Testing entanglement and Bell inequalities in H $\rightarrow$ ZZ

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Main goals

1. Reconstruct the spin density matrix, ρ_{ZZ}, from angular observables in the decay $H \to ZZ^* \to \ell_1^+ \ell_1^- \ell_2^+ \ell_2^-$ (Quantum Tomography).
2. State a necessary and sufficient condition for entanglement in ρ_{ZZ} by taking into account symmetries of the system.
3. Give a novel optimization method for the violation of Bell inequalities for ρ_{ZZ}.

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⋮

Preliminaries

- Bipartite quantum system:

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \quad p_i \geq 0, \quad \sum_i p_i = 1.$$

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- CGLMP Bell inequality (A_1, A_2 and B_1, B_2 observables acting respectively on $\mathcal{H}_A$ and $\mathcal{H}_B$):

$$I_3(P(A_i = k, B_j = l)) = \langle \mathcal{O}_{Bell}(A_i,\, B_j) \rangle_\rho \leq 2.$$

Quantum Tomography

- **Spin density matrix ρ_{ZZ}?**

  In our case, $\mathcal{H}_A = \mathcal{H}_B = \mathcal{H}_{Spin}$ $(s_A = s_B = 1 \Rightarrow d_A = d_B = 3)$.

  We use the irreducible tensor operators $\{T^{L_1}_{M_1} \otimes T^{L_2}_{M_2}\}$ (transforming under rotations as the spherical harmonics) as a basis:

$$T^{L_1}_{M_1}, T^{L_2}_{M_2} \in \left\{\mathbb{I}_3; T^1_1, T^1_0, T^1_{-1}; T^2_2, T^2_1, T^2_0, T^2_{-1}, T^2_{-2}\right\} .$$

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Hence:

$$\rho_{ZZ} = \frac{1}{9} \sum C_{L_1,M_1,L_2,M_2}\, T^{L_1}_{M_1} \otimes T^{L_2}_{M_2}.$$

$$\mathrm{Tr}\left\{T^L_M\ \left(T^L_M\right)^\dagger\right\} = 3 \Longrightarrow C_{L_1,M_1,L_2,M_2} = \left\langle \left(T^{L_1}_{M_1} \otimes T^{L_2}_{M_2}\right)^\dagger \right\rangle_{\rho_{ZZ}}$$

To extract the coefficients, we use the cross section of $ZZ^* \to \ell_1^+ \ell_1^- \ell_2^+ \ell_2^-$:

$$\frac{1}{\sigma}\frac{d\sigma}{d\Omega_1 d\Omega_2} = \left(\frac{3}{4\pi}\right)^2 \left\langle \Gamma_1^T\left(\Omega_1\right) \otimes \Gamma_2^T\left(\Omega_2\right)\right\rangle_{\rho_{ZZ}}$$

where $\Gamma_j(\Omega_j) = \Gamma_j(\theta_j, \varphi_j)$ are the decay density matrices of each Z boson.

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Noticing that $\mathrm{Tr}\left\{ T_M^L \, \Gamma^T \right\} = B_L \; Y_L^M(\theta, \varphi)$, with B_L a constant:

$$\frac{1}{\sigma} \frac{d\sigma}{d\Omega_1 d\Omega_2} = \frac{1}{(4\pi)^2} \sum C_{L_1, M_1, L_2, M_2} B_{L_1} B_{L_2} \; Y_{L_1}^{M_1}(\Omega_1) Y_{L_2}^{M_2}(\Omega_2).$$

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Finally,

Quantum Tomography of ρ_{ZZ}

$$\int \frac{1}{\sigma}\frac{d\sigma}{d\Omega_1 d\Omega_2} Y_{L_1}^{M_1}(\Omega_1)^* Y_{L_2}^{M_2}(\Omega_2)^* d\Omega_1 d\Omega_2 = \frac{B_{L_1} B_{L_2}}{(4\pi)^2} C_{L_1 M_1 L_2 M_2}.$$

Symmetries of the system

- Momentum J_z conserved. In particular $J_z = 0$.
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In particular,

$$|\psi_{\beta=1}\rangle = |\psi_s\rangle = \frac{1}{\sqrt{3}} \left(|{+}{-}\rangle - |00\rangle + |{-}{+}\rangle\right) \text{ and } \rho_{\beta=1} = \rho_s.$$

Singlet State (Maximally Entangled)

On the other hand, from the Lorentz structure of the HZZ SM vertex, $\propto \eta_{\mu\nu} H Z^{\mu} Z^{\nu}$,

$$|\psi_\beta\rangle = \eta_{\mu\nu}\; e^{\mu}_{\sigma}(m_{Z_1}, \vec{k})\; e^{\nu}_{\lambda}(m_{Z_2}, -\vec{k})\; |\vec{k}, \sigma\rangle_A |{-\vec{k}}, \lambda\rangle_B,$$

where σ, λ represent spin states and

$$e^{\mu}_{\sigma}(m, \vec{k}) = \begin{pmatrix} 0 & \frac{|\vec{k}|}{m} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & -\frac{\sqrt{|\vec{k}|^2+m^2}}{m} & 0 \end{pmatrix}.$$

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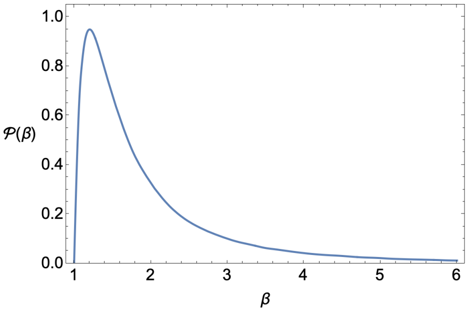
Comparing both results for $|\psi_\beta\rangle$, an analytical expression for β is obtained:

$$\beta = 1 + \frac{m_H^2 - (m_{Z_1} + m_{Z_2})^2}{2 m_{Z_1} m_{Z_2}}, \quad \beta \geq 1.$$

In practice, we have a probability distribution $\mathcal{P}(\beta)$ for the parameter β:



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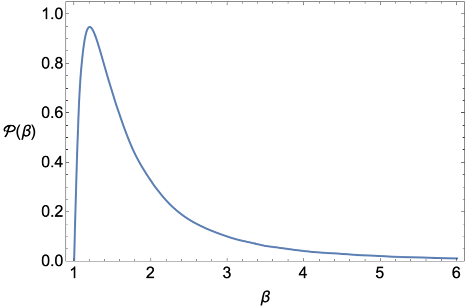

Figure 1: Probability distribution of β.

Thus, the final density matrix is

$$\rho_{ZZ} = \int d\beta \; \mathcal{P}(\beta)\rho_\beta = \rho_{ZZ}\left(C_{2,0,0,0}, C_{2,1,2,-1}, C_{2,2,2,-2}\right).$$

$\mathcal{P}(\beta)$ peaked in $\beta \approx 1 \Longrightarrow \rho_{ZZ} \backsim \rho_\beta \backsim \rho_s \rightarrow$ **Max. Ent.**

Testing Entanglement

- **Peres-Horodecki criterion**

> Given a density matrix ρ describing a bipartite system, if the matrix $\rho' = (\mathbb{I} \otimes T_B)\,\rho$ has at least one negative eigenvalue, then ρ is entangled.

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This criterion is only a sufficient condition for:

$$dim\,\mathcal{H}_A = dim\,\mathcal{H}_B = d \geq 3 \text{ (our case).}$$

However, due to the symmetries mentioned and the structure of ρ_{ZZ}:

$$\rho_{ZZ} \text{ entangled } \iff C_{2,1,2,-1} \neq 0 \text{ or } C_{2,2,2,-2} \neq 0.$$

Testing Bell inequalities

We recall that $\rho_s = |\psi_s\rangle \langle\psi_s|$, with $|\psi_s\rangle = \frac{1}{\sqrt{3}} (|+-\rangle - |00\rangle + |-+\rangle)$.

This state has a U(3) symmetry in the sense:

$$\left\langle (U \otimes U^*)^\dagger \mathcal{O}_{Bell} (U \otimes U^*) \right\rangle_{\rho_s} = \langle \mathcal{O}_{Bell} \rangle_{\rho_s} \quad \text{for } U \in U(3).$$

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We denote the **known** optimal Bell operator for the singlet state as $\mathcal{O}^s_{Bell}$, and in the $\{T^{L_1}_{M_1} \otimes T^{L_2}_{M_2}\}$ basis is given by:

Optimal Bell operator for ρ_s

$$\mathcal{O}^s_{Bell} = \frac{4}{3\sqrt{3}} \left(T^1_1 \otimes T^1_{-1}\right) + \frac{2}{3} \left(T^2_2 \otimes T^2_{-2}\right) + h.c.$$

For $\rho_\beta = |\psi_\beta\rangle \langle\psi_\beta|$, where $|\psi_\beta\rangle = \frac{1}{\sqrt{2+\beta^2}} \left(|{+}{-}\rangle - \beta\,|00\rangle + |{-}{+}\rangle\right)$, the $U(3)$ symmetry is broken to $U(2) \otimes U(1)$:

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When $\beta \approx 1$, a good analytical approximation is given by deforming $\mathcal{O}^s_{Bell}$ in the broken part of the initial symmetry group:

$$\left\langle \mathcal{O}^\beta_{Bell}\right\rangle_{\rho_\beta} \approx \max_{U\in U(3)/(U(2)\otimes U(1))} \left\langle (U\otimes U^*)^\dagger \mathcal{O}^s_{Bell}(U\otimes U^*)\right\rangle_{\rho_\beta}.$$

The final optimal matrix is $U_0 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$.

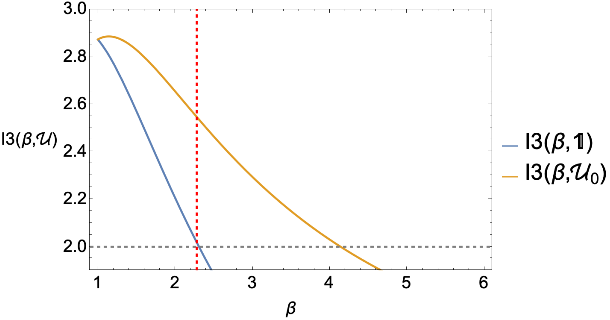

Figure 2: Functions $(I_3(\beta, \mathbb{I}_3), I_3(\beta, U_0))$, local-realistic upper bound (gray line) and mean value of β with respect to $\mathcal{P}(\beta)$ (red line).

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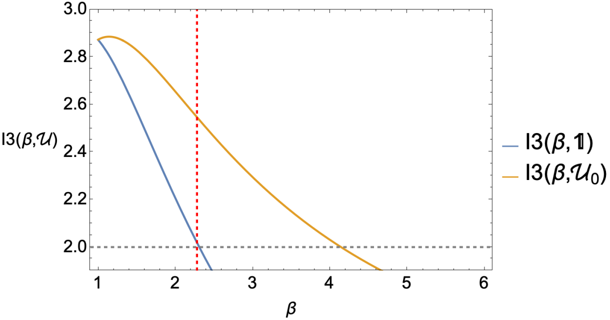

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Taking this value of U_0 we evaluate the violation of the Bell Ineq. for ρ_{ZZ}:

$$I_3 = I_3\left(C_{2,0,0,0}, C_{2,1,2,-1}, C_{2,2,2,-2}\right).$$

Numerical Resuts

- **LHC Run 2+3**

	min m_{Z_2}			
	0	10 GeV	20 GeV	30 GeV
N	450	418	312	129
$C_{2,1,2,-1}$	-0.98 ± 0.31	-0.97 ± 0.33	-1.05 ± 0.38	-1.06 ± 0.61
$C_{2,2,2,-2}$	0.60 ± 0.37	0.64 ± 0.38	0.74 ± 0.43	0.82 ± 0.63
I_3	2.66 ± 0.46	2.67 ± 0.49	2.82 ± 0.57	2.88 ± 0.89

Table 1: Values $C_{2,1,2,-1}$, $C_{2,2,2,-2}$ and I_3 obtained from 1000 pseudo experiments with $L = 300$ fb^{-1}.

- **HL-LHC**

	min m_{Z_2}			
	0	10 GeV	20 GeV	30 GeV
N	4500	4180	3120	1290
$C_{2,1,2,-1}$	-0.95 ± 0.10	-1.00 ± 0.10	-1.04 ± 0.12	-1.04 ± 0.19
$C_{2,2,2,-2}$	0.60 ± 0.12	0.64 ± 0.12	0.74 ± 0.14	0.83 ± 0.20
I_3	2.63 ± 0.15	2.71 ± 0.16	2.81 ± 0.18	2.84 ± 0.28

Table 2: Same as Table 1, for $L = 3$ ab^{-1}.

Conclusions

- The decay channel $H \to ZZ^* \to \ell_1^+ \ell_1^- \ell_2^+ \ell_2^-$ is an excellent way to probe the quantum nature of high energy physics:
  - Run 2+3: ρ_{ZZ} entangled in more than 2σ and $I_3 > 2$ in more than 1σ.
  - HL-LHC: ρ_{ZZ} entangled in more than 5σ and $I_3 > 2$ in more than 3σ.
- The quantum tomography formalism developed is practical and generalizable for other kinds of processes.
- Entanglement criteria as well as optimal violations of Bell inequalities can be extracted and implemented taking into account the symmetries of the system.

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Thank you for listening!