ERG 2024

The inviscid fixed point
of the
multi-dimensional Burgers-KPZ equation

L. Gosteva, M. Tarpin, N. Wschebor, L. Canet

Co-funded by
the European Union

Outline

1. The model: Burgers-KPZ equation
2. Motivation: the new scaling regime found in numerics
3. FRG study
    3.1. Simple approximation: confirm the existence of the “Inviscid” UV fixed point
    3.2. Large-momentum approximation: find *z* at this fixed point

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The model: Burgers-KPZ equation

Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \underbrace{\nu\nabla^2 h}_{\text{Relaxation}} + \underbrace{\frac{\lambda}{2}(\nabla h)^2}_{\text{Growth}} + \underbrace{\eta}_{\text{Noise}}$$

Video: Club De Montana Calahorra, Facebook

The model: Burgers-KPZ equation

Burgers equation

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \underbrace{\nu\nabla^2\mathbf{v}}_{\text{Dissipation}} + \underbrace{\mathbf{f}}_{\text{Random forcing}}$$

$$\xleftrightarrow[\mathbf{v} = -\lambda\nabla h]{\text{Mapping}}$$

$$\boxed{\begin{aligned}\nabla\times\mathbf{v} &= 0\\ \mathbf{f} &= -\lambda\nabla\eta\end{aligned}}$$

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$$\partial_t h = \underbrace{\nu\nabla^2 h}_{\text{Relaxation}} + \underbrace{\frac{\lambda}{2}(\nabla h)^2}_{\text{Growth}} + \underbrace{\eta}_{\text{Noise}}$$

$$\langle\eta(t,\mathbf{x})\eta(t',\mathbf{x}')\rangle = 2D\delta(t-t')\delta^d(\mathbf{x}-\mathbf{x}') \xrightarrow{\text{Fourier}} 2D\delta(t-t')$$

$$\langle f_\alpha(t,\mathbf{x})f_\beta(t',\mathbf{x}')\rangle \xrightarrow{\text{Fourier}} \underbrace{2D\lambda^2}_{\mathcal{D}}\underbrace{p_\alpha p_\beta}_{\text{acts on small scales}}\delta(t-t')$$

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The model: Burgers-KPZ equation

What do we know so far?

Kardar-Parisi-Zhang (KPZ) equation

$$\partial_t h = \underbrace{\nu \nabla^2 h}_{\text{Relaxation}} + \underbrace{\frac{\lambda}{2}(\nabla h)^2}_{\text{Growth}} + \underbrace{\eta}_{\text{Noise}}$$

Scaling properties of correlation functions at large *t*, *x* :

$$C(t,\vec{x}) = \langle h(t,\vec{x})h(0,0)\rangle$$

$$C(t,\vec{x}) = x^{2\chi}\, F(t/x^{z})$$

Roughness exponent

Scaling function

Dynamical exponent

The model: Burgers-KPZ equation

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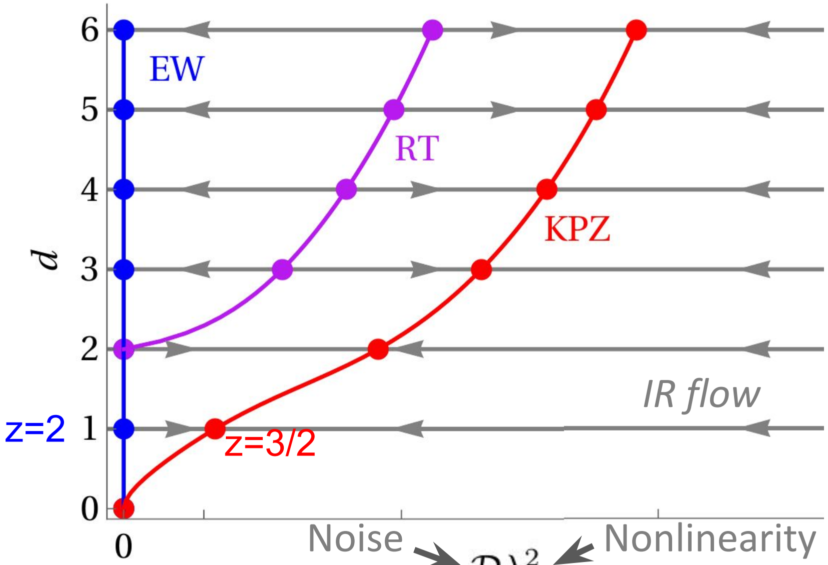
Flow diagram of the Burgers-KPZ equation:



Noise

Nonlinearity

$$g \equiv \frac{\mathcal{D}\lambda^2}{\nu^3}$$

Viscosity

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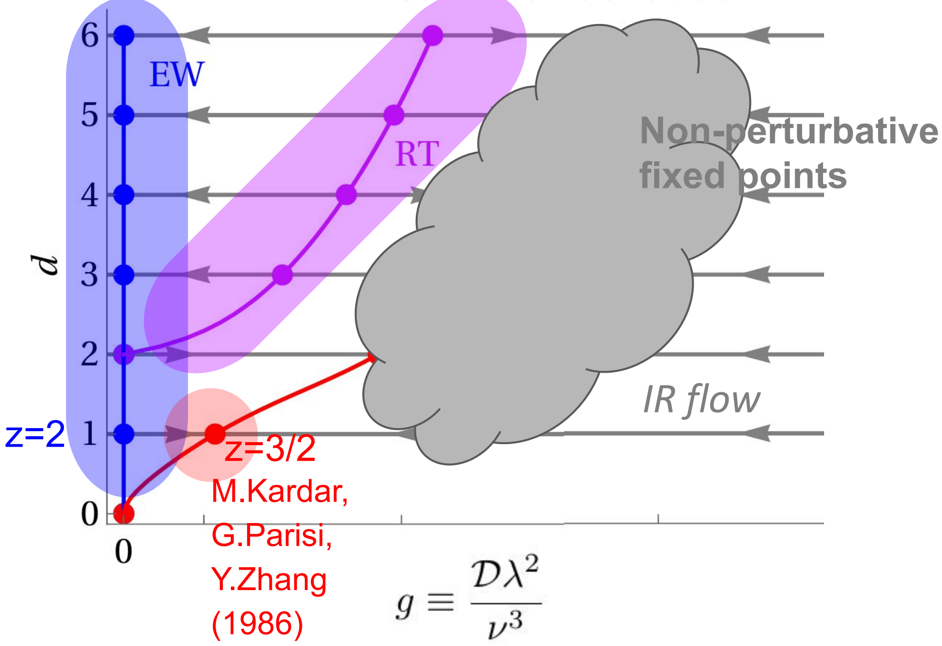
Perturbative fixed points:

K.Wiese (1998):
Summation to all orders



M.Kardar,
G.Parisi,
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(1986)

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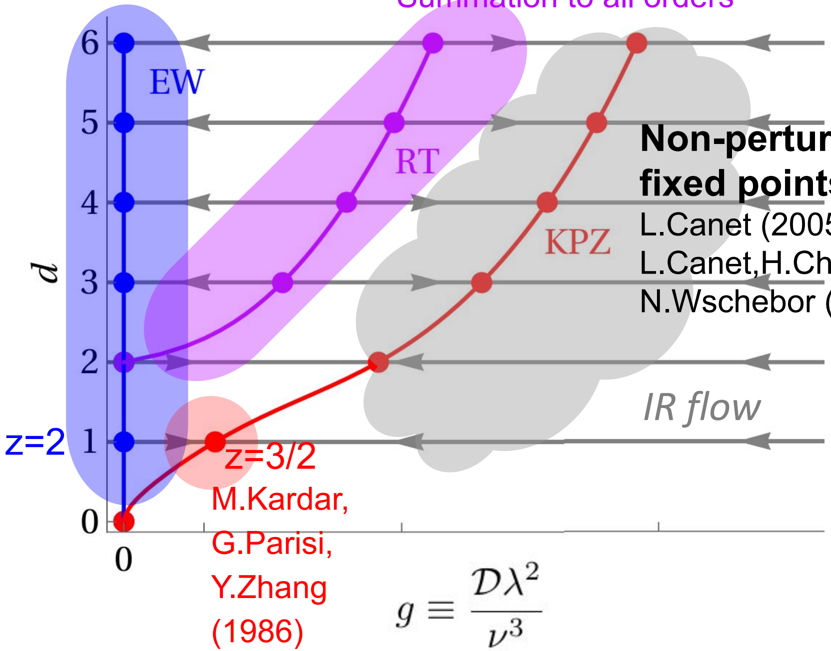

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Non-perturbative fixed points
L.Canet (2005);
L.Canet,H.Chaté,B.Delamotte,
N.Wschebor (2010)

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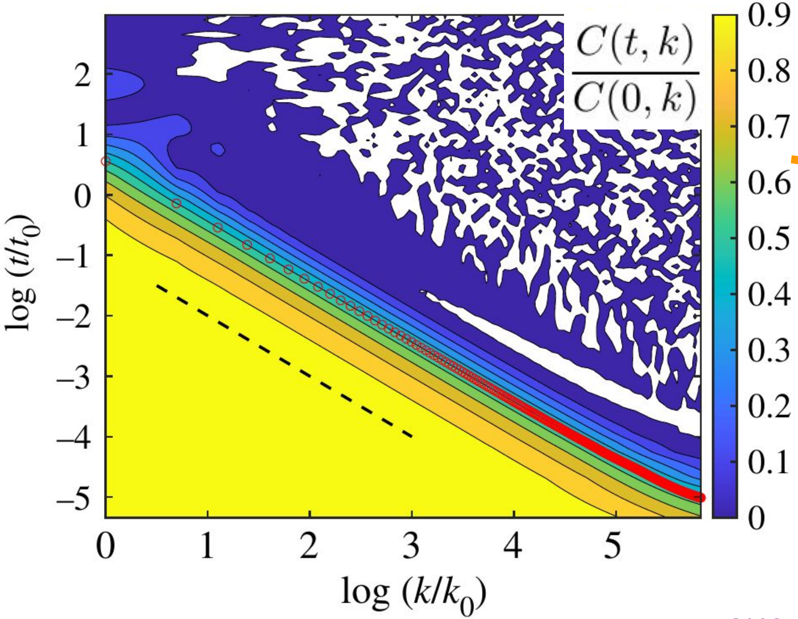
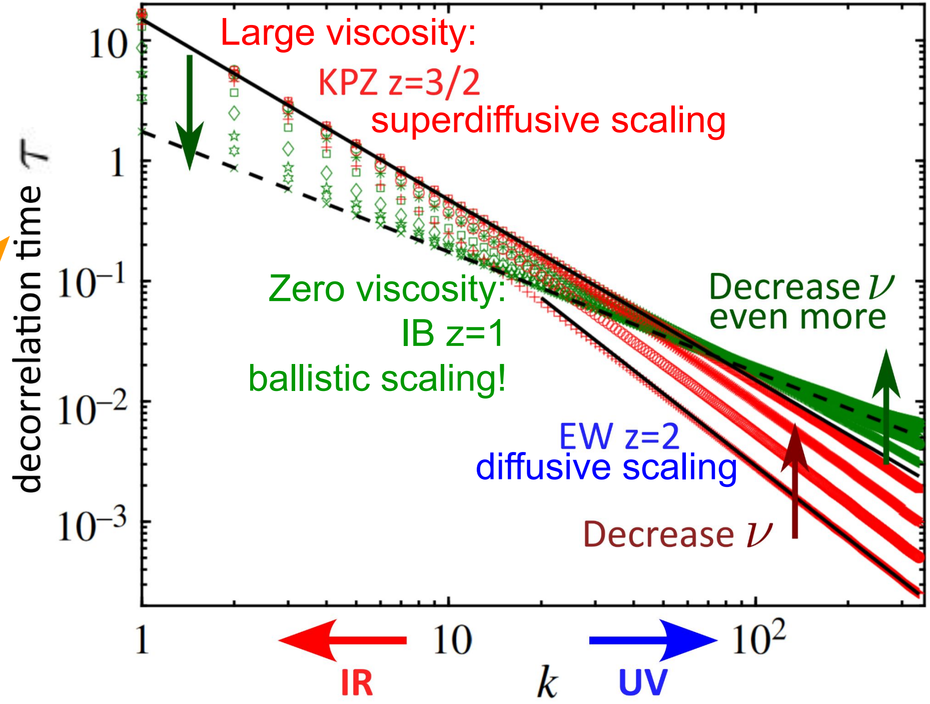
Cartes, Tirapegui, Pandit, Brachet (2022):
The Galerkin-truncated Burgers equation: crossover from inviscid-thermalized to KPZ scaling

$$\boxed{\partial_t v + \lambda v \partial_x v = \nu \partial_x^2 v + f}$$

corr. function $C(t-t', x-x') \equiv \langle v(t,x) v(t',x') \rangle$

$\tau \sim k^{-z}$





Motivation: the new scaling regime found in numerics (1D)

The z=1 scaling had been already explained within FRG:

- ➔ in 1D Burgers-KPZ equation [Fontaine, Vercesi, Brachet, Canet (2023)]
- ➔ in 2D and 3D Navier-Stokes [Canet, Delamotte, Wschebor (2016);
  Tarpin, Canet, Wschebor (2018)]

Is it general?
Let's consider **d-dimensional** Burgers-KPZ equation!

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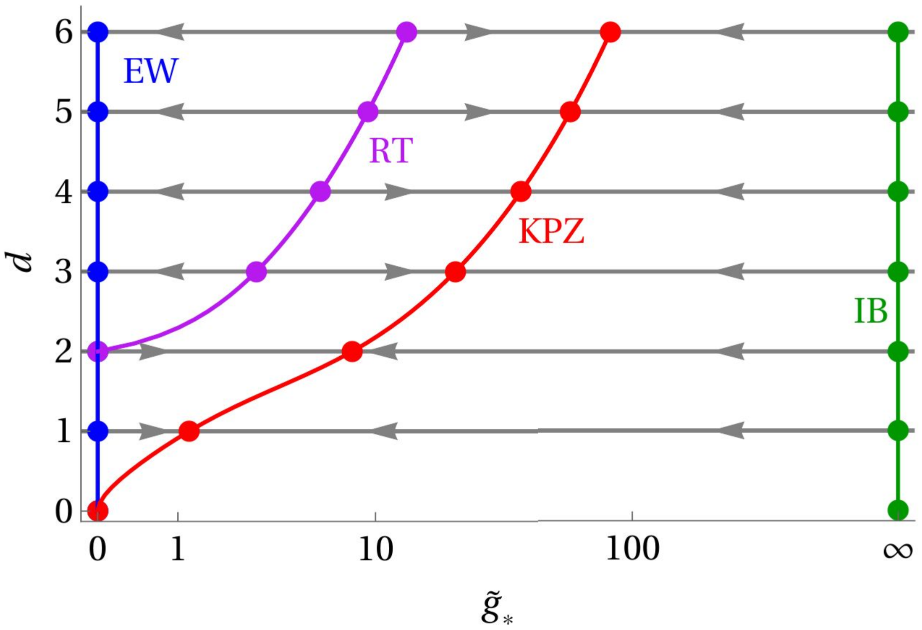
Simple approximation: confirm the existence of the “Inviscid” UV fixed point

Complete flow diagram of the Burgers-KPZ equation:



The “Simple approximation”:

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta$$

$$\updownarrow$$

$$\mathcal{S}_{\mathrm{KPZ}}[h,\bar{h}] = \int_{t,\mathbf{x}} \left\{ \bar{h}\left[\partial_t h - \frac{\lambda}{2}(\nabla h)^2 - \nu \nabla^2 h\right] - D\,\bar{h}^2 \right\}$$

$$\downarrow$$

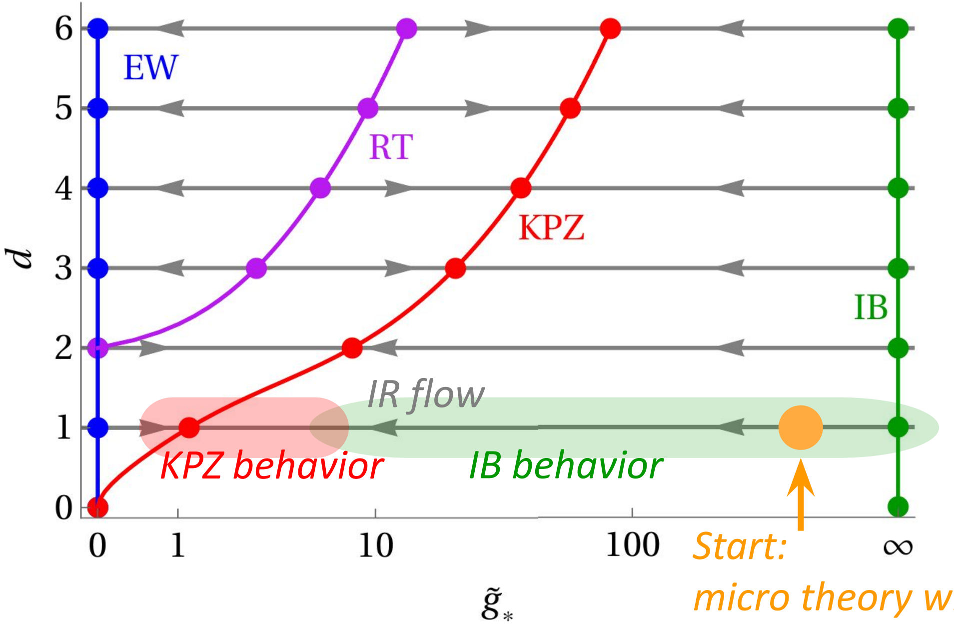
$$\Gamma_\kappa[h,\bar{h}] = \int_{t,\mathbf{x}} \left\{ \bar{h}\left[\mu_\kappa \partial_t h - \frac{\lambda_\kappa}{2}(\nabla h)^2 - \nu_\kappa \nabla^2 h\right] - D_\kappa\,\bar{h}^2 \right\}$$

$\mu_\kappa = \mu,\ \lambda_\kappa = \lambda$ – from symmetries

$$g_\kappa = \frac{\lambda^2 D_\kappa}{\nu_\kappa^3}$$

Simple approximation: confirm the existence of the “Inviscid” UV fixed point

Complete flow diagram of the Burgers-KPZ equation:



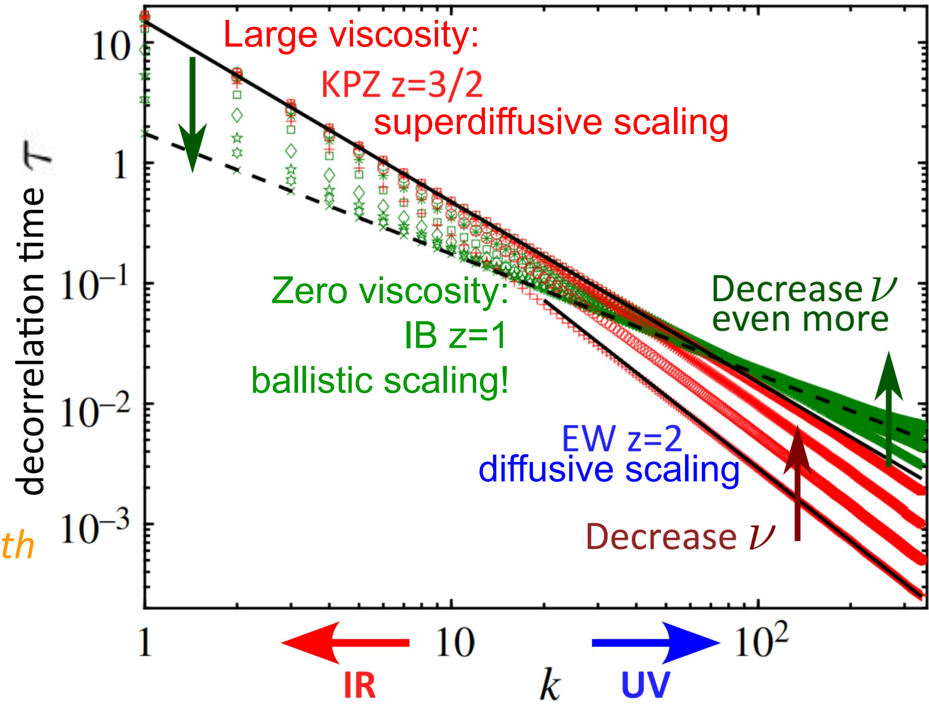
KPZ behavior

IB behavior

Start:
micro theory with small viscosity

$$g_\kappa = \frac{\lambda^2 D_\kappa}{\nu_\kappa^3}$$

$$\tau \sim k^{-z}$$



Large viscosity:
KPZ z=3/2
superdiffusive scaling

Zero viscosity:
IB z=1
ballistic scaling!

Decrease ν even more

EW z=2
diffusive scaling

Decrease ν

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Technical details

Construction of the MSRJD action in d dimensions (dD)

- ➔ in KPZ formulation:

$$\partial_t h = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta \quad\longleftrightarrow\quad \mathcal{S}_{\mathrm{KPZ}}[h,\bar{h}] = \int_{t,x}\left\{\bar{h}\left[\partial_t h - \frac{\lambda}{2}(\nabla h)^2 - \nu\nabla^2 h\right] - D\,\bar{h}^2\right\}$$

- ➔ in Burgers formulation: we need **additional fields**! In 3D:

$$\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = \nu\nabla^2\mathbf{v} + \mathbf{f} \quad\longleftrightarrow\quad \mathcal{S}[\Phi] = \int_{t,\mathbf{x}}\left\{\bar{\mathbf{v}}\cdot\left[\partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\nabla^2\mathbf{v}\right] - \mathcal{D}(\nabla\cdot\bar{\mathbf{v}})^2\right.$$

$$\left. + \underbrace{\bar{\mathbf{v}}\cdot(\nabla\times\mathbf{w}) + \bar{\mathbf{w}}\cdot(\nabla\times\mathbf{v})}_{\text{To preserve irrotationality}} + \underbrace{\bar{\theta}\nabla\cdot\mathbf{w} + \bar{\mathbf{w}}\cdot\nabla\theta}_{\text{To remove zero modes}}\right\}$$

d − 1 pairs of Lagrange multipliers in dD

Technical details

Extended symmetries and the related Ward identities

1. Fully-gauged shift symmetries of the auxiliary fields

$$\varphi(t,\mathbf{x}) \to \varphi(t,\mathbf{x}) + \varepsilon_\varphi(t,\mathbf{x}),\ \ \varphi \in \{\mathbf{w},\bar{\mathbf{w}},\theta,\bar\theta\} \quad \longrightarrow \quad \frac{\delta\Gamma[\Psi]}{\delta\Psi_i} = \frac{\delta S[\Psi]}{\delta\Psi_i}$$

2. Time-gauged shift symmetry of the response field - in Burgers formulation only!

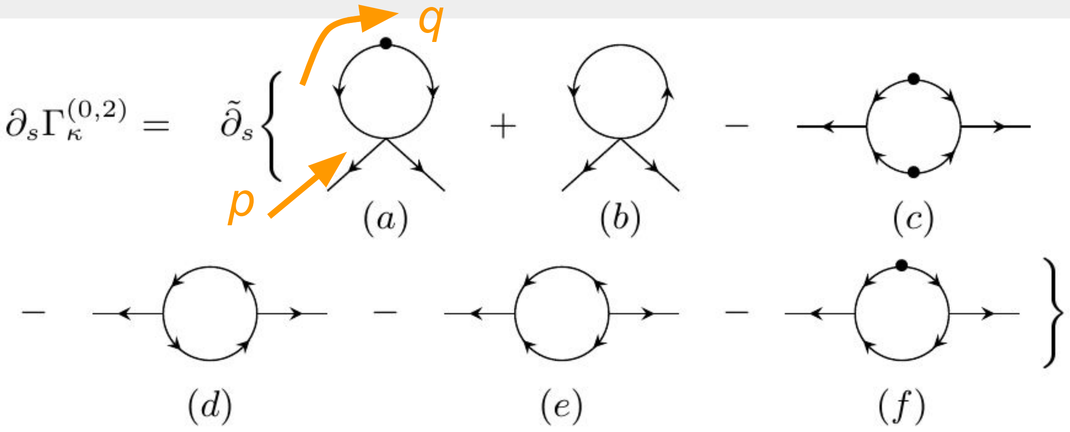
$$\begin{cases} \bar v_\alpha(t,\mathbf{x}) \to \bar v_\alpha(t,\mathbf{x}) + \bar\varepsilon_\alpha(t) \\ \bar w_\alpha(t,\mathbf{x}) \to \bar w_\alpha(t,\mathbf{x}) + \epsilon_{\alpha\beta\gamma}\bar\varepsilon_\beta(t) v_\gamma(t,\mathbf{x}) - \bar\varepsilon_\alpha(t)\theta(t,\mathbf{x}) \end{cases} \quad \longrightarrow \quad \bar\Gamma^{(m,n+1)}_{\alpha_1\cdots\alpha_{m+n+1}}(\omega_1,\mathbf{p}_1;\omega_2,\mathbf{p}_2;\dots,\underbrace{\omega_{k>m},0;}_{\bar{\mathbf{u}}}\dots) = 0$$

3. Time-gauged Galilean symmetry

$$\begin{cases} v_\alpha(t,\mathbf{x}) \to v_\alpha(t,\mathbf{x}) - \partial_t\varepsilon_\alpha(t) + \varepsilon_\beta(t)\partial_\beta v_\alpha(t,\mathbf{x}) \\ \varphi(t,\mathbf{x}) \to \varphi(t,\mathbf{x}) + \varepsilon_\beta(t)\partial_\beta\varphi(t,\mathbf{x}),\ \ \varphi \in \{\bar{\mathbf{v}},\mathbf{w},\bar{\mathbf{w}},\theta,\bar\theta\} \end{cases} \quad \longrightarrow \quad \bar\Gamma^{(m+1,n)}(\underbrace{\omega,0;}_{\mathbf{u}}\omega_1,\mathbf{p}_1;\dots;\omega_{m+n},\mathbf{p}_{m+n}) = \dots\bar\Gamma^{(m,n)}\dots$$

Large-momentum approximation: idea

$$\left.\begin{array}{l}|\mathbf{p}| \gg \kappa \\ |\mathbf{q}| \lesssim \kappa\end{array}\right\} \longrightarrow |\mathbf{q}| \ll |\mathbf{p}|$$



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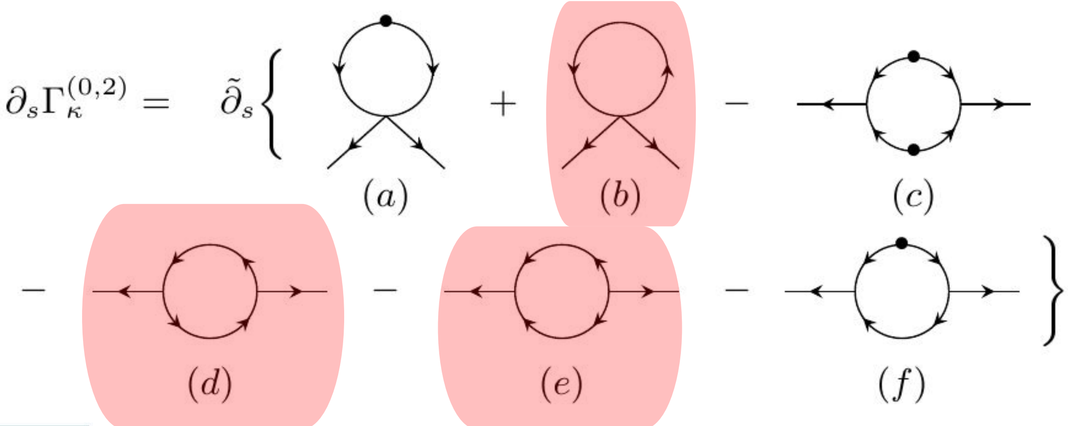

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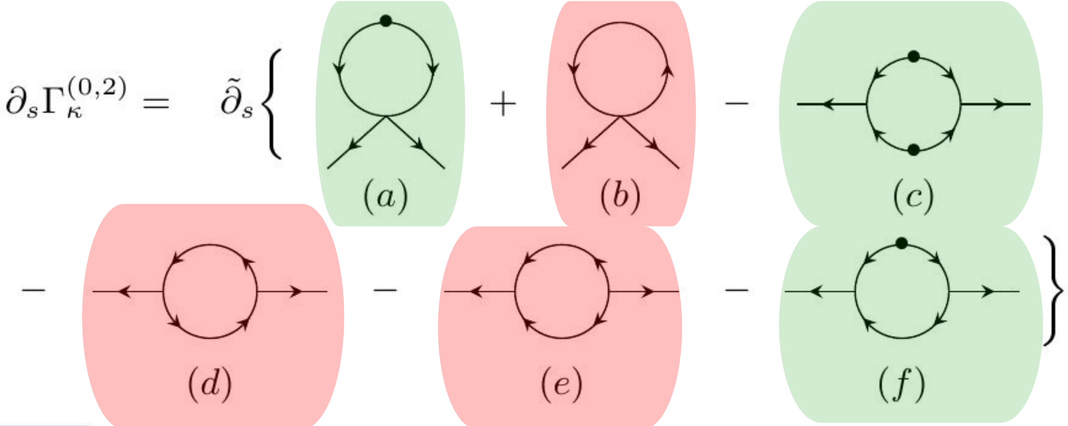

Closure of the flow equations in the large-momentum limit!

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Large-momentum approximation: results

- Symmetries of the model
- Large-momentum expansion

→ The RG equation is simplified:

$$\partial_\kappa C_\kappa(t,\mathbf{p}) = \frac{1}{d}\, p^2 C_\kappa(t,\mathbf{p}) \int_\omega \frac{\cos(\omega t)-1}{\omega^2}\, \tilde{\partial}_s \int_{\mathbf{q}} C_\kappa(\omega,\mathbf{q})$$

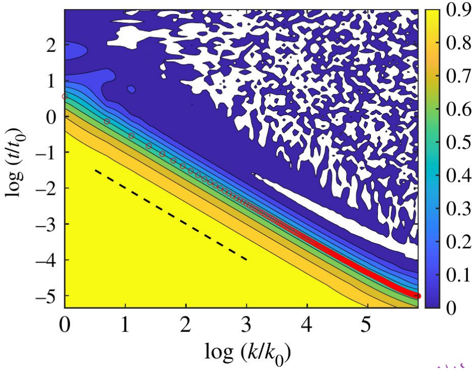
where $C(t-t',\mathbf{x}-\mathbf{x}') \equiv \left\langle \mathbf{v}(t,\mathbf{x})\mathbf{v}(t',\mathbf{x}')\right\rangle_{||}$

The solution at the fixed point:

$$C(t,\mathbf{p}) = C(0,\mathbf{p}) \times \begin{cases} \exp\left(-\mu_0 (pt)^2\right), & t \ll \tau_c \\ \exp\left(-\mu_\infty p^2 |t|\right), & t \gg \tau_c \end{cases}$$

$$pt^{1/z} \equiv pt \longrightarrow z = 1$$ ✔

exact result



Conclusion

1) Confirmed the existence of the **“Inviscid” UV fixed point in dD** ✓

   by integrating the RG equation numerically

1) Found ***z*=1 scaling** at this fixed point ✓

   by solving the RG equation at the fixed point analytically

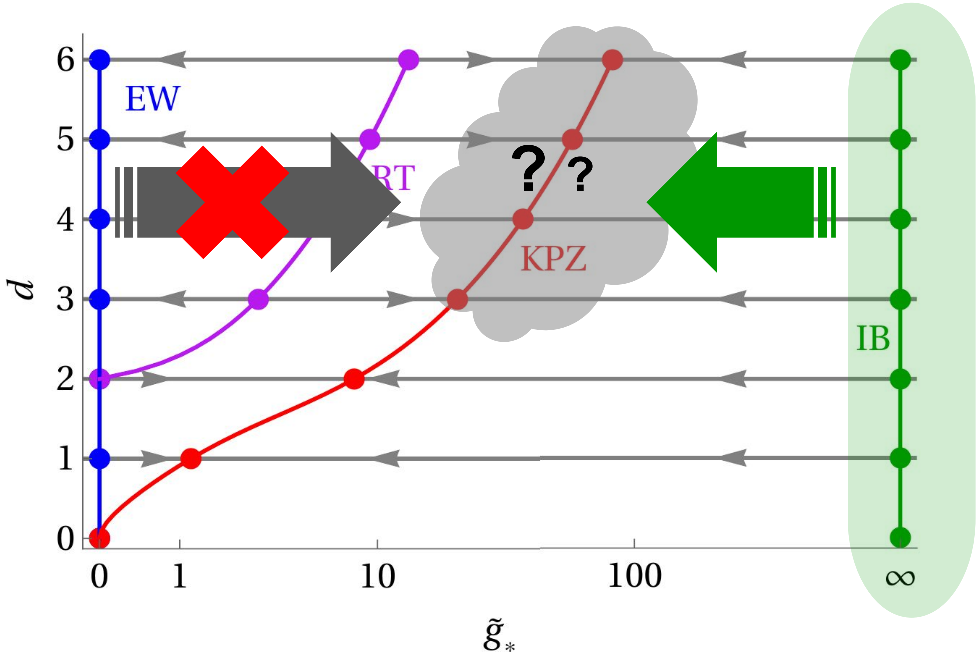
   using only
   - symmetries of the model
   - large-momentum expansion

   This result does not depend on
   - ○ scale of forcing
   - ○ dimension

   → *superuniversal scaling!*

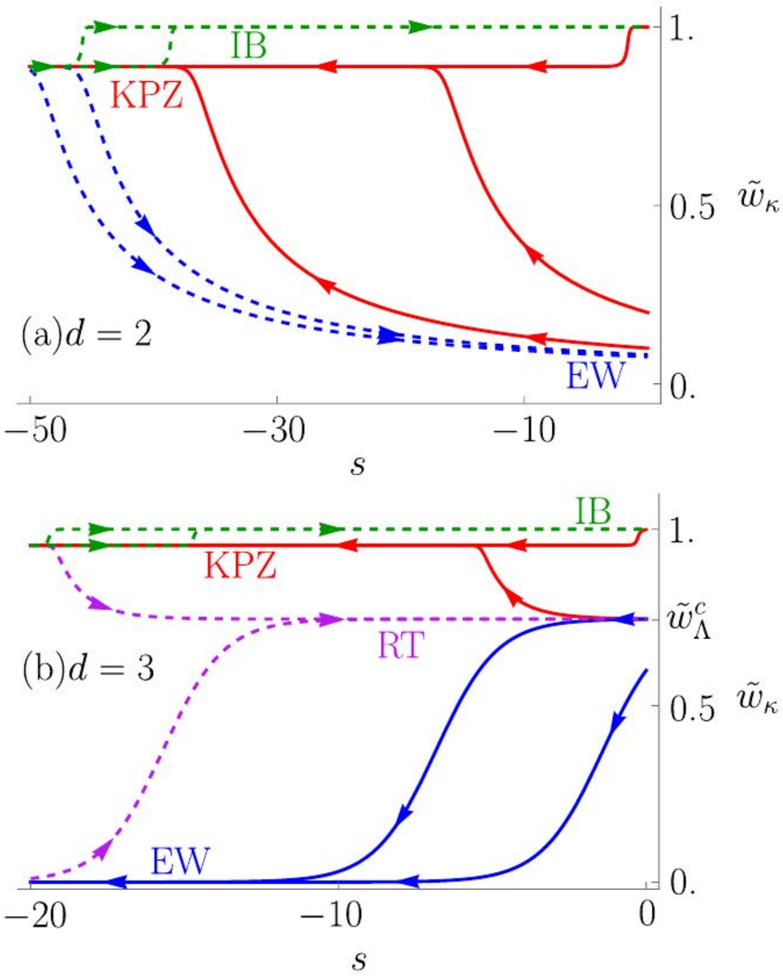
L. Gosteva, M. Tarpin, N. Wschebor, L. Canet (arxiv.org/abs/2406.14030)

Outlook

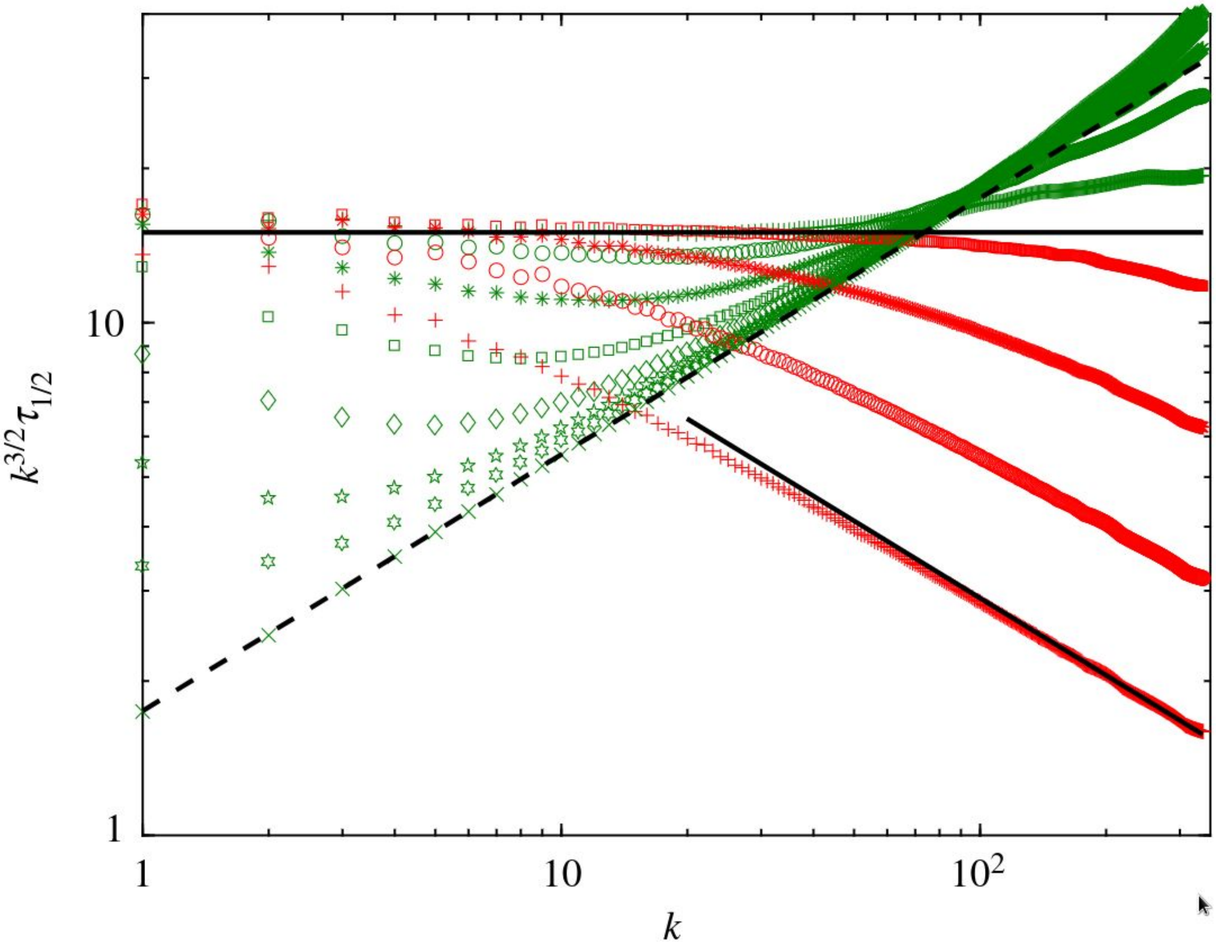


Large-momentum approximation: exact z=1

Thank you for your attention!



$$\tilde{w}_\kappa = \tilde{g}_\kappa/(1 + \tilde{g}_\kappa)$$



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Motivation: the new scaling regime found in numerics (1D)

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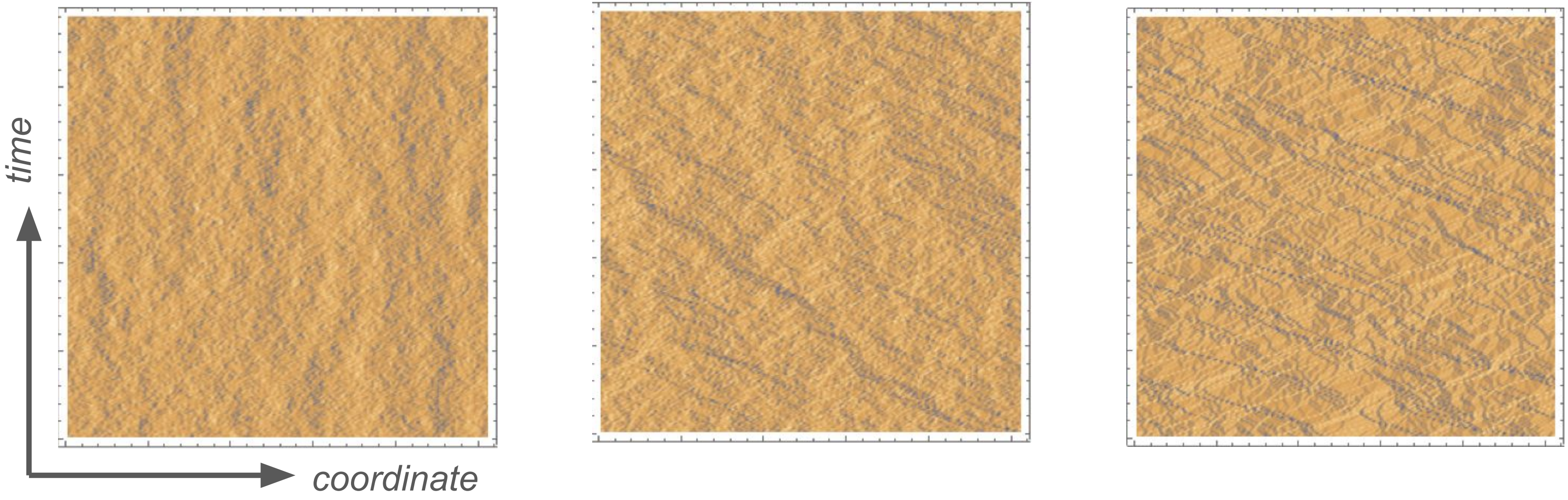
Velocity profiles in 1D



EW: z=2

KPZ: z=3/2

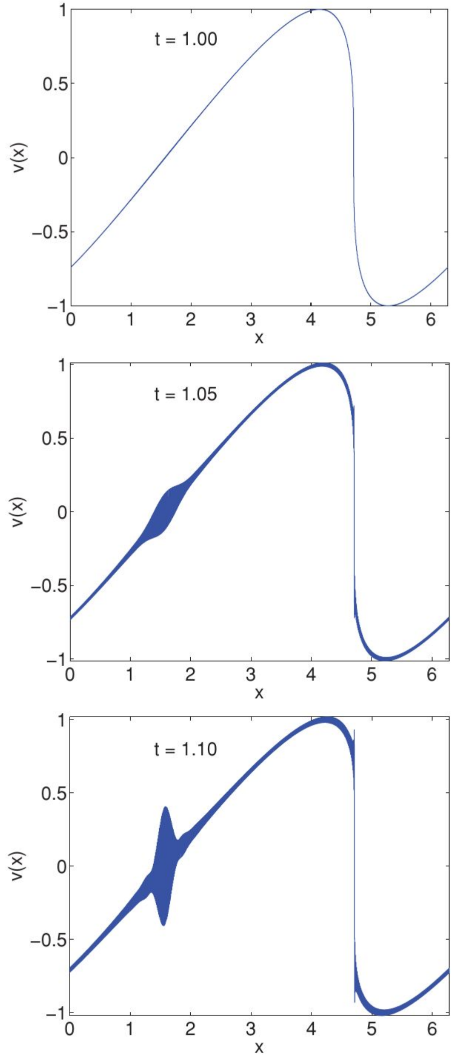
Inviscid Burgers: z=1

$$g \equiv \frac{\mathcal{D}\lambda^2}{\nu^3}$$

$g = 0$ SMALLER VISCOSITY $g \to \infty$

Pre-Motivation: “tygers” in inviscid hydrodynamical equations

Ray, Frisch, Nazarenko, Matsumoto, Resonance phenomenon for the Galerkin-truncated Burgers and Euler equations (2011) : 1D Burgers, 2D Euler



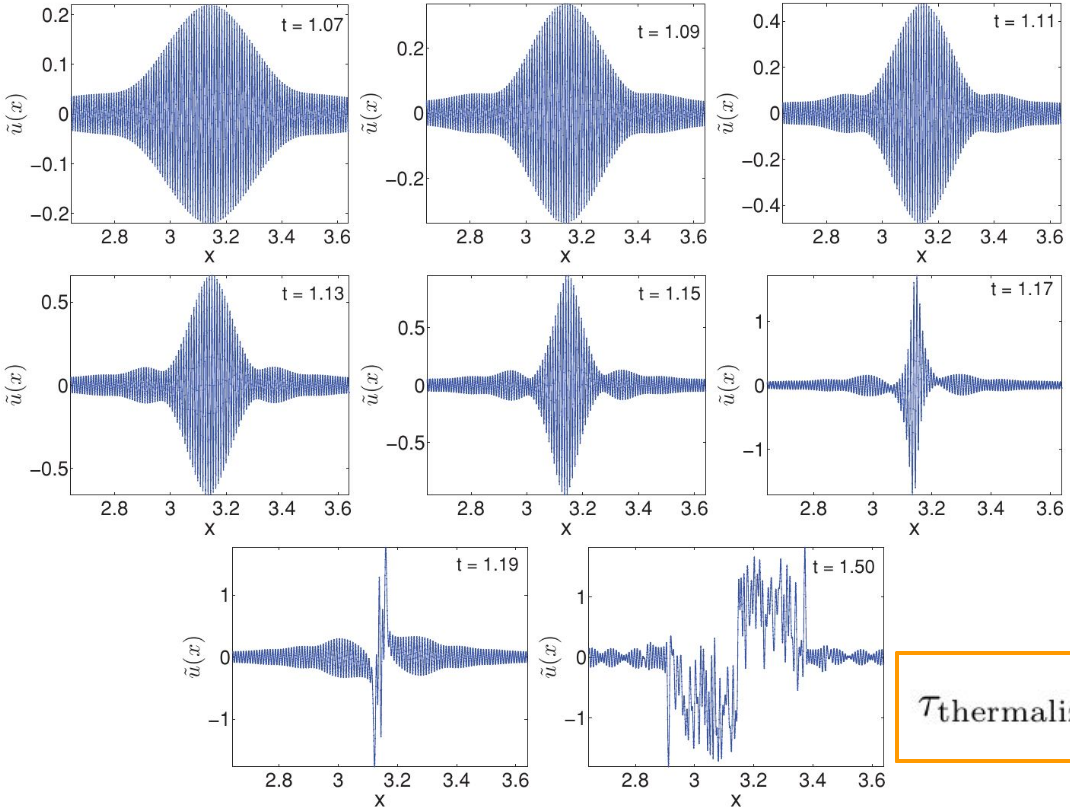


FIG. 5. (Color online) Evolution of the tyger (discrepancy) for same conditions as in Fig. 1: growth, thinning, asymmetrization, collapse, and chaotization.

$$\tau_{\text{thermalization}} \sim k^{-1}$$