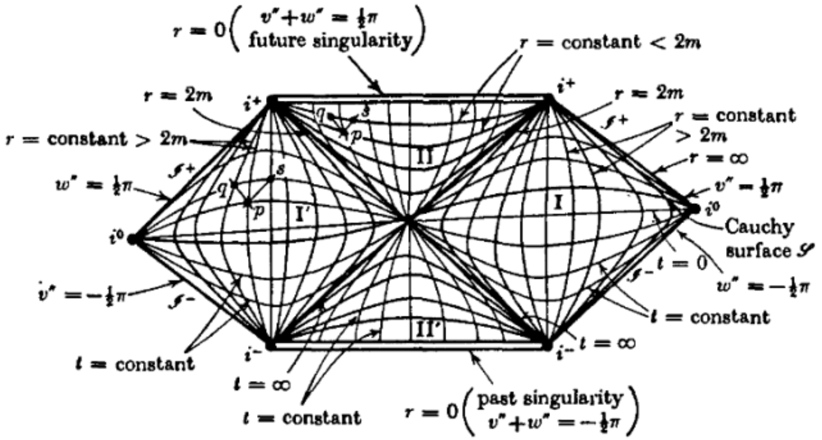
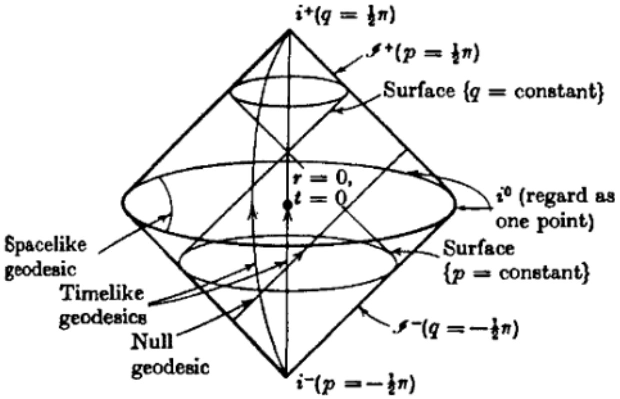
Xmas Theoretical Physics Workshop
University of Athens
22/12/2023

Lessons from DLCQ for gravity





Hawking & Ellis

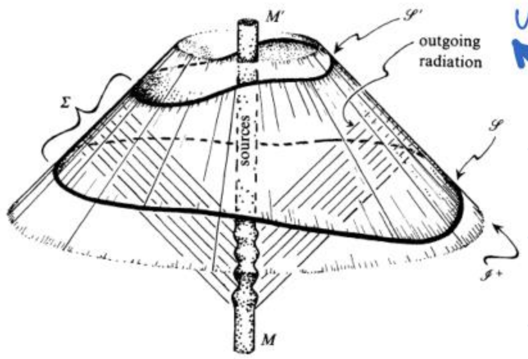
Glenn Barnich

Physique théorique et mathématique

Université libre de Bruxelles &
International Solvay Institutes

Gravity with boundary conditions

Particular case: asymptotically flat spacetimes at null infinity



u

Holographic screen to study GW

Penrose & Rindler Vol II

Characteristic initial value problem

Solution space, free data at $\mathscr{I}^+$: $\boxed{\overset{\circ}{\psi}_2 + \overline{\overset{\circ}{\psi}_2}\,, \quad \overset{\circ}{\psi}_1\,, \quad \overset{\circ}{\sigma}}$ undetermined u- dependence

$\dot{\overset{\circ}{\sigma}}$ news

additional data to construct solutions

$$\psi_0 = \sum_{n \geq 0} \psi_0^n(\xi, \bar{\xi}, u_0)\, r^{-5-n}$$

BMS transformations

Transformation of relevant free data at $\mathscr{I}^+$ $\qquad s = (\mathcal{Y}, \bar{\mathcal{Y}}, T)\,, \qquad f = T + \frac{1}{2}u(\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}})$

$$\delta_s \sigma^0 = \left[f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + \frac{3}{2}\eth\mathcal{Y} - \frac{1}{2}\bar{\eth}\bar{\mathcal{Y}} \right]\sigma^0 - \eth^2 f$$

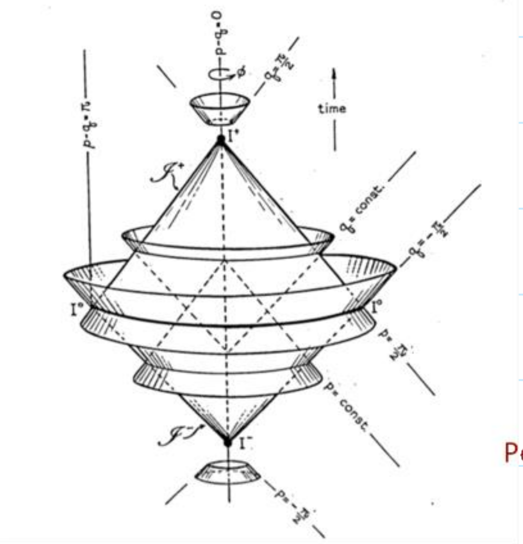
$$\delta_s \dot{\sigma}^0 = \left[f\partial_u + \mathcal{Y}\eth + \bar{\mathcal{Y}}\bar{\eth} + 2\eth\mathcal{Y} \right]\dot{\sigma}^0 - \frac{1}{2}\eth^2(\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}})$$

$\left\{ \begin{array}{l} \text{EM tensor} \\ \text{Schwarzian derivative} \end{array} \right.$

$$\delta_s \psi_2^0 = \left[\;''\quad ''\quad ''\; + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar{\eth}\bar{\mathcal{Y}} \right]\psi_2^0 + 2\eth f\, \psi_3^0$$

$$\delta_s \psi_1^0 = \left[\;''\quad ''\quad ''\; + 2\eth\mathcal{Y} + \bar{\eth}\bar{\mathcal{Y}} \right]\psi_1^0 + 3\eth f\, \psi_2^0$$

Important matching conditions
at spatial infinity



Penrose, Les Houches 1963

BMS charge algebra

$$J^u_s = -\frac{1}{8\pi G}\Big\{ \big[\underbrace{\psi^0_2 + \bar\psi^0_2}_{BH} + \underbrace{\sigma^0 \dot{\bar\sigma}^0 + \bar\sigma^0 \dot\sigma^0}_{\text{gravitons}}\big] f + \big[\psi^0_1 + \sigma^0 \eth \bar\sigma^0 + \tfrac{1}{2}\eth(\sigma^0\bar\sigma^0)\big] \mathcal{Y} + \big[\bar\psi^0_1 + \bar\sigma^0 \bar\eth \sigma^0 + \tfrac{1}{2}\bar\eth(\sigma^0\bar\sigma^0)\big]\bar{\mathcal{Y}}\Big\}$$

$$\Theta^u_s(\delta X) = \frac{1}{8\pi G}\big[\dot{\bar\sigma}^0 \delta\sigma^0 + \dot\sigma^0 \delta\bar\sigma^0\big] f$$

charges $\quad Q_s = \int_{S^2,\, u = cte} \frac{i}{R^2}\frac{d\zeta\, d\bar\zeta}{P_s \bar P_s} J^u_s \qquad \Theta_s[\delta X] = \int_{S^2,\, u = cte} \frac{i}{R^2}\frac{d\zeta\, d\bar\zeta}{P_s \bar P_s} \Theta^u_s[\delta X]$

algebra $\quad \delta_{s_1} Q_{s_2} + \Theta_{s_2}[\delta_{s_1} X] = -Q_{[s_1, s_2]}$

Charge algebra with non-standard "modified" Poisson brackets

non-conservation of BMS_4 charges

$$\frac{d}{du} Q_s = -\underbrace{\int_{S^2, u = cte} \frac{i}{R^2\, 8\pi G}\frac{d\zeta\, d\bar\zeta}{P_s \bar P_s}\big[\dot{\bar\sigma}^0 \delta_s \sigma^0 + \dot\sigma^0 \delta_s \bar\sigma^0\big]}_{\text{generalizes mass loss}}$$

fluxes

Motivation & Contents

1) Quantization on null surfaces - front form of dynamics

2) Simplest example : (Massless) boson in 1+1 dimension

3) Well-known results in instant form

4) Dirac algorithm & characteristic initial value problem

5) Puzzles in single front formulation

6) Double front formulation, Peierls bracket & matching conditions

with Majumdar, Speziale, Tan in progress

Light-cone Lagrangian analysis

Simplest system: massless boson $\quad S = \frac{1}{2}\int dx^0 dx^1 \, \partial_\mu \phi \, \partial^\mu \phi \qquad ds^2 = (dx^0)^2 - (dx^1)^2$

Light-cone coordinates: $\quad x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}} \qquad S = \int dx^+ dx^- \, \partial_+ \phi \, \partial_- \phi \; , \qquad \partial_+ \partial_- \phi = 0$

Solution: $\quad \phi = \phi_+^S(x^+) + \phi_-^S(x^-) \qquad$ initial conditions $\quad \phi_+^S(x^+) = \phi(x^+, c^-) \qquad \phi_-^S(x^-) = \phi(c^+, x^-)$

matching condition $\quad \phi_+^S(c^+) = \phi_-^S(c^-) \quad$ at $\begin{cases} \text{intersection} \\ \text{corners} \end{cases}$

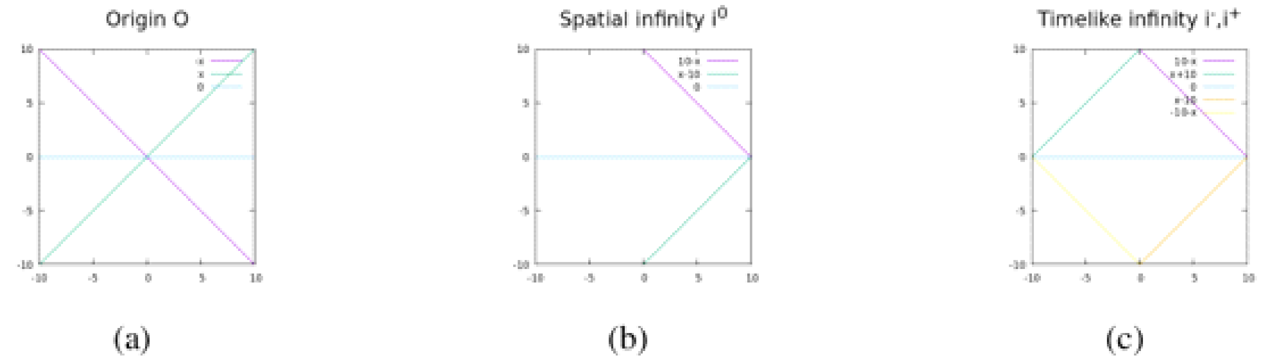


Figure 2: Intersecting initial value null lines

Lesson 0:

$\begin{cases} u, r \\ v, r \end{cases}$ at $\begin{matrix} \mathcal{J}^+ \\ \mathcal{J}^- \end{matrix} \longleftrightarrow u, v$

Kehrberger

Symmetries & currents

$$\frac{\delta \mathcal{L}}{\delta \phi}\delta\phi + \partial_\mu j^\mu = 0$$

- constant shift $\quad \delta\phi = c \qquad j^\mu = \partial^\mu \phi$

- conformal trsf

$$\begin{cases} \delta\phi = \xi^\rho \partial_\rho \phi \qquad j^\mu = T^\mu{}_\rho \xi^\rho \\ T_{\mu\rho} = \partial_\mu\phi\partial_\rho\phi - \frac{1}{2}\eta_{\mu\rho}\partial_\nu\phi\partial^\nu\phi \\ \partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \partial_\rho \xi^\rho \eta_{\mu\nu} \end{cases}$$

light-cone coordinates

$$\begin{cases} \partial_\pm \xi^\mp = 0 \\ T_{\pm\pm} = (\partial_\pm\phi)^2 , \quad T_{\pm\mp} = 0 \qquad j^\pm = \xi^\mp T_{\mp\mp} \end{cases}$$

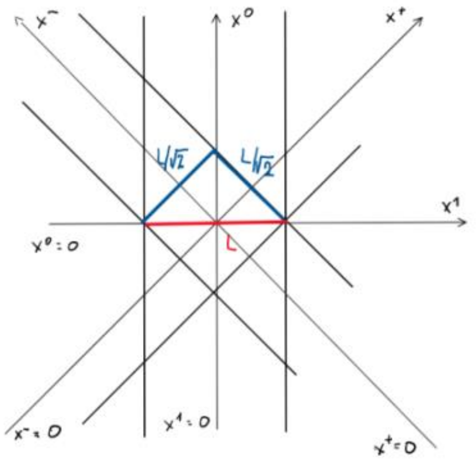
- chiral shifts $\quad \delta^\pm \phi = \epsilon^\pm(x^\pm)$

$$\begin{cases} j^+_{\epsilon^+} = \partial_-\phi\,\epsilon^+ , \quad j^-_{\epsilon^+} = \partial_+\phi\,\epsilon^+ - \phi\,\partial_+\epsilon^+ \\ j^+_{\epsilon^-} = \partial_-\phi\,\epsilon^- - \phi\,\partial_-\epsilon^- , \quad j^-_{\epsilon^-} = \partial_+\phi\,\epsilon^+ \end{cases}$$

∞ # of global symmetries

Topology & boundary conditions

standard spatial cylinder $\quad x^1 \sim x^1 + L \,, \quad \forall\, x^0$



$\Leftrightarrow$ entangled periodicities $\quad (x^+, x^-) \sim (x^+ + L_+ \,, x^- \sim x^- - L_-) \,, \quad L_\pm = \dfrac{L}{\sqrt{2}}$

general solution

$$\phi = \underbrace{\bar{\phi}(0) + \bar{\Pi}^0_0(0)\left(\frac{x^+ + x^-}{\sqrt{2}\,L}\right)}_{=\ \frac{x^0}{L}} + \phi_R(x^-) + \phi_L(x^+)$$

particle zero mode

no zero modes

NB: solution not periodic in x^-, $\quad x^- \sim x^- + L_- \,, \quad \forall\, x^+$

Discrete light-cone quantization (DLCQ):

- time $= x^+$ (front form) & impose $x^- \sim x^- + L_-$, $\forall x^+$

$\Leftrightarrow$ work on null cylinder

$\Rightarrow$ misses particle zero mode, sector $\pi_0^0 |0\rangle = 0$.

Progress of Theoretical Physics, Vol. 56, No. 1, July 1976

The Problem of $P^+=0$ Mode in the Null-Plane Field Theory and Dirac's Method of Quantization

Toshihide MASKAWA and Koichi YAMAWAKI*

Department of Physics, Kyoto University, Kyoto 606[†)]
**Research Institute for Fundamental Physics, Kyoto University, Kyoto 606*

(Received January 7, 1976)

The null-plane quantization is studied with the emphasis on the $P^+=0$ mode, by using Dirac's quantization for constrained systems. This mode is eliminated from the Hilbert space and the physical vacuum can be defined in a kinematical way. It enables us to construct the physical Fock space kinematically. Poincaré invariance is also studied in detail.

N·H ELSEVIER

PHYSICS REPORTS

Physics Reports 301 (1998) 299–486

Quantum chromodynamics and other field theories on the light cone

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Received October 1997; editor: R. Petronzio

take inspiration from instant form : well adapted to spatial cylinder , integrate by parts in space.

$$S_H = \int dx^0 dx^1 \, \mathcal{L}^0_H \,, \quad \mathcal{L}^0_H = \pi^0 \partial_0 \phi - \mathcal{H} \qquad \mathcal{H} = \frac{1}{2}\left((\pi^0)^2 + (\partial_1 \phi)^2\right)$$

[formulation with auxiliary field π^1 : $S'_H = \int dx^0 dx^1 \, \mathcal{L}'_H \,, \quad \mathcal{L}'_H = \pi^\mu \partial_\mu \phi - \frac{1}{2}\pi^\mu \pi_\mu \qquad \partial_\rho \phi - \pi_\rho = 0 \qquad \partial_\mu \pi^\mu = 0$

$\pi^1 \approx -\partial_1 \phi$]

periodic boundary conditions $x^1 \sim x^1 + L$ $\qquad \{\phi(x^1), \pi^0(y^1)\}_0 = \delta^P(x^1, y^1) = \frac{1}{L}\sum_{n \in \mathbb{Z}} e^{ik(x^1 - y^1)} \qquad k = \frac{2\pi}{L} n$

decomposition $\quad \phi(x^0, x^1) = \bar{\phi}_0(x^0) + \tilde{\phi}_0(x^0, x^1) \,, \quad \pi^0(x^0, x^1) = \frac{\bar{\pi}^0_0}{L}(x^0) + \tilde{\pi}^0_0(x^0, x^1)$

zero mode : free particle $\quad S_P = \int dx^0 \left[\bar{\pi}^0_0 \partial_0 \bar{\phi}_0 - \frac{1}{2L}(\bar{\pi}^0_0)^2\right] \qquad \{\bar{\phi}_0, \bar{\pi}^0_0\}_0 = 1$

$\{\tilde{\phi}_0(x^1), \tilde{\pi}^0_0(y^1)\} = \delta^P(x^1, y^1) - \frac{1}{L}$ $\quad (*)$

split off-shell $\tilde{\phi}_0, \tilde{\pi}^0_0$ into left & right chiral fields without zero modes HT, Chiral forms

$$\begin{cases} \phi^L = \frac{1}{2}\left[\tilde{\Phi}_0 + \int_0^{x^1} dy^1\, \tilde{\pi}^0_0 - \frac{1}{L}\int_0^L dy^1 \int_0^{y^1} dz^1\, \tilde{\pi}^0_0\right] \\ \phi^R = \frac{1}{2}\left[\tilde{\Phi}_0 - \int_0^{x^1} dy^1\, \tilde{\pi}^0_0 + \frac{1}{L}\int_0^L dy^1 \int_0^{y_1} dz^1\, \tilde{\pi}^0_0\right] \end{cases} \Rightarrow \begin{cases} \partial_1 \phi^L = \frac{1}{2}\left(\partial_1 \tilde{\Phi}_0 + \tilde{\pi}^0_0\right) \\ \partial_1 \phi^R = \frac{1}{2}\left(\partial_1 \tilde{\Phi}_0 - \tilde{\pi}^0_0\right) \end{cases}$$

$$S_H = S^P + \underbrace{\int_{x^0_i}^{x^0_f} dx^0 \int_{-L/2}^{L/2} dx^1 \left[\partial_1\phi^L \partial_0 \phi^L - (\partial_1 \phi^L)^2\right]}_{\text{left chiral bosons } S^L} + \underbrace{\int_{x^0_i}^{x^0_f} dx^0 \int_{-L/2}^{L/2}\left[-\partial_1\phi^R\, \partial_0\phi^R - (\partial_1 \phi^R)^2\right]}_{\text{right chiral bosons } S^R} + \underbrace{\frac{1}{2}\left[\int_0^L dx^1 \left[\partial_1\phi^L\phi^R - \phi^L\partial_1\phi^R\right]\right]_{x^0_i}^{x^0_f}}_{\text{boundary term } S^B}$$

$$\partial_1^x\, \omega^P(x^1,y^1) = -\partial_1^y\, \omega^P(x^1,y^1) = \delta^P(x^1, y^1) - \frac{1}{L} \qquad \omega^P(x^1,y^1) = \sum_{n>0} \frac{1}{\pi n} \sin k(x^1-y^1) = \frac{1}{2}\epsilon(x^1-y^1) - \frac{x^1-y^1}{L}$$

$$\epsilon(x) = \sum_{n>0} \frac{2(1-\cos \pi n)}{\pi n} \sin kx \qquad \frac{x}{L} = -\sum_{n>0} \frac{\cos \pi n}{\pi n} \sin kx$$

(x) after change of variables $\begin{cases} \{\phi^R(x^1), \phi^R(y^1)\}_0 = \frac{1}{4}\epsilon(x^1-y^1) - \frac{x^1-y^1}{2L} = -\{\phi^L(x^1), \phi^L(y^1)\}_0 \\ \{\phi^L(x^1), \phi^R(y^1)\}_0 = 0 \end{cases}$

• Conformal transformations $\quad \partial_0 \xi^0 = \partial_1 \xi^1 , \quad \partial_1 \xi^0 = \partial_0 \xi^1$

$$\delta_\xi^H \phi = \xi^0 \pi^0 + \xi^1 \partial_1 \phi , \quad \delta_\xi^H \pi^0 = \partial_1 \left(\xi^0 \partial_1 \phi + \xi^1 \pi^0 \right) \qquad \frac{\delta \mathcal{L}_H}{\delta \phi} \delta_\xi^H \phi + \frac{\delta \mathcal{L}_H}{\delta \pi^0} \delta_\xi^H \pi^0 + \partial_\mu j_\xi^\mu = 0$$

$$j_\xi^0 = \xi^0 \mathcal{H} + \xi^1 \mathcal{P} \qquad j_\xi^1 = -\xi^0 \mathcal{P} - \xi^1 \mathcal{H} - \left(\xi^0 \partial_1 \phi + \xi^1 \pi^0 \right)\left(\partial_0 \phi - \pi^0 \right) \qquad \mathcal{P} = \pi^0 \partial_1 \phi$$

$$Q_\xi = \int_{-L/2}^{L/2} dx^1 \, j_\xi^0 \qquad \{ Q_{\xi_1} , Q_{\xi_2} \} = Q_{[\xi_1,\xi_2]}$$

canonical realization of conformal algebra

Decomposition $\quad \delta_\xi \bar{\phi}_0 = \frac{\bar{\pi}^0}{L^2} \int_0^L dy^1 \xi^0 + \frac{1}{L} \int_{-L/2}^{L/2} dy^1 \left[\xi^0 \tilde{\pi}^0_0 + \xi^1 \partial_1 \tilde{\phi}_0 \right] , \qquad \delta_\xi \bar{\pi}^0_0 = 0$

$$\delta_\xi \tilde{\phi}_0 = \xi^0 \tilde{\pi}^0_0 + \xi^1 \partial_1 \tilde{\phi}_0 - \frac{1}{L} \int_{-L/2}^{L/2} dy^1 \left(\xi^0 \tilde{\pi}^0_0 + \xi^1 \partial_1 \tilde{\phi}_0 \right) + \left(\xi^0 - \frac{1}{L} \int_{-L/2}^{L/2} dx^1 \xi^0 \right) \frac{\bar{\pi}^0_0}{L}$$

$$\delta_\xi \tilde{\pi}^0_0 = \partial_1 \left(\xi^0 \partial_1 \tilde{\phi}_0 + \xi^1 \tilde{\pi}^0_0 \right) + \frac{1}{L} \partial_1 \xi^1 \, \bar{\pi}^0_0$$

canonical generator $Q_\xi = \int_{-L/2}^{L/2} dx^1 \left[\xi^0 \tilde{\mathcal{H}} + \xi^1 \tilde{\mathcal{P}}\right] + \frac{\bar{\Pi}_0^0}{L} \int_{-L/2}^{L/2} dx^1\, \xi^0 \left[\tilde{\Pi}_0^0 + \xi^1 \partial_1 \tilde{\Phi}_0\right] + \frac{(\bar{\Pi}_0^0)^2}{2L} \int_{-L/2}^{L/2} dx^1 \xi^0$

$$= \frac{1}{\sqrt{2}} \int_{-L/2}^{L/2} dx^1 \left[\xi^+ (\partial_1 \phi^L)^2 - \xi^- (\partial_1 \phi^R)^2\right] + \boxed{\frac{\bar{\Pi}_0^0}{\sqrt{2} L} \int_{-L/2}^{L/2} dx^1 \left[\xi^+ \partial_1 \phi^L - \xi^- \partial_1 \phi^R\right] + \frac{(\bar{\Pi}_0^0)^2}{2L} \int_{-L/2}^{L/2} dx^1 \xi^0}$$

no mixing (i) if there is no zero mode sector

but then no modular invariant partition function

(ii) only for $\xi^0 = c^0$, $\xi^1 = c^1$ constants (spacetime translations)

$\int_{-L/2}^{L/2} dx^1 \tilde{\Pi}_0^0 = 0 = \int_{-L/2}^{L/2} dx^1 \partial_1 \tilde{\Phi}_0$

$$H = \int_{-L/2}^{L/2} dx^1 (\partial_1 \phi^L)^2 + \int_{-L/2}^{L/2} dx^1 (\partial_1 \phi^R)^2 + \frac{1}{2L} \left(\bar{\Pi}_0^0\right)^2 , \quad P = -\int_{-L/2}^{L/2} dx^1 (\partial_1 \phi^L)^2 + \int_{-L/2}^{L/2} dx^1 (\partial_1 \phi^R)^2$$

- mode expansion of on-shell fields

$$\phi = \bar{\Phi}_0(0) + \frac{x^0}{L} \bar{\Pi}_0^0(0) + \underbrace{\sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(a_k e^{-ik_- x^-} + c.c.\right)}_{\phi_R(x^+)} + \underbrace{\sum_{n>0} \frac{1}{\sqrt{4\pi n}} \left(\tilde{a}_k e^{-ik_+ x^+} + c.c.\right)}_{\phi_L(x^-)}$$

- quantization $Z(\beta, \alpha) = \mathrm{Tr}\, e^{-\beta \hat{H} + i\alpha \hat{P}} = \frac{1}{\sqrt{\tau_2}} \frac{1}{\eta(\tau)} \frac{1}{\eta(\bar{\tau})}$ $\qquad \tau = \frac{\alpha + i\beta}{L}$ $\Rightarrow$ reproduce in front form

particle $\quad$ right movers $\quad$ left movers

Exercise:

- start from $\mathcal{L} = \partial_+ \phi \, \partial_- \phi$
- use x^+ as time
- perform Hamiltonian analysis in front form
- (Poisson) bracket realization of conformal algebra?
- quantization & partition function?

Single front Hamiltonian analysis

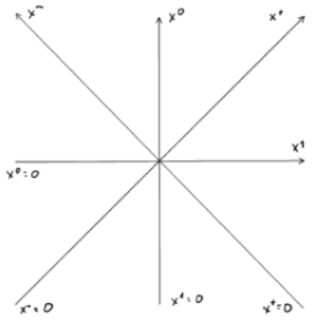


Figure 1: Coordinate axes

Conventions x^+ : "time" evolution direction (Kogut-Spencer)

$$\det\begin{pmatrix}\frac{\partial x^+\partial x^-}{\partial x^0\partial x^1}\end{pmatrix} = -1$$ { reversal of orientation (rotation of $-\frac{\pi}{4}$ + reflection of x^-)

primary constraint $g^+ = \pi^+ - \partial_- \phi$, $H_c \approx 0$

first order action $S_H = \int dx^+ \int_{-L/2}^{L/2} dx^- \mathcal{L}_H^+$, $\mathcal{L}_H^+ = [\pi^+\partial_+\phi - \lambda^+(\pi^+ - \partial_-\phi)]$

$$\partial_+\pi^+ + \partial_-\lambda^+ = 0 , \quad \pi^+ = \partial_-\phi , \quad \lambda^+ = \partial_+\phi$$

right mover left mover

fixed on-shell

determined on-shell

NB: A_0 : Lagrange multiplier for Gauss law $A_0 = \frac{1}{\Delta}[\partial_0 \vec{\nabla}\cdot\vec{A} + j^0]$ from unphysical dof.

Dirac analysis $\quad \{ \phi(x^+,x^-), \pi^+(x^+,y^-) \} = \delta(x^-,y^-)$

constraints $\quad G^+[\lambda^+] = \int_0^{L_-} dx^- \, g^+(\lambda^+) \qquad \{ G^+[\lambda^{+1}], G^+[\lambda^{+2}] \}_t = \int_{-L/2}^{L/2} dx^- (\lambda^{+2} \partial_- \lambda^{+1} - (1 \leftrightarrow 2))$

$H_c \approx 0$ no secondary constraints but restrictions on Lagrange multipliers

$$\{ g^+, G^+[\lambda^+] \} = -2\partial_- \lambda^+ \approx 0 \Rightarrow \partial_- \lambda^+ = 0 \Rightarrow \lambda^+ = \bar{\lambda}^+_+(x^+)$$

$\Rightarrow$ zero mode of constraint $\tilde{g}^+_+ = \int_0^{L_-} dy^- g^+$, $\quad G^+[\bar{\lambda}^+] = \tilde{g}^+_+ \bar{\lambda}^+$ is first class

$$\begin{cases} \partial_+ \phi = \{ \phi, G^+(\bar{\lambda}^+) \}_t = \bar{\lambda}^+_+ \\ \partial_+ \pi^+ = \{ \pi, G^+(\bar{\lambda}^+) \}_t = 0 \end{cases} \Rightarrow \begin{cases} \phi(x^+,x^-) = \int_{c^+}^{x^+} dy^+ \bar{\lambda}^+(y^+) + \phi(c^+,x^-) \\ \pi^+ = \pi^+(x^-) \end{cases} \Bigg\| \ \partial_- \phi = \pi^+ \Rightarrow \phi(x^+,x^-) = \int_{c^-}^{x^-} dy^- \pi^+(y^-) + \phi(x^+,c^-)$$

Matching at $x^+ = c^+$ or $x^- = c^-$ $\qquad \phi(x^+,x^-) = \int_{c^+}^{x^+} dy^+ \bar{\lambda}^+(y^+) + \int_{c^-}^{x^-} dy^- \pi^+(y^-) + \phi(c^+,c^-)$

$\boxed{\text{Lesson 1: Free data } \bar{\lambda}^+_+(x^+) \text{ at } x^- = c^-, \ \pi^+(x^-) \text{ at } x^+ = c^+, \ \phi(c^+,c^+) \text{ at corner}}$

characteristic initial value problem

Puzzle 1: first class constraint $G[\epsilon^+]$, $\epsilon^+(x^+)$ generates global but not a gauge symmetry!

$$\delta_{\epsilon^+} \Phi = \{\phi, G[\epsilon^+]\}_t = \epsilon^+ , \quad \delta_{\epsilon^+} \pi^+ = \{\pi^+, G[\epsilon^+]\}_t = 0$$

$$\delta_{\epsilon^+} a^+ = \partial_t \epsilon^+$$

Resolution:

REVIEWS OF MODERN PHYSICS VOLUME 21, NUMBER 3 JULY, 1949

Forms of Relativistic Dynamics

P. A. M. Dirac
St. John's College, Cambridge, England

A similar difficulty arises, in a less serious way, with the front form of theory. Waves moving with the velocity of light in exactly the direction of the front cannot be described by physical conditions on the front, and some extra variables must be introduced for dealing with them.

Dirac 1949 "Front form of dynamics"

1.2. FIRST-CLASS CONSTRAINTS AS GENERATORS OF GAUGE TRANSFORMATIONS

1.2.1. Transformations That Do Not Change the Physical State. Gauge Transformations

The presence of arbitrary functions v^a in the total Hamiltonian tells us that not all the q's and p's are observable. In other words, although the physical state is uniquely defined once a set of q's and p's is given,

the converse is not true—*i.e.*, there is more than one set of values of the canonical variables representing a given physical state. To see how this conclusion comes about, we notice that if we give an initial set of canonical variables at the time t_1 and thereby completely define the physical state at that time, we expect the equations of motion *to fully determine the physical state at other times.* Thus, by definition, any ambiguity in the value of the canonical variables at $t_2 \neq t_1$ should be a physically irrelevant ambiguity.

Henneaux & Teitelboim

(Primary) first class constraints generate gauge symmetries in instant form under the assumption that initial data uniquely fix the physical state.

not the case in front form

left mover $\phi^S(x^+)$ does not intersect $x^+ = c^+$

Assumption: fields periodic in x^-, $\forall x^+$

Zero mode & chiral boson sectors

$$\phi(x^+,x^-) = \bar{\Phi}_+(x^+) + \tilde{\phi}(x^+,x^-)$$

$$J^+ = \bar{J}^+_+(x^+) + \tilde{J}^+_+ \,, \quad \begin{cases} \bar{J}^+_+ = \frac{1}{L_-}\int_{-L_-/2}^{L_-/2} dx^- \, J^+(x^+,x^-) \\ \int_{-L_-/2}^{L_-/2} dx^- \, \tilde{J}^+_+ = 0 \end{cases}$$

idem for $\pi^+(x^+,x^-) = \dfrac{\bar{\pi}^+_+(x^+)}{L_-} + \tilde{\pi}^+(x^+,x^-)$

$$\{\bar{\Phi}_+, \bar{\pi}^+_+\}_+ = 1, \quad \{\tilde{\Phi}_+(x^-), \tilde{\pi}^+_+(y^-)\}_+ = \delta(x^-,y^-) - \frac{1}{L_-}$$

constraints $\bar{g}^+_+ = \bar{\pi}^+_+$ first class $\quad \tilde{g}^+_+ = \tilde{\pi}^+_+ - \partial_-\tilde{\Phi}_+$ second class

solve in the action

$$S_R = \int_{-\infty}^{+\infty} dx^+ \, L^+_R \,, \quad L^+_R = \bar{\pi}^+_+ \partial_+\bar{\Phi}_+ - \bar{J}^+_+\bar{\pi}^+_+ + \int_{-L_-/2}^{L_-/2} dx^- \, \tilde{\mathcal{L}}^+ \,, \quad \tilde{\mathcal{L}}^+ = \partial_-\tilde{\Phi}_+ \partial_+\tilde{\Phi}_+$$

looks like pure gauge dof

but information on left mover

finite volume-analog of principal value prescription

Dirac brackets $\{\tilde{g}^+_+(x^-), \tilde{g}^+_+(y^-)\}_+ = 2\,\partial^x_-\delta(x^-,y^-) \Rightarrow$

$$\begin{cases} \{\tilde{\Phi}_+(x^-), \tilde{\Phi}_+(y^-)\}^* = -\frac{1}{4}\epsilon(x^--y^-) + \dfrac{x^--y^-}{2L_-} & (x) \\ \{\tilde{\Phi}_+(x^-), \tilde{\pi}^+_+(y^-)\}^* = \frac{1}{2}\left[\delta(x^-,y^-) - \frac{1}{L_-}\right] & (xx) \\ \{\tilde{\pi}^+_+(x^-), \tilde{\pi}^+_+(y^-)\}^* = \frac{1}{2}\partial^x_-\delta(x^-,y^-) & (xxx) \end{cases}$$

(x) primitive of (xx) without zero-mode

Maskawa & Yamakawi 1976

Dynamics $H_R = \tilde{\partial}^+_+ \bar{\pi}^+_+$ only in the zero-mode sector = left movers

conformal symmetries : left chiral half $Q^{+*}_{\xi^+} = \bar{\pi}^+_+ \tilde{\partial}^+_+ \xi^+$ act only on zero-modes

$$\delta_{\xi^+} \tilde{\phi}_+ = \{\tilde{\phi}_+ , Q^{+*}_{\xi^+}\}^*_+ = 0$$

right chiral half $Q^{-*}_{\xi^-} = \int_0^{L_-} dx^- \, \xi^- \left[(\partial_- \tilde{\phi}_+)^2 + \bar{\pi}^+_+ \partial_- \tilde{\phi}_+ \right]$

$$\delta_{\xi^-} \tilde{\phi}_+ = \xi^- \partial_- \tilde{\phi}_+ - \frac{1}{L_-} \int_0^{L_-} dy^- \, \xi^- \partial_- \tilde{\phi}_+ \, , \qquad \delta_{\xi^-} \bar{\phi}_+ = \frac{1}{L_-} \int_0^{L_-} dy^- \, \xi^- \partial_- \tilde{\phi}_+$$ mixes sectors !

Puzzle 2: No representation of conformal algebra

$$\{ Q^{+*}_{\xi_1^+} , Q^{+*}_{\xi_2^+} \}^*_+ = 0 \neq Q^{+*}_{[\xi_1^+ , \xi_2^+]}$$ left chiral half

$$\{ Q^{+*}_{\xi_1^-} , Q^{+*}_{\xi_2^-} \}^*_+ = Q^{+*}_{[\xi_1^- , \xi_2^-]}$$ only if there is no zero mode sector

right chiral half

Preliminary attempt at quantization $Z(\beta,\alpha) = \mathrm{Tr}\, e^{-\beta \hat{H} + i\alpha \hat{P}}$

Boundary conditions x^- :

$\boxed{\text{periodic } x^- \sim x^- + L_- \text{ (finite-volume analog of Christodoulou Klainerman) Discrete light-cone quantization (DLCQ) box in a null coordinate; clean separation of zero-mode and oscillator sector}}$

Mode expansion $\tilde{\Phi}_+(x^-) = \sum_{n>0} \frac{1}{\sqrt{2 k_- L_-}} \left(a_{k_-} e^{-i k_- x^-} + c.c. \right) = \phi_R(x^-)$, $k_- = \frac{2\pi n}{L_-}$

$$\{ a_{k_-} , a^*_{k'_-} \}^*_+ = -i\, \delta_{n,n'} \iff (x), (xx), (xxx)$$

$$-\beta Q^+_{J_0} - i\alpha Q^+_{J_1} = \frac{i\tau L}{\sqrt{2}} H^{+R} - \frac{i\bar{\tau} L}{\sqrt{2}} H^{+*} \qquad \tau = \frac{\alpha + i\beta}{L}$$

$$H^{+R} = \int_{-L_-/2}^{L_-/2} dx^- \left(\partial_- \tilde{\Phi}_+ \right)^2 = \frac{1}{2} \sum_{n>0} k_- \left(a^*_{k_-} a_{k_-} + a_{k_-} a^*_{k_-} \right), \qquad H^{+*} = \bar{J}^+_+ \,\bar{\Pi}^+_+$$

no contribution if quantized as a pure gauge dof

$$\hat{H}^{+R} = E_0^{+R} + \sum_{n>0} k_- \hat{a}^{+}_{k_-} \hat{a}_{k_-} \qquad E_0^{+R} = \frac{\pi}{L_-} \sum_{n>0} n = -\frac{2\pi}{24 L_-}$$ Casimir energy

partition function $\quad Z(\tau, \bar{\tau}) = \dfrac{1}{\eta\left(q\left(\frac{L\tau}{\sqrt{2} L_-}\right)\right)}$

the contribution from the left mover & particle zero mode is missing

Results on the other front "time" x^- exchange the roles of left (+) and right (−)

$$S = \int dx^- \int_0^{L_+} dx^+ \, \mathcal{L}_H^- \,, \quad \mathcal{L}_H^- = \pi^- \partial_- \phi - J^- (\pi^- - \partial_+ \phi)$$

$\boxed{\bar{J}_-(x^-) \text{ analog of news}}$

$$\tilde{\phi}_-(x^+) = \sum_{n>0} \frac{1}{\sqrt{2 k_+ L_+}} \left(\tilde{a}_{k_+} e^{-i k_+ x^+} + c.c. \right) = \phi^L(x^+) \qquad \{ \tilde{a}_{k_+}, \tilde{a}_{k'_+} \} = -i \, \delta_{n,n'} \qquad k_+ = \frac{2\pi}{L_+} n$$

other chiral half of partition function $\quad Z(\tau, \bar{\tau}) = \dfrac{1}{\eta\left(q\left(\frac{L\tau}{\sqrt{2} L_+}\right)\right)}$

Double front Hamiltonian analysis

renaming Lagrange multipliers $\lambda^+ = \pi^-$, $\lambda^- = \pi^+$

$$S_H = \int dx^+ dx^- \left[\pi^+ \partial_+ \phi + \pi^- \partial_- \phi - \pi^- \pi^+\right] \qquad \pi^+ = \partial_- \phi \,, \quad \pi^- = \partial_- \phi \,, \quad \partial_+ \pi^+ + \partial_- \pi^- = 0$$

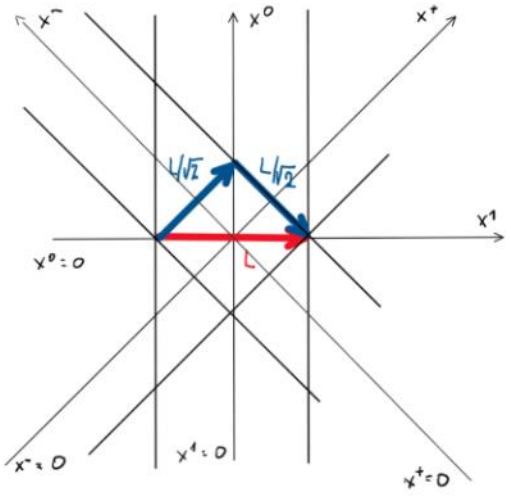
standard instant form periodicity $\Longleftrightarrow$ entangled periodicities in 2 null coord.

$$x^1 \sim x^1 + L \quad \Longleftrightarrow \quad (x^+, x^-) \sim (x^+ + L_+ , x^- - L_-) \qquad L_\pm = \frac{L}{\sqrt{2}}$$

Lesson 2

Sectors $\quad \phi(x^+, x^-) = \bar{\phi}_\pm (x^\pm) + \tilde{\phi}_\pm (x^+, x^-) \; ; \quad \pi^\pm (x^+, x^-) = \frac{1}{L_\mp} \bar{\pi}^\pm_\pm (x^\pm) + \tilde{\pi}^\pm_\pm (x^+, x^-)$

not independent $\quad \frac{1}{L_+} \int_0^{L_+} dx^+ \, \bar{\phi}_+(x^+) = \frac{1}{L_-} \int_0^{L_-} dx^- \, \bar{\phi}_-(x^-) \qquad \int_0^{L_\pm} dx^\pm \, \bar{\pi}^\pm_\pm (x^\pm) = \int_0^{L_\mp} dx^\mp \, \bar{\pi}^\pm_\mp (x^\mp)$



Conserved currents & Stoke's theorem

$$\partial_\mu j^\mu \approx 0$$

$$j = d^{n-1}x_\mu \, j^\mu , \qquad dj = \partial_\mu j^\mu d^n x \qquad \int_{\partial V} j = \int_V dj \approx 0$$

$$j = dx^1 j^0 - dx^0 j^1 = dx^- j^+ - dx^+ j^-$$

$$j^\pm = -\frac{1}{\sqrt{2}}(j^0 \pm j^1) \qquad j'^\mu(x') = \left(\det \frac{\partial x}{\partial x'}\right)\frac{\partial x'^\mu}{\partial x^\nu} j^\nu(x)$$

$$Q = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx^1 \, j^0 \Big|_{x^0=0} \approx -\int_{-L_+/2}^{L_+/2} dx^+ \, j^- \Big|_{x^-=\frac{L_-}{2}} - \int_{-L_-/2}^{L_-/2} dx^- \, j^+ \Big|_{x^+=\frac{L_+}{2}}$$

NB: if $\partial_+ j^+ = 0 = \partial_- j^-$ separately, the intersection point does not matter
integrals may be evaluated at any $x^\pm = c^\pm$

Conserved symplectic $(2, n-1)$ form

first variational formula $\quad d^n x \; \delta \mathcal{L} = d^n x \; \delta\phi^i \frac{\delta \mathcal{L}}{\delta \phi^i} + d_H a \qquad a = d^{n-1}x_\mu \, a^\mu$

second variational formula $\quad 0 = - d^n x \; \delta\phi^i \, \delta \frac{\delta \mathcal{L}}{\delta \phi^i} + d_H \sigma \qquad \sigma = (-)^{n-1} \delta a = d^{n-1}x_\mu \underbrace{\delta a^\mu}_{\sigma^\mu}$

$\partial_\mu \sigma^\mu \approx 0 \qquad$ linearized field equations

$a = dx^- \pi^+ \delta\phi - dx^+ \pi^- \delta\phi \qquad \sigma = dx^- \underbrace{\delta\pi^+ \delta\phi}_{\sigma^+} - dx^+ \underbrace{\delta\pi^- \delta\phi}_{\sigma^-}$

$\partial_+ \sigma^+ \not\approx 0 \not\approx \partial_- \sigma^-$

non-vanishing Poisson brackets $\quad \{\phi(L_+, x^-), \pi^+(L_+, y^-)\}_+ = \delta(x^-, y^-) \qquad \{\phi(x^+, 0), \pi^-(y^+, 0)\}_- = \delta(x^+, y^+)$

Lesson 3: the Lagrange multipliers are the canonical momenta along rather than off the front

$\lambda^+ = \pi^-, \quad \lambda^- = \pi^+$

missing bracket for quantization

more rigorous inversion: Peierls bracket

Peierls bracket

$$\phi(x^+,x^-) = \bar{\phi}_0(0) + \left(\frac{x^+ + x^-}{\sqrt{2}L}\right)\bar{\pi}_0^0(0) + \phi^R(x^-) + \phi^L(x^+),$$

General solution on spatial cylinder, entangled periodicity, but not separate periodicities !

$$\tilde{G}(x^0,x^1) = G^+(x^0,x^1) - G^-(x^0,x^1),$$

Difference of advanced and retarded propagator, Pauli-Jordan commutation function

$$(\partial_0^2 - \partial_1^2)\tilde{G}(x^0,x^1) = 0, \quad \tilde{G}(0,x^1) = 0, \quad \partial_0\tilde{G}(x^0,x^1)|_{x^0=0} = -\delta(x^1),$$

Solution to the homogeneous equations, initial conditions determined by canonical equal time commutation relations

$$\begin{aligned}\tilde{G}(x^0,x^1) &= -\int_{-\infty}^{+\infty} dk^1 \frac{1}{4\pi k^1}[\sin k_1(x^0+x^1) + \sin k^1(x^0-x^1)] \\ &= -\frac{1}{4}[\varepsilon(x^0+x^1) + \varepsilon(x^0-x^1)] = -\frac{1}{2}\varepsilon(x^0)\theta(x_\mu x^\mu), \\ &= \int_{-\infty}^{+\infty} dk^0 \int_{-\infty}^{+\infty} dk^1 \frac{1}{2\pi i} e^{-ik_\mu x^\mu}\delta(k_\mu k^\mu)\varepsilon(k^0).\end{aligned}$$

$$\{\phi(0,0),\ \phi(0,x^1)\} = 0, \quad \{\phi(0,0),\ \partial_0\phi(0,x^1)\} = -\delta(x^1)$$

$$\{\phi(x^+,x^-),\ \phi(y^+,y^-)\} = -\frac{1}{4}\left[\varepsilon(x^+-y^+) + \varepsilon(x^- - y^-)\right]$$

in empty space

Reproduces correctly all equal-time brackets on the two different fronts

$$\{\phi(x^+,x^-),\phi(x^+,y^-)\} = -\frac{1}{4}\varepsilon(x^- - y^-),$$
$$\{\phi(x^+,x^-),\phi(y^+,x^-)\} = -\frac{1}{4}\varepsilon(x^+ - y^+),$$
$$\{\phi(x^+,x^-),\pi^+(x^+,y^-)\} = \frac{1}{2}\delta(x^-,y^-),$$
$$\{\phi(x^+,x^-),\pi^-(y^+,x^-)\} = \frac{1}{2}\delta(x^+,y^+),$$
$$\{\phi(x^+,x^-),\pi^-(x^+,y^-)\} = 0 = \{\phi(x^+,x^-),\pi^+(y^+,x^-)\},$$
$$\{\pi^+(x^+,x^-),\pi^+(x^+,y^-)\} = \frac{1}{2}\delta'(x^-,y^-),$$
$$\{\pi^-(x^+,x^-),\pi^-(y^+,x^-)\} = \frac{1}{2}\delta'(x^+,y^+),$$
$$\{\pi^-(x^+,x^-),\pi^+(x^+,y^-)\} = 0 = \{\pi^+(x^+,x^-),\pi^-(y^+,x^-)\}.$$

Shift and conformal symmetries: On-shell non-vanishing charges on one of the fronts

$$Q^+_{\epsilon^+} = \int_{-L_-/2}^{L_-/2} dx^-(\pi^+ - \partial_-\phi)\epsilon^+ \approx 0,$$
$$Q^-_{\epsilon^+} = \int_{-L_+/2}^{L_+/2} dx^+(\pi^- + \partial_+\phi)\epsilon^+ \approx \int_{-L_+/2}^{L_+/2} dx^+ 2\partial_+\phi\epsilon^+,$$
$$Q^+_{\epsilon^-} = \int_{-L_-/2}^{L_-/2} dx^-(\pi^+ + \partial_-\phi)\epsilon^- \approx \int_{-L_-/2}^{L_-/2} dx^- 2\partial_-\phi\epsilon^-,$$
$$Q^-_{\epsilon^-} = \int_{-L_+/2}^{L_+/2} dx^+(\pi^- - \partial_+\phi)\epsilon^- \approx 0.$$

$$Q'^+_\xi \approx \int_{-L_-/2}^{L_-/2} dx^-\xi^-(\partial_-\phi)^2 = Q'^+_{\xi^-}, \quad Q'^-_\xi \approx \int_{-L_+/2}^{L_+/2} dx^+\xi^+(\partial_+\phi)^2 = Q'_{\xi^+}.$$

To be done: adapt to torus topology !

include particle sector

Gibbons-Hawking terms

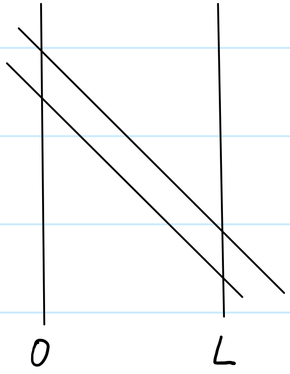
improved generators

Correct representation of the conformal algebra, separately on the two fronts

$$\{Q'^+_{\xi_1^-},Q'^+_{\xi_2^-}\} = Q'^+_{[\xi_1^-,\xi_2^-]}, \quad \{Q'^-_{\xi_1^+},Q'^-_{\xi_2^+}\} = Q'^-_{[\xi_1^+,\xi_2^+]}, \quad \{Q'^+_{\xi_1^-},Q'^-_{\xi_2^+}\} \approx 0,$$

Matching & equivalent off-shell descriptions of a theory ?

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} dx^0 \int_0^L dx^1\, \partial_\mu\phi\,\partial^\mu\phi \iff \begin{cases} S_H^+ = \displaystyle\int_{-\infty}^{+\infty} dx^+ \left[2\left(\bar\pi_+^+\,\partial_+\bar\Phi_+ - \frac{1}{L_-}\bar\pi_+^-\,\bar\pi_+^+\right) + \int_{x^+-2L_-}^{x^+} dx^- \left(\tilde\pi_+^+\,\partial_+\tilde\Phi_+ - \tilde\pi_+^-\,\tilde g_+^+\right)\right] \\ S_H^- = \displaystyle\int_{-\infty}^{+\infty} dx^- \left[2\left(\bar\pi_-^-\,\partial_-\bar\Phi_- - \frac{1}{L_+}\bar\pi_-^+\,\bar\pi_-^-\right) + \int_{x^-}^{x^-+2L_+} dx^+ \left(\tilde\pi_-^-\,\partial_-\tilde\Phi_- - \tilde\pi_-^+\,\tilde g_-^-\right)\right] \end{cases}$$



$$S \iff \frac{1}{2}S_H^+ + \frac{1}{2}S_H^- = S^P + \tilde S^R[\tilde\Phi_+, \tilde\pi_+^\pm] + \tilde S^L[\tilde\Phi_-, \tilde\pi_-^\mp]$$

$$\tilde S^R = \int_{-\infty}^{+\infty} dx^+ \int_0^{L_-} dx^- \left[\tilde\pi_+^+\,\partial_+\tilde\Phi_+ - \tilde\pi_+^-\,\tilde g_+^+\right]$$ right movers | chiral bosons

$$\tilde S^L = \int_{-\infty}^{+\infty} dx^- \int_0^{L_+} dx^+ \left[\tilde\pi_-^-\,\partial_-\tilde\Phi_- - \tilde\pi_-^+\,\tilde g_-^-\right]$$ left movers | chiral bosons

$$S^P = \int_{-\infty}^{+\infty} dx^+ \left[\bar\pi_+^+\,\partial_+\bar\Phi_+ - \frac{1}{L_-}\bar\pi_+^-\,\bar\pi_+^+\right] + \int_{-\infty}^{+\infty} dx^- \left[\bar\pi_-^-\,\partial_-\bar\Phi_- - \frac{1}{L_+}\bar\pi_-^+\,\bar\pi_-^-\right]$$

old problem : massless KG $\iff$ left + right chiral bosons up to zero mode on-shell field $\phi(x^+,x^-) = \phi(0,0) + \frac{x^- \bar{\pi}^+_+(L_+)}{L_-} + \frac{x^+ \bar{\pi}^-_-(0)}{L_+} + \int_0^{x^-} dy^- \; \tilde{\pi}^+_+(L_+, y^-) + \int_0^{x^+} dy^+ \; \tilde{\pi}^-_-(y^+, 0)$

$\partial_\mu \pi^\mu \approx 0$ conserved current but $\pi'^\mu(x') = \left|\det \frac{\partial x}{\partial x'}\right| \frac{\partial x'^\mu}{\partial x^\nu} \pi^\nu(x)$

$\partial_+ \pi^+ \approx 0$ & $\partial_- \pi^- \approx 0$ $\quad \pi^\pm = \pm \frac{\pi^0 \pm \pi^1}{\sqrt{2}}$

Stokes' theorem

$$\bar{\pi}^0_0 = \int_{-L/2}^{L/2} dx^1 \, \pi^0 = \int_{-L_+/2}^{L_+/2} dx^+ \, \pi^- \Big|_{x^- = \frac{L_-}{2}} + \int_{-\frac{L_-}{2}}^{L_-/2} dx^- \, \pi^+ \Big|_{x^+ = L_+/2} = \bar{\pi}^-_-\left(\frac{L_-}{2}\right) + \bar{\pi}^+_+\left(\frac{L_+}{2}\right) = \boxed{\bar{\pi}^-_-(0) + \bar{\pi}^+_+(0)}$$

$$\phi(x^+,x^-) = \phi(0,0) - \phi^R(0) - \phi^L(0) + \frac{x^- \bar{\pi}^+_+(0)}{L_-} + \frac{x^+ \bar{\pi}^-_-(0)}{L_+} + \phi_R(x^-) + \phi_L(x^+)$$

entangled periodicity iff matching condition $\bar{\pi}^+_+(0) = \bar{\pi}^-_-(0) = \frac{1}{2}\bar{\pi}^0_0$

to be completed: (i) prove that zero mode sector corresponds to single free particle

$\Rightarrow$ correct partition function $Z(\zeta,\bar{\zeta}) = \frac{1}{\sqrt{\zeta_2}} \frac{1}{\eta(q(\zeta))} \frac{1}{\eta(q(\bar{\zeta}))}$ particle zero mode

(ii) show that conformal trsf are generated by Noether charge in Peierls bracket and that the Peierls bracket of charges form a realization of the algebra. with particle zero mode usual DLCQ : massive case

$$S = \int dx^+ dx^- \left(\partial_+ \phi \partial_- \phi - \frac{1}{2} m^2 \phi^2 \right)$$

- no chiral shift symmetries
- conformal → Poincaré

$$\partial_+ \xi^- = \partial_- \xi^+ = 0$$

$$\partial_+ \xi^+ + \partial_- \xi^- = 0$$

$$\xi^+ = a^+ + \omega x^+ , \quad \xi^- = a^- + \omega x^- , \quad T_{\pm\pm} = (\partial_\pm \phi)^2$$

$$T_{\pm\mp} = \frac{m^2}{2} \phi^2$$

Dirac algorithm

$$S_H = \int dx^+ dx^- \left(\pi^+ \partial_+ \phi - \underbrace{\frac{m^2}{2} \phi^2}_{= H_c} - \lambda^+ (\pi^+ - \partial_- \phi) \right)$$

conservation of constraints $\quad \partial_- \lambda^+ = -\frac{m^2}{2} \phi \quad \Rightarrow \quad \bar{\Phi}_+ = 0$ secondary constraint

$$\Rightarrow \quad \bar{\lambda}^+_+ = 0$$

all constraints $\bar{\phi}^+, \bar{\pi}^+_+, \tilde{\pi}^+_+ - \partial_- \tilde{\Phi}_+$ are second class reduced theory : free data $\tilde{\phi}_+(0, x^-)$

only data on a single null hypersurface is needed $x^+ = 0$

$$H^R = \int_{-L/2}^{L/2} dx^- \, \frac{m^2}{2} \tilde{\phi}_+^2$$

$$\{ \tilde{\phi}_+(x^-), \tilde{\phi}_-(y^-) \}_+^* = -\frac{1}{4} \epsilon(x^- - y^-) + \frac{x^- - y^-}{2L_-}$$

Mode expansion $\tilde{\phi}_+(x^-) = \sum_{n>0} \frac{1}{\sqrt{2k_- L_-}} \left(a_{k_-} e^{-ik_- x^-} + c.c. \right) = \phi_R(x^-)$ $\quad k_- = \frac{2\pi n}{L_-}$

$$\{ a_{k_-}, a_{k'_-}^\dagger \}_+^* = -i \, \delta_{n,n'}$$

Instant form $\phi(x^0, x^1) = \bar{\Phi}_0 + \frac{x^0}{L} \bar{\pi}_0 + \sum_{n \in \mathbb{Z}^*} \frac{1}{\sqrt{2|k|L}} \left(a_k e^{-ik_\mu x^\mu} + c.c. \right)$

$k = \frac{2\pi}{L} n = -k_1$

$k_0 = \sqrt{k^2 + m^2}$

Bogoliubov trf to compare

$\boxed{\text{Massless \& massive cases are very different}}$