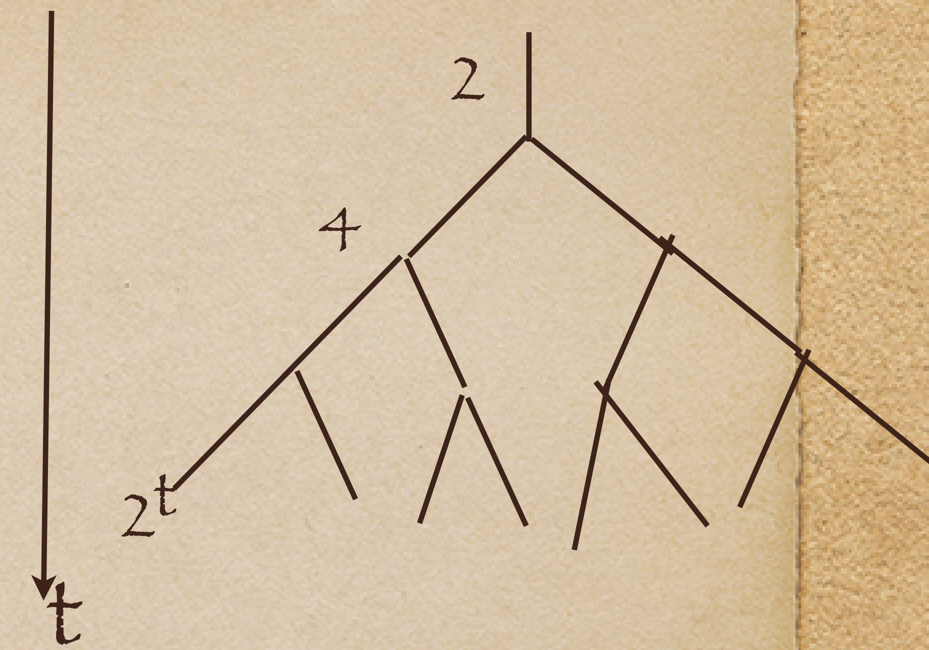
A bit of cascade theory

Simple toy:



At each 1 radiation length, a particle split into two.
In each step, each particle loses energy, ε.
Then, at t, the total energy lost so far is $\sim \varepsilon \times 2^{t+1}$. If this becomes the incident energy, E_0, no more cascade can develop. Thus, the maximum number of particles and the depth of the maximum are:

$$N_{max} \propto \frac{E_0}{\varepsilon}$$

$$T_{max} \propto \log(\frac{E_0}{\varepsilon})$$

This holds even for more complex real showers.

Diffusion Eq.

Rossi: High Energy Particles
Nishimura: HANDBUCH DER PHYSIK

$\pi(E,t)$: **number of e in** $(E, E+dE)$ **at t in radiation length**
$\gamma(E,t)$**: number of** γ

Complete screening: v=E'/E

Radiation + Pair creation + constant dE/dt

Approx. A

Approx. B

Change of the number in Δt.

Decrease: $\int_0^E \pi(E,t)\varphi(E,E')dE'\Delta t = \int_0^1 \pi(E,t)\varphi_B(v)dv\Delta t$

Increase: $\int_E^\infty \pi(E',t)\varphi(E',E'-E)dE'\Delta t = \int_0^1 \frac{1}{1-v}\pi(\frac{E}{1-v},t)\varphi_B(v)dv\Delta t$

Increase: $\int_E^\infty \gamma(E',t)2\varphi(E',E)dE'\Delta t = 2\int_0^1 \gamma(\frac{E}{v},t)\varphi_p(v)\frac{dv}{v}\Delta t$

Ionization energy loss: $\pi(E+\varepsilon\Delta t,t)$ lose energy $\varepsilon\Delta t$ and fall in $\pi(E,t+\Delta t)$ so that the net loss during Δt is

$$\pi(E,t+\Delta t)-\pi(E,t)=$$
$$\pi(E,t)+\varepsilon\frac{\partial\pi(E,t)}{\partial E}\Delta t-\pi(E,t)=\varepsilon\frac{\partial\pi(E,t)}{\partial E}\Delta t$$

Hence

$$\frac{\partial\pi}{\partial t} = -\int_0^1 \left[\pi(E,t) - \frac{1}{1-v}\pi(\frac{E}{1-v},t)\right]\varphi_B(v)dv$$
$$+2\int_0^1 \gamma(\frac{E}{v},t)\varphi_p(v)\frac{dv}{v} + \varepsilon\frac{\partial\pi(E,t)}{\partial E}$$

Similarly we get

$$\frac{\partial\gamma}{\partial t} = \int_0^1 \pi(\frac{E}{v},t)\varphi_B(v)\frac{dv}{v} - \sigma_0\gamma(E,t)$$

where $\sigma_0 = \int_0^1 \varphi(v)dv \sim 7/9 = 0.77$

The solution of these eq. can be obtained by using integral transformation (Mellin transformation). We show first a solution for the number of electrons with energy greater than E for the case of electron incident with energy E_0.

For Approxiamtion A, the main term can be written as

$$\Pi_A \equiv \int_E \pi(E,t)dE = \frac{1}{2\pi i}\int ds \frac{1}{s}(\frac{E_0}{E})^s H_1(s) e^{\lambda_1(s)t}$$

where $\lambda_1(s)$ and $H_1(s)$ are functions derived from integrating $\varphi_B(v), \varphi_P(v)$: They include $A(s) = \int_0^1 (1-(1-v)^2)\varphi_B(v)dv$, $B(s) = 2\int_0^1 v^s \varphi_P(v)dv$, $\int_0^1 v^s \varphi_B(v)dv$

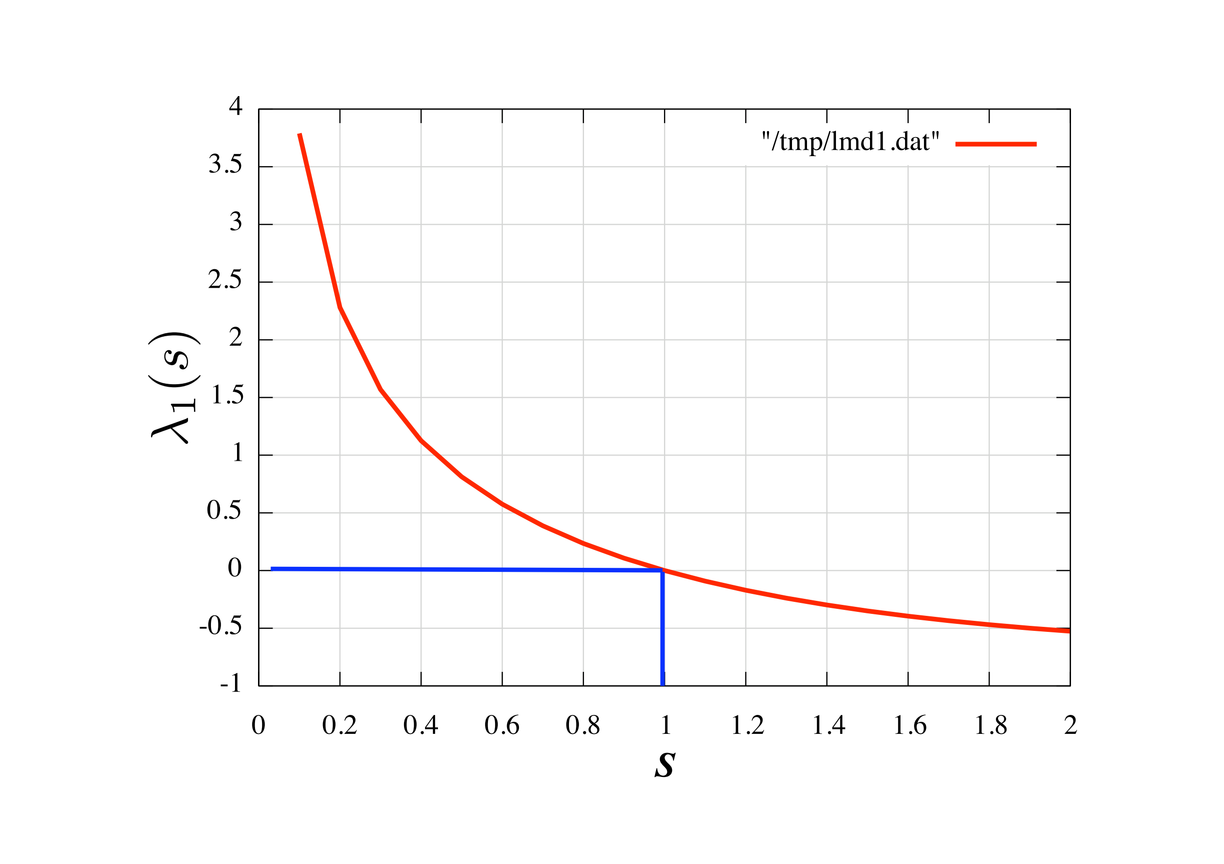
The integration (path is running along the imaginary axis) may be evaluated by the saddle point method (note: H_1 is slowly varying function and other part is collected as $\exp(\lambda_1 + \log(E_0/E) - \log s)$):

$$\Pi_A = \frac{H_1(s)(\frac{E_0}{E})^s e^{\lambda_1(s)t}}{s\sqrt{2\pi(\lambda_1''(s)t + \frac{1}{s^2})}}$$

where s is the solution of

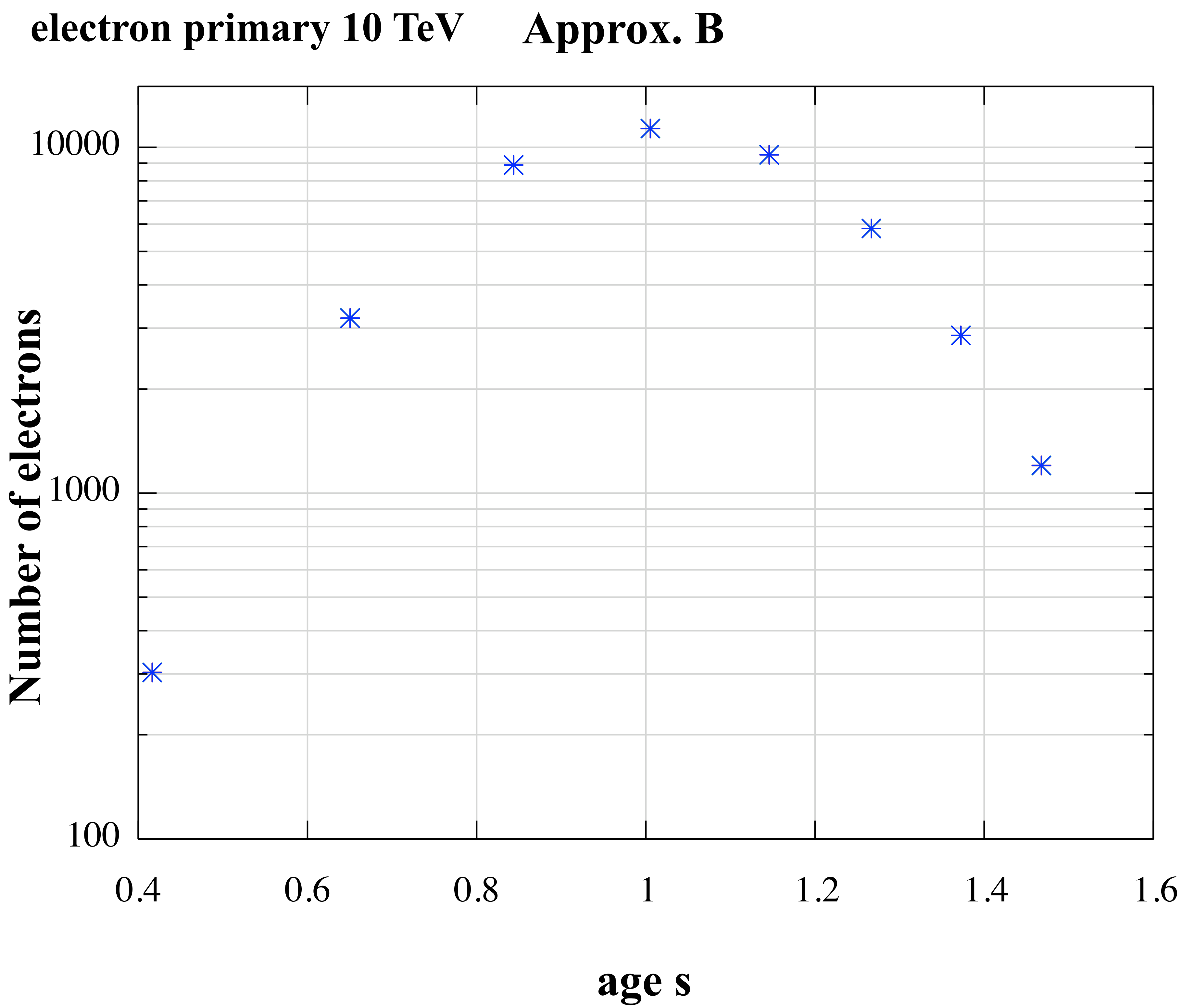
$$\lambda_1'(s)t + \log(\frac{E_0}{E}) - \frac{1}{s} = 0$$

Transition of Π is governed by $e^{\lambda_1(s)t}$. λ_1 is positive at $s < 1$, 0 at $s = 1$ and negative at $s > 1$.
Therefore, at $s = 1$ the shower maximum is realized: s is called *age parameter*.

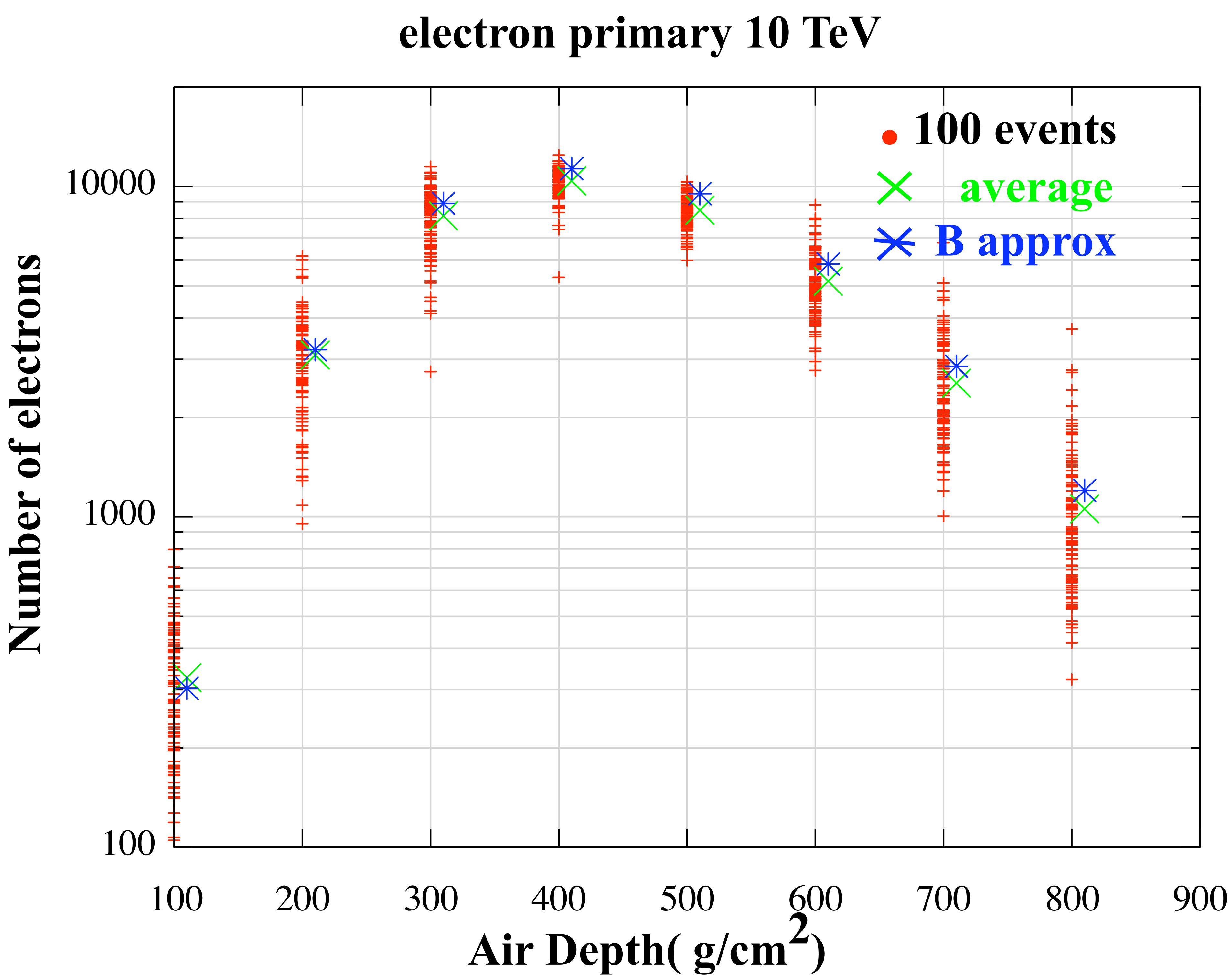


For Approximation B, treatment is more complex, but similar expression can be obtained. The total number of electrons (i.e, $E \to 0$), Π_B can be expressed in a similar form as Π_A by replacing $E \to \varepsilon$. The function corresponding to $H_1(s)$ becomes much more complex (expressed via H_1, Γ and some implicit function of λ_1 etc). However, the transition is still governed by $e^{\lambda_1(s)t}$ so that concept of age is still valid.

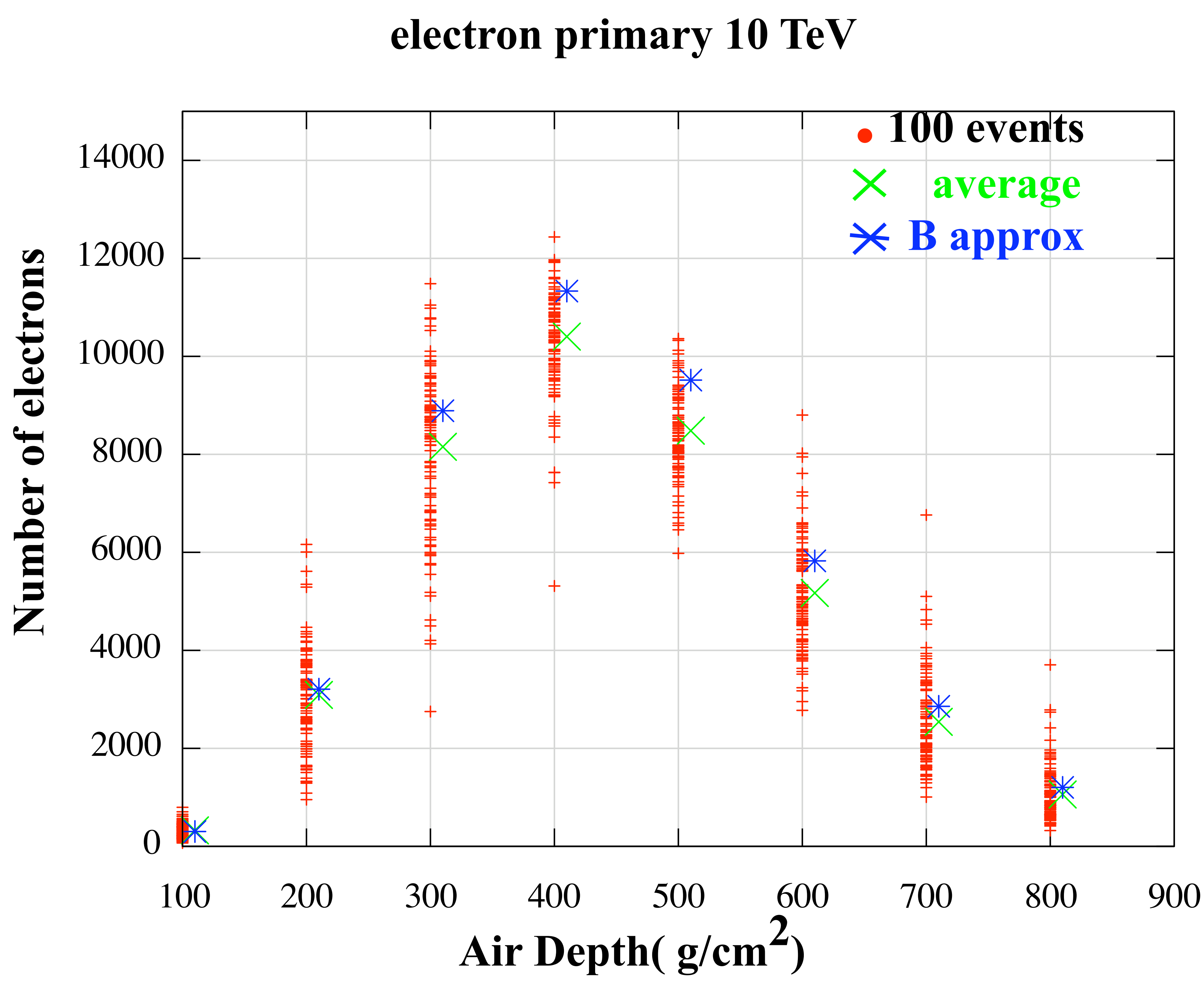
The *track length* $\int \Pi_B(t)dt$ is very close to $\frac{E_0}{\varepsilon}$. At high energies and for low Z material (say, Air), approximation B is not so bad.



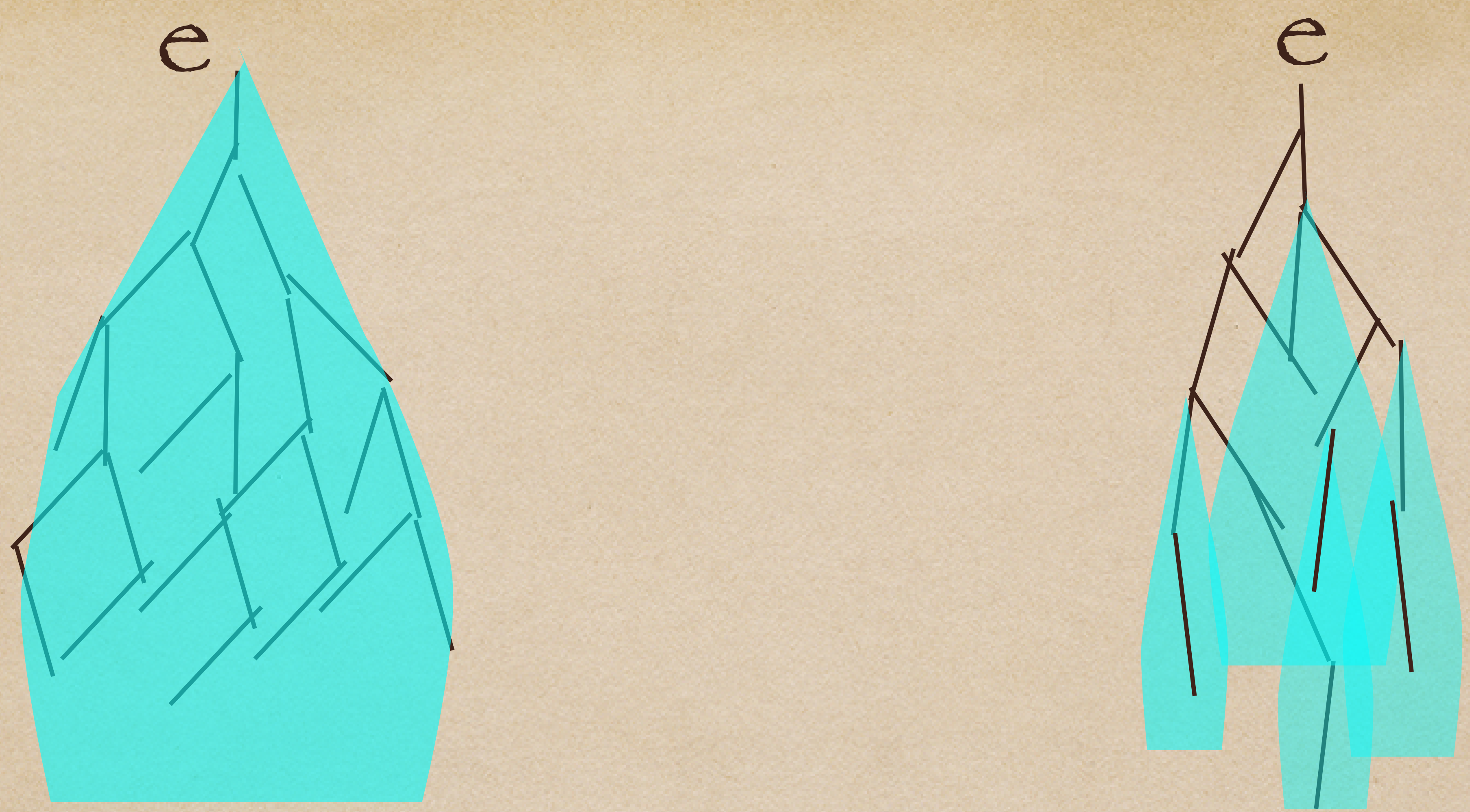
electron primary 10 TeV
Approx. B
Number of electrons
10000
1000
100
0.4
0.6
0.8
1
1.2
1.4
1.6
age s



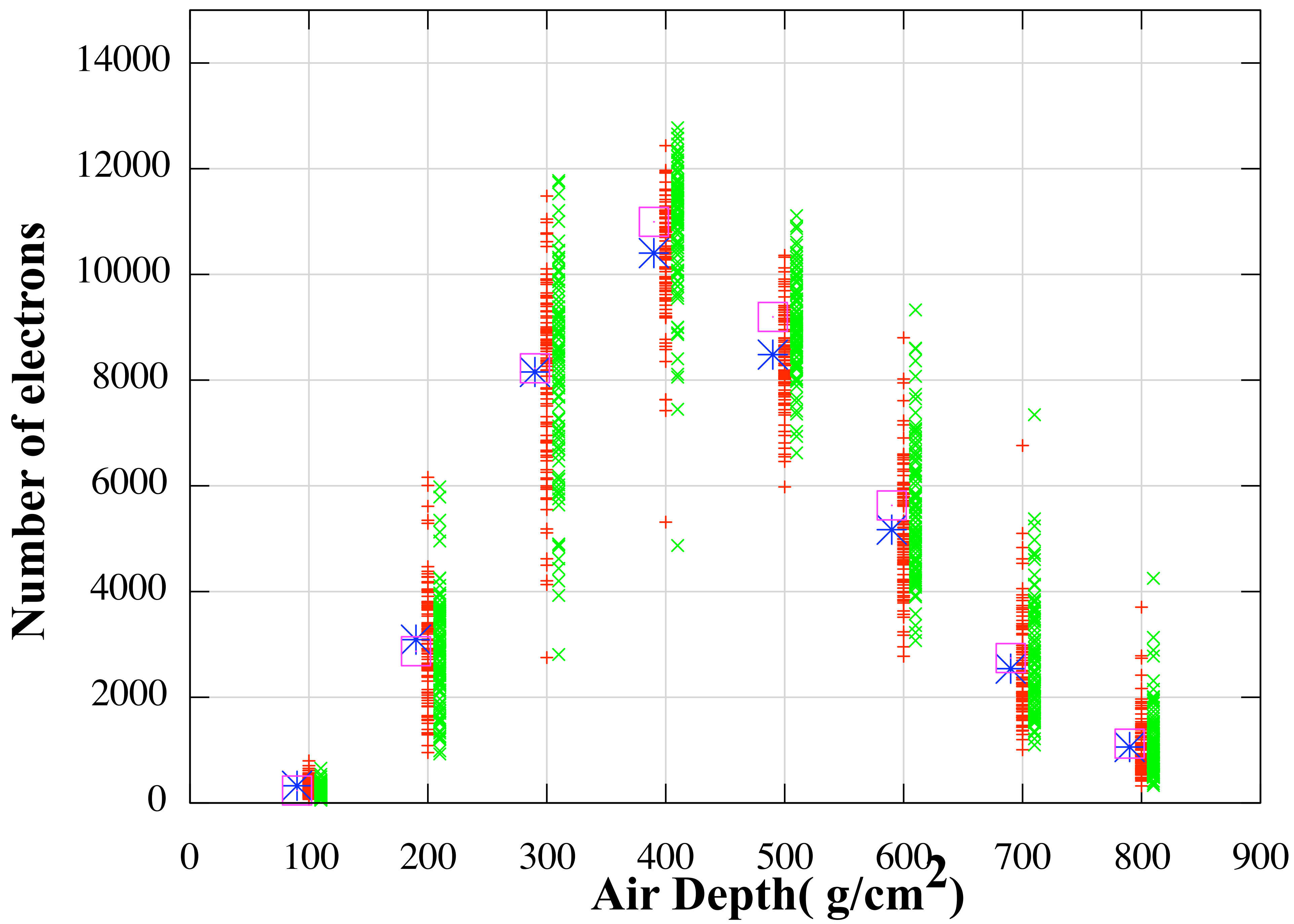
electron primary 10 TeV
100 events
average
B approx
Number of electrons
10000
1000
100
100
200
300
400
500
600
700
800
900
Air Depth(g/cm2)

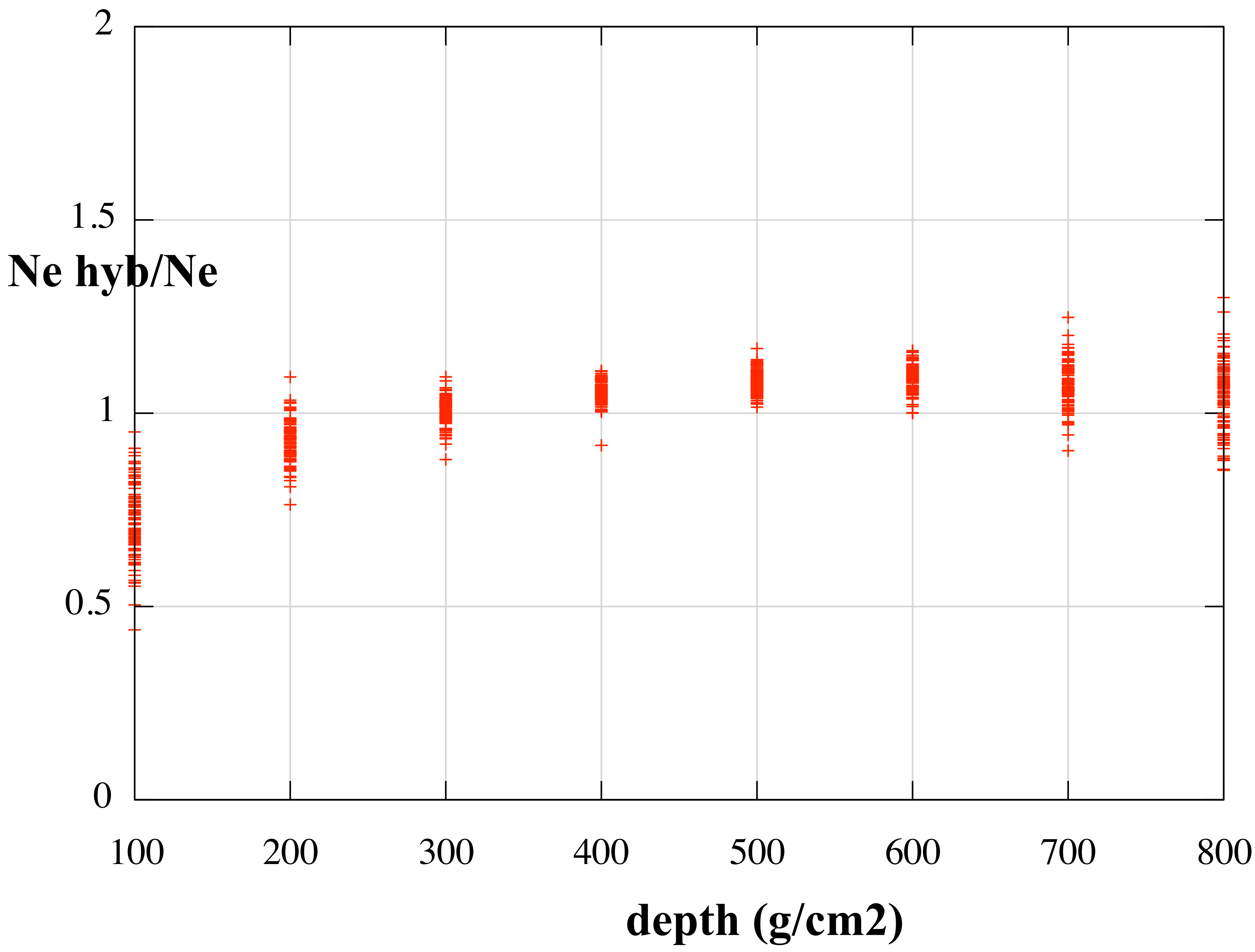


electron primary 10 TeV
100 events
average
B approx
Number of electrons
Air Depth(g/cm2)
0
2000
4000
6000
8000
10000
12000
14000
100
200
300
400
500
600
700
800
900



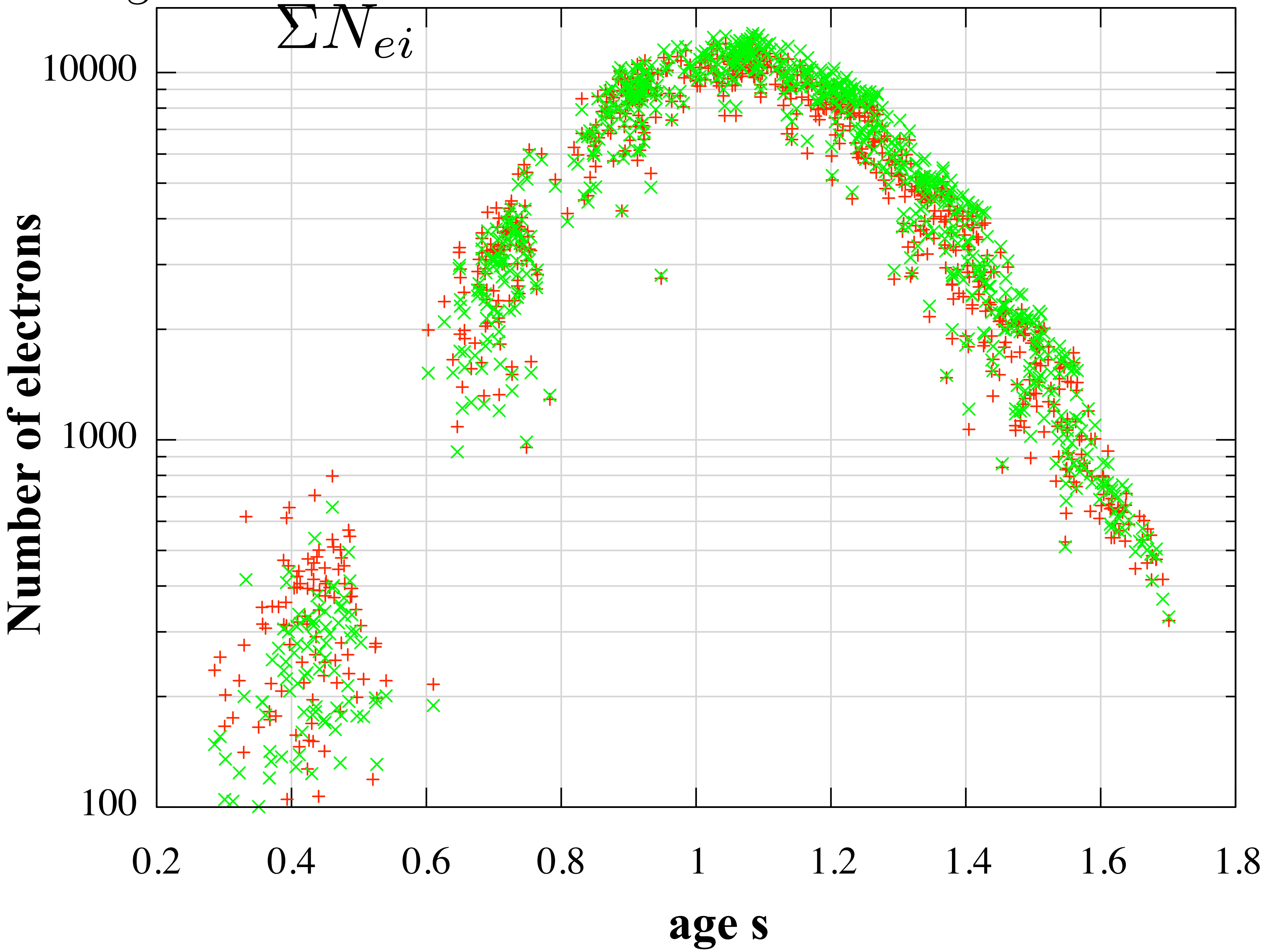
We wait for Ee becomes < say, $E_0/100$

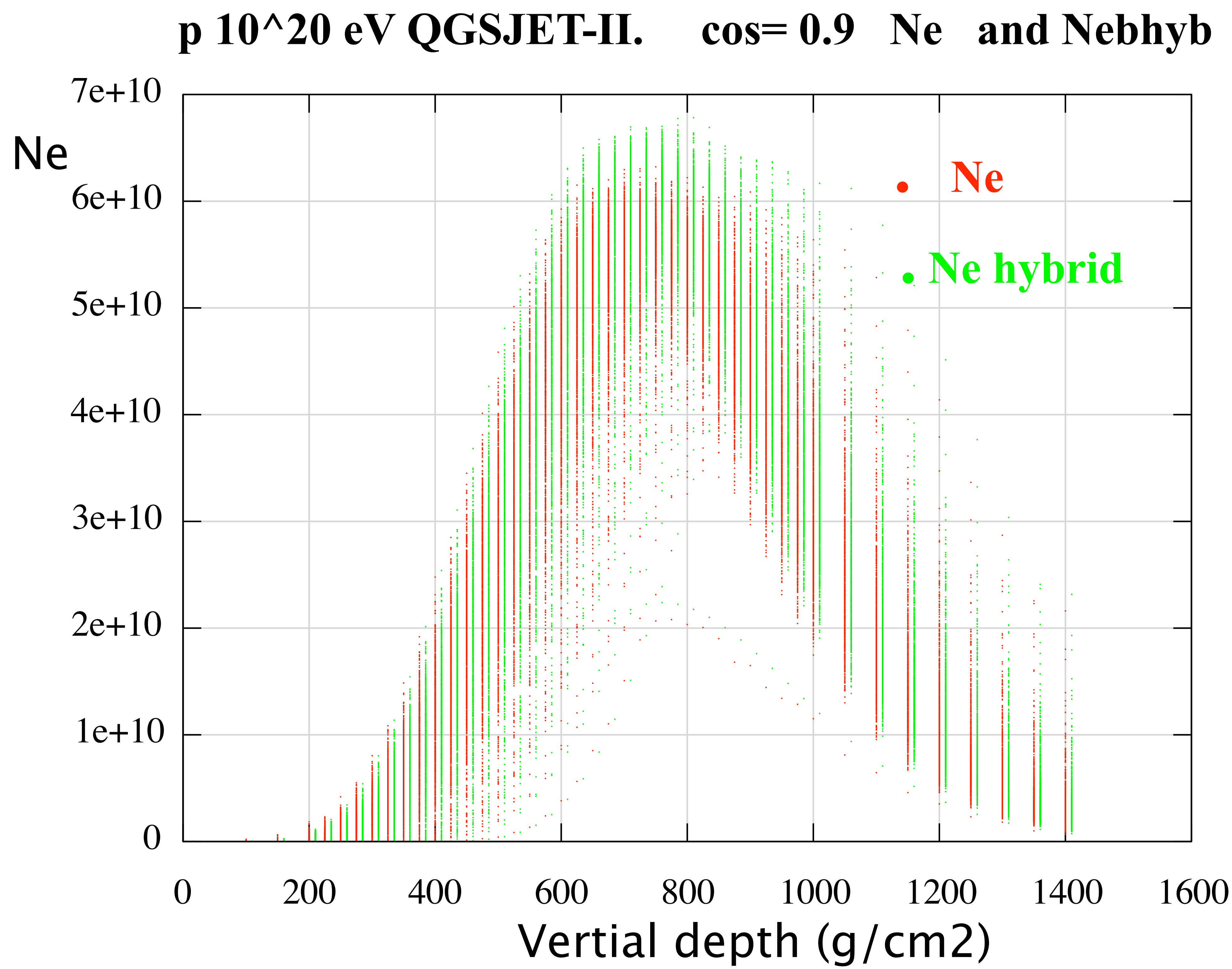

Number of electrons
14000
12000
10000
8000
6000
4000
2000
0
0
100
200
300
400
500
600
700
800
900
Air Depth(g/cm2)

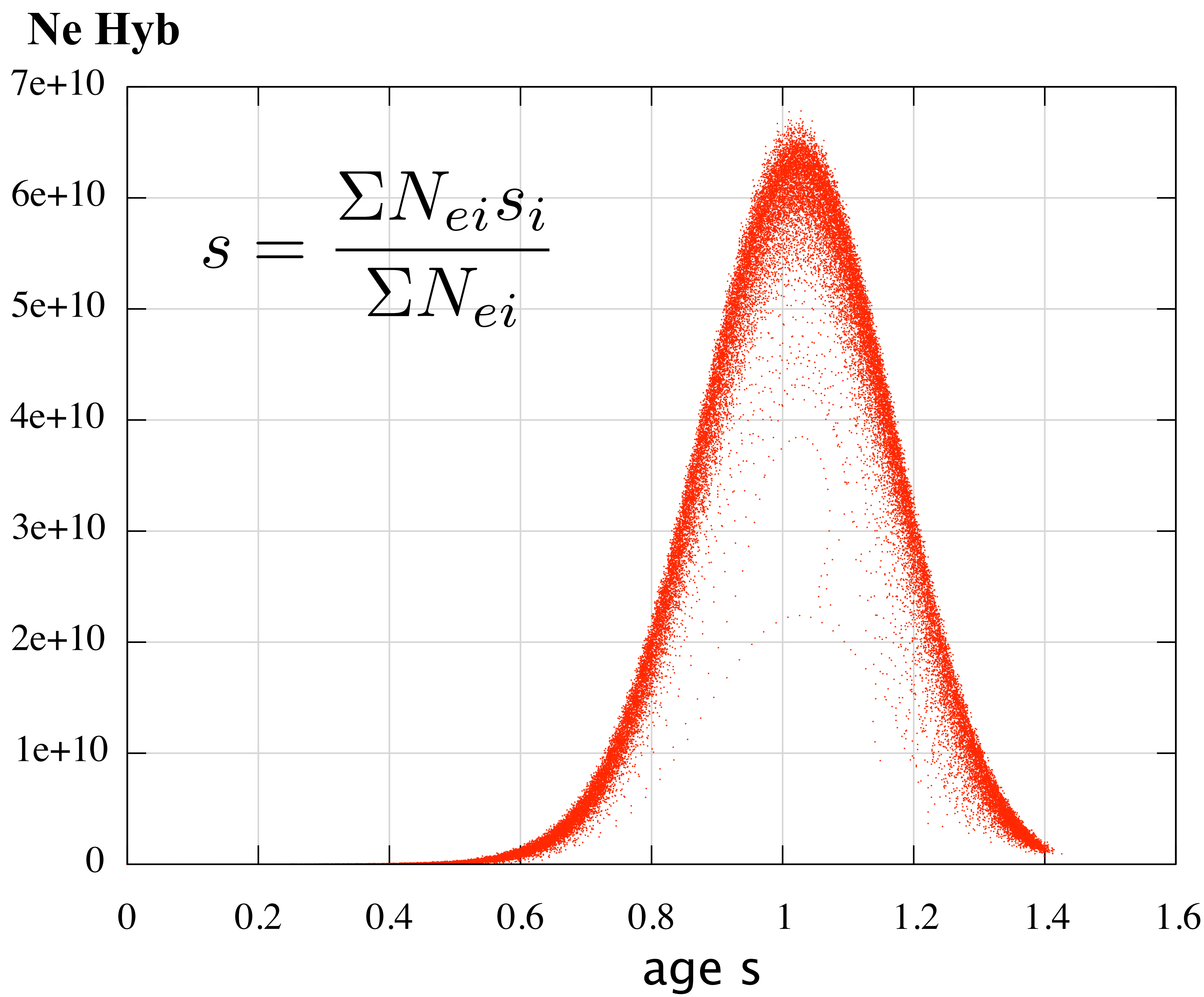


2
1.5
Ne hyb/Ne
1
0.5
0
100
200
300
400
500
600
700
800
depth (g/cm2)

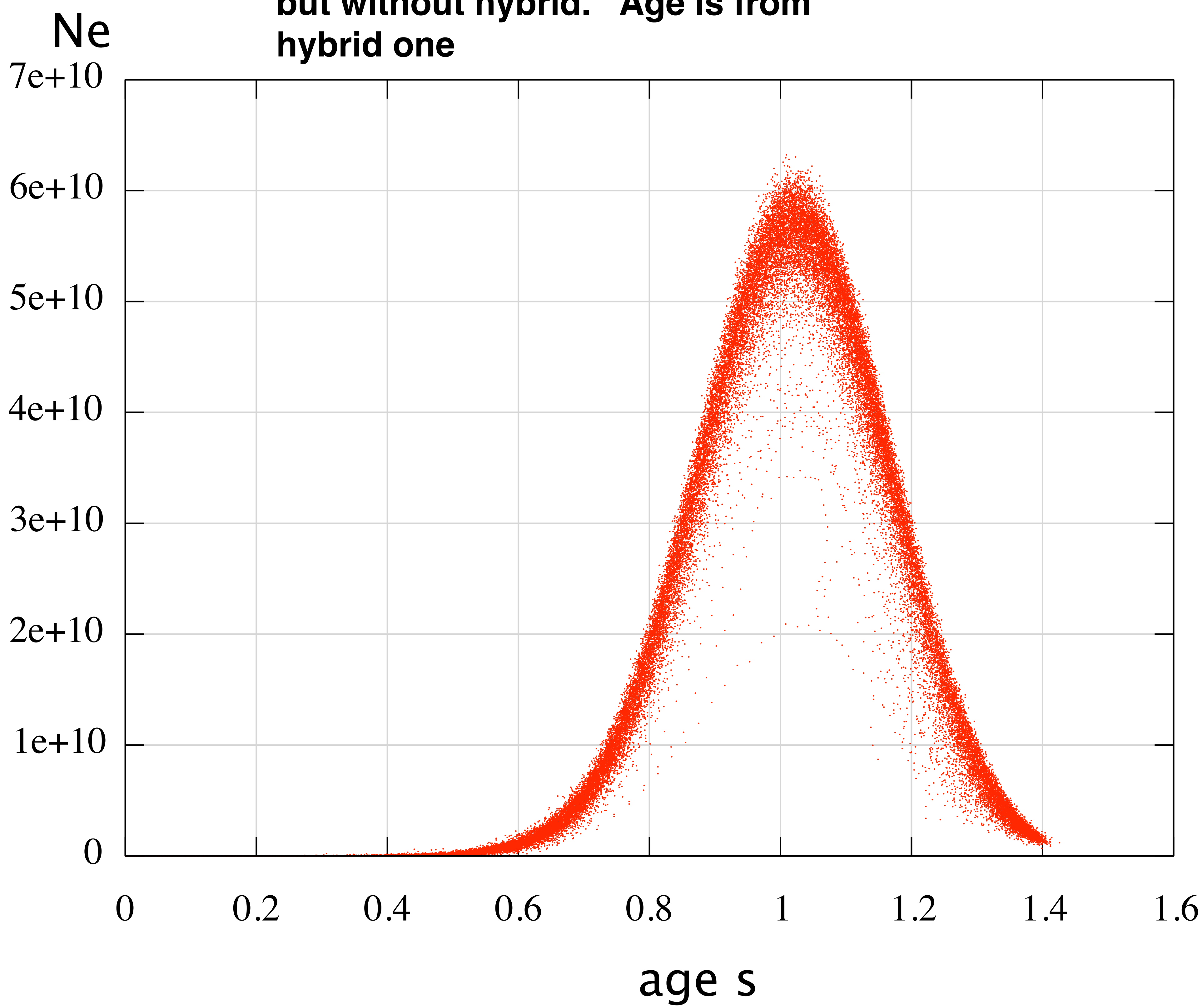
$$s = \frac{\Sigma N_{ei} s_i}{\Sigma N_{ei}}$$

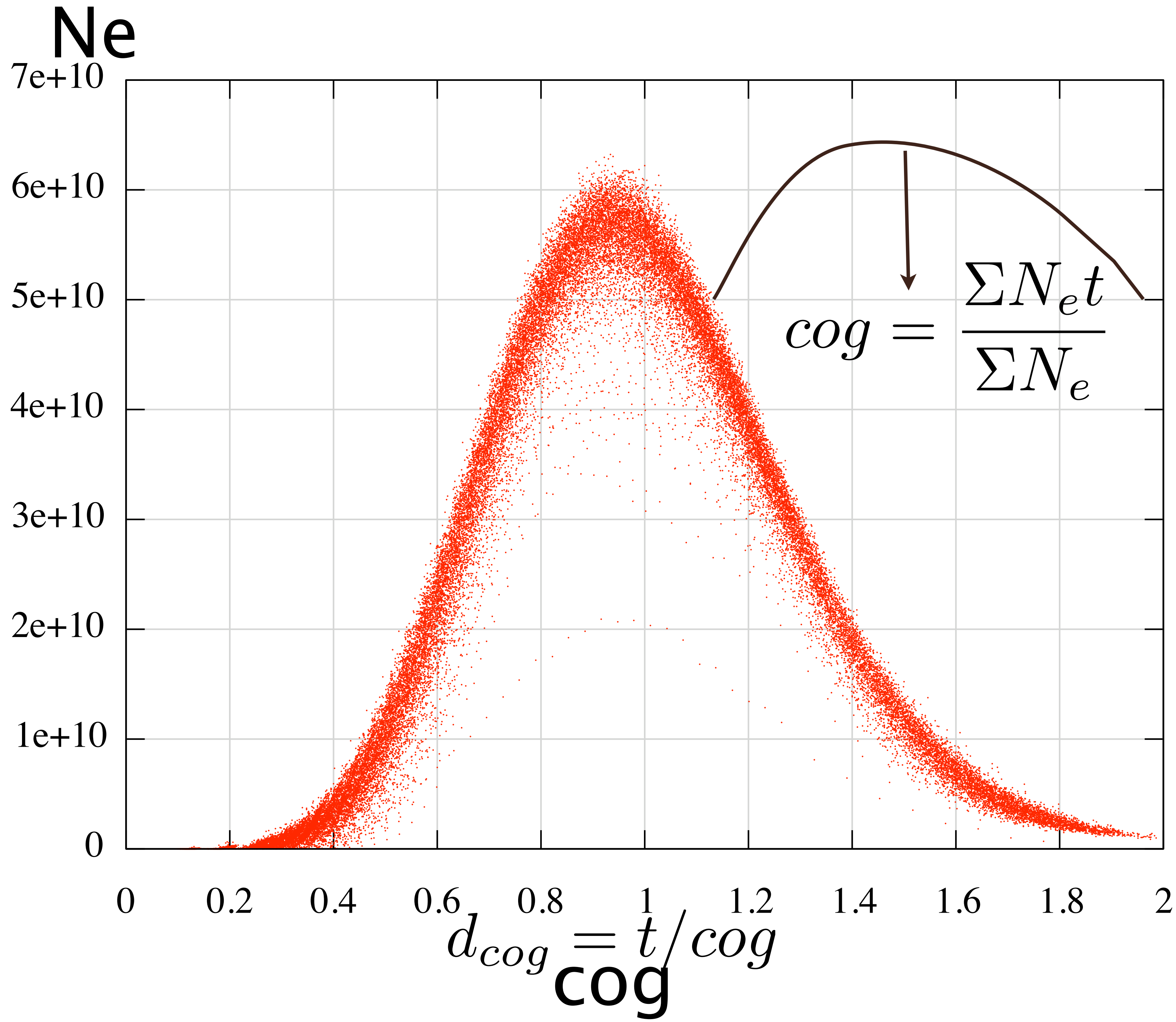


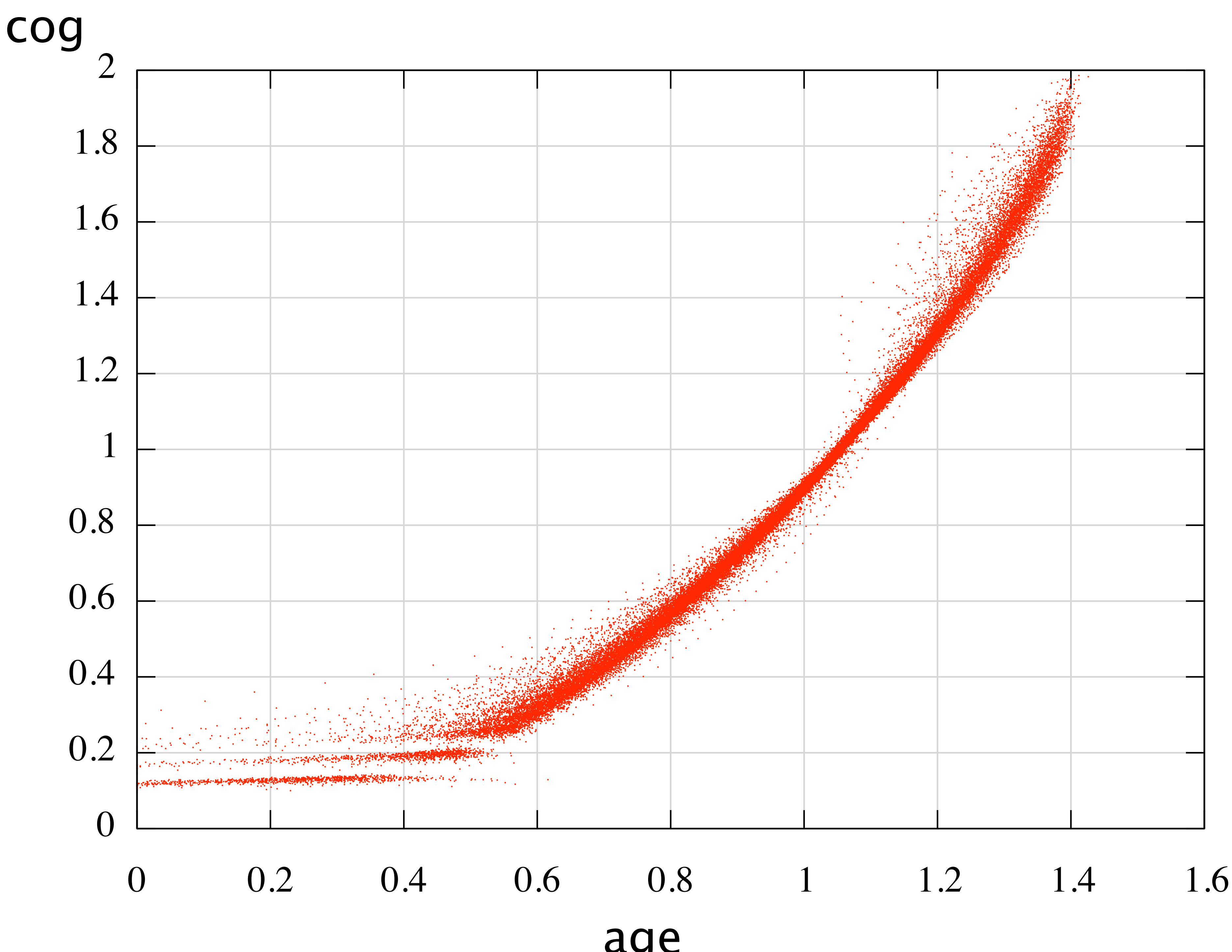

p 10^20 eV QGSJET-II. cos= 0.9 Ne and Nebhyb
Ne
7e+10
6e+10
5e+10
4e+10
3e+10
2e+10
1e+10
0
Ne
Ne hybrid
0
200
400
600
800
1000
1200
1400
1600
Vertial depth (g/cm2)


Ne Hyb
$s = \frac{\Sigma N_{ei} s_i}{\Sigma N_{ei}}$
7e+10
6e+10
5e+10
4e+10
3e+10
2e+10
1e+10
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
age s

Same showers as the previous one but without hybrid. Age is from hybrid one




Ne
7e+10
6e+10
5e+10
4e+10
3e+10
2e+10
1e+10
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
$cog = \frac{\Sigma N_e t}{\Sigma N_e}$
$d_{cog} = t/cog$
cog


cog
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
age

Ionization Energy Loss (-dE/dx)

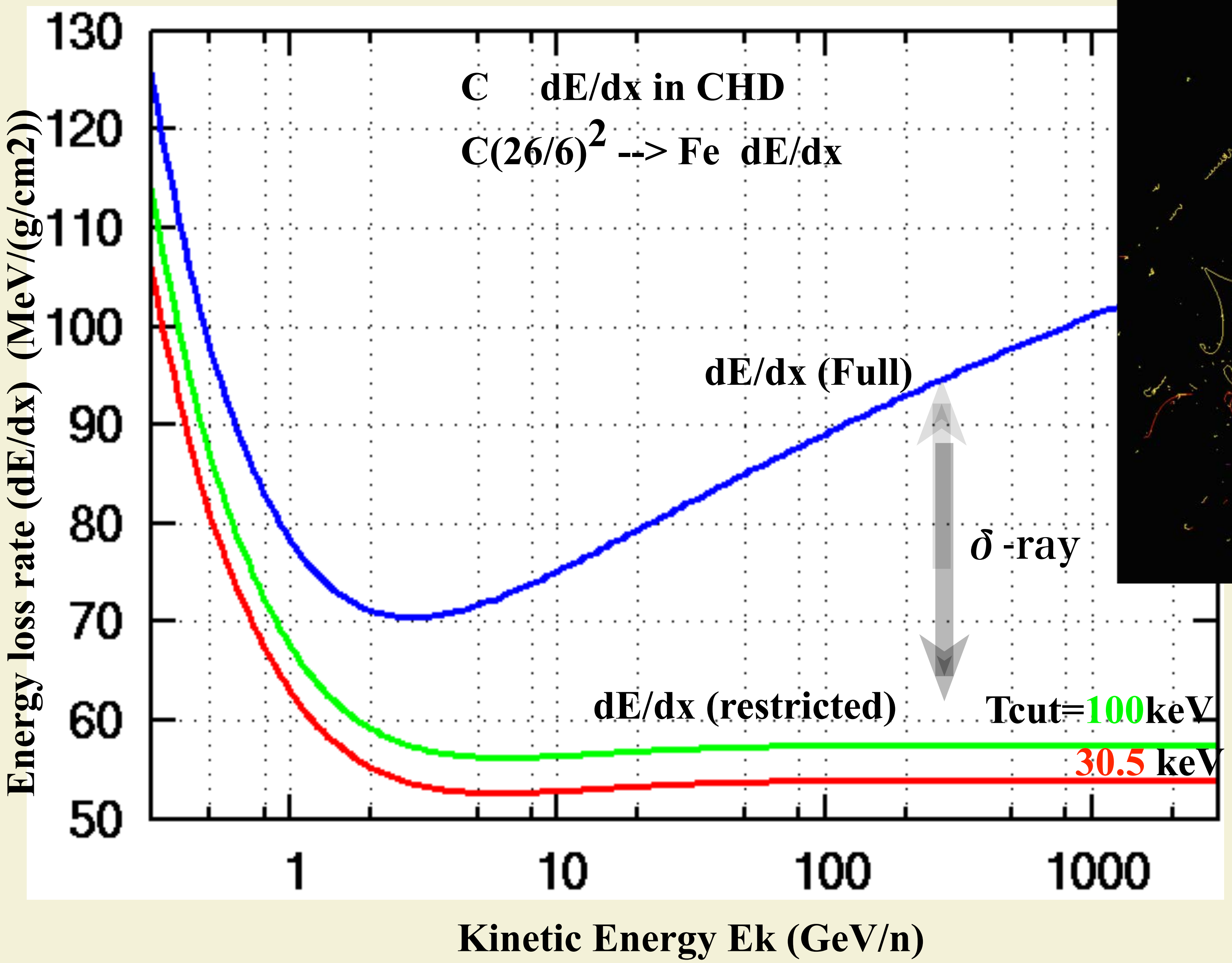
Bethe-Bloch:

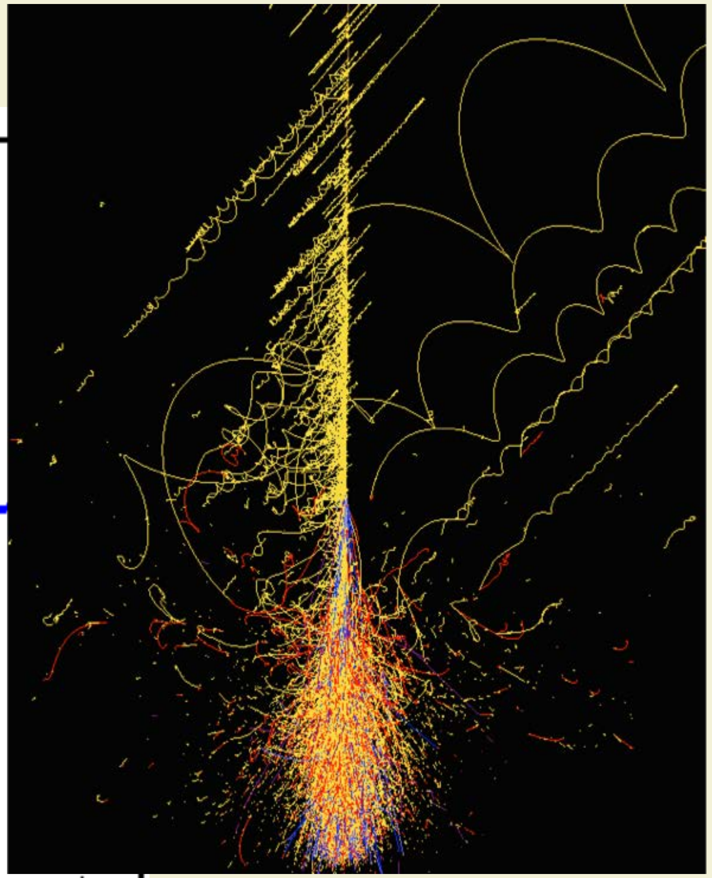
$$-\frac{dE}{dx}(<\eta) = \frac{2Cm}{\beta^2}\left[\ln\frac{2m\beta^2\eta}{(1-\beta^2)I^2(Z)} - \cdots\right]$$

$I(Z)$: Ionization potential: Very roughly
$I \sim \frac{1}{2}m\alpha^2 Z = 13.6Z$ eV
but is a complex Z function.
$(\alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} = \frac{1}{137})$

It is important to note:

- function of β only.
- So is p/m or E/m, Ek/m





e-: Moller scat

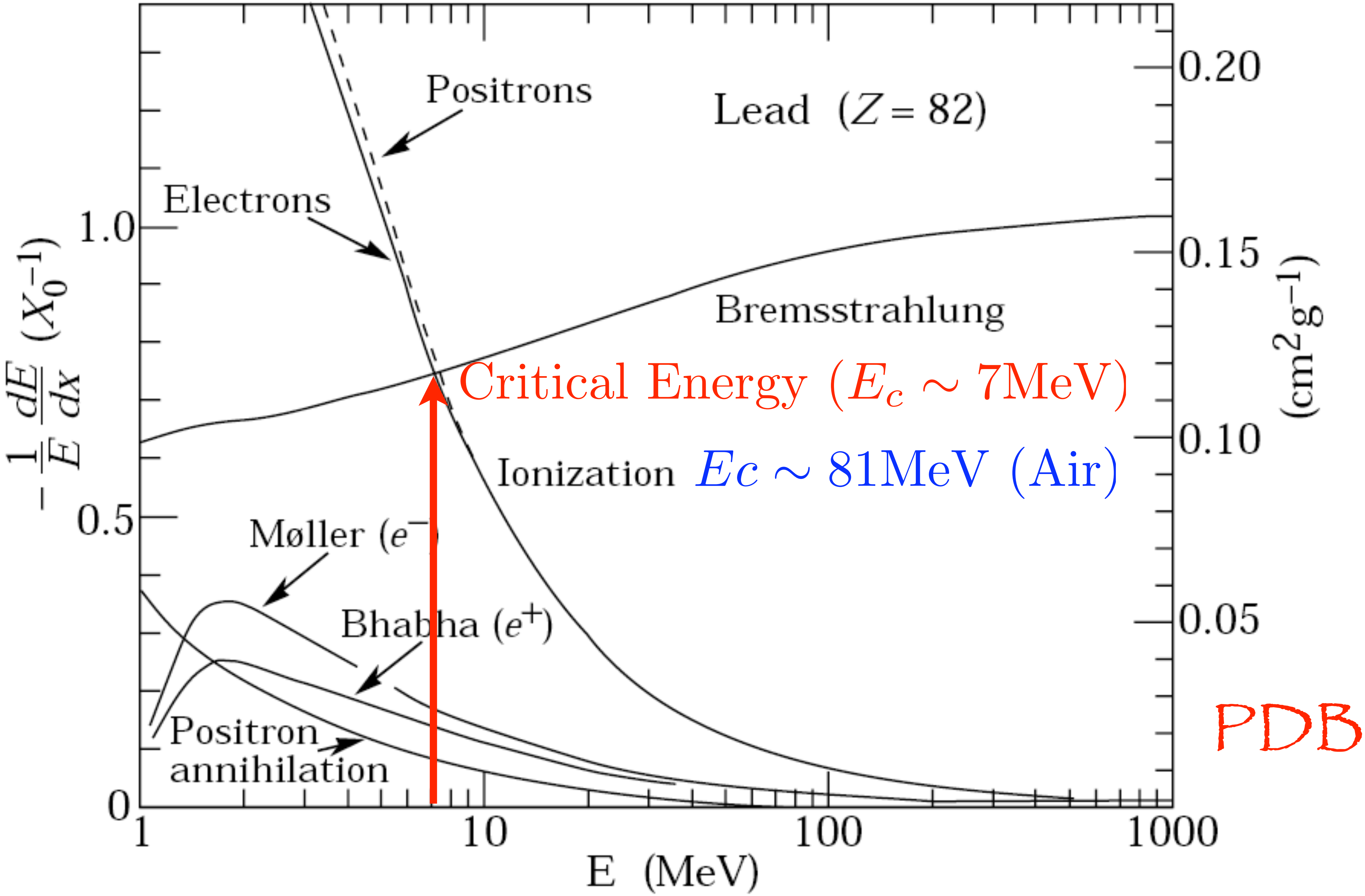
e+: Bhabha scat

p,mu,pi (spin dependce)

Knock-on

$$\frac{2Cm}{\beta^2}\frac{dE'}{E'^2}$$

Fractional Energy Loss / r.l



Lateral spread

- Source of spread
  - **Multiple scattering**
  - **Pt at particle production (hadron int & compton, brems,pair,inelastic scattering)**
  - **Magnetic field, Electric field**

Multiple Satt. $<\theta>\sim \frac{E_s}{E}\sqrt{t}$

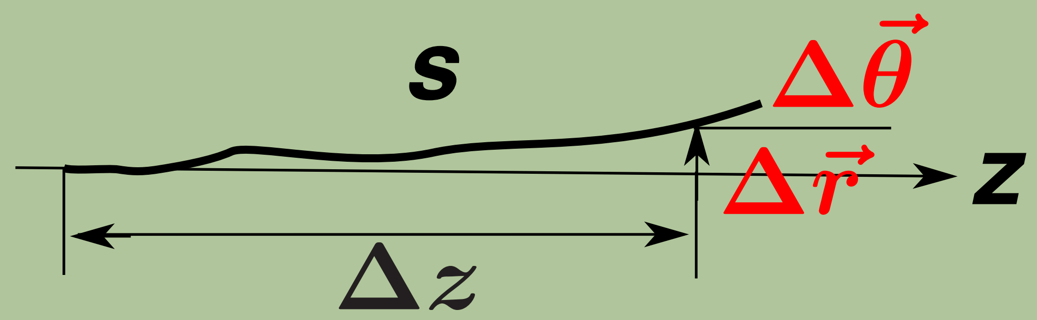
After traversing 1 radiation length, spread becomes $\sim X_0 <\theta>= \frac{E_s}{E} X_0$

Typical energy is critical energy, E_c, so that we may measure the lateral spread in terms of Moliere unit:

$\frac{E_s}{E_c} X_0 \sim 90$ m.

It is said we may use X_0 2 r.l above the observation depth.

Multiple Scattering



s vs Δz

- **Fermi** $(\Delta\vec{\theta}, \Delta\vec{r})$ **Rossi text: High Energy Particles**

  Consts: PDB (Lynch & Dahl, NIM B51(1991). But not recommended one.

- **Goudsmit & Saunderson** $\Delta\vec{\theta}$ **Phys. Rev. vol.57 (1940)**

  Phys. Rev. vol.58 (1940); there is a letter in between.

  Difficult for numerical applications

- **Lewis** $(\Delta\vec{\theta}, \Delta\vec{r})$ **Phys. Rev. vol.72 (1950). Formalism.**

Difficult for numerical applications.

- **Moliere=>Bethe.** $\Delta\vec{\theta}$ **Phys. Rev. vol.89 (1953)**

  physical translation. small angle approx. relaxed

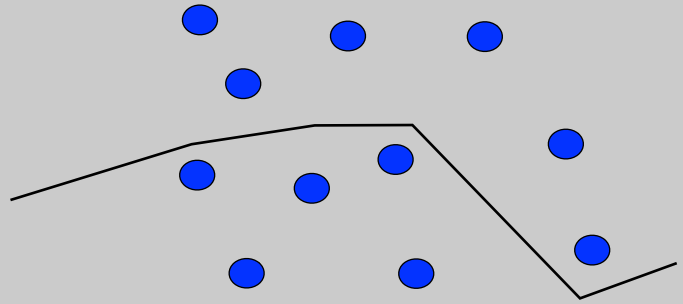
Multiple Scattering

Many E.M theories: ready for use bef. 1960

But M.S theory continuously developed even in 2000s.

Where is difficulty ?

Elastic Coulomb Scattering

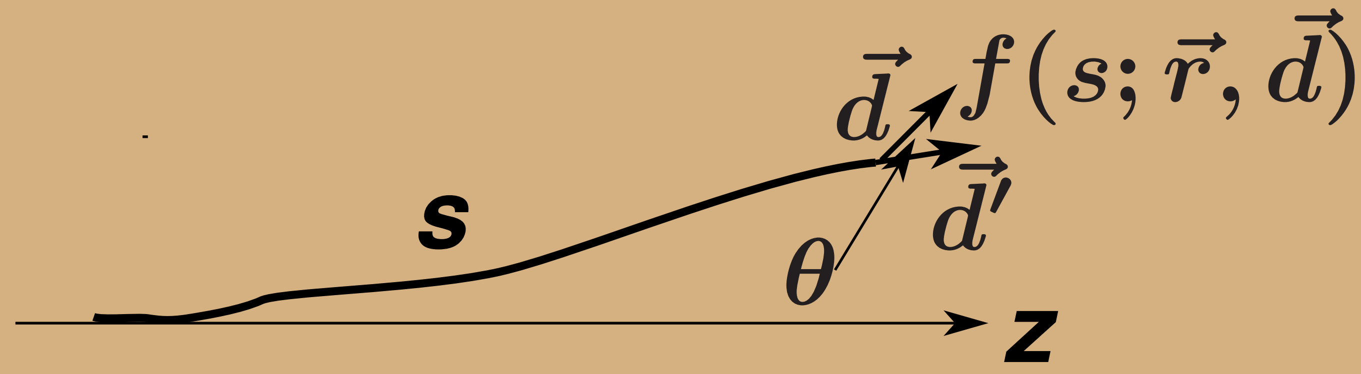


$$\frac{d\sigma}{d\Omega}$$ DCS

- **Getting reliable DCS is difficult**
  - **Partial Wave Analysis (of Schrödinger & Dirac Eq.) is thought to be most reliable, but above 1 MeV, more than 10^4 terms must be added and becomes unstable.**
    - **Born approximation**
    - **Eikonal approximation etc**
  - **How to treat the screening effect (expression of atom)**
  - **Nuclear size effect (finiteness of nucleus)**

    - F.Salvat: Phys. Rev. A43(1991)
    - J.Fernandes-Varea, R.Mayol &F.Salvat NIMB 82(1993)
    - F.Salvat & R.Mayol, CPC, 74(1993)
    - R.Mayol & F.Salvat, ATOMIC DATA AND NUCLEAR DATA TABLES 65, 55–154 (1997)
    - F.Salvat, A,Jabloinski & C.Powell, CPC 165 (2005)

•DCS to M.C.S



$$\frac{\partial f}{\partial s} + \vec{d}\cdot\nabla f = N\int \left(f(s;\vec{r},\vec{d}') - f(s;\vec{r},\vec{d})\right)\frac{d\sigma(\theta)}{d\Omega}d\Omega$$

Lewis

$$f(0,\vec{r},\vec{d}) = \frac{1}{\pi}\delta(\vec{r})\delta(1-\cos\chi) \qquad \chi\text{: polar angle of } \vec{d}$$

$$\text{Angular Disit: } F(s;\chi) = \int f(s;\vec{r},\vec{d})d\vec{r}$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{4\pi}\exp(-s/\lambda_{el,\ell})P_\ell(\cos\chi)$$

P_ℓ: Legendre Pol. $\quad \lambda_{el,\ell}$: ℓ-th transport m.f.p

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = (3x^2-1)/2$$

$\lambda_{el,\ell}$: ℓ-th transport m.f.p

ℓ-th **Transport xs:**

$$\sigma_{el,\ell} \equiv \int (1 - P_\ell(\cos\theta)) \frac{d\sigma_{el}}{d\Omega} d\Omega$$

$$\mu = (1 - \cos(\theta))/2$$

$$N\lambda_{el,\ell}\sigma_{el,\ell} = 1$$

$$<\mu> = \frac{1}{2}\sigma_{el,1}/\sigma_{el}$$

$$6\sigma_{el}(<\mu> - <\mu^2>) = \sigma_{el,2}$$

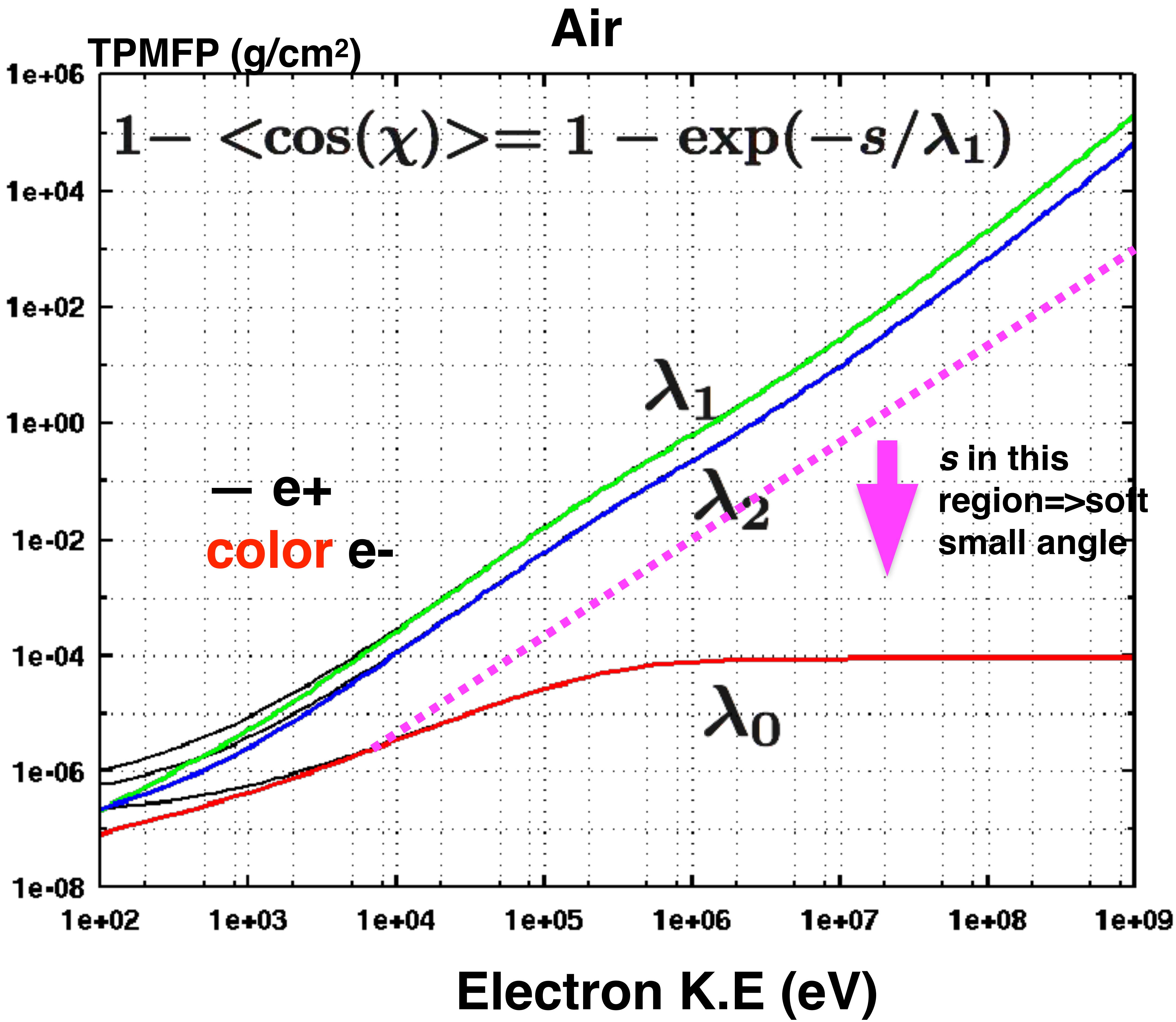
$$<\cos\chi> = \exp(-s/\lambda_{el,1})$$

$$<\cos^2\chi> = (1 + 2\exp(-s/\lambda_{el,2}))/3$$

$$<z> = 2\pi \int z f(s;\vec{r},\vec{d}) d\cos(\chi) d\vec{r} = \lambda_{el,1}(1 - \exp(-s/\lambda_{el,1}))$$

$<x^2 + y^2>$, $<z\cos(\chi)>$ etc

function of $\lambda_{el,1}, \lambda_{el,2}$



Air
TPMFP (g/cm²)
$1-<\cos(\chi)>= 1 - \exp(-s/\lambda_1)$
λ_1
λ_2
λ_0
— e+
color e-
s in this region=>soft small angle
1e+06
1e+04
1e+02
1e+00
1e-02
1e-04
1e-06
1e-08
1e+02
1e+03
1e+04
1e+05
1e+06
1e+07
1e+08
1e+09
Electron K.E (eV)

•Salvat, Fernandes, Mayol… (Spain)
•Kawrakow, Bielajew… (Canada)
•Uerban (Hungary)

Mixed simulation of the multiple elastic scattering…
EBenedito, J.Fernandes, F.Salvat NIM B174(2001)
On the theory and simulation of multiple elastic scattering of electrons.
J.Fernandes, R.Mayol, J.Baro, F.Salvat. NIM B (1993).
Pedagogical including summary of G.S, Lewis, Moliere theories.
PENELOPE-2011; A code system for M.C simulation of electron and photon transport
F.Salvat, J.Fernandes, J.Sempau. Data Bank, NEA/NSC/DOC/(2011)5

Electron transport; lateral and longitudinal correlation algorithm.
I.Kawrakow, NIM B114 (1996)
On the condensed history technique for electron transport.
I.Kawrakow, A.Bielajew. NIM B147 (1998)

A Model for Multiple Scattering in Geant4
L.Urbán, CERN-OPEN-2006-077

• Mixed simulation 1

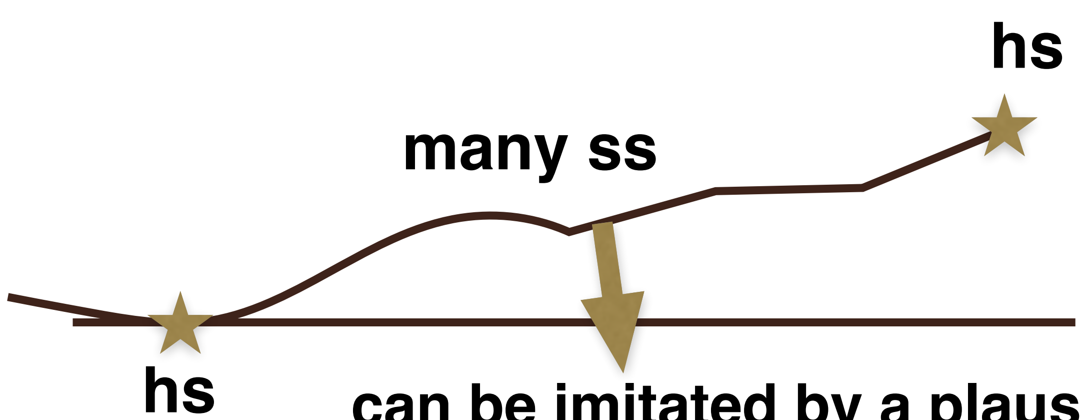
dE/dx | **M.C.S**

high E delta-ray | **hard large angle single scattering**

sampling individual event

low E delta, excitation | **soft small angle multiple scant.**

macroscopic continuous process



can be imitated by a plausible distribution with the same $\langle\cos(\chi)\rangle$, $\langle\cos^2(\chi)\rangle$ **as the correct ones.**

Spain group: random-hinge method (蝶番)

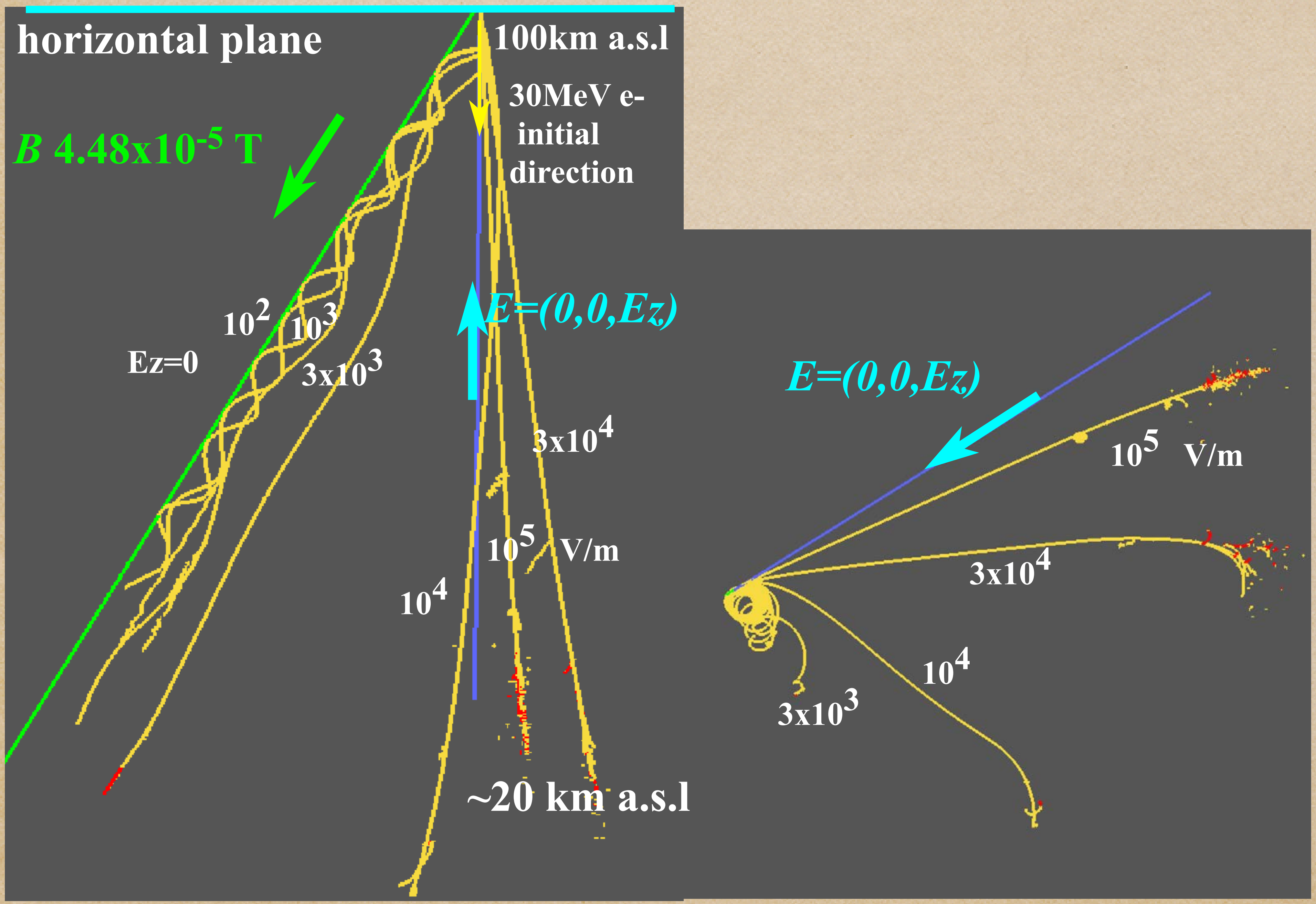
Lateral spread

Experimental problem: No test has been done over 2~3 M.U while we observe even at 10 M.U

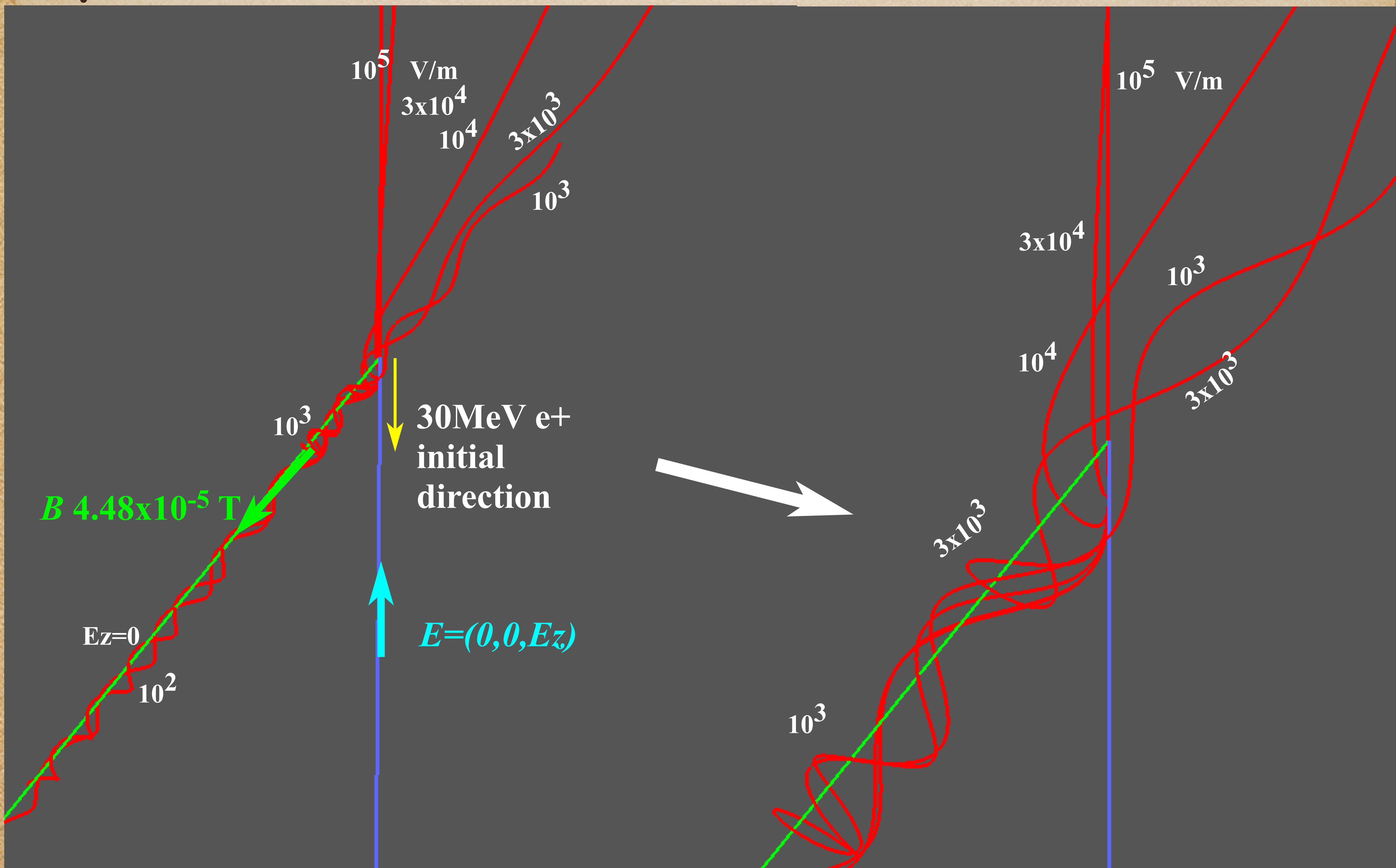
Electric field

- Simple filed (E(r,t): Specified by parameters
- Complex one: The user may supply a subroutine for giving E(r,t).

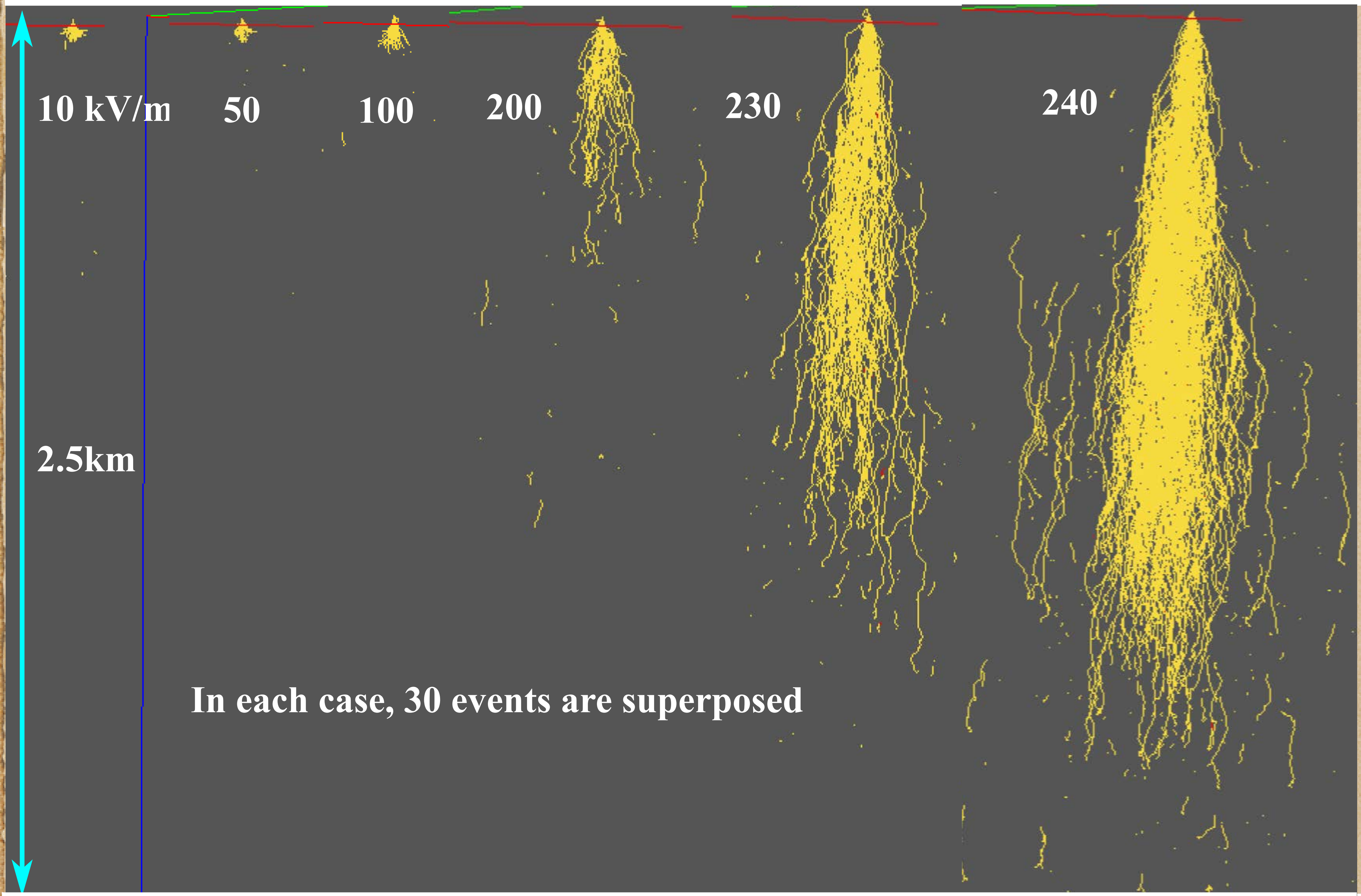
- Test at almost no air hight, (B, E)
- electron 30 MeV



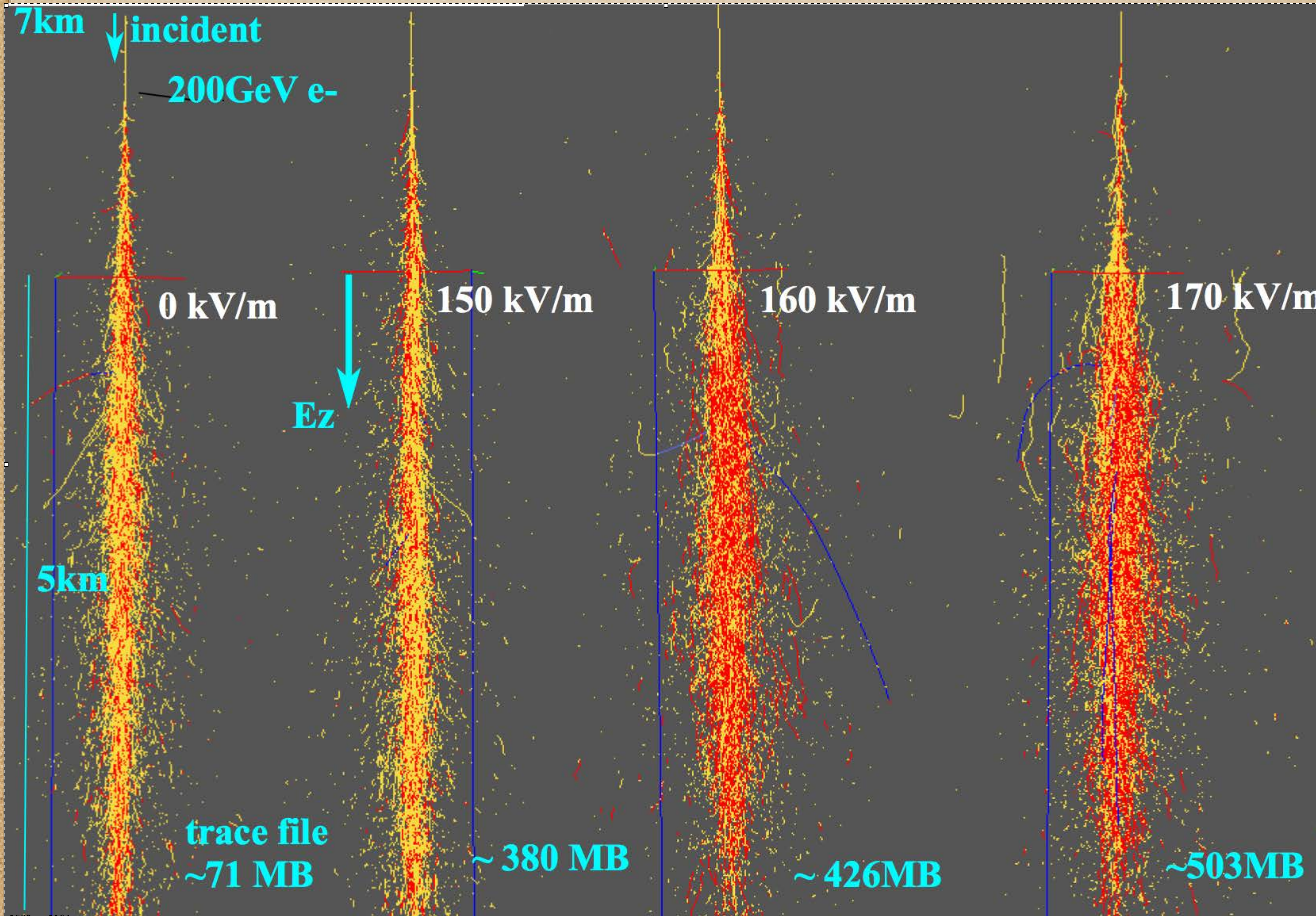
- Test at almost no air hight, (B, E)
- positron 30MeV



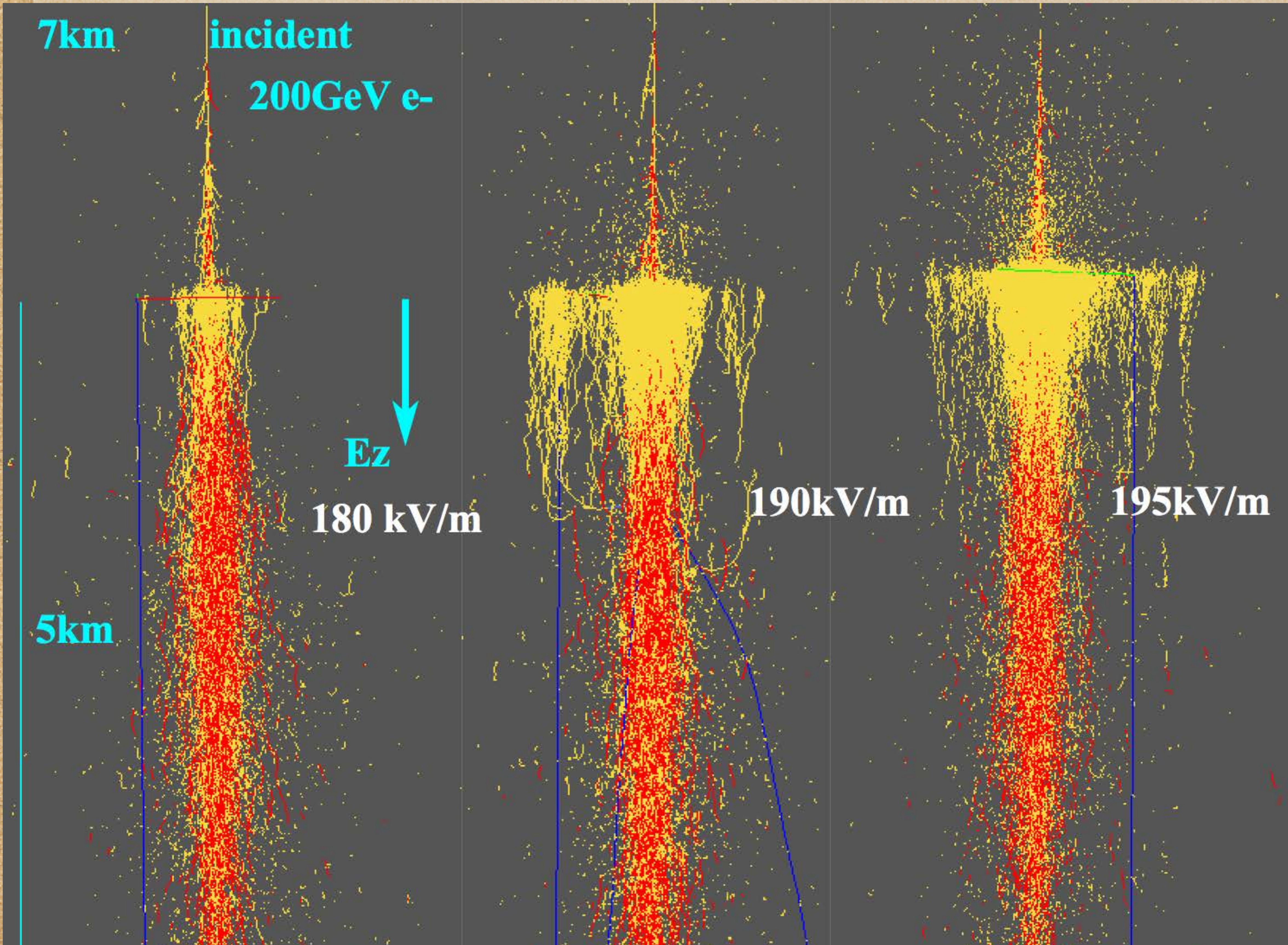
• Test in air   10MeV e-



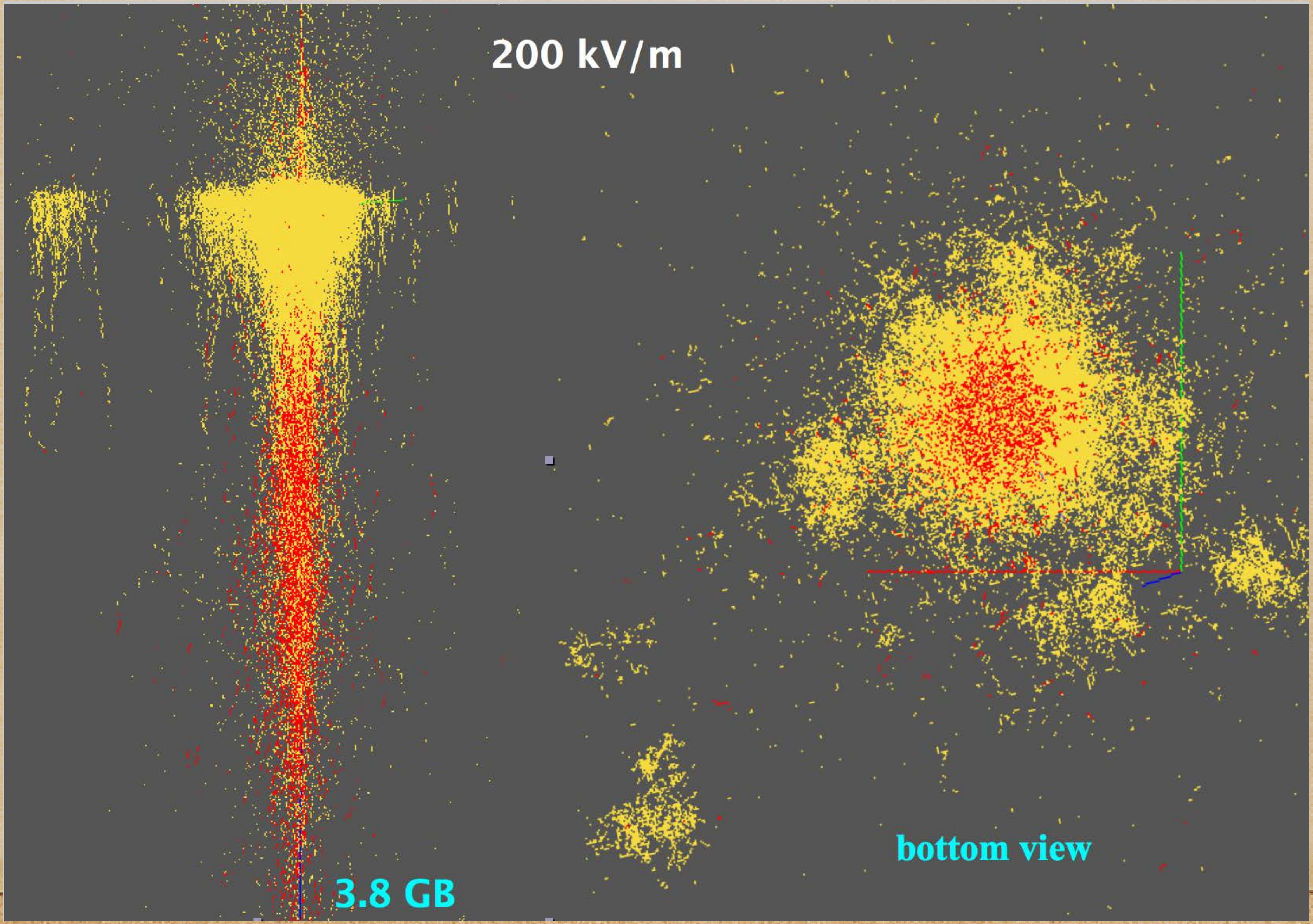
• Test in air 200 GeV e-



- Test in air   200 GeV e-



- Test in air   200 GeV e-



- In actual applications, it is important to

take into account the time information of electric field (duration time), as well as r-dependence.