Improved tests of entanglement and Bell inequalities with LHC tops

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Tops are good candidates to test entanglement and Bell inequalities at high energy

See Afik's presentation in this conference

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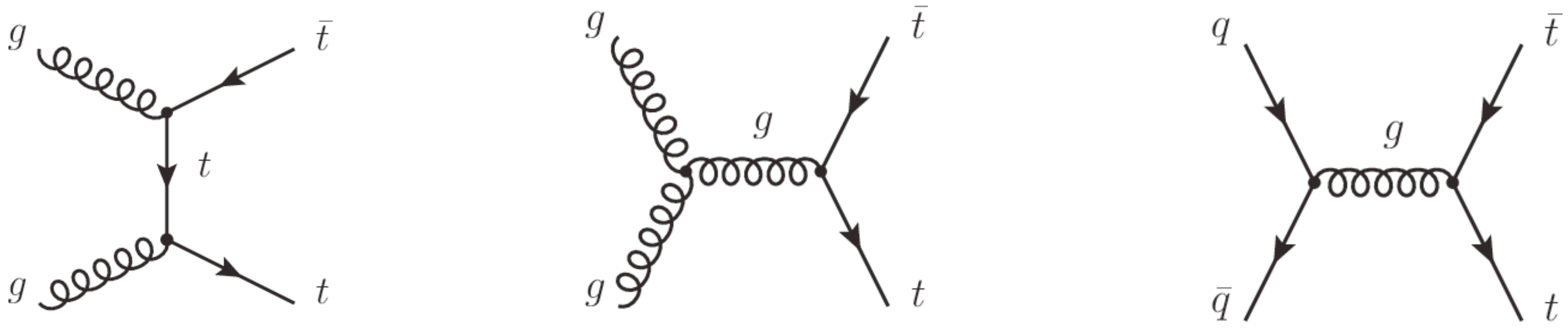
A pretty large amount of work has been devoted to this subject in the last 1-2 years:

Afik and Nova, 2021, 2022
Fabbrichesi, Floreanini and Panizzo 2021
Severi, Boschi, Maltoni and Sioli 2021
Aoude, Madge, Maltoni and Mantani, 2022
…

Top pairs are copiously produced at the LHC through gluon fusion and $q\bar{q}$ annihilation



The generated $t\bar{t}$ system is generically in a (mixed) entangled state of spin, i.e. a 2-qubit system, suitable to test entanglement & Bell inequalities

Afik and Nova, 2021

How to measure entanglement and (violation of) Bell inequalities at the LHC with tops

(Brief review)

In general, an entangled (not necessarily pure) state of two subsystems, Alice and Bob, is described by a density matrix,

$$\rho_{\text{ent}} \neq \rho_{\text{sep}} = \sum_n p_n \; \rho_n^A \otimes \rho_n^B$$

with $\sum p_n = 1$, and ρ Hermitian, positive semi-definite and $\mathrm{Tr}\,\rho = 1$

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with $\sum p_n = 1$, and ρ Hermitian, positive semi-definite and $\mathrm{Tr}\,\rho = 1$

For two qubits ρ can be expressed in a basis of Hermitian operators,

$$\left\{I^A, \sigma_1^A, \sigma_2^A, \sigma_3^A\right\} \otimes \left\{I^B, \sigma_1^B, \sigma_2^B, \sigma_3^B\right\}$$

$$\rho = \frac{1}{4}\left(I \otimes I + \sum_{i=1,2,3} \left(B_i^{+}\, \sigma_i \otimes I + B_i^{-}\, I \otimes \sigma_i \right) + \sum_{i,j=1,2,3} C_{ij}\, \sigma_i \otimes \sigma_j \right)$$

Hermiticity and $\mathrm{Tr}\,\rho = 1$ are automatically, but not semi-positivity

$$\rho = \frac{1}{4}\left(I \otimes I + \sum_{i=1,2,3} (B_i^+ \sigma_i \otimes I + B_i^- I \otimes \sigma_i) + \sum_{i,j=1,2,3} C_{ij}\, \sigma_i \otimes \sigma_j \right)$$

Hermiticity and $\mathrm{Tr}\,\rho = 1$ are automatically, but not semi-positivity

Moreover $\quad P, CP \;\Rightarrow\; B_i^{\pm} = 0, \quad C_{ij} = C_{ji}$

The conditions for the semi-positivity of ρ are complicated, but they get quite simplified once one takes into account that in the helicity basis (see later) $C_{13} \simeq C_{23} \simeq 0$

<u>Semi-positivity</u>

$$\boxed{\begin{array}{l} |C_{11} + C_{22}| \leq 1 - C_{33} \\ \text{and} \\ |4C_{12}^2 + (C_{11} - C_{22})^2|^{1/2} \leq 1 + C_{33} \end{array}}$$

Entanglement

Mathematically, a necessary and sufficient condition for entanglement in joint systems of two qubits is provided by the Peres-Horodecki criterion:

First construct ρ^{T_2}

$$\rho \longrightarrow \rho^{T_2}$$

by transposing only the indices associated to the Bob (or Alice) Hilbert space.

Then, if ρ^{T_2} is not a legal density matrix, in particular it has some negative eigenvalues, then the system is entangled

Entanglement

In our case

$$\rho = \frac{1}{4}\begin{bmatrix} 1+B_3^+ + B_3^- + C_{33} & B_1^- + C_{31} - i(B_2^- + C_{32}) & B_1^+ + C_{13} - i(B_2^+ + C_{23}) & C_{11} - C_{22} - i(C_{12} + C_{21}) \\ B_1^- + C_{31} + i(B_2^- + C_{32}) & 1 + B_3^+ - B_3^- - C_{33} & C_{11} + C_{22} + i(C_{12} - C_{21}) & B_1^+ - C_{13} - i(B_2^+ - C_{23}) \\ B_1^+ + C_{13} + i(B_2^+ + C_{23}) & C_{11} + C_{22} - i(C_{12} - C_{21}) & 1 - B_3^+ + B_3^- - C_{33} & B_1^- - C_{31} - i(B_2^- - C_{32}) \\ C_{11} - C_{22} + i(C_{12} + C_{21}) & B_1^+ - C_{13} + i(B_2^+ - C_{23}) & B_1^- - C_{31} + i(B_2^- - C_{32}) & 1 - B_3^+ - B_3^- + C_{33} \end{bmatrix}$$

$$\rho^{T_2} = \frac{1}{4}\begin{bmatrix} 1+B_3^+ + B_3^- + C_{33} & B_1^- + C_{31} + i(B_2^- + C_{32}) & B_1^+ + C_{13} - i(B_2^+ + C_{23}) & C_{11} + C_{22} + i(C_{12} - C_{21}) \\ B_1^- + C_{31} - i(B_2^- + C_{32}) & 1 + B_3^+ - B_3^- - C_{33} & C_{11} - C_{22} - i(C_{12} + C_{21}) & B_1^+ - C_{13} - i(B_2^+ - C_{23}) \\ B_1^+ + C_{13} + i(B_2^+ + C_{23}) & C_{11} - C_{22} + i(C_{12} + C_{21}) & 1 - B_3^+ + B_3^- - C_{33} & B_1^- - C_{31} + i(B_2^- - C_{32}) \\ C_{11} + C_{22} - i(C_{12} - C_{21}) & B_1^+ - C_{13} + i(B_2^+ - C_{23}) & B_1^- - C_{31} - i(B_2^- - C_{32}) & 1 - B_3^+ - B_3^- + C_{33} \end{bmatrix}$$

The eigenvalues of ρ^{T_2} are complicated expressions of $B_i^\pm$, C_{ij}, but, again things get simplified for $B_i^\pm = 0$, $C_{ij} = C_{ji}$ (due to P, CP) and $C_{13} \simeq C_{23} \simeq 0$ (helicity basis):

Negativity of ρ^{T_2} (Entanglement) $\Rightarrow$

$$|C_{11} + C_{22}| > 1 + C_{33}$$

or

$$|4C_{12}^2 + (C_{11} - C_{22})^2|^{1/2} > 1 - C_{33}$$

Entanglement

Semi-positivity of ρ and "negativity" of ρ^{T_2} (entanglement) get further simplified if $C_{12}^2 \ll C_{11}^2, C_{22}^2$ (as it is the case):

$$\boxed{\text{and}\quad \begin{array}{l} |C_{11}+C_{22}| \le 1 - C_{33} \\ |C_{11}-C_{22}| \le 1 + C_{33} \end{array}} \qquad \boxed{\text{or}\quad \begin{array}{l} |C_{11}+C_{22}| > 1 + C_{33} \\ |C_{11}-C_{22}| > 1 - C_{33} \end{array}}$$

Semi-positivity of ρ "negativity" of ρ^{T_2} (**entanglement**)

<u>Sufficient</u> (and almost necessary) conditions

Note they require $C_{11}, C_{22}, C_{33} \neq 0$

Hence, the straightforward way to show entanglement is to measure the C_{ij} and check these relations

Bell (CHSH) inequalities

The physical consequence of entanglement that departs from classical intuition is the violation of Bell inequalities, an impossible result in any local-realistic (“classical") theory of nature.

(except perhaps in Everett’s interpretation of QM!, see Timpson’s talk)

CHSH

Alice (Bob) chooses to measure certain (bi-valued) observables, $A, A' \ (B, B')$. Then, classically,

$$\left| \langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle \right| \leq 2 \qquad \text{(CHSH)}$$

The goal is to choose A, A', B, B' s.t. CHSH is experimentally maximally violated.

Bell (CHSH) inequalities

Defining

$$A = a_i \sigma_i^A, \quad A' = a'_i \sigma_i^A, \quad B = b_i \sigma_i^B, \quad B' = b'_i \sigma_i^B$$

$$\underbrace{\left| \vec{a} \cdot C(\vec{b} - \vec{b'}) \ + \ \vec{a'} \cdot C(\vec{b} + \vec{b'}) \right|} \leq 2 \qquad \text{(CHSH)}$$

The maximal value of the l.h.s. is $2\sqrt{\lambda_1 + \lambda_2}$, with λ_1, λ_2 the largest eigenvalues of $C^T C$

Horodecki, Horodecki, Horodecki '95

This corresponds to $\vec{a}, \vec{a'}, \vec{b}, \vec{b'}$ that depend on C itself.

Bell (CHSH) inequalities

Hence, the optimal strategy seems to (experimentally) determine C_{ij} and the largest eigenvalues of C^TC. If they satisfy $\lambda_1 + \lambda_2 > 1$, CHSH are violated

see Fabbrichesi, Floreanini, Panizzo '21

However, to implement this approach is a subtle matter:

The fact that C^TC is positive implies that the uncertainties in λ_1, λ_2 are likely biased to values larger than the real ones

Severi, Boschi, Maltoni, Sioli 2021

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Severi, Boschi, Maltoni, Sioli 2021

For example, for a 2×2 matrix

$$\begin{pmatrix} a & \delta c \\ \delta c & b \end{pmatrix} \qquad a > b > 0,\ \delta c \sim 0$$

The largest eigenvalue is

$$\lambda_1 = \frac{1}{2}\left(a - b + \sqrt{(a+b)^2 + 4(\delta c)^2}\right) \geq a$$

even if the uncertainty of δc is symmetrically distributed around $\delta c = 0$ (the true value)

Bell (CHSH) inequalities

Thus, implementing the $\lambda_1 + \lambda_2 > 1$ criterion and derive statistical uncertainties on CHSH violation is a subtle task.

see Fabbrichesi, Floreanini, Panizzo '21,
Severi, Boschi, Maltoni, Sioli '21

An easier approach is to choose fixed $\vec{a}, \vec{a}', \vec{b}, \vec{b}'$ directions in

$$\left| \vec{a} \cdot C(\vec{b} - \vec{b}') \; + \; \vec{a}' \cdot C(\vec{b} + \vec{b}') \right| \leq 2 \qquad \text{(CHSH)}$$

Namely ,

$$a_k = \delta_{ki}, \qquad a'_k = \delta_{kj},$$

$$b_i = -\,b'_i = \pm\frac{1}{\sqrt{2}}, \qquad b_j = b'_j = \pm\frac{1}{\sqrt{2}}, \qquad b_{k\neq i,j} = 0$$

$$\boxed{|\,C_{ii} \pm C_{jj}\,| > \sqrt{2}}$$

Bell (CHSH) inequalities

$$\boxed{|C_{ii} \pm C_{jj}| > \sqrt{2}}$$

~~CHSH~~

★ Simple linear (non-biased) combinations

★ Opens the possibility to design dedicated observables to directly measure $C_{ii} \pm C_{jj}$ and thus the CHSH violation

How far is this simple criterion from $\lambda_1 + \lambda_2 > 1$?

From the eigenvalue interlacing theorem:

$$\sqrt{\lambda + \lambda'} \geq \sqrt{(C^2)_{ii} + (C^2)_{jj}} \geq \sqrt{(C_{ii})^2 + (C_{jj})^2} \geq \frac{1}{\sqrt{2}} |C_{ii} \pm C_{jj}| \quad ✓$$

The $\geq$ are close to equalities up to $\mathcal{O}\left(\frac{C_{ij}^2}{|C_{ii}| + |C_{jj}|}, \frac{(|C_{ii}| - |C_{jj}|)^2}{|C_{ii}| + |C_{jj}|} \right)$ corrections

Aguilar-Saavedra, JAC

How to strengthen the sensitivity to entanglement and CHSH violation

Two strategies:

☆ Select gg-fusion events at the tt threshold

☆ Design dedicated experimental observables to directly measure entanglement and CHSH violation

How to strengthen the sensitivity to entanglement and CHSH violation

$\beta-$cut

At the tt **threshold** the tt state generated by gluon fusion is close to the singlet state

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle)$$

which is maximally entangled. In contrast, $q\bar{q}$ production leads to a separable tt state.

How to strengthen the sensitivity to entanglement and CHSH violation

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An upper cut on the velocity of the tt system:

$$\beta \equiv \left| \frac{p_t^z + p_{\bar{t}}^z}{E_t + E_{\bar{t}}} \right| \leq \beta^{\mathrm{cut}}$$

reduces the $q\bar{q}$ production, due to the different PDFs of valence quarks and sea anti-quarks inside the protons, thus enhancing th entanglement

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (entanglement)

Consider the conditions for entanglement:

$$\boxed{\begin{array}{ll} & |C_{11} + C_{22}| > 1 + C_{33} \\ \text{or} & \\ & |C_{11} - C_{22}| > 1 - C_{33} \end{array}}$$

which, depending on the values of C_{ii} leads to a different optimal observable:

$$-C_{11} - C_{22} - C_{33} > 1$$

$$C_{11} + C_{22} - C_{33} > 1$$

$$C_{11} - C_{22} + C_{33} > 1$$

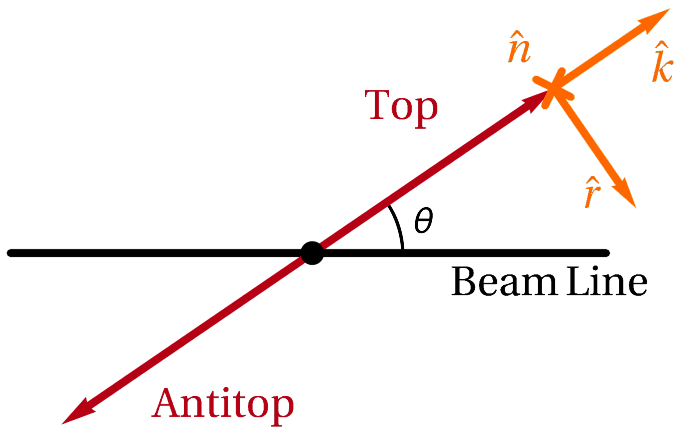
$$-C_{11} + C_{22} + C_{33} > 1$$

Can we measure these observables in a direct way?

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (entanglement)

Here it is convenient to work in the helicity basis, $\overrightarrow{k}, \vec{r}, \overrightarrow{n}$



(From Severi et al.)

Then, given the predicted (SM) values of the various C_{ij} in the different kinematical regions, the most useful observable for entanglement is

$$E \equiv |C_{kk} + C_{rr}| - C_{nn} - 1 > 0$$

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (entanglement)

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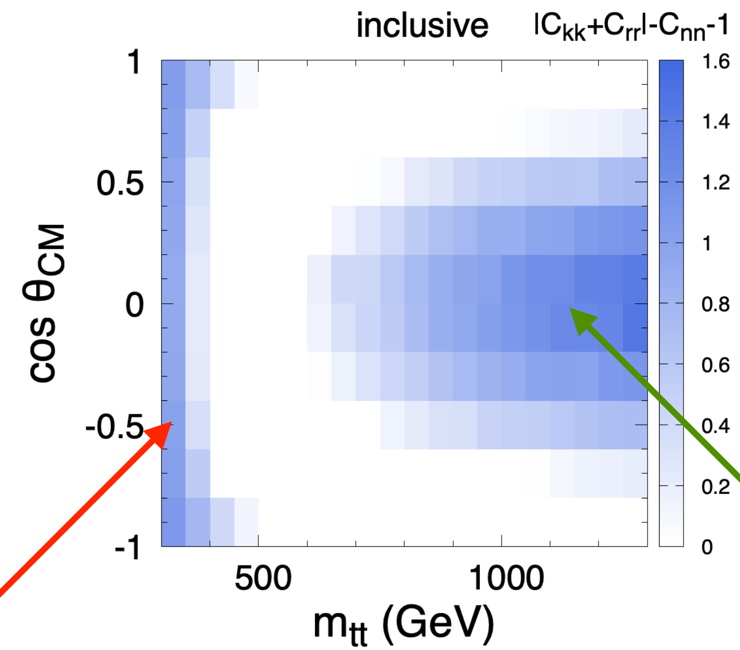

Threshold region,

$$E = -(C_{kk} + C_{rr} + C_{nn}) - 1 > 0$$

Boosted region,

$$E = C_{kk} + C_{rr} - C_{nn} - 1 > 0$$

How to strengthen the sensitivity to entanglement and CHSH violation

$C_{kk} + C_{rr} + C_{nn}$ can be measured by

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta_{ab}} = \frac{1}{2}\left(1 + \alpha_a \alpha_b D \cos\theta_{ab}\right)$$

with

$$D = \frac{1}{3}(C_{kk} + C_{rr} + C_{nn}) \qquad \cos\theta_{ab} \equiv \hat{p}_a \cdot \hat{p}_b$$

Bernreuther, Si, 2015

How to strengthen the sensitivity to entanglement and CHSH violation

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Bernreuther, Si, 2015

Similarly $C_{kk} + C_{rr} - C_{nn}$ can be measured by

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\theta'_{ab}} = \frac{1}{2}\left(1 + \alpha_a \alpha_b D_3 \cos\theta'_{ab}\right)$$

with

$$D_3 = \frac{1}{3}(C_{kk} + C_{kk} - C_{nn}) \qquad \cos\theta'_{ab} \equiv \hat{p}_a \cdot \hat{p}'_b$$

Aguilar-Saavedra, JAC, 2022

Reflected in the k-r plane

How to strengthen the sensitivity to entanglement and CHSH violation

Improvement in entanglement detection (only statistical uncertainty):

$\beta-$cut

LHC Run 2 139 fb^{-1}

threshold [β]	E :	0.559 ± 0.017	$\longrightarrow$	0.679 ± 0.019	1.27 ×
boosted [D_3]	E :	0.671 ± 0.069	$\longrightarrow$	0.663 ± 0.056	1.23 ×

dedicated observable

The improvement is more important for the boosted region, as it has less statistics, and systematic and statistical uncertainties are probably comparable:

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (CHSH)

These CHSH tests depend just on two parameters

$$\boxed{B \equiv |\, C_{ii} \pm C_{jj} \,| - \sqrt{2} > 0}$$

In the helicity basis, the best choices are

$$\begin{cases} C_{kk} + C_{nn} & \text{at threshold,} \\ C_{rr} - C_{nn} & \text{at the boosted region} \end{cases}$$

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (CHSH)

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but the beam basis $\overrightarrow{\hat{x}}$, $\overrightarrow{\hat{y}}$, $\vec{\hat{z}}$ works slightly better

$$\downarrow$$

$$C_{xx} + C_{yy}$$

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (CHSH)

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$$\boxed{B \equiv |\, C_{ii} \pm C_{jj} \,| - \sqrt{2} > 0}$$

In the helicity basis, the best choices are

$$\begin{cases} C_{kk} + C_{nn} & \text{at threshold,} \quad \text{but the beam basis } \overrightarrow{\hat{x}}, \overrightarrow{\hat{y}}, \vec{\hat{z}} \text{ works slightly better} \longrightarrow C_{xx} + C_{yy} \\ C_{rr} - C_{nn} & \text{at the boosted region} \end{cases}$$

$C_{ii} \pm C_{jj}$ can be measured with the observable

$$A_{\pm} = \frac{N(\cos(\varphi_a \mp \varphi_b) > 0) - N(\cos(\varphi_a \mp \varphi_b) < 0)}{N(\cos(\varphi_a \mp \varphi_b) > 0) + N(\cos(\varphi_a \mp \varphi_b) < 0)} \;\propto\; C_{11} \pm C_{22}$$

How to strengthen the sensitivity to entanglement and CHSH violation

Dedicated observables (CHSH)

Improvement in CHSH detection:

Setting systematics aside, there is an improvement of the statistical uncertainty of the `CHSH violation indicators´ $B = |C_{ii} \pm C_{jj}| - \sqrt{2}$

LHC Run 2+3 300 fb^{-1}

threshold [β,A_+]	$B:$	0.021 ± 0.053	$\longrightarrow$	0.121 ± 0.045	6.8 ×
boosted [A_-]	$B:$	0.218 ± 0.141	$\longrightarrow$	0.208 ± 0.125	1.13 ×

In all cases the statistics are small, therefore an improvement of the statistical sensitivity is very welcome.

HL-LHC 3 ab^{-1}

threshold [β,A_+]	$B:$	0.024 ± 0.017	$\longrightarrow$	0.124 ± 0.013	6.8 ×
boosted [A_-]	$B:$	0.218 ± 0.041	$\longrightarrow$	0.208 ± 0.036	1.13 ×

Summary and Conclusions

- Top pairs produced at the LHC looks a promising system to test entanglement and CHSH violation at high energy

- We have explored two strategies to strengthen the detection of both phenomena:

  - ☆ Select gg-fusion events at the tt threshold ($\beta-$cut)

  - ☆ Design dedicated experimental observables to directly measure indicators of entanglement and CHSH

- We have only analyzed the improvement on the stat. uncertainty. The results are quite promising and it seems feasible to detect entanglement in Run 2 and CHSH violation at HL LHC.

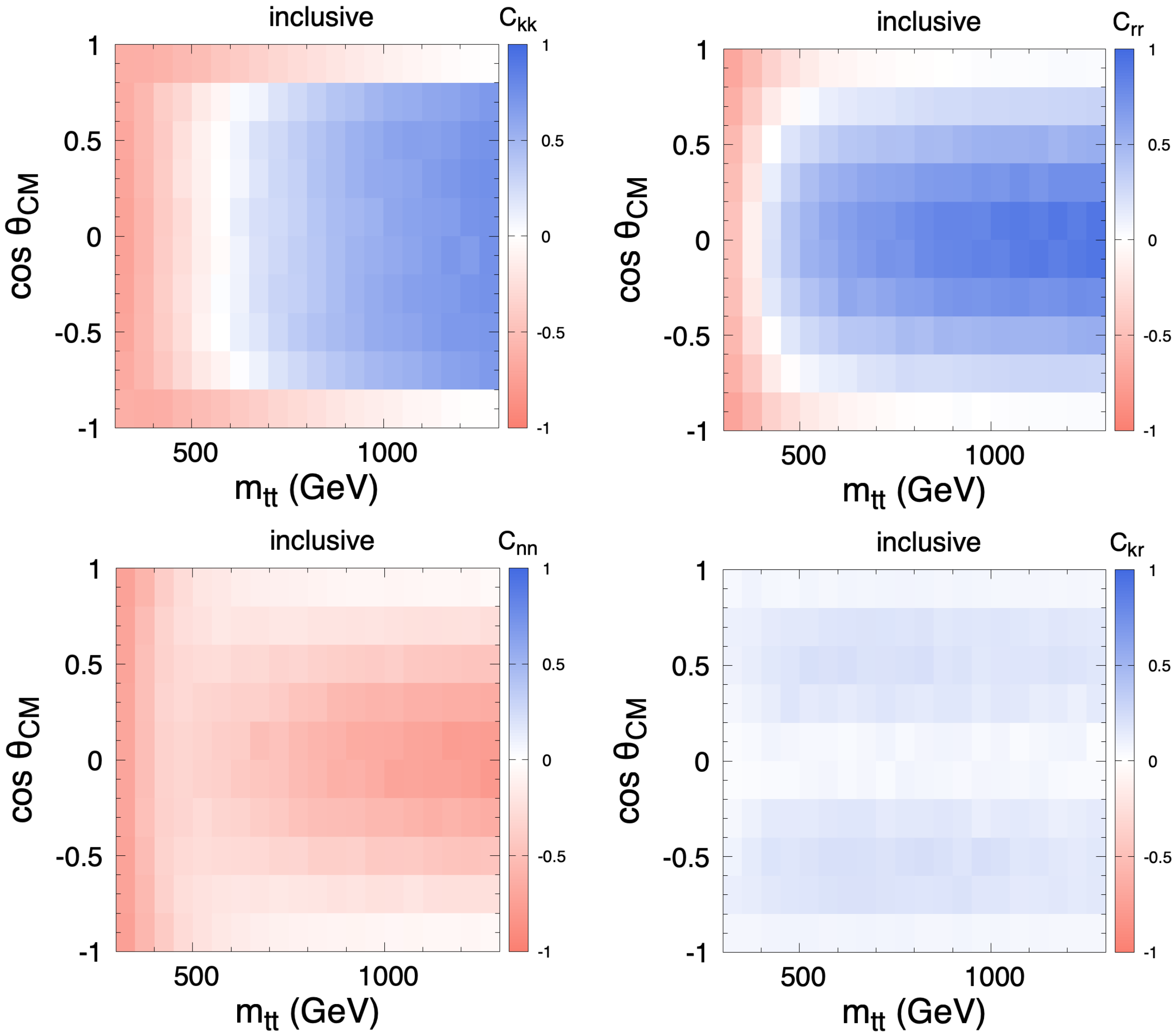

Figure 1: Dependence on $m_{t\bar{t}}$ and $\theta_{\rm CM}$ of the spin correlation coefficients C_{kk}, C_{rr}, C_{nn} and C_{kr}, at the parton level without kinematical cuts.

Entanglement

	$m_{t\bar{t}}$	$\lvert\cos\theta_{\rm CM}\rvert$	β	σ	N
Threshold $\not\beta$	≤ 390	–	–	3.59 pb	108000
Threshold β	≤ 390	–	≤ 0.9	2.76 pb	83000
Boosted	≥ 800	≤ 0.6	–	310 fb	9400

Table 1: Kinematical cuts on $m_{t\bar{t}}$ (in GeV), $\cos\theta_{\rm CM}$ and β used to optimise the figure of merit S_E. The fourth column gives the tree-level cross section with the corresponding cuts, and the fifth column the expected number of events after reconstruction (see the text for details).

	C_{kk}	C_{rr}	C_{nn}	C_{kr}	C_{xx}	C_{zz}
Threshold $\not\beta$	-0.619	-0.372	-0.568	0.080	-0.606	-0.356
Threshold β	-0.668	-0.417	-0.593	0.071	-0.632	-0.415
Boosted	0.51	0.64	-0.52	0.15	-0.05	0.76

Table 2: Parton-level values of the spin correlation coefficients in the helicity and beamline bases with the kinematical selection in Table 1. In the helicity basis $C_{kr} = C_{rk}$, and in the beamline basis $C_{xx} = C_{yy}$. The rest of coefficients are below 0.01.

	Threshold $\not\beta$	Threshold β	Boosted
Individual	0.560 ± 0.020	0.680 ± 0.022	0.671 ± 0.069
Direct	0.559 ± 0.017	0.678 ± 0.019	0.663 ± 0.056

Table 3: Values of the entanglement indicator E in (40) obtained from 200 pseudo-experiments with $L = 139$ fb^{-1} and the kinematical cuts in Table 1. The row labeled as 'individual´ presents results from individual determinations of C_{kk}, C_{rr} and C_{nn}. The row labeled as 'direct' corresponds to results obtained measuring either D (at threshold) or D_3 (in the boosted regime).

CHSH viol.

	$m_{t\bar{t}}$	$\lvert\cos\theta_{\mathrm{CM}}\rvert$	β	σ	N
Threshold $\not\beta$	≤ 353	–	–	303 fb	19600
Threshold β	≤ 353	–	≤ 0.8	181 fb	11700
Boosted	≥ 1000	≤ 0.2	–	23.3 fb	1500

Table 4: Kinematical cuts on $m_{t\bar{t}}$ (in GeV), $\cos\theta_{\mathrm{CM}}$ and β used to optimise the figure of merit S_B. The fourth column gives the tree-level cross section with the corresponding cuts, and the fifth column the expected number of events after reconstruction (see the text for details).

	C_{kk}	C_{rr}	C_{nn}	C_{kr}	C_{xx}	C_{zz}
Threshold $\not\beta$	-0.677	-0.562	-0.712	0.067	-0.719	-0.506
Threshold β	-0.743	-0.640	-0.761	0.052	-0.767	-0.602
Boosted	0.659	0.874	-0.760	0.037	-0.043	0.878

Table 5: Parton-level values of the spin correlation coefficients in the helicity and beamline bases with the kinematical selection in Table 4. In the helicity basis $C_{kr} = C_{rk}$, and in the beamline basis $C_{xx} = C_{yy}$. The rest of coefficients are below 0.01.

	Threshold $\not\beta$	Threshold β	Boosted
Individual	0.021 ± 0.053	0.119 ± 0.074	0.218 ± 0.141
Direct	0.027 ± 0.035	0.121 ± 0.045	0.208 ± 0.125

Table 6: Values of the ~~CHSH~~ indicators $B_{1,2}$ in (42), (43) obtained from 1000 pseudo-experiments with with $L = 300$ fb^{-1} and the kinematical cuts in Table 4. The row labeled as 'individual' gives the results from individual determinations of spin correlation coefficients. The row labeled as 'direct' corresponds to results obtained measuring azimuthal asymmetries (see the text for details).

	Threshold $\not\beta$	Threshold β	Boosted
Individual	0.024 ± 0.017	0.120 ± 0.021	0.218 ± 0.041
Direct	0.027 ± 0.010	0.124 ± 0.013	0.210 ± 0.036

Table 7: The same as Table 6, for $L = 3000$ fb^{-1}.

In this work we mainly use as reference system to express the $t\bar{t}$ density matrix (4) the helicity basis, with vectors $(\hat{r}, \hat{n}, \hat{k})$ defined as

- K-axis (helicity): $\hat{k}$ is a normalised vector in the direction of the top quark three-momentum in the $t\bar{t}$ rest frame.
- R-axis: $\hat{r}$ is in the production plane and defined as $\hat{r} = \text{sign}(y_p)(\hat{p}_p - y_p\hat{k})/r_p$, with $\hat{p}_p = (0,0,1)$ the direction of one proton in the laboratory frame, $y_p = \hat{k} \cdot \hat{p}_p$, $r_p = (1-y_p^2)^{1/2}$. The definition for $\hat{r}$ is the same if we use the direction of the other proton $-\hat{p}_p$.
- N-axis: $\hat{n} = \hat{k} \times \hat{r}$ is orthogonal to the production plane and can also be written as $\hat{n} = \text{sign}(y_p)(\hat{p}_p \times \hat{k})/r_p$, which again is independent of the proton choice.