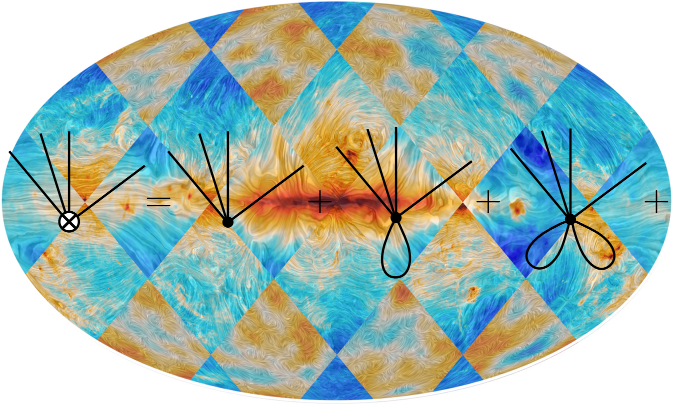
# Stochastic dynamics from QFT

Spyros Sypsas

Chulalongkorn U., Thailand

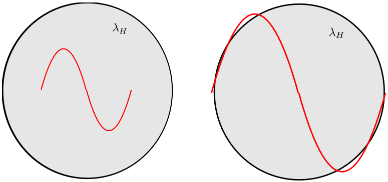


# The problem

A random process in an open system: a quantum field in the Poincare patch of (classical) de Sitter space:

$$\varphi, V(\varphi) \;\Big|\; ds^2 = a^2(t)(-dt^2 + dx^2),\; a(t) = e^{Ht}.$$

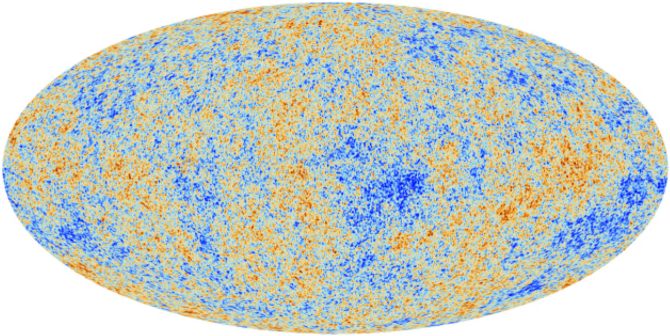
A Fourier mode of wavelength $\lambda$ will stretch $\lambda \to a(t)\lambda$.




- ⋆ Q: What is the statistics (PDF) of long modes?
- ⋆ Methods: QFT (Feynman graphs) = Stochastic (Langevin eq)

# A physical system

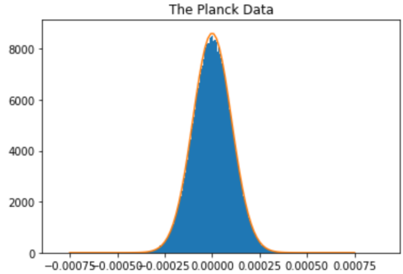
The night sky through a strong telescope:

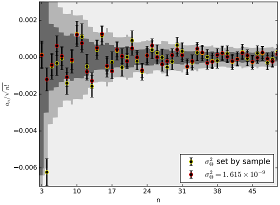


# Planck Legacy Archive

Launcher | The Planck PD | Code | Python 3

```
In [9]: pyfits.open('COM_CMB_IQU-smica_1024_R2.02_full.fits').info()

Filename: COM_CMB_IQU-smica_1024_R2.02_full.fits
No.    Name         Type      Cards   Dimensions   Format
0    PRIMARY     PrimaryHDU      16   ()
1    COMP-MAP    BinTableHDU     61   12582912R x 5C   [E, E, E, B, B]
2    BEAMTF      BinTableHDU     45   4001R x 2C   [E, E]

In [183]: CMBdata2.info()

Filename: COM_CMB_IQU-smica-field-Int_2048_R2.01_full.fits
No.    Name      Ver    Type      Cards   Dimensions   Format
  0  PRIMARY       1 PrimaryHDU      16   ()
  1  COMP-MAP      1 BinTableHDU     52   50331648R x 2C   ['E', 'B']
  2  BEAMTF        1 BinTableHDU     41   4001R x 1C   [E]

In [10]: CMBdata[1].data

Out[10]: FITS_rec([(-9.2010232e-05,  6.4709297e-08, -6.5706149e-07, 0, 0),
          (-8.0415215e-05, -9.1876444e-09, -6.9431684e-07, 0, 0),
          (-8.9856963e-05,  7.3611503e-08, -6.8458655e-07, 0, 0), ...,
          (-3.4241329e-04, -1.4987631e-06, -6.5261207e-07, 0, 0),
          (-3.3521044e-04, -1.5905293e-06, -6.4830010e-07, 0, 0),
          (-3.8269864e-04, -1.5804380e-06, -6.8047660e-07, 0, 0)],
         dtype=(numpy.record, [('I_STOKES', '>f4'), ('Q_STOKES', '>f4'), ('U_STOKES', '>f4'), ('TMASK', 'u1'),
('PMASK', 'u1')]))
```




$$\mathrm{P}(\delta T) \sim e^{S[\delta\phi]}$$

Quadratic action (linear dynamics) $\Rightarrow$ Gaussian statistics

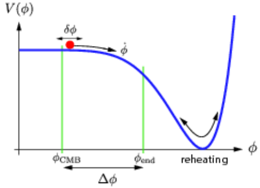
Interactions: non-Gaussian contributions

# A simple realization: cosmic inflation

A bunch of problems with cosmological observations (including the previous picture) point to an early phase of de Sitter space:

$$ds^2 = a^2(t)(-dt^2 + dx^2)$$

$$G_{\mu\nu}(g) = T_{\mu\nu}(\phi)$$

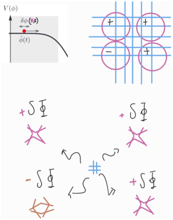



$$\delta\phi \sim \delta T$$

$\delta\phi$ causes tiny $\mathcal{O}(10^{-5})$ temperature anisotropies in the CMB!

# Perturbations

$$\phi(t,x) = \phi_0(t) + \delta\phi(t,x)$$



Studying field theory in dS space is physically relevant

# $\mathcal{V}(\varphi)$ on dS

$$S = \int \mathrm{d}^3x\,\mathrm{d}t\, a^3(t) \left[\frac{1}{2}\dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{2a^2(t)} - \mathcal{V}(\varphi)\right]$$

We will be interested in single-point statistics (i.e. histograms marginalised over $\boldsymbol{x}$)

$$\langle \varphi^n \rangle = \langle \varphi(\boldsymbol{x}_1)\cdots\varphi(\boldsymbol{x}_n)\rangle\Big|_{\boldsymbol{x}_1=\cdots=\boldsymbol{x}_n}$$

$$\rho(\varphi) = \int \mathrm{d}J\, z(J) e^{-iJ\varphi}$$

Using $z = e^w$, and $w(J) = -\frac{1}{2}\sigma^2 J^2 + \sum_{n=1}^{\infty} \frac{i^n}{n!}\langle\varphi^n\rangle_c J^n$, we obtain

$$\rho(\varphi) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2}}}{\sqrt{2\pi}\sigma} \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{\langle\varphi^{n_1}\rangle_c}{n_1!\sigma^{n_1}} \cdots \frac{\langle\varphi^{n_N}\rangle_c}{n_N!\sigma^{n_N}} \mathrm{He}_{n_1+\cdots+n_N}(\varphi/\sigma)$$

Let us focus on linear order:

$$\rho(\varphi, t) = \frac{e^{-\frac{1}{2}\frac{\varphi^2}{\sigma^2(t)}}}{\sqrt{2\pi\sigma^2(t)}}\left[1 + \Delta(\varphi, t)\right]$$

with

$$\Delta(\varphi, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\left\langle \varphi^n(t) \right\rangle_c}{\sigma^n} \mathrm{He}_n(\varphi/\sigma)$$

Perturbative QFT = Gaussian variables=

Feynman rules = Wick's theorem

So we can compute these $n$-pt cumulants using any PT scheme that we like: SK, *in*-*in*, wavefunction of the universe:

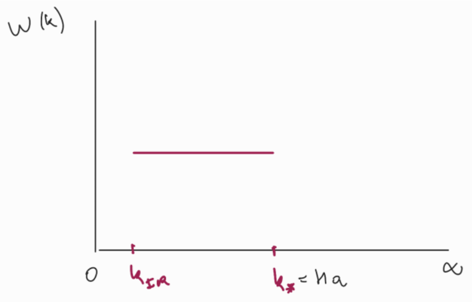
$$\left\langle \varphi^n(t) \right\rangle_c \propto \mathcal{V} + \mathcal{O}(\mathcal{V}^2)$$

We want the statistics of long modes; $\varphi = \varphi_S + \varphi_L$:

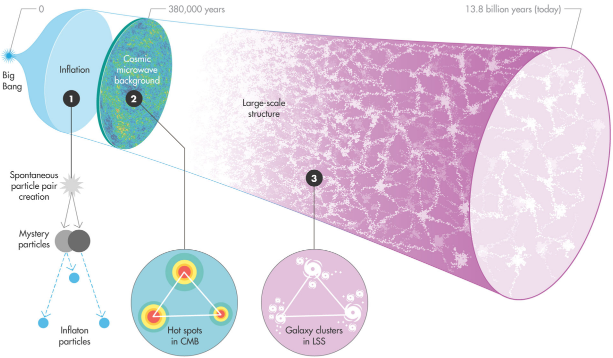
$$\varphi_L(x,t) = \int_k e^{ikx} W(k) \tilde{\varphi}_k(t)$$
$$\varphi_S(x,t) = \int_k e^{ikx} \Big[1 - W(k)\Big] \tilde{\varphi}_k(t)$$

with

$$W(k) = \theta(k_*(t) - k) \times \theta(k - k_{\rm IR}) \quad | \quad \sigma^2(t) = \left(\frac{H}{2\pi}\right)^2 \int \frac{\mathrm{d}k}{k} W(k,t)$$



CMB temperature fluctuations reflect density fluctuations and large-scale structure:




Non-Gaussianity/signatures of nontrivial physics in LSS

Let us compute the moments and plug them in:

$$\left\langle \varphi^n(\tau) \right\rangle_c = -\frac{8\pi}{H^4}\,\mathrm{Im}\left\{\int_0^\infty \mathrm{d}x \int_{\tau_\mathrm{i}}^{\tau} \mathrm{d}\bar{\tau}\,\frac{x^2}{\bar{\tau}^4}\lambda_n \left[g(x,\bar{\tau},\tau)\right]^n\right\}$$

with $g$ the propagator. Upon using

$$\mathcal{O}_\varphi \mathrm{He}_n(\varphi/\sigma) = -n\mathrm{He}_n(\varphi/\sigma)$$

the PDF can then be resummed:

$$\Delta(\varphi,\tau) = \frac{8\pi}{H^4}\mathrm{Im}\int_0^\infty \mathrm{d}x \int_{\tau_\mathrm{i}}^{\tau}\mathrm{d}\bar{\tau}\,\frac{x^2}{\bar{\tau}^4}\left(\frac{g(x,\bar{\tau},\tau)}{\sigma^2}\right)^{-\mathcal{O}_\varphi} e^{-\frac{\sigma^2}{2}\frac{\partial^2}{\partial\varphi^2}}\mathcal{V}(\varphi)$$

such that $\rho = \rho_0\left[1+\Delta\right]$

This is the PDF of long modes

But let us look at a simpler version to make contact with the title!

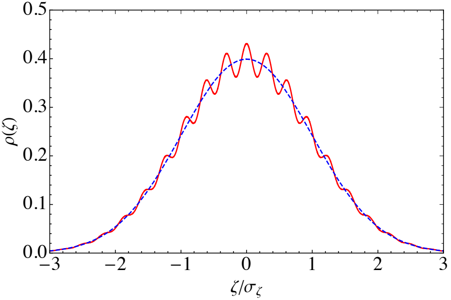
$$\Delta(\varphi, t) = \frac{\Delta N}{3H^2}\left(\mathcal{V}''_{\rm ren}(\varphi) - \frac{\varphi}{\sigma^2}\mathcal{V}'_{\rm ren}(\varphi)\right)$$

and we then take a time derivative of the PDF:

$$\dot{\rho} = \frac{H^3}{8\pi^2}\frac{\partial^2}{\partial\varphi^2}\left[\rho\left(1 - \frac{2\Delta N}{3H^2}\mathcal{V}''_{\rm ren}\right)\right] + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(\rho\mathcal{V}'_{\rm ren}\right)$$

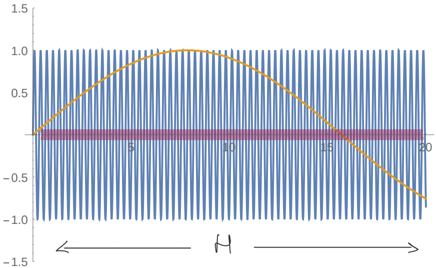
Fokker-Planck equation

An axionic example of this PDF (for $\mathcal{V} = 1 - \cos$)



# The stochastic approach: Starobinski-Yokoyama 90's

Long modes: nonlin but classical
Short modes: lin but QM




How to model the effect of short modes at large scales?

$$\hat{\varphi}_k = \varphi_{\rm lin}\hat{a}_k + \varphi^*_{\rm lin}\hat{a}^\dagger_k \rightarrow \hat{\xi}_k = \varphi_{\rm lin}(\hat{a}_k - \hat{a}^\dagger_k)$$

# The renormalized potential

Separate the IR modes

$$\varphi = \varphi_S + \varphi_L + \varphi_{\rm IR}$$

and integrate them out:

$$\mathcal{V}_L(\bar{\varphi}) = \langle \Psi_{\rm IR} | \mathcal{V}(\bar{\varphi} + \varphi_{\rm IR}) | \Psi_{\rm IR} \rangle$$

with

$$|\Psi_{\rm IR}\rangle = \int \mathcal{D}\varphi_{\rm IR} \Psi(\varphi_{\rm IR}) |\varphi_{\rm IR}\rangle$$

with $|\varphi_{\rm IR}\rangle$ an IR field-eigenstate and $|\Psi(\varphi_{\rm IR})|^2$ the Gaussian. This leads directly to

$$\boxed{\mathcal{V}_L(\bar{\varphi}) = \mathcal{V}_{\rm ren}(\bar{\varphi})}$$

The EOM for long modes now becomes a stochastic Langevin equation: for $\Delta t \gg 1/H$,

$$\dot{\varphi}_L + \frac{1}{3H}\mathcal{V}'_{\rm ren}(\varphi_L) = H\hat{\xi}(t)$$

where $\hat{\xi}(t)$ is a Gaussian stochastic noise representing the short-mode bath:

$$\hat{\xi} \equiv H^{-1}\int_k \left(\frac{\mathrm{d}}{\mathrm{d}t}W(k)\right)\,\tilde{\varphi}_k$$

The linear solution for $\varphi_S$ is

$$\varphi_S(k,\tau) \simeq \varphi_0(k,\tau)\left(1 - \frac{1}{3}\Delta N H^{-2}\mathcal{V}''_{\rm ren}\right)$$

From this one gets the previous FP equation

## To sum-up

- ⋆ Statistics of long scalar modes on dS
- ⋆ resummation of Feynman diagrams = stochastic formalism
- ⋆ Important details :

  $\mathcal{V} \to \mathcal{V}_{\rm ren}$

  $\Delta N$ cumulative diffusion

# Thank you for your attention!