Quantum systems as manipulators of information:

Entanglement, complexity, and phases of matter

Georgios Styliaris

MAX PLANCK INSTITUTE
OF QUANTUM OPTICS

Xmas Theoretical Physics Workshop @Athens 2023

Friday, Dec 22, 2023

The New York Times

Google Claims a Quantum Breakthrough That Could Change Computing

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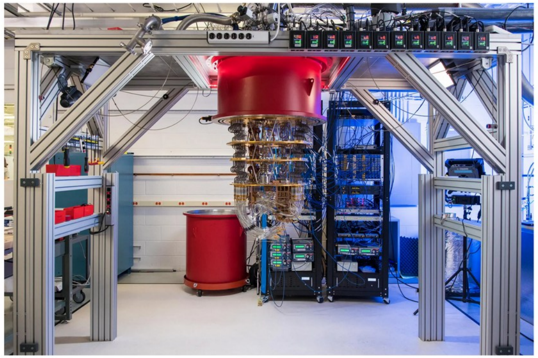

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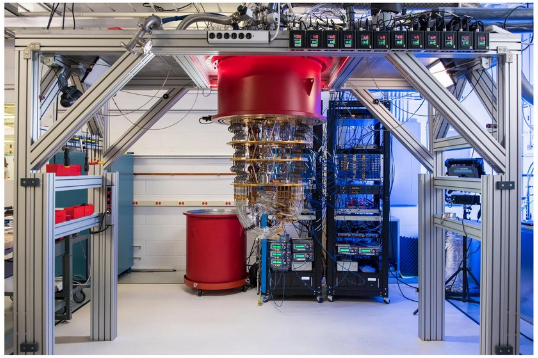

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REVIEWS OF MODERN PHYSICS, VOLUME 90, OCTOBER–DECEMBER 2018

APS Medal for Exceptional Achievement in Research: Invited article on entanglement properties of quantum field theory*

Edward Witten
School of Natural Sciences, Institute for Advanced Study, Einstein Drive, Princeton, New Jersey 08540, USA

(published 23 October 2018)

These are notes on some entanglement properties of quantum field theory, aiming to make accessible a variety of ideas that are known in the literature. The main goal is to explain how to deal with entanglement when—as in quantum field theory—it is a property of the algebra of observables and not just of the states.

DOI: 10.1103/RevModPhys.90.045003

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Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence

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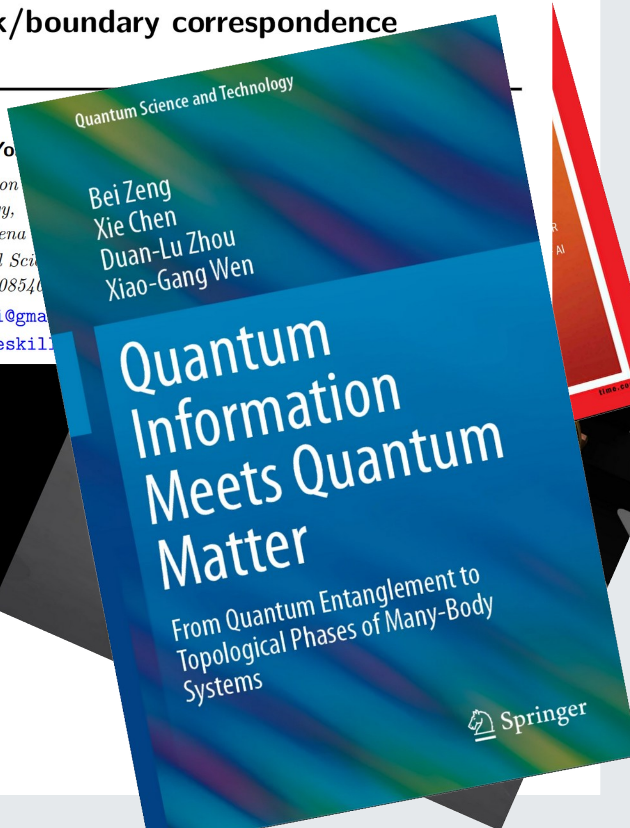
ANNUAL REVIEWS
Annual Review of Condensed Matter Physics

Spacetime from Entanglement

Brian Swingle[1,2]
[1] Department of Physics, Harvard University, Cambridge, Massachusetts 02142, USA
[2] Department of Physics, University of Maryland, College Park, Maryland 20742, USA;
email: bswingle@umd.edu



Quantum information theory

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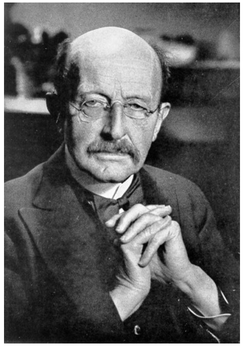
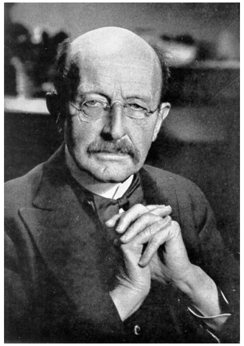


Quantum
Theory

Quantum formalism:

- Spin
- Entanglement

Quantum information theory



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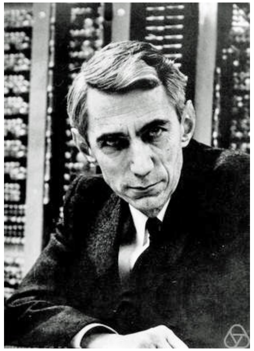

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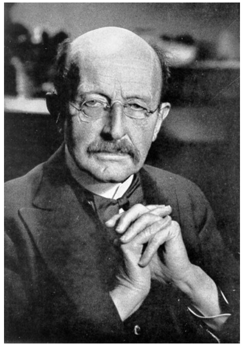

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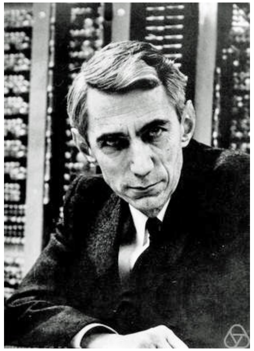

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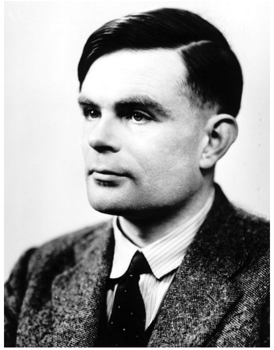

Computer
Science

Quantum formalism:

- Spin
- Entanglement

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- Communication

- Complexity theory

Factoring and quantum mechanics

Problem *(Integer factorization)*:
Given a composite integer, find a factor.

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Example:

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Factoring on a "classical" computer:

- Best current algorithm has $f_{factoring}$ which grows *faster than any polynomial.*
- Very strong theoretical evidence that $f_{factoring}$ cannot be any polynomial

Factoring and quantum mechanics

Peter Shor

On quantum hardware, I can factor any integer in less than $O(n^3)$ elementary quantum operations!

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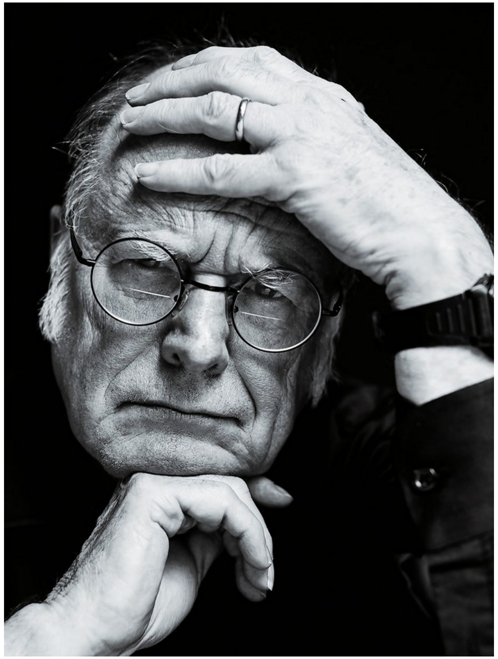
Extended Church-Turing thesis:

All “reasonable” models of computation yield the same class of problems that can be computed in polynomial time

Quantum Teleportation

Quantum degrees of freedom can be entangled



Charles Bennett

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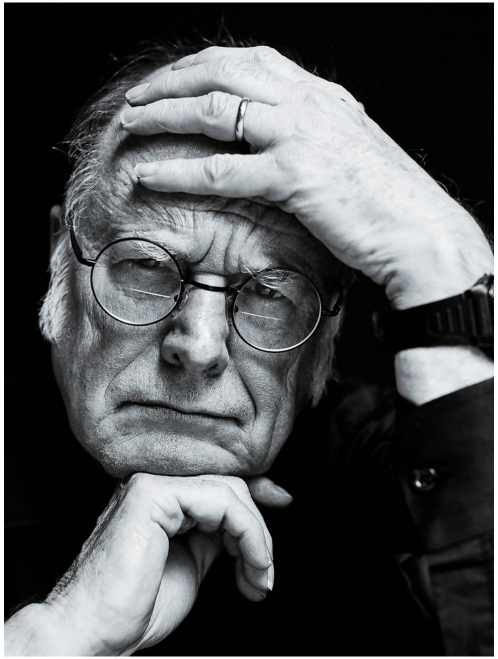

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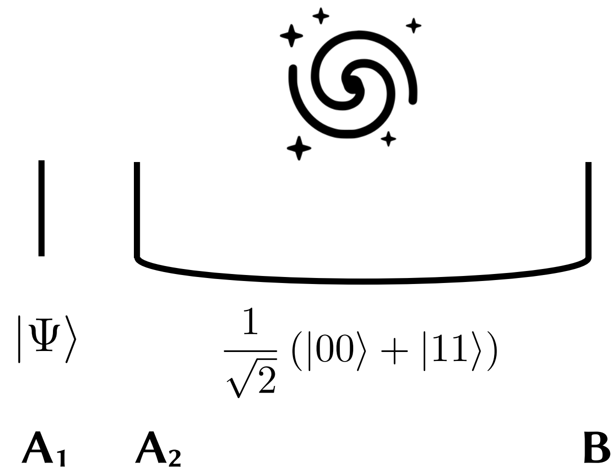
Quantum Teleportation (Bennett, Wiesner '92)

- Shared **entangled** pair + 2 **classical bits** ⟶ Teleportation of **1 qubit**!

 +  ⟶ 

Quantum Teleportation

- Alice has two qubits A_1, A_2
- Bob has a single qubit B



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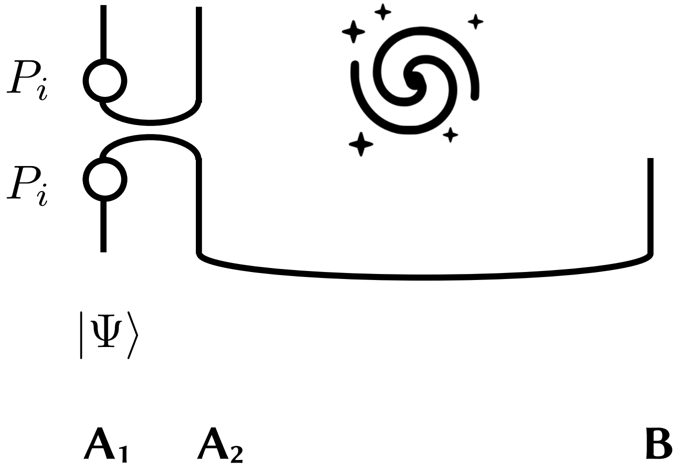

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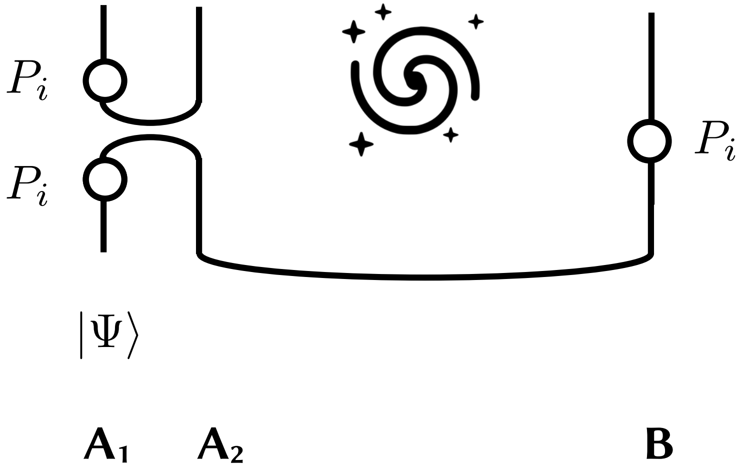

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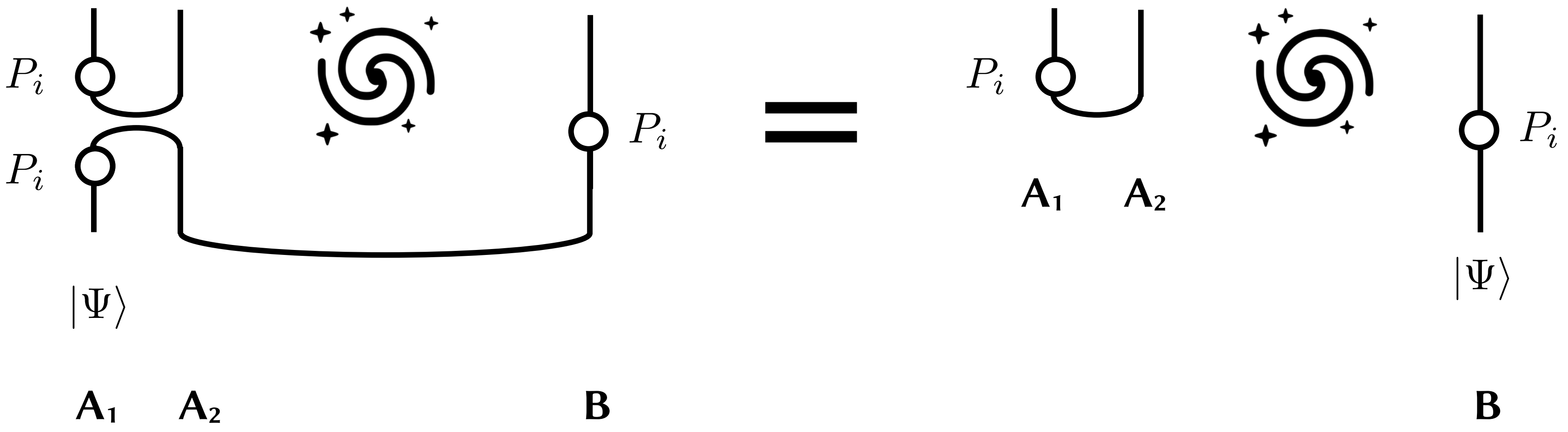

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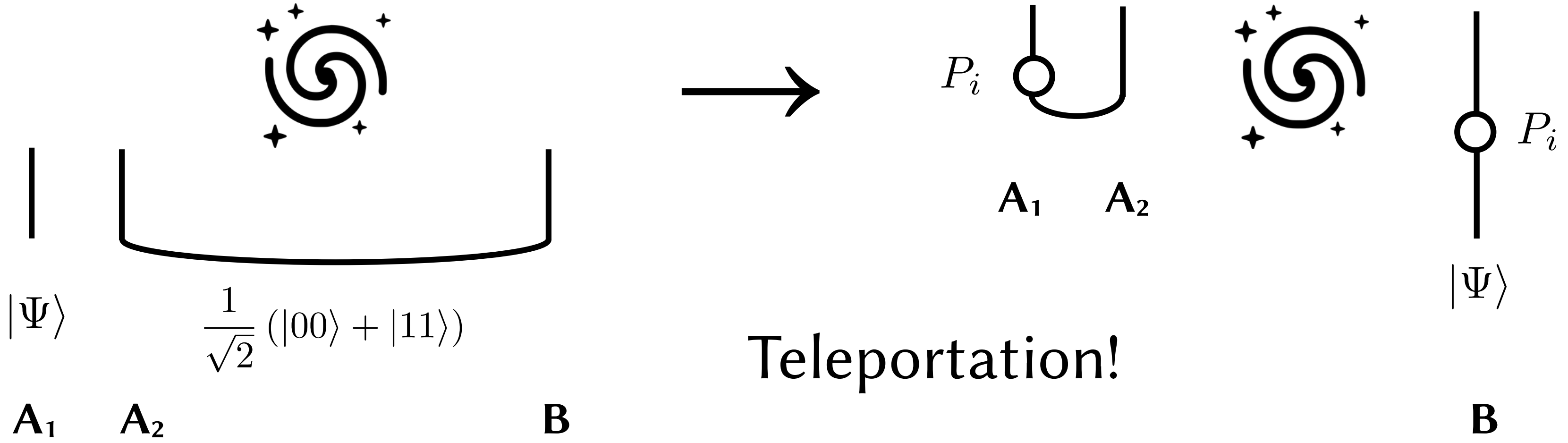

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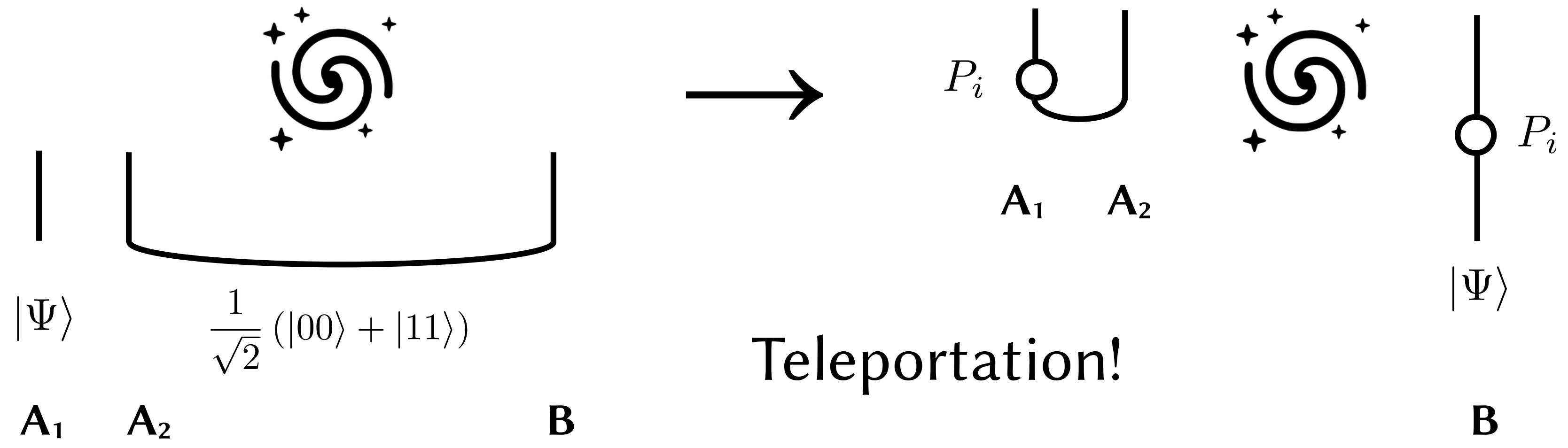
Local Operations &

Classical Communication **(LOCC):**

- Local Measurements
- Local unitary evolution
- Conditioning operations on measurement outcomes



Entanglement, complexity, and phases of matter

From computational complexity to quantum state complexity:

- Complexity is a feature of the entanglement structure

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- The relevant “corner” of the Hilbert space

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Collapsing the wavefunction:

- Connecting distinct phases without phase transitions by LOCC

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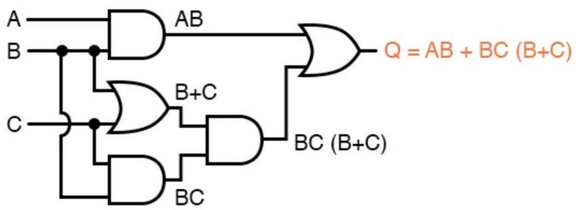
Hilbert space is large because of **entanglement.**
Classify states according to their entanglement properties.

Quantifying Complexity: Quantum Circuits

- **Boolean** function of N variables: $f : \{0,1\}^{\times N} \to \{0,1\}$
- Decomposition over **elementary building blocks**



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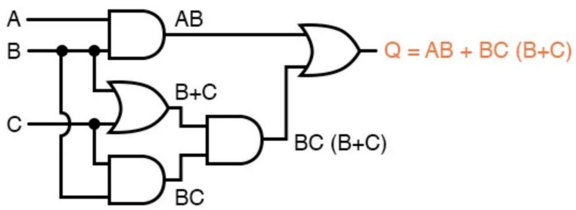

- **Quantum** process is a **Unitary** $U : (\mathbb{C}^2)^{\otimes N} \to (\mathbb{C}^2)^{\otimes N}$
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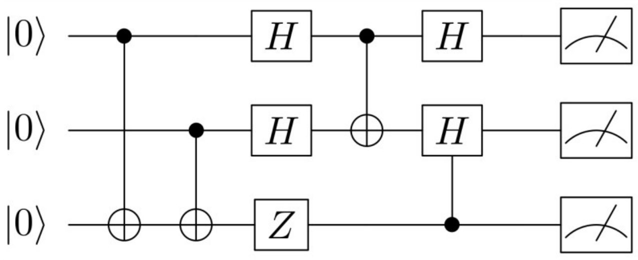

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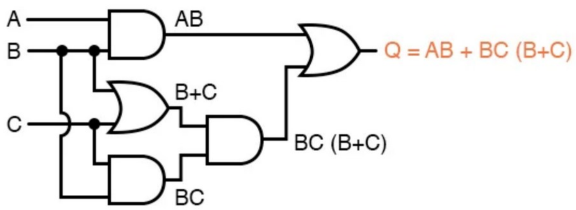

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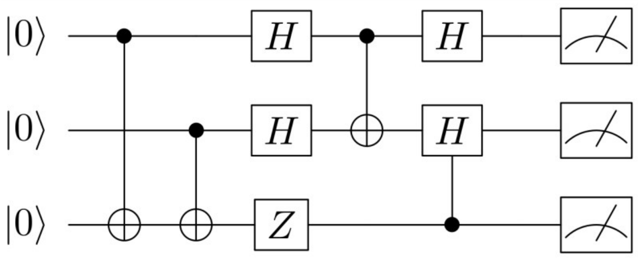

Definition *(State complexity):* Shortest possible #layers to prepare $|\psi\rangle$ from a product state.

Typical state complexity is exponential

Consider a (uniformly) randomly chosen state $|\psi\rangle$ of N qubits. How complex is it?

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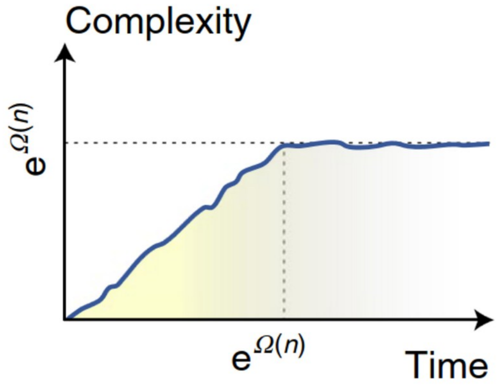

Random circuits have no shortcuts!

Brown & Susskind, PRD '18
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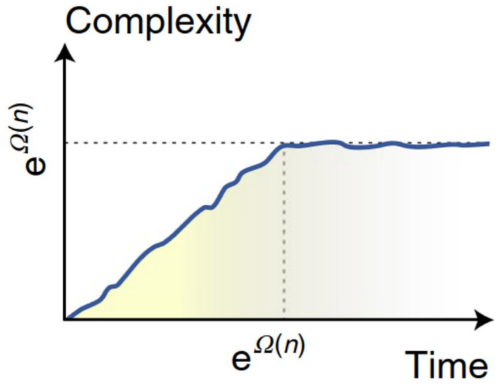

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PRL **106,** 170501 (2011) PHYSICAL REVIEW LETTERS week ending 29 APRIL 2011

Quantum Simulation of Time-Dependent Hamiltonians and the Convenient Illusion of Hilbert Space

David Poulin,[1] Angie Qarry,[2,3] Rolando Somma,[4] and Frank Verstraete[2]

“[…] The overwhelming majority of states in Hilbert space are not physical as they can only be produced after an exponentially long time”

Brown & Susskind, PRD ‘18
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Entanglement area law states

For a meaningful classification, we need to **restrict** to states with *reasonable complexity*

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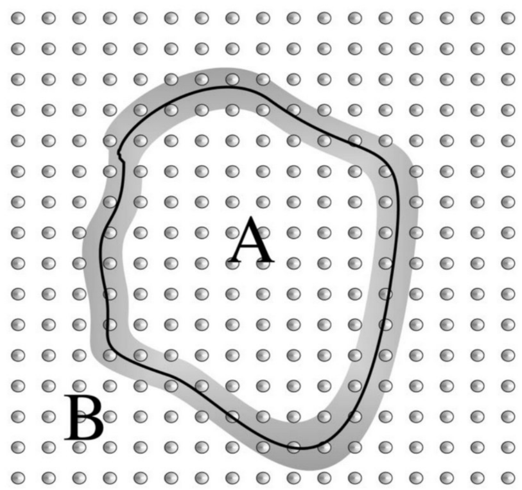
Definition *(entanglement area law states):*

$S(A:B) \leq c|\partial A|$ for all regions A

where

$$S(A:B) = -\mathrm{Tr}\,(\rho_A \log \rho_A)$$

$$\rho_A = \mathrm{Tr}_B \left|\psi_{AB}\right\rangle \left\langle\psi_{AB}\right|$$



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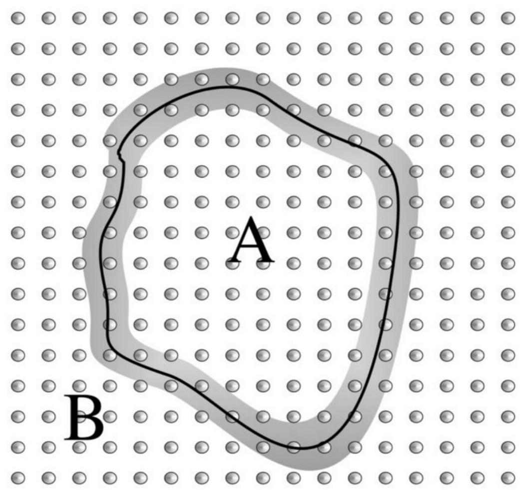

Area law states:

- Describe *low-energy physics* when interactions are local, i.e., short-ranged

Hastings J. Stat. Mech. '07

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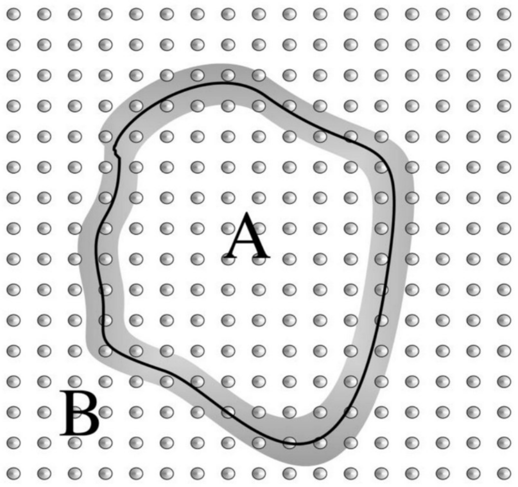

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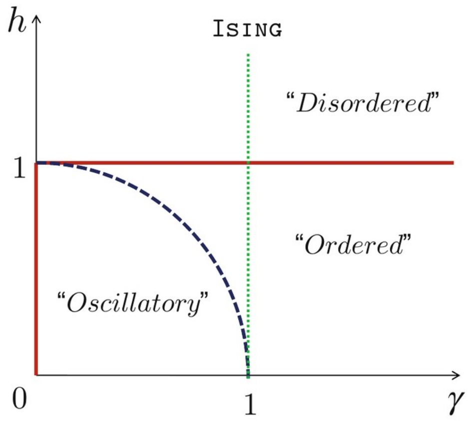
- They admit a tensor-network description. The necessary memory only scales *linearly* in the system size

Hastings J. Stat. Mech. '07
Cirac et al., RMP '22

Phases of quantum matter via complexity



Phase diagram separates regions in parameter space where **distinct states have a common property**

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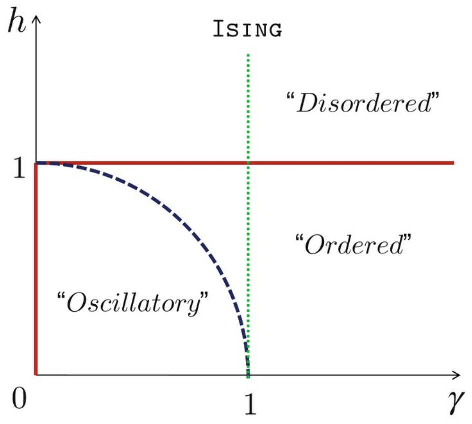

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$$|\Psi_1\rangle_N \sim |\Psi_2\rangle_N$$

Phase = Equivalence class

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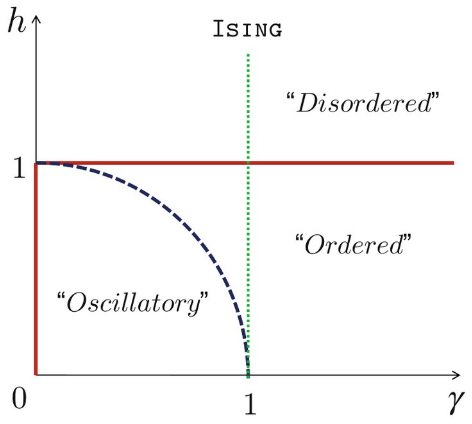

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Hastings, Wen, PRB '05
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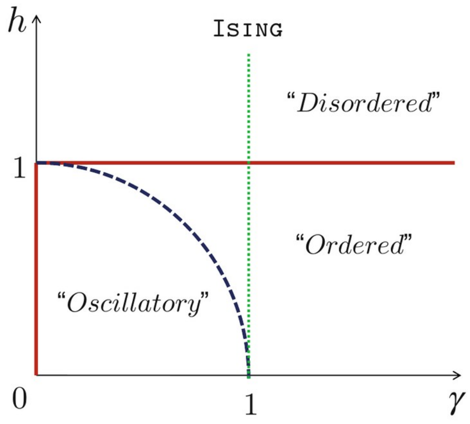

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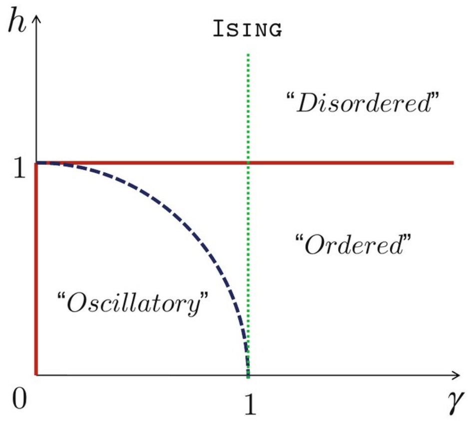

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States in the **trivial** phase are **feasible** to prepare in a quantum simulator

Hastings, Wen, PRB '05
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Classifying topological phases for 1D area law states

Definition (Topological phase): Two translation invariant quantum states are in the same topological phase if there exists a *log-depth local quantum circuit connecting them*, i.e.,

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Example:

$$|\mathrm{Trivial}\rangle_N = |00...0\rangle \quad \Longleftrightarrow \quad |\mathrm{GHZ}\rangle_N = \frac{1}{\sqrt{2}}\left(|0...0\rangle + |1...1\rangle\right)$$

Complexity phase transition

Classifying topological phases for 1D area law states

Is it possible to **connect phases** without a blowup in the complexity?

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Theorem *(Classification* of topological phases in 1D including LOCC)
[Piroli, GS, Cirac, PRL '21]:

All 1D area law states can be connected with log-depth circuits using LOCC

…So what?

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- Topological classification is *not stable* with respect to LOCC
  - What is easy and what is hard **depends on if measurements are considered!**

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***No** complexity phase transition with access to LOCC*

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- Explicit protocol to create states on a digital quantum device (Malz,* GS,* Wei,* Cirac, PRL '23):
  - 1-round of measurements and log(N) circuit depth, or
  - log log (N) rounds of measurements and log log(N) circuit depth

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  - 1-round of measurements and log(N) circuit depth, or
  - log log (N) rounds of measurements and log log(N) circuit depth
  - Exponential improvement over state of the art using LOCC

Lu et al., PRX Quantum ‘22

…So what?

All 1D area law states can be connected with log-depth circuits using LOCC

- Structure of 1D area law states:
  - **Exact complexity** of log (N) for all states in the trivial phase
  - **Symmetry-protected** version (Gunn, GS, Kraft, Kraus '23)

Summary

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Thank you!