Gauge Invariance at Large Charge

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Based on: *"Gauge Invariance at Large Charge", Antipin, Bednyakov, Bersini, Panopoulos, Pikelner, Phys. Rev. Lett.* ***130*** *(2023)*

Large Charge & Semiclassics

- Compute correlation functions without the use of Feynman diagrams
- Obtain information for the spectrum of the QFT in strongly coupled regions

Goal

Calculate contributions to the anomalous dimensions the large charge $\mathcal{O}_Q$ scalar operators by using CFT data

$$\langle \mathcal{O}_Q^\dagger(x_f)\mathcal{O}_Q(x_i)\rangle_{CFT} = \frac{1}{|x_{fi}|^{2\Delta_Q}}$$

Spontaneous Symmetry Breaking (SSB)

Spontaneous symmetry breaking is one of the most interesting processes in Quantum Field Theory and applies to many systems in nature (Standard Model, magnets ...)

- SSB occurs when we choose a minimum of a quantity in our model and this minimun breaks a part of an original theory

$$G \to H, \qquad \mathcal{L}_G \to \mathcal{L}_H, \qquad G/H: \text{ broken generators}$$

Goldstone theorem

The number of broken generators corresponds to the number of massless fields that appear in the action after SSB.

We can systematically construct the most general action after SSB using CCWZ method!

What we compute with Large Charge Expansion?

Using Large Charge Expansion we can evaluate anomalous dimensions to infinite order in perturbation theory by using Effective Field Theory. *(Hellerman, Orlando, Reffert, Watanabe 2015)*

Main Idea

- We compute anomalous dimensions of charged scalar operators exploiting a $U(1)$ global symmetry of the action. This is achieved by fixing the charge of the Hamiltonian

$$H \to H - \mu Q$$

- We compute the lowest lying operators with $Q \gg 1$
- One can show the general formula

$$\Delta_Q = c_1 Q^{\frac{d}{d-1}} + c_2 Q^{\frac{d-2}{d-1}} + \text{ quantum corrections...}$$

Large Charge is blind to the operator identification!

- Map the flat space theory on the cylinder $\mathbb{R} \times S^{D-1}$

Operator/State Correspondence

Every operator of a CFT corresponds to state on Hilbert space

$$\mathcal{O}_{\Delta_Q} \leftrightarrow |Q\rangle, \qquad \frac{\Delta_Q}{R} \leftrightarrow E_Q$$

- One can show that the time evolution operator is expressed as

$$\langle Q|e^{-HT}|Q\rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\Phi e^{-S_{\text{eff}} + \text{charge fixing}}$$

Expanding around classical solutions → SSB (superfluid phase)

$$\langle Q|e^{-HT}|Q\rangle = \underbrace{e^{-S[\Phi = \text{v.e.v}]}}_{\Delta_{-1}} \times \underbrace{\frac{1}{\mathcal{Z}} \int \mathcal{D}(\delta\Phi)\, e^{-S^{(2)}[\delta\Phi]}}_{\propto \Delta_0}$$

Since

$$\langle Q|e^{-HT}|Q\rangle \overset{T\to\infty}{\sim} e^{-E_Q T} = e^{-\frac{\Delta_Q^{\text{cyl}} T}{R}}.$$

at the $T \to \infty$ saddle point approximation gives the energy on the cylinder

$$e^{-\frac{\Delta_Q^{\text{cyl}} T}{R}} = \frac{1}{\mathcal{Z}} \int_{T\to\infty} \mathcal{D}\Phi\, e^{-S_{\text{eff}}+\text{charge fixing}}$$

Result:

The results obtained from semiclassical approximation on the cylinder are matched with the flat space computations on the Wilson Fisher (WF) fixed point

The energy in terms of bare and renormalized quantities is given by

$$\begin{aligned} RE_{\phi^n} =& \frac{1}{\lambda_0} e_{-1}(\lambda_0 Q, D) + e_0(\lambda_0 Q, D) + \lambda_0 e_1(\lambda_0 Q, D) + \ldots \\ =& \frac{1}{\lambda} \bar{e}_{-1}(\lambda Q, RM, D) + \bar{e}_0(\lambda Q, RM, D) + \lambda \bar{e}_1(\lambda Q, RM, D) + \ldots, \end{aligned}$$

At the fixed point λ_* the scaling dimension Δ_Q is organized as

$$\boxed{\Delta_{\phi^n} = \frac{1}{\lambda_*} \Delta_{-1}(\lambda_* Q) + \Delta_0(\lambda_* Q) + \lambda_* \Delta_1(\lambda_* Q) + \ldots}$$

$(\bar{\phi}\phi)^2$ at WF fixed point on flat space

The action is

$$\mathcal{L} = \partial\bar{\phi}\partial\phi + \frac{\lambda_0}{4}(\bar{\phi}\phi)^2, \quad \phi = Z_\phi \phi_r, \quad \lambda_0 = \mu^\epsilon \lambda \, Z_\lambda$$

The β-function in $4-\epsilon$ is given by

$$\frac{d\lambda}{d\log\mu} \equiv \beta(\lambda) = -\epsilon\lambda + \beta_4(\lambda)$$

and the WF fixed point fixed point λ_* is reached by imposing $\beta(\lambda) = 0$ and we express the value of the coupling in terms of ϵ

$$\frac{\lambda_*}{(4\pi)^2} = \frac{\epsilon}{5} + \frac{3}{25}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

Scaling dimension

$$\Delta_{\phi^Q} = Q\left(\frac{d}{2} - 1\right) + Q\left[\frac{\lambda_*}{16\pi^2}\frac{Q-1}{2} - \left(\frac{\lambda_*}{16\pi^2}\right)^2 \frac{2Q^2 - 2Q - 1}{4}\right]$$

Three steps

1. Use Weyl map

$$\int d^D x |\partial\phi|^2 \to \int d^D x \sqrt{-g}\Big(|\partial\phi|^2 + c_D \mathcal{R}|\phi|^2\Big)$$

($\mathcal{R}$ is Ricci scalar curvature)

2. Express $\phi(x) = \frac{\rho(x)}{\sqrt{2}} e^{i\chi(x)}$
3. Impose charge fixing condition

The action reads

$$S_{\text{eff}} = \int d^D x \sqrt{-g}\Big(\frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2 + \frac{1}{2}m^2\rho^2 + \frac{\lambda_0}{24}\rho^4 + \frac{i\,Q}{R^{d-1}\Omega_{d-1}}\dot{\chi}\Big)$$

- Equations of Motion (EOM)

$$-\nabla^2\rho + \rho[(\partial\chi)^2 + m^2] + \frac{\lambda}{6}\rho^3 = 0\,,$$
$$\nabla_\mu(\rho^2 g^{\mu\nu}\partial_\nu\chi) = 0\,,$$
$$\rho^2\dot{\chi} = -\frac{iQ}{R^{D-1}\Omega_{D-1}}$$

- Write dynamical fields → (classical solutions+variations)

$$\rho(x) = f + r(x), \quad \chi(x) = -i\mu + \frac{1}{\sqrt{f}}\,\pi(x)$$

where μ is the chemical potential...

- Use the representation

$$\langle Q|e^{-HT}|Q\rangle = e^{-S_{\rm cl}(f,\mu)} \times \frac{1}{\mathcal{Z}}\int \mathcal{D}r\mathcal{D}\pi\, e^{-S^{(2)}(r,\pi)}$$

Classical contributions Δ_{-1}

Plugging the classical solutions to S_{cl} we get

$$S_{\text{cl}} = \frac{Q}{2}\left(\frac{3}{2}\mu + \frac{1}{2}\frac{m^2}{\mu}\right)$$

Use EOM and solve for the critical point $(*)$

$$\mu(\mu^2 - m^2) = \frac{\lambda_0\, Q}{4R^{D-1}\Omega_{D-1}} \rightarrow \boxed{R\mu_* = \frac{3^{1/3} + \left[9\frac{\lambda_* Q}{(4\pi)^2} - \sqrt{81\frac{(\lambda_* Q)^2}{(4\pi)^4} - 3}\right]^{2/3}}{3^{2/3}\left[9\frac{\lambda_* Q}{(4\pi)^2} - \sqrt{81\frac{(\lambda_* Q)^2}{(4\pi)^4} - 3}\right]^{1/3}}}$$

Solution

$$4\,\Delta_{-1} = \frac{3^{\frac{2}{3}}\left(x + \sqrt{-3 + x^2}\right)^{\frac{1}{3}}}{3^{\frac{1}{3}} + \left(x + \sqrt{-3 + x^2}\right)^{\frac{2}{3}}} + \frac{3^{\frac{1}{3}}\left(3^{\frac{1}{3}} + \left(x + \sqrt{-3 + x^2}\right)^{\frac{2}{3}}\right)}{\left(x + \sqrt{-3 + x^2}\right)^{\frac{1}{3}}}$$

Small and Large limits

$$\frac{\Delta_{-1}}{\lambda_*} = \begin{cases} Q\left[1 + \frac{1}{2}\left(\frac{\lambda_* Q}{16\pi^2}\right) - \frac{1}{2}\left(\frac{\lambda_* Q}{16\pi^2}\right)^2 + \mathcal{O}\left(\frac{(\lambda_* Q)^3}{(4\pi)^6}\right)\right], & \lambda_* Q \ll (4\pi)^2, \\ \frac{8\pi^2}{\lambda_*}\left[\frac{3}{4}\left(\frac{\lambda_* Q}{8\pi^2}\right)^{4/3} + \frac{1}{2}\left(\frac{\lambda_* Q}{8\pi^2}\right)^{2/3} + \mathcal{O}(1)\right], & \lambda_* Q \gg (4\pi)^2. \end{cases}$$

Identification of operator:

The scaling dimension obtained from semiclassics reproduces the scaling dimension of ϕ^Q operator at the WF fixed point in flat space.

Next to Leading Order

The quadratic action takes the form

$$S^{(2)} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{D-1} \Big[\frac{1}{2} r(-\nabla^2) r + \frac{1}{2}\pi(-\nabla^2)\pi - 2i\mu r \partial_\tau \pi + (\mu^2 - m^2) r^2 \Big]$$

- $\omega_\pm(\ell) = J_\ell + 3\mu^2 + m^2 \pm \sqrt{4J_\ell^2 \mu^2 + (3\mu^2 - m^2)^2}$
- $e_0(\lambda_0 Q, D) = \frac{R}{2} \sum_{\ell=0}^{\infty} n_\ell \left[\omega_+(\ell) + \omega_-(\ell) \right]$
- $-\nabla^2 = -\partial_\tau^2 + \left(-\nabla^2_{S^{D-1}} \right)$

The sum diverges and is regularized using ζ-regulatization by subtracting systematically the infinite part concluding to

$$\Delta_0 = -\frac{15\mu_* R^4 + 6\mu_*^2 R^2 - 5}{16} + \frac{1}{2} \sum_{\ell=1}^{\infty} \underbrace{\sigma(\ell)}_{\sum_\ell^\infty - \text{ infinite part}} + \frac{\sqrt{3\mu^2 R^2 - 1}}{\sqrt{2}}$$

Small & Large limits

$$\Delta_0 = \begin{cases} -3\left(\frac{\lambda_* Q}{16\pi^2}\right) + \frac{1}{2}\left(\frac{\lambda_* Q}{16\pi^2}\right)^2 + \mathcal{O}\left(\frac{(\lambda_* Q)^3}{(4\pi)^6}\right) & \lambda_* Q \ll (4\pi)^2, \\ \left[\alpha + \frac{5}{24}\log\left(\frac{\lambda_* n}{8\pi^2}\right)\right]\left(\frac{\lambda_* n}{8\pi^2}\right)^{4/3} & \\ \quad + \left[\beta - \frac{5}{36}\log\left(\frac{\lambda_* n}{8\pi^2}\right)\right]\left(\frac{\lambda_* n}{8\pi^2}\right)^{2/3} + \mathcal{O}(1)\,, & \lambda_* Q \gg (4\pi)^2\,. \end{cases}$$

$$\alpha = -0.5753315(3), \qquad \beta = -0.93715(9).$$

Gauge Invariance at Large Charge

Consider the action

$$S = \int d^4x \left(\frac{1}{4} F_{\mu\nu}^2 + (D_\mu \phi)^\dagger D_\mu \phi + \frac{\lambda}{24} (\bar{\phi}\phi)^2 \right)$$

Equations of Motion

$$-D^\mu D_\mu \phi + m^2 \bar{\phi} + \frac{\lambda_0}{12} (\bar{\phi}\phi)\bar{\phi} = 0\,, \qquad \partial_\mu F^{\mu\nu} = J^\nu,$$

Fixed Point

$$\lambda_* = \frac{3}{20} \left(19\epsilon \pm i\sqrt{719}\epsilon \right)\,, \qquad e_*^2 = 24\pi^2 \epsilon$$

- The 2-pt function

$$G_\phi(x_f - x_i) \equiv \langle \phi^\dagger(x_f)\phi(x_i)\rangle$$

violates gauge invariance and we should search for gauge invariant correlator for the WF (Kleinert, Schakel '02, '05) .

Gauge invariant correlators

	Schwinger Line	Dirac Line
2-pt function	$\langle \phi^\dagger(x_f) e^{-ie\int_{x_i}^{x_f} dx^\mu A_\mu(x)} \phi(x_i)\rangle$,	$\langle \phi^\dagger(x_f) e^{-ie\int_{x_i}^{x_f} d^d x A_\mu(x) J^\mu} \phi(x_i)\rangle$
Critical exponent	$\eta_s = -\frac{3}{4\pi} e_*$	$\eta_D = -\frac{3}{8\pi} e_*$

Note that $\eta_D > \eta_s$

More on Dirac line (Kleinert, Schakel '05)

The current has the form $J_\mu = J'_\mu(z - x_f) - J'_\mu(z - x_i)$ where

$$J'_\mu(z) = -i \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu}{k^2} e^{ik\cdot z} = -\frac{\Gamma(D/2-1)}{4\pi^{D/2}} \partial_\mu \frac{1}{z^{D-2}} \ .$$

Based on the definitions above

$$G_D(x_f - x_i) = \langle \bar{\phi}_{nl}(x_f) \phi_{nl}(x_i) \rangle$$

where

$$\phi^Q_{nl}(x) \equiv e^{-ieQ \int d^D z J'_\mu(z-x) A^\mu(x)} \phi(x)$$

has been proposed as the *non-local order parameter* for the superconducting phase transition.

In Landau gauge $\phi_{nl}(x)$ reduces to $\phi(x)$

Semiclassics for Abelian Higgs model

- Parametrizing $\phi = \rho(x)e^{i\chi(x)}$ and expanding around v.e.v

  $$\rho(x) = f + r(x), \quad \chi(x) = -i\mu + \frac{1}{\sqrt{f}}\,\pi(x), \quad A_\mu(x) = 0 + A_\mu(x)$$

  the classical action is not affected and gives the same Δ_{-1}.

- **Essential difference appears at 1-loop order**

The quadratic action of fluctuation is given by

$$S^{(2)} = \int d^D x \sqrt{-g} \Big(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu r)^2 + \frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2} 2(m^2 - \mu^2) r^2 - 2 i \mu r \partial_\tau \pi + e f \partial_\mu \pi A^\mu - 2 i e \mu f r A_0 + \frac{1}{2} (ef)^2 A_\mu A^\mu \Big)$$

- ***After Higgs mechanism there is a local residual symmetry***

$$\delta r = 0, \quad \delta \pi = f\, \alpha(x), \quad \delta A_\mu = -\frac{1}{e} \partial_\mu \alpha(x)$$

- Gauge Fixing via R_ξ-gauge

$$S^{(2)} \to S^{(2)} + \frac{1}{2} \int d^D x \sqrt{g}\, G^2, \qquad G^2 = \frac{1}{\xi} \left(\nabla_\mu A^\mu + e f \pi \right)^2$$

The computation reduces to

$$\frac{1}{\mathcal{Z}_{\mathcal{A}}} \int \mathcal{D}A_\mu \mathcal{D}\Phi \, e^{-\int d^D x \left(\mathcal{L}_{\text{1-loop}}(r,\pi,A)+\frac{1}{2}G^2\right)} \det\left(\frac{\delta G}{\delta \alpha}\right)$$

The quadratic part of the exponential is written

$$\begin{aligned}\mathcal{L}_{\text{1-loop}} =& \frac{1}{2} A_\mu \left(-g^{\mu\nu}\nabla^2 + \mathcal{R}^{\mu\nu} + \left(1-\frac{1}{\xi}\right)\nabla^\mu\nabla^\nu + (ef)^2 g^{\mu\nu}\right) A_\nu \\ &+ \frac{1}{2}\begin{pmatrix} r & \pi \end{pmatrix} \begin{pmatrix} -\nabla^2 + 2(\mu^2 - m^2) & -2i\mu\partial_\tau \\ 2i\mu\partial_\tau & -\nabla^2 + \frac{1}{\xi}e^2 f^2 \end{pmatrix} \begin{pmatrix} r \\ \pi \end{pmatrix} \\ &- 2if\mu r A^0 + ef\left(1-\frac{1}{\xi}\right) A_\mu \partial^\mu \pi\end{aligned}$$

- $\mathcal{R}^{\mu\nu} = \frac{D-2}{R^2} g^{\mu\nu}$ *is the Ricci tensor on* $\boldsymbol{R} \times S^{D-1}$

- Represent determinant via *ghosts* :

$$\det\left(\frac{\delta G}{\delta\alpha}\right) = \exp\left(-\int \bar{c}\left(-\nabla^2 + (ef)^2\right)c\right)$$

- Split the gauge field $A^i = B^i + C^i$ as

$$\nabla_i B^i = 0 \quad (\text{kernel of } \nabla^i), \qquad C^i = \nabla^i f \quad (\text{image of } \nabla^i)$$

$-\nabla^2_{S^{D-1}}$	**scalars**	**vectors**
eigenvalues	$\frac{1}{R^2}\ell(\ell+D-2)$	$\frac{1}{R^2}\left(\ell\left(\ell+D-2\right)-1\right)$
degeneracies	n_b	n_A

$$n_b(\ell) = \frac{(2\ell+D-2)\Gamma(\ell+D-2)}{\Gamma(D-1)\Gamma(\ell+1)}$$

$$n_A(\ell) = \frac{\ell(\ell+D-2)(2\ell+D-2)\Gamma(\ell+D-3)}{\Gamma(\ell+2)\Gamma(D-2)}$$

- $\det B = -\partial_\tau^2 - \nabla^2_{S^{D-1}} + \frac{D-2}{R^2} + (ef)^2$

- Scalar field matrix (r, π, A_0, B_i)

$$\begin{pmatrix} -\omega^2 + J_\ell^2 + 2(\mu^2 - m^2) & -2i\mu\omega & -2ie\mu f & 0 \\ 2i\mu\omega & -\omega^2 + J_\ell^2 + \frac{1}{\xi} e^2 f^2 & -ef\left(1 - \frac{1}{\xi}\right)\omega & -ief\left(1 - \frac{1}{\xi}\right)|J_\ell| \\ -2ie\mu f & ef\left(1 - \frac{1}{\xi}\right)\omega & -\frac{1}{\xi}\omega^2 + J_\ell^2 + (ef)^2 & i\left(1 - \frac{1}{\xi}\right)\omega|J_\ell| \\ 0 & ief\left(1 - \frac{1}{\xi}\right)|J_\ell| & i\left(1 - \frac{1}{\xi}\right)\omega|J_\ell| & -\omega^2 + \frac{1}{\xi} J^2_{\ell(s)} + (ef)^2 \end{pmatrix}$$

giving

$$\boxed{\xi \det \Phi = (\omega + \omega_+^2)(\omega + \omega_-^2)(\omega + \omega_1^2)^2}$$

Field	ω_ℓ	ℓ_0
B_i	$\sqrt{J^2_{\ell(v)} + (d-2) + e^2 f^2}$	1
C_i	$\sqrt{J^2_\ell + e^2 f^2}$	1
$(c, \bar{c})$	$\sqrt{J^2_\ell + e^2 f^2}$	0
A_0	$\sqrt{J^2_\ell + e^2 f^2}$	0
ϕ	$\sqrt{J^2_\ell + 3\mu^2 - m^2 + \frac{1}{2}e^2 f^2 \pm \sqrt{\left(3\mu^2 - m^2 - \frac{1}{2}e^2 f^2\right)^2 + 4J^2_\ell \mu^2}}$	0

Table: The fields and their energies as a function of the chemical potentials with a nonvanishing VEV for $\phi, \bar{\phi}$. Note that J^2_ℓ are the Laplacian scalar eigenvalues and $J^2_{\ell(v)}$ are the vector eigenvalues.

Following the standard steps of infinity cancelations we obtain the 1-loop correction

$$\Delta_0 = \frac{1}{16}\left(-15\mu^4 - 6\mu^2 + 8\sqrt{6\mu^2-2} + 5\right) + \frac{1}{2}\sum_{l=1}\sigma(\ell) - \frac{3e^2\left(\mu^2-1\right)\left(3e^2\left(7\mu^2+5\right)+16\pi^2 g\left(5-9\mu^2\right)\right)}{2048\pi^4 g^2}$$

where

$$\sigma(\ell) = \left(\sqrt{\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} + 3\mu^2 + \ell(\ell+2) - 1 - \sqrt{\left(\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} - 3\mu^2 + 1\right)^2 + 4\ell(\ell+2)\mu^2}}\right.$$
$$\left. + \sqrt{\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} + 3\mu^2 + \ell(\ell+2) - 1 + \sqrt{\left(\frac{3e^2\left(\mu^2-1\right)}{16\pi^2 g} - 3\mu^2 + 1\right)^2 + 4\ell(\ell+2)\mu^2}}\right)(\ell+1)^2$$
$$+ 2\ell(\ell+2)\sqrt{\frac{3e^2\left(\mu^2-1\right)}{8\pi^2 g} + \ell(\ell+2) + 1} - \frac{-5\mu^4 + 10\mu^2 + 16l^2(l+1)(l+2) + 8l(l+1)\mu^2 - 5}{4l}$$
$$+ \frac{9e^2\left(\mu^2-1\right)\left(3e^2\left(\mu^2-1\right) - 16\pi^2 g\left(\mu^2 + 2l(l+1) - 1\right)\right)}{512\pi^4 g^2 l}$$

Result

Operator Identification

Results on the cylinder reproduce loop computations on flat space at the WF fixed point of the Dirac operator in Landau gauge $\partial^\mu A_\mu = 0$*.*

This was unexpected due to $\eta_D > \eta_S$ **!**

Infinite order results for charged sectors of the Standard Model

(Antipin, Bersini, P.P, Sannino, Wang, arXiv:2312.12963)

We obtain for the Standard Model the anomalous dimensions of composite Higgs operators.

In the presence of non-Abelian $SU(2)_W$ gauge fields semiclassical approach is characterized by a *vector condensate*

Details:

- LO contributions from three generations of quarks and $SU(2)_W \times U(1)_Y$ gauge bosons to the scaling dimensions.
- LO and NLO at the global $SU(2)$ limit and operator identification is demonstrated.

The results are checked against perturbation theory up to three loops and are found to be in perfect agreement!

Thank you!