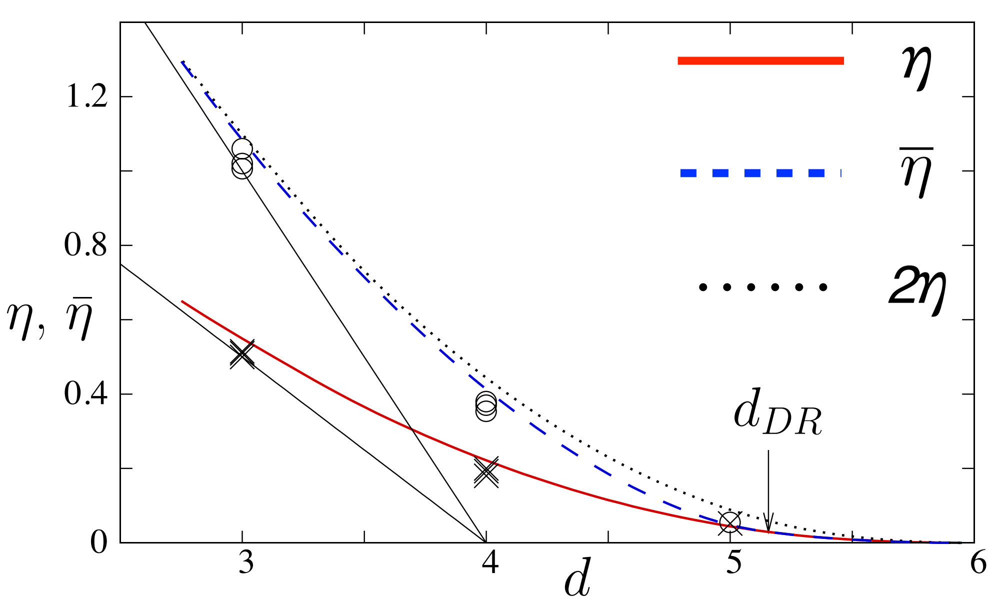
Oddities and need for a functional description

- In $d_{\rm DR}$, coalescence of two cuspless fixed points and emergence of a cuspy fixed point below $d_{\rm DR}$ thru a boundary-layer mechanism:

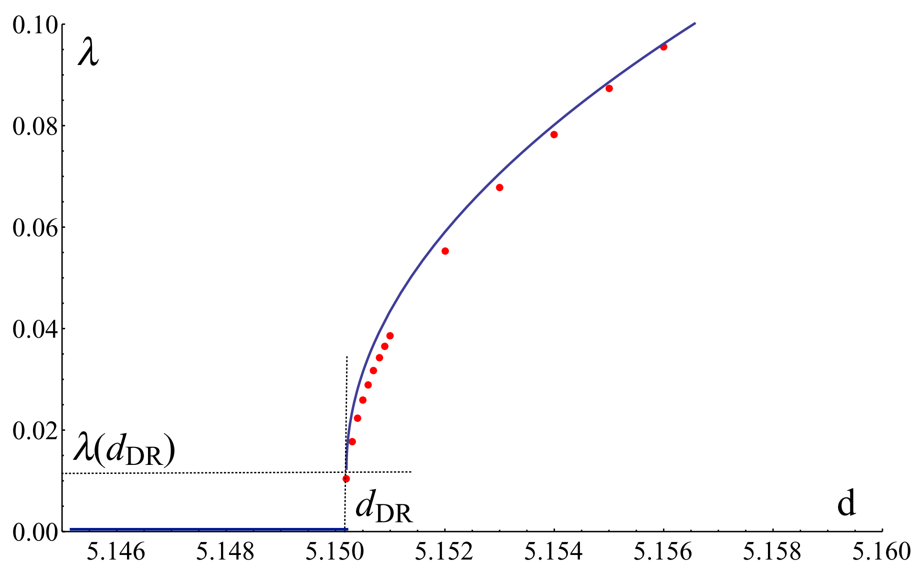
$$d \to d_{\rm DR}^- : \delta(\varphi, \delta\varphi) = \delta_0(\varphi) + (d_{\rm DR} - d)^\mu \, f(\varphi, \frac{|\delta\varphi|}{(d_{\rm DR} - d)^{\mu/2}}) + \cdots$$

- Cannot be pictured in a simple diagram with few coupling constants. Some quantities and exponents are continuous, some are discontinuous as a function of d.

Anomalous field dimensions are continuous



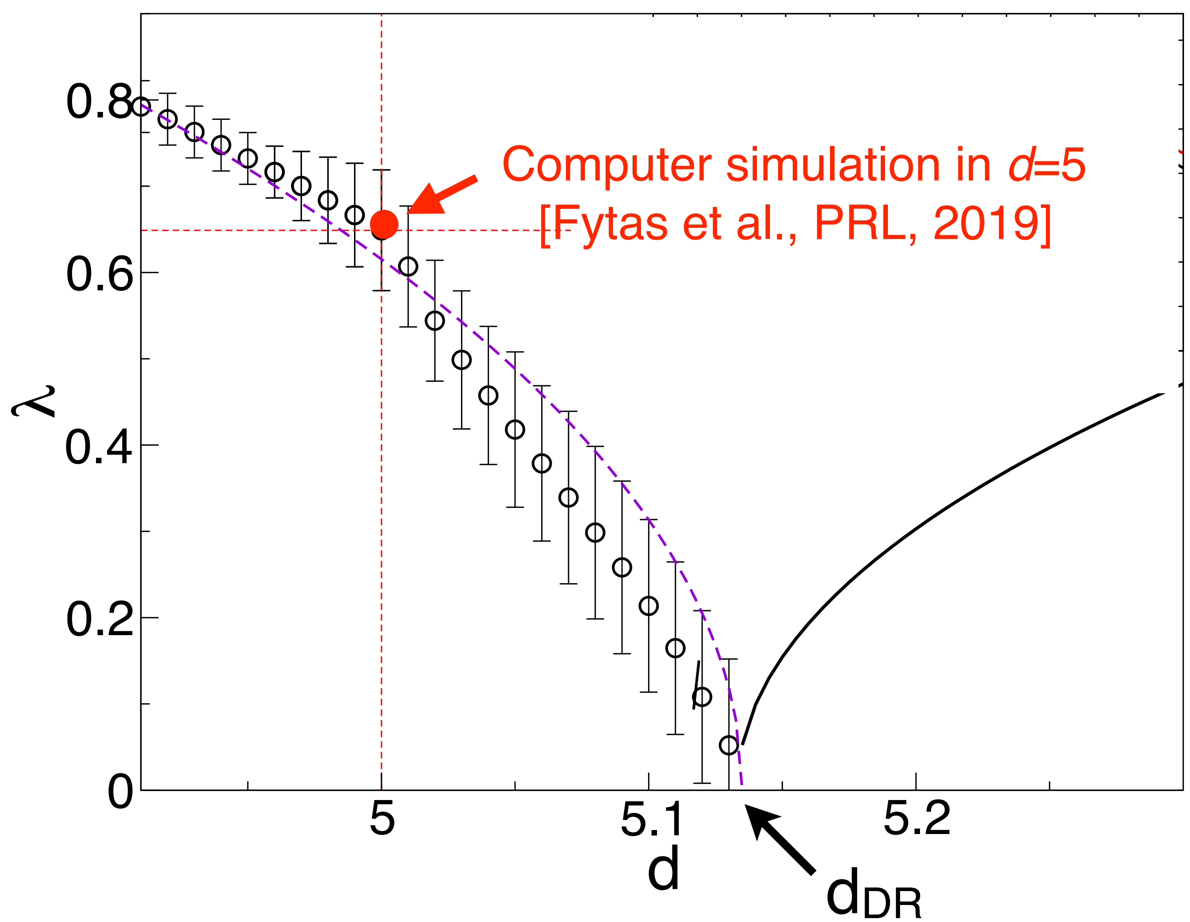
Eigenvalue of the cuspy perturbation is discontinuous



Anomalous increase of the correction-to-scaling exponent below $d_{DR} \approx 5.1$

As a result of the boundary-layer mechanism, anomalous square-root behavior:

NP-FRG result for the lowest irrelevant eigenvalue (correction to scaling) λ near d_{DR}



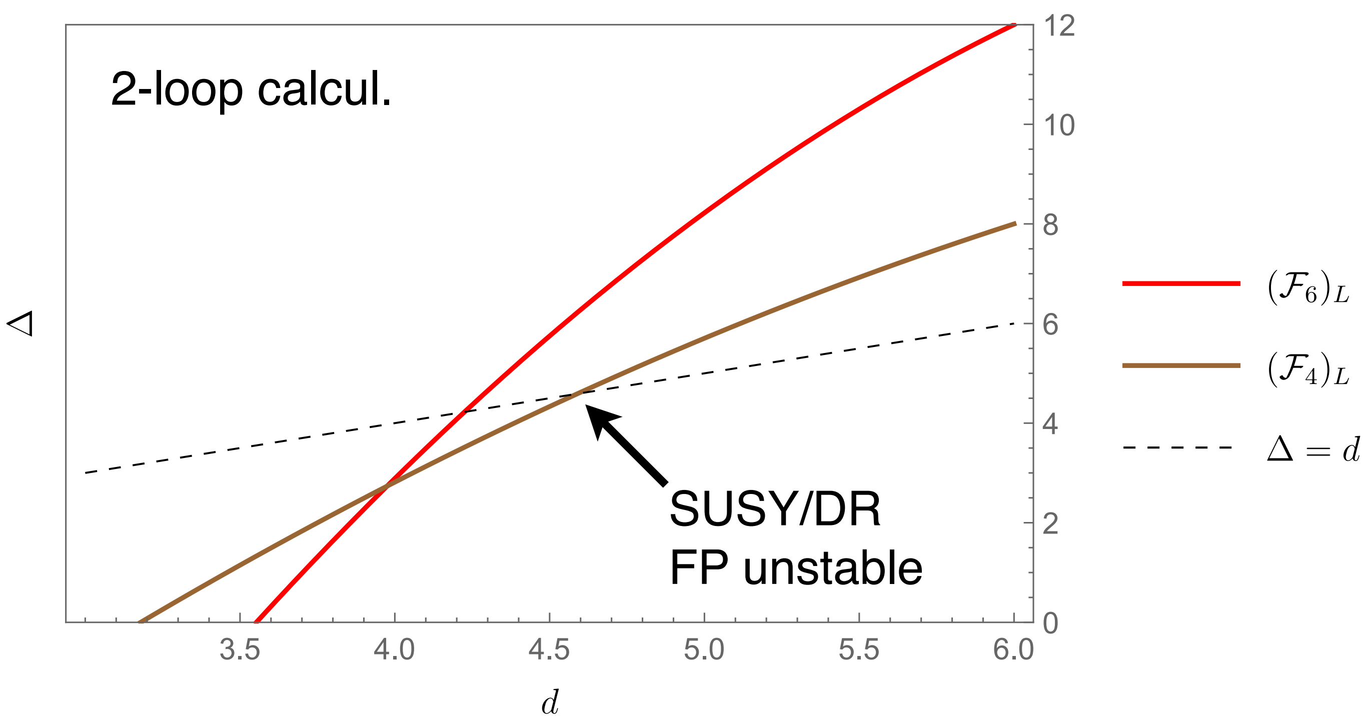
[Balog, G.T., Tissier, PRE, 2020]

The need to be nonperturbative…

Slava Rychkov and his collaborators solved a 40-year-old problem posed by the Parisi-Sourlas conjecture

Press release – 8 September 2022

- Perturbative RG description of destabilization of the SUSY/DR fixed point by irrelevant operators in *d*=6 that become less so as ϵ=6-*d* increases. [Rychkov, Trevisani et al., 2021-2024]



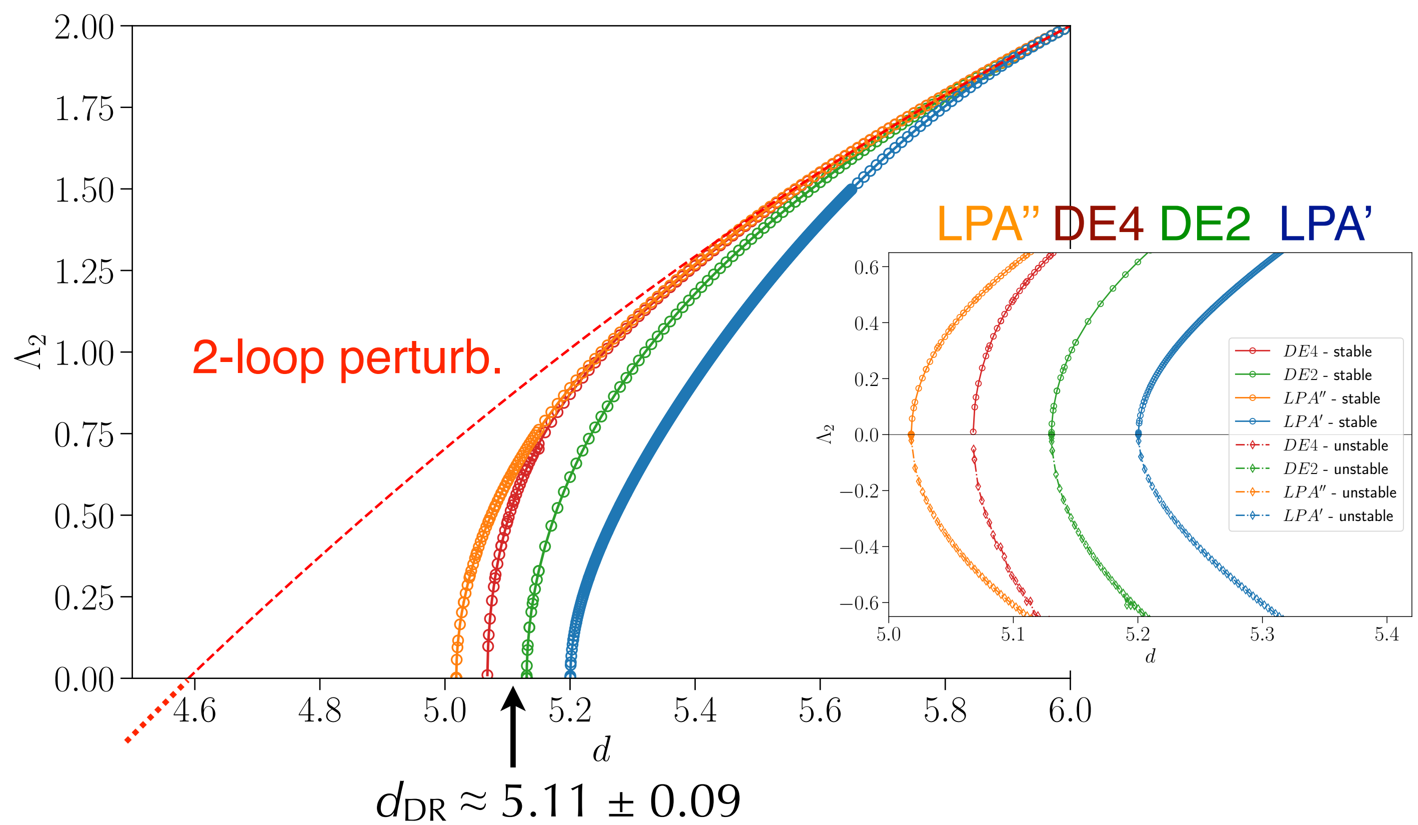
- Already in FRG description:

$$\delta(\varphi, \delta\varphi) = \delta_0(\varphi)$$
$$+ (1/2)\delta_2(\varphi)\delta\varphi^2 \quad \to \mathcal{F}_4$$
$$+ 1/(4!)\delta_4(\varphi)\delta\varphi^4 \quad \to \mathcal{F}_6$$
$$+ \cdots$$

The need to be nonperturbative…

- Perturbative RG is intrinsically unable to capture the disappearance of the SUSY/DR cuspless fixed point at d_{DR}. Focus on eigenvalue Λ_2 (associated with δ_2 and the most dangerous operator $\mathcal{F}_4$) above d_{DR}

$$\Lambda_2 = \Delta_{\mathcal{F}_4} - d$$



- Check robustness of NP-FRG results by study of successive approximation orders => Rapid apparent convergence (from LPA′ to LPA″ to DE2 to DE4): $d_{\mathrm{DR}} \approx 5.11 \pm 0.09$ [G.T., Tissier, Balog, 2024]

Conclusion

- Many collective phenomena in disordered systems require a **functional RG** (functional in the order-parameter fields).
  ➡ For random-field, random-manifold and alike models, solution can be obtained through a combined truncation of the cumulants series and the derivative expansion.

- Resulting zero-temperature fixed-point theories are unusual (e.g., nonanalyticities in the functional dependence of the 1-PI cumulants).
  ➡ In random-field models disappearance of the SUSY/dim.-red. fixed point and emergence of a "cuspy" fixed point is highly unconventional and can only be described via a **functional and nonperturbative** RG.

- Exponentially rare events and Griffiths phenomena: Functional but whole distributions?