Quantum state tomography and applications in weak decays

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Workshop on the foundational tests of Quantum Mechanics at the LHC

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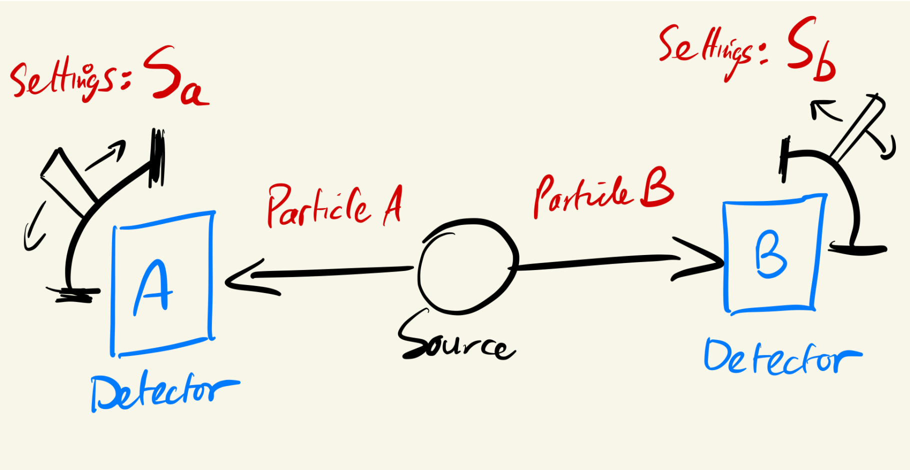
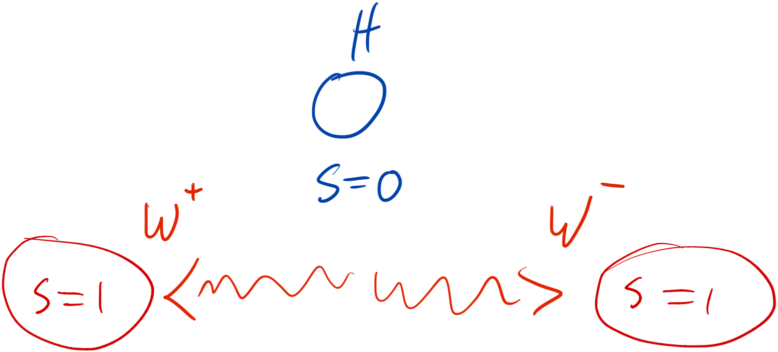
AJB, Phys.Lett.B 825 (2022) 136866 — 2106.01377 [hep-ph]
R.Ashby-Pickering, AJB, A.Wierzchucka — 2209.13990 [quant-ph]

The textbook Bell experiment



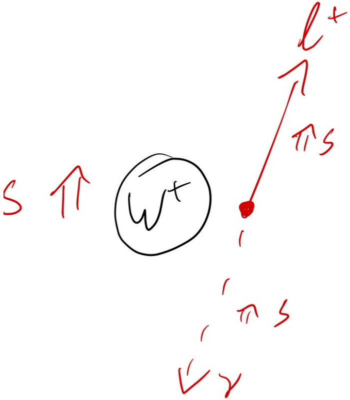
Find expectation values of various correlations between measurement outcomes

A Higgs boson decay



≈maximally entangled spin state

Particles with weak decays are their own polarimeters



The charged leptons are emitted in directions that strongly depend on the spin of the parent W boson

More precisely

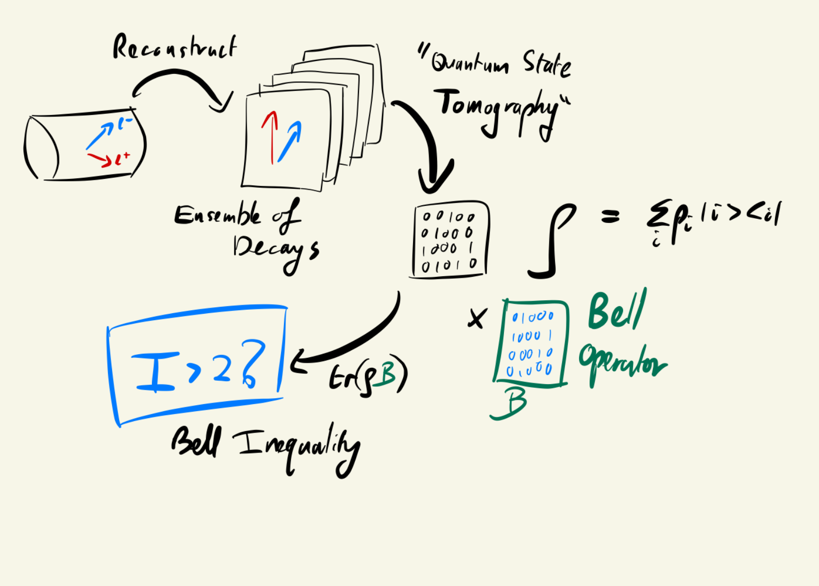
The probability density function for a $W^{\pm}$ boson to emit its charged lepton in the $\hat{\mathbf{n}}$ direction is:

$$p(\ell^{\pm}_{\hat{\mathbf{n}}}; \rho) = \frac{3}{4\pi} \operatorname{tr}(\rho \, \Pi_{\pm,\hat{\mathbf{n}}}) \, ,$$

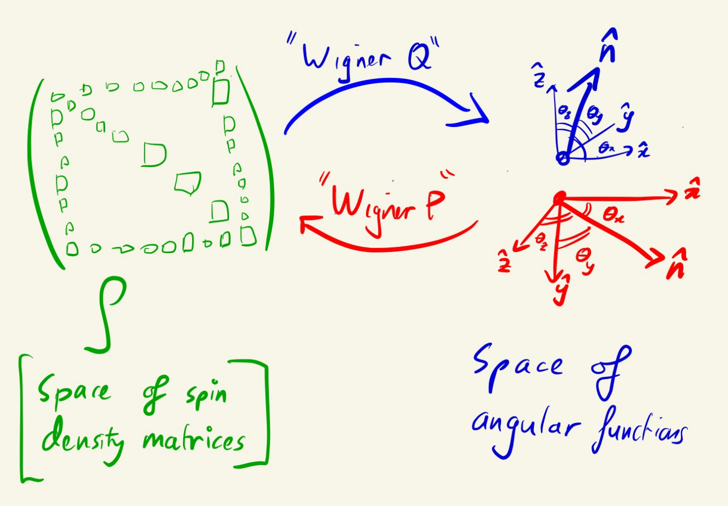
where ρ is the W boson spin density matrix and the projection operator is

$$\Pi_{\pm,\hat{\mathbf{n}}} \equiv |\pm_{\hat{\mathbf{n}}}\rangle \langle \pm_{\hat{\mathbf{n}}}|$$

observe decay $\Longleftrightarrow$ measure spin


Reconstruct
"Quantum State Tomography"
Ensemble of Decays
ρ = Σ_i p_i |i><i|
x
Bell Operator
B
I > 2?
tr(ρB)
Bell Inequality

A useful formalism



Also true for e.g. $W^{\pm}$, Z^0, t, τ

Transforming between the spaces

The Wigner-Weyl formalism for spin

Operator $\to$ function

$$\Phi_A^Q(\hat{\mathbf{n}}) = \langle \hat{\mathbf{n}} | A | \hat{\mathbf{n}} \rangle$$

Wigner Q symbols

Function $\to$ operator

$$A = \frac{2j+1}{4\pi} \int \mathrm{d}\Omega_{\hat{\mathbf{n}}} \, |\hat{\mathbf{n}}\rangle \, \Phi_A^P(\hat{\mathbf{n}}) \, \langle \hat{\mathbf{n}}| \,,$$

Wigner P symbols

Parameterise ρ

Symmetrically for qutrits in terms of the Gell-Mann matrices λ_i

Single vector boson

$$\rho = \frac{1}{3} I_3 + \sum_{i=1}^{8} a_i \lambda_i ,$$

a_i : 8 real parameters $(3^2 - 1)$

- Generalised Gell-Mann matrices $\lambda_i^{(d)}$ exist for any spin
- For spin-half ($d = 2$) they are the Pauli matrices and we get the Bloch sphere
- For $d = 3$ they are the eight generators of SU(3)

Other parameterisations, e.g. Cartesian Tensors are good alternatives

Parameterise ρ – bipartite system

Symmetrically for qutrits in terms of the Gell-Mann matrices λ_i

Single vector boson

$$\rho = \frac{1}{3} I_3 + \sum_{i=1}^{8} a_i \lambda_i,$$

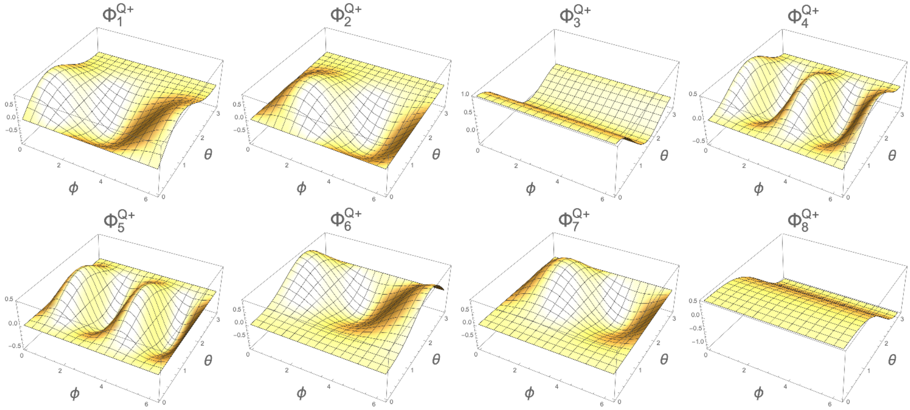
a_i : 8 real parameters $(3^2 - 1)$

Two vector bosons

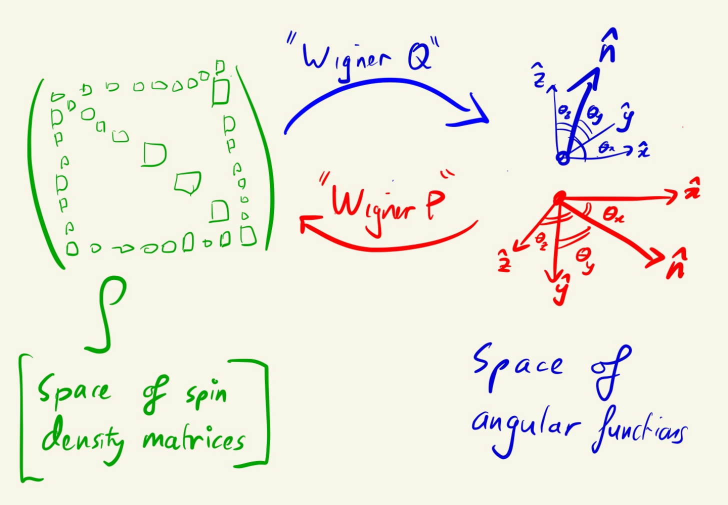
$$\rho = \frac{1}{9} I_3 \otimes I_3 + \sum_{i=1}^{8} a_i \, \lambda_i \otimes \frac{1}{3} I_3 + \sum_{j=1}^{8} b_j \, \frac{1}{3} I_3 \otimes \lambda_j + \sum_{i,j=1}^{8} c_{ij} \, \lambda_i \otimes \lambda_j,$$

8+8+64 = 80 real parameters $(9^2 - 1)$

Angular distributions for each parameter

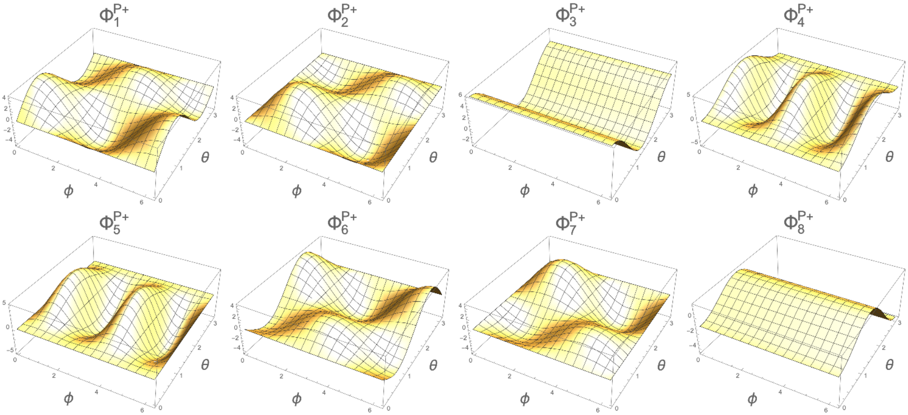


Wigner Q functions for the eight Gell-Mann matrices



"Wigner Q"
"Wigner P"
Space of spin density matrices
Space of angular functions

To get back to the density matrix



Wigner *P* functions for the eight Gell-Mann matrices

Extracting the parameters experimentally

Parameters of ρ are the experimentally-measurable classical averages of the Wigner P functions

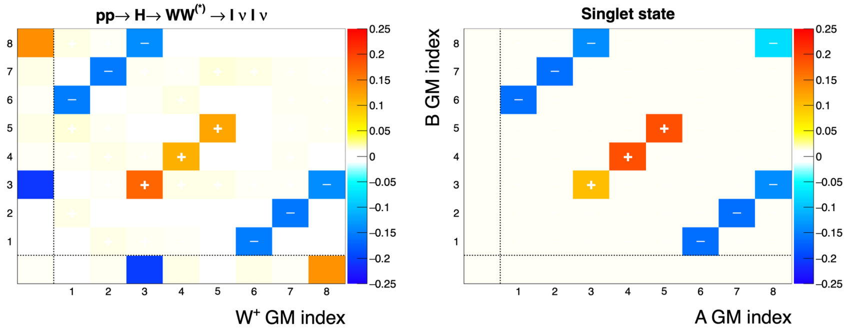
$$\hat{a}_i = \frac{1}{2}\left\langle \Phi_i^P(\hat{\mathbf{n}}_1) \right\rangle_{\mathrm{av}}$$

$$\hat{b}_i = \frac{1}{2}\left\langle \Phi_i^P(\hat{\mathbf{n}}_2) \right\rangle_{\mathrm{av}}$$

$$\hat{c}_{ij} = \frac{1}{4}\left\langle \Phi_i^P(\hat{\mathbf{n}}_1)\,\Phi_j^P(\hat{\mathbf{n}}_2) \right\rangle_{\mathrm{av}}$$

Quantum State Tomography example

Higgs boson decays



Density matrix parameters from simulated Higgs boson decays to vector bosons (Madgraph, no background)

Testing a Bell Inequality

The CGLMP Qutrit inequality

Collins Gisin Linden Massar Popescu (2002)

The optimal Bell inequality for pairs of **qutrits**

CGLMP function

$$\begin{aligned}\mathcal{I}_3 = P(A_1 = B_1) + P(B_1 = A_2 + 1) \\ + P(A_2 = B_2) + P(B_2 = A_1) \\ - P(A_1 = B_1 - 1) - P(B_1 = A_2) \\ - P(A_2 = B_2 - 1) - P(B_2 = A_1 - 1).\end{aligned}$$

$P(A_i = B_j + k)$ is the probability that A_i and B_j differ by k mod 3

CGLMP limits?

In a local realist theory

$$\mathcal{I}_3 \leq 2$$

In QM

$$\mathcal{I}_3^{\mathrm{QM}} \leq 1 + \sqrt{11/3} \approx 2.9149$$

In QM for a maximally entangled state

$$\mathcal{I}_3^{\mathrm{QM,singlet}} \leq 4/(6\sqrt{3} - 9) \approx 2.8729$$

Testing the CGLMP inequality

Knowing elements of ρ calculate

$$\mathcal{I}_3 = \mathrm{tr}(\rho\, \mathcal{B}^{xy}_{\mathrm{CGLMP}})$$

where the CGLMP operator is

$$\mathcal{B}^{xy}_{\mathrm{CGLMP}} = -\frac{2}{\sqrt{3}}\left(S_x \otimes S_x + S_y \otimes S_y\right) + \lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5$$

where

$$S_x = \frac{1}{\sqrt{2}}(\lambda_1 + \lambda_6) \quad \text{and} \quad S_y = \frac{1}{\sqrt{2}}(\lambda_2 + \lambda_7).$$

$H \to W^+W^-$ simulated results

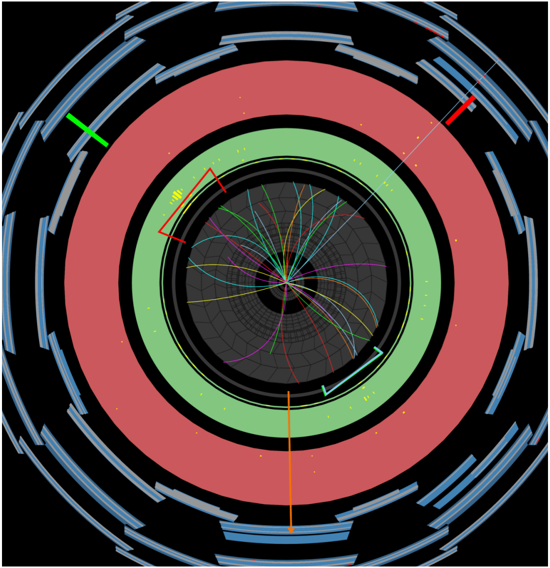
In idealised, numerical simulation of $H \to W^+W^-$, with finite width effects and relativistic effects:

$$\mathcal{I}_3 \approx 2.6$$

Summary

- Weak decays are wonderful quantum probes
- Quantum spin self measurement via chiral weak decays
- Spin density matrix can be reconstructed from angular distributions ('tomography')
- Expect entanglement and even Bell inequality violation

EXTRAS



Weak gauge bosons measure their own spin

SU(2) weak force is chiral: $\gamma^\mu(1-\gamma^5)$

W boson

$$W^+ \to \ell_R^+ + \nu_L$$
$$W^- \to \ell_L^- + \bar{\nu}_R$$

Decay of a $W^\pm$ boson is equivalent to a projective (von Neumann) quantum measurement of its spin along the axis of the emitted lepton

Z boson

Z bosons also have spin-sensitive decays

Left, right couplings determined by electroweak mixing
Equivalent to a non-projective quantum measurement

Qutrits

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}\big(\,|+\rangle\,|-\rangle - |0\rangle\,|0\rangle + |-\rangle\,|+\rangle\,\big)$$

- This is a maximally entangled state of two **qutrits**
- $|\psi\rangle_{AB} \in (\mathbb{C}^{3})^2$
- Basis for each qutrit $\{0, 1, 2\}$

[On the board: qutrits vs 3-state systems]

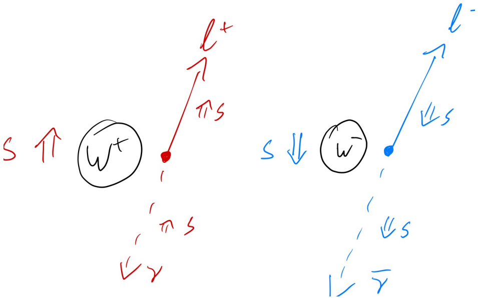
In case you're curious

The CGLMP operator is[1]

$$\mathcal{B}^{xy}_{\mathrm{CGLMP}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 2 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 0 & 2 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[1] after a minor tweak – see 2106.01377

Getting the directions right



- ℓ^+ is emitted preferentially along spin direction (of W^+)
  ℓ^- is emitted preferentially against spin direction (of W^-)
- For $H \to W^+W^-$ the $W^\pm$ spins are in different directions
- So the two leptons prefer to go in the same direction as each other

Spin in the $H \to W^+W^-$ decay

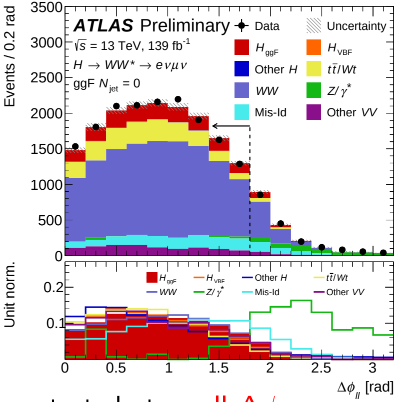
The Higgs boson is a scalar, while $W^\pm$ bosons are massive vector bosons, each with three possible spin states (qutrits)

- $H \to W^+W^-$ decays produce pairs of W bosons with zero total angular momentum
- In the narrow-width and non-relativistic approximations:

$$|\psi_s\rangle = \frac{1}{\sqrt{3}}\left(|+\rangle\,|-\rangle - |0\rangle\,|0\rangle + |-\rangle\,|+\rangle \right)$$

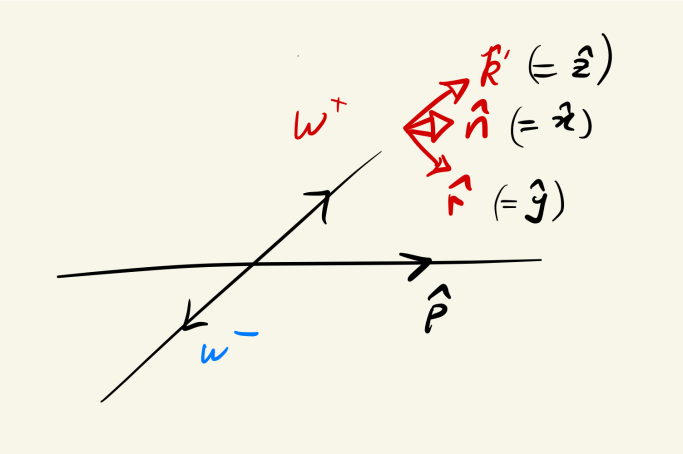
This is a maximally entangled state of two qutrits

$\ell^+\ell^-$ azimuthal correlations observed in $H \to W^+W^-$



- Higgs signal concentrated at small $\Delta\phi_{\ell\ell}$
- Used e.g. in discovery searches

Coordinates



Angular distributions are measured in the rest-frame of the parent gauge boson

Measuring Bell expectation values directly

CGLMP (qutrit) inequality from data for *WW*

$$\mathcal{I}_3 = \mathrm{tr}(\rho \mathcal{B}^{xy}_{\mathrm{CGLMP}}) = \frac{8}{\sqrt{3}} \left\langle \xi_x^+ \xi_x^- + \xi_y^+ \xi_y^- \right\rangle_{\mathrm{av}}$$
$$+ 25 \left\langle \left((\xi_x^+)^2 - (\xi_y^+)^2\right) \left((\xi_x^-)^2 - (\xi_y^-)^2\right) \right\rangle_{\mathrm{av}}$$
$$+ 100 \left\langle \xi_x^+ \xi_y^+ \xi_x^- \xi_y^- \right\rangle_{\mathrm{av}}$$

Is this Bell inequality violated in data?

$$\mathcal{I}_3 \leq 2?$$