# On the detectability of higher harmonics with LISA

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based on arXiv:2303.03142, in collaboration with
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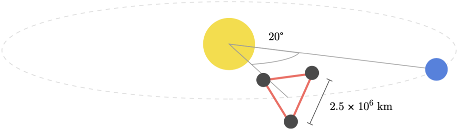
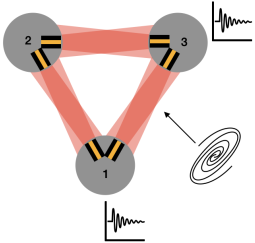
# LISA: Laser Interferometer Space Antenna

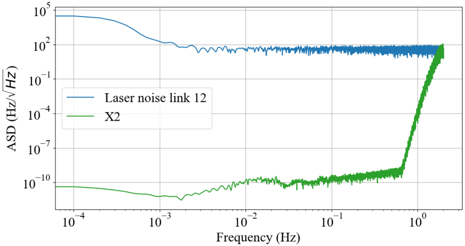
- 3 space-craft constellation separated by 2.5 million km




- Emitted laser ∼ 2 W ⟶ received laser∼ 300 pW
- 6 links response
- The combination of links allows for multiple interferometers

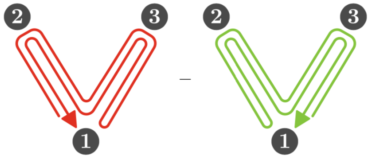
# LISA




*Generated with PyTDI*[1]

- Laser noise cancellation

Time Delay Interferometry (TDI)




Quasi-orthonormal combination

$$A = \frac{1}{\sqrt{2}}(Z - X)$$

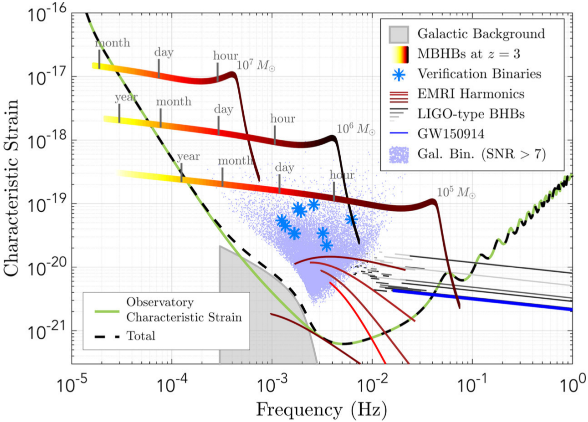
$$E = \frac{1}{\sqrt{6}}(X - 2Y + Z)$$

$$T = \frac{1}{\sqrt{2}}(X + Y + Z)$$

[1] *PyTDI*,

# Motivation

- SMBHB sources with high SNR $\longrightarrow$ Test General Relativity

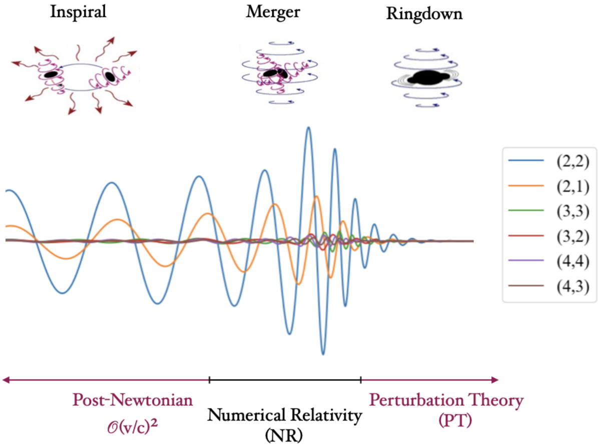



*credit: LISA consortium proposal 2017*[2]

[2] P. AMARO-SEOANE *et al.*, *arXiv e-prints*, arXiv:1702.00786 (fév. 2017).

# Compact Object Binary Waveform

$$h(t,\Xi,\theta,\varphi) = \sum_{lm} h_{lm}(t,\Xi)\ {}_{-2}Y^{lm}(\theta,\varphi)$$



# Are we able to disentangle modes?

`Lisabeta` software[3] $\longrightarrow$ Introduce LISA response to the waveform

- IMR (Inspiral-Merger-Ringdown) Phenomenological waveforms in frequency domain
- Pass the waveform through LISA $\longrightarrow$ LISA response can be integrated as a transfer function in Fourier's domain for each harmonic

$$\mathcal{H}^I_{lm}(f,\Xi,\Theta) = \mathcal{T}^I_{lm}(f,\Theta)h_{lm}(f,\Xi)$$

where $\mathcal{T}^I_{lm}(f,\Theta)$ includes:

- Orbits of the constellation
- Delays with respect to SSB's frame and to LISA's CoM frame
- TDI (Time delay interferometry)

Signal:

- Inject a signal with `PhenomHM`[4] (IMR waveform) with 6 harmonics: (l,m) = (2, 2), (2, 1), (3, 3), (3, 2), (4, 4), (4, 3)
- Random source with a SNR $\sim$ 748

[3] S. Marsat *et al.*, *Physical Review D* **103** (2021).

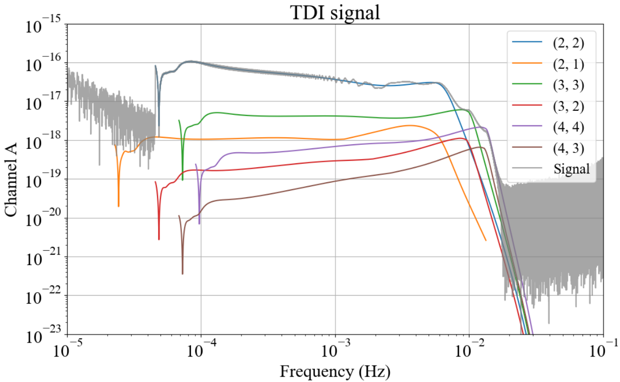
[4] L. London *et al.*, *Physical Review Letters* **120** (2018).

# Signal and Bayesian analysis

Compare different models $M_i(l,m)$ with different number of modes with a Bayesian analysis.

- Dynesty[5] : nested sampler → estimate model evidence

$$\mathcal{Z} = \int_{\Theta} \mathcal{L}(\theta)\pi(\theta)d\theta \;\rightarrow\; \text{Bayes factor:} \quad \mathcal{B} = \frac{\mathcal{Z}_i}{\mathcal{Z}_j}$$

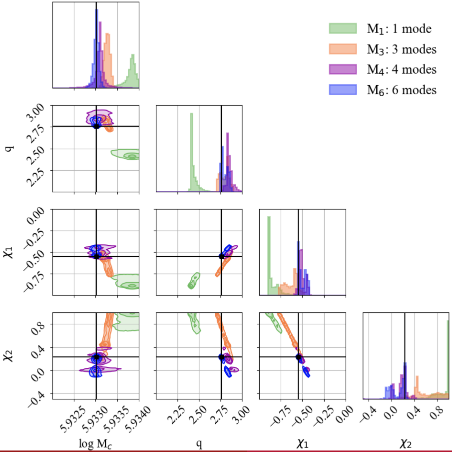



[5] J. S. Speagle, *Monthly Notices of the Royal Astronomical Society* **493**, 3132-3158 (avr. 2020).

# Results with noise

Parameter estimation with different models $M_i(l, m)$

| | | | |
|---|---|---|---|
| (2, 2) | $M_1$ | (2, 2), (3, 3), (4, 4), (2, 1) | $M_4$ |
| (2, 2), (3, 3) | $M_2$ | (2, 2), (3, 3), (4, 4), (2, 1), (3, 2) | $M_5$ |
| (2, 2), (3, 3), (4, 4) | $M_3$ | (2, 2), (3, 3), (4, 4), (2, 1), (3, 2), (4, 3) | $M_6$ |




| Parameter | True value | Estimated value $M_6$ in noisy data |
|---|---|---|
| $\log M_c$ ($M_\odot$) | 5.93302 | $5.93304^{+0.00009}_{-0.00010}$ |
| q | 2.759 | $2.759^{+0.013}_{-0.023}$ |
| $\chi_1$ | -0.549 | $-0.548^{+0.011}_{-0.021}$ |
| $\chi_2$ | 0.232 | $0.231^{+0.057}_{-0.030}$ |

Bayes factor $\longrightarrow \log \mathcal{B}$

| | |
|---|---|
| $\log(\mathcal{Z}_1/\mathcal{Z}_6)$ | -6873 |
| $\log(\mathcal{Z}_2/\mathcal{Z}_6)$ | -1015 |
| $\log(\mathcal{Z}_3/\mathcal{Z}_6)$ | -259 |
| $\log(\mathcal{Z}_4/\mathcal{Z}_6)$ | -134 |
| $\log(\mathcal{Z}_5/\mathcal{Z}_6)$ | -100 |

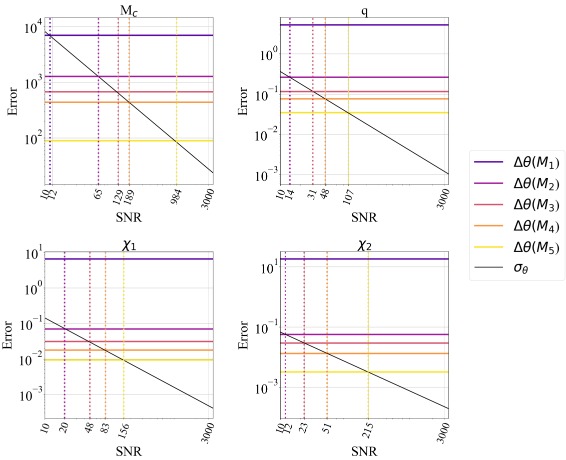
# Modelling error

Using an incorrect template results in a systematic modelling error ($\Delta\theta_i$). If the statistical error ($\sigma_{\theta_i} \propto \mathrm{SNR}^{-1}$) is smaller than the modelling error, the bias in the parameters becomes relevant.

$$\begin{cases} \sigma_{\theta_i} = \sqrt{\Gamma_{ii}^{-1}} \\ \Delta\theta_i = \sum_j \Gamma_{ij}^{-1} \left( \frac{\partial \mathcal{H}}{\partial \theta_j} \middle| \delta \mathcal{H} \right) \end{cases}$$

$$\Gamma_{ij} = \left( \frac{\partial \mathcal{H}}{\partial \theta_i} \middle| \frac{\partial \mathcal{H}}{\partial \theta_j} \right), \quad \text{where}$$

$$(a|b) = 4\mathrm{Re} \int_0^\infty \frac{a(f) b^*(f)}{S_n(f)} df$$



# Conclusions

- We are able to discriminate waveform models and therefore their modes with a Bayesian analysis.
- We see how the use of an incorrect template of modes causes bias in the source parameter estimation.
- This bias can lead to misinterpretation in GR tests.
- Given a certain SNR we can constrain the number of modes needed to estimate the parameters without significant bias, in the case of a waveform with 6 modes.

Next steps:

- Repeat this analysis using only the ringdown to study sensitivity to the remnant's quasi-normal modes .
- Issues:
  - start of the ringdown
  - connection of the amplitude with the progenitors
  - mixing of the modes
  - non-linearity
- Addition of glitches

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*Thank you!*

# *Back up slides*

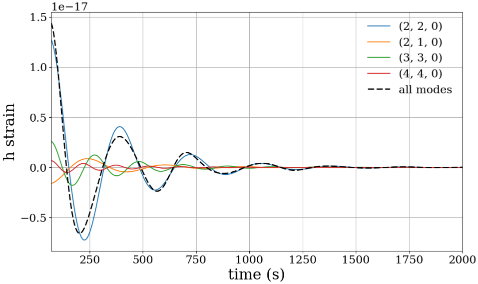
# Ringdown model

Emission of GW's from a perturbed BH as a sum of QNM's:

$$h_{lmn}(t,\Xi) = A_{lmn}(\Xi) e^{-t/\tau_{lmn}} \cos(\omega_{lmn} t + \phi_{lmn}(\Xi))$$

$$h(t,\Xi,\theta,\varphi) = \sum_{lmn} h_{lmn}(t,\Xi)\ {}_{-2}S^{lmn}(j\tilde{\omega}_{lmn},\theta,\varphi)$$

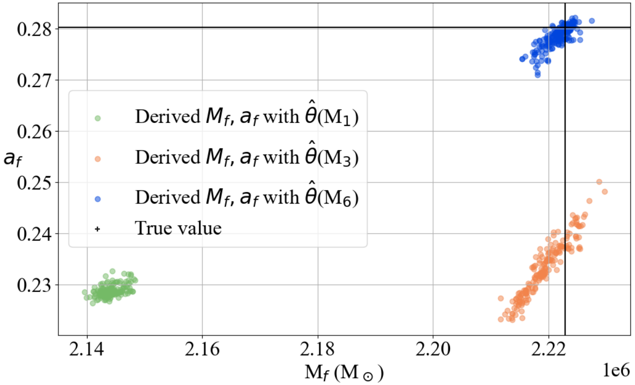
$$\underbrace{\tilde{\omega}_{lmn}}_{\text{complex freq}} = \underbrace{\omega_{lmn}}_{\text{freq}} + \underbrace{i/\tau_{lmn}}_{\text{damping time}} \longrightarrow \tilde{\omega}_{lmn} = \tilde{\omega}_{lmn}(M,j)$$




$Mass = 5 \times 10^6\ M_\odot,\ D_L = 1\ Gpc$

# Testing GR

Comparing final mass and spin derived from parameters from models $M_i$ with ringdown parameters from $\tilde{\omega}_{lmn}(M,j)$

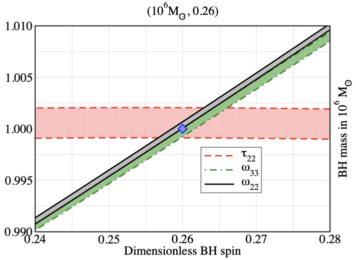


# Testing No-hair theorem with QNM

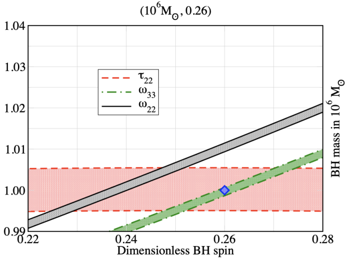
Knowing values of $\tilde{\omega}_{lm}$ one can find the mass and spin through a parametrization[6,7] :

$$M\omega = f_1 + f_2(1-j)^{f_3}$$
$$Q = q_1 + q_2(1-j)^{q_3}$$
$$Q_{lmn} = \omega_{lmn}\tau_{lmn}/2$$

$$\omega_{lmn}^{nonGR} = \omega_{lmn}(1+\delta\omega_{lmn})$$
$$\tau_{lmn}^{nonGR} = \tau_{lmn}(1+\delta\tau_{lmn})$$




Mass = $10^6$ $M_\odot$, $j$ = 0.26




Mass = $10^6$ $M_\odot$, $j$ = 0.26

[6] E. Berti *et al.*, *Physical Review D* **73** (2006).

[7] S. Gossan *et al.*, *Physical Review D* **85** (2012).

# Modes and quasi-normal modes

We can write the GW emission from a rotating BH's ringdown as:

$$h(t,\theta,\varphi) = \frac{1}{r}\sum_{l,m,n} h_{lmn}(t)\,{}_{-2}S_{lmn}(j_f\,\tilde{\omega}_{lmn},\theta,\varphi)$$

$${}_{-2}S_{lmn}(\theta,\varphi) = e^{im\varphi}\ {}_{-2}S_{lm}\left(j_f\tilde{\omega}_{lmn},\cos\theta\right)$$

$${}_{-2}Y_{lm}(\theta,\varphi) = e^{im\varphi}\ {}_{-2}Y_{lm}(\theta)$$

- Press and Teukolsky noted:

$$\boxed{{}_{-2}S_{lmn} = {}_{-2}Y_{lm}} + j_f\,\tilde{\omega}_{lmn}\sum_{l\neq l'} {}_{-2}Y_{l'm}c_{l'lm} + \mathcal{O}(j_f\,\tilde{\omega}_{lmn})^2$$

  where $c_{l'lm}$ are related to Clebsch-Gordan coefficients
- The modes on NR can be written in terms of spherical harmonics ${}_{-2}Y_{lm}$, which is a complete and orthogonal set.

$$\Psi^{NR}_{l'm}(t) \simeq \sum_{l,n} A_{lmn}\sigma_{l'lmn}e^{i\,\tilde{\omega}_{lmn}t}$$

where[8]

$$\sigma_{l'lmn} = \int_{\Omega} {}_{-2}S_{lm}(j_f\,\tilde{\omega}_{lmn},\theta,\varphi)\,{}_{-2}\overline{Y}_{l'm}(\theta,\varphi)d\Omega$$

[8] L. London *et al.*, *Physical Review D* **90** (2014).

# Ringdown model

A perturbation of a Schwarzschild BH[9] :

$$\Psi''(x) + (w^2 - V)\Psi(x) = 0$$

$$\Psi(w,x) \sim e^{iwx} \quad \text{as} \quad x \to -\infty, \qquad \Psi(w,x) \sim e^{-iwx} \quad \text{as} \quad x \to \infty$$

$$\hat{\Psi}(s,x) = \int_0^\infty e^{-st}\Psi(t,x)dt$$

$$\hat{\Psi}''(s,x) + (-s^2 - V(x))\hat{\Psi}(s,x) = I(s,x)$$

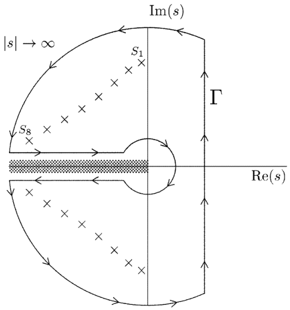
$$\hat{\Psi}(s,x) = \int_{-\infty}^{\infty} G(s,x,x')I(s,x')dx'$$

$$G(s,x,x') = \frac{1}{W(s)}\hat{\Psi}_-(s,x_{min})\hat{\Psi}_+(s,x_{max})$$

QNM's $\rightarrow \tilde{\omega}_{lmn} = \omega_{lmn} + i/\tau_{lmn}$

We can write the solution as:

$$\Psi_{lmn}(t) = A_{lmn}e^{-i\tilde{\omega}_{lmn}}$$




*credit: H-P. Nollert*

[9] H.-P. Nollert, *Classical and Quantum Gravity* **16**, R159-R216 (1999).

# Parametrized deviations

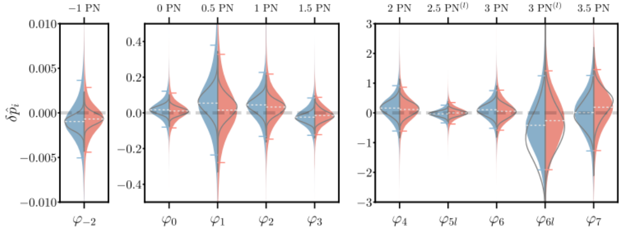
The waveform of a compact binary in frequency domain is represented as

$$\tilde{h}(f) = A(f)\, e^{i\phi(f)}$$

where $A(f)$ denotes amplitude and $\phi(f)$ the phase in the inspiral regime. While the amplitude is unperturbed, the phase could present fractional deviations in the form of

$$\phi_i(f) = (1 + \varphi_i)\phi_i \qquad i = N^{th}/2$$

If GR is correct, $\phi_i(f)$ would vanishes for $N^{th}$ PN order.
In the post-merger and ringdown regime deviations are denoted by $\{\alpha_i, \beta_i\}$




*credits LVK Collaboration*[10]

[10] R. Abbott *et al.*, *Phys. Rev. D* **103**, 122002 (juin 2021).

# SNR

The SNR builds up in time and frequency:

$$\rho^2 = \sum_{lm} \sum_{l'm'} \sum_{ch} 4\,\mathrm{Re} \int \frac{\mathcal{H}^{ch}_{lm}(f) \mathcal{H}^{ch}_{l'm'}{}^{*}(f)}{S_n(f)} df,$$

$$(lm|l'm') = \sum_{ch} 4\,\mathrm{Re} \int \frac{\mathcal{H}^{ch}_{lm}(f) \mathcal{H}^{ch}_{l'm'}{}^{*}(f)}{S_n(f)} df.$$

Then the squared SNR can be written as

$$\rho^2 = \sum_{lm} \sum_{l'm'} (lm|l'm').$$

$$\mathcal{H}^{a}_{lm}(f) = h_{lm}(f) \cdot \frac{i\sqrt{2}\sin(2\pi f L)}{e^{-2i\pi f L}} \cdot \Big[(1 + z(f))(\mathcal{T}^{lm}_{13}(f) + \mathcal{T}^{lm}_{31}(f)) - \mathcal{T}^{lm}_{32}(f) - z(f)\mathcal{T}^{lm}_{23}(f) \;\; \mathcal{T}^{lm}_{12}(f) - z(f)\mathcal{T}^{lm}_{21}(f)\Big],$$

$$\mathcal{H}^{e}_{lm}(f) = h_{lm}(f) \cdot \frac{i\sqrt{2}\sin(2\pi f L)}{\sqrt{3}e^{-2i\pi f L}} \cdot \Big[(1 - z(f))(\mathcal{T}^{lm}_{31}(f) + \mathcal{T}^{lm}_{13}(f)) + (2 + z(f))(\mathcal{T}^{lm}_{21}(f) - \mathcal{T}^{lm}_{23}(f)) + (1 + 2z(f))(\mathcal{T}^{lm}_{12}(f) - \mathcal{T}^{lm}_{32}(f))\Big],$$

where $z(f) = e^{2i\pi f L}$

# Bayes factor

⟶ Bayesian method:

$$P(\Theta|D,M) = \frac{\overbrace{P(D|\Theta,M)}^{likelihood}\overbrace{P(\Theta|M)}^{prior}}{\underbrace{P(D|M)}_{evidence}}$$

⟶ Focus on estimating the evidence

$$\mathcal{Z} = \int_{\Omega_\Theta} P(D|M)d\Theta = \int_{\Omega_\Theta} \mathcal{L}(\Theta)\pi(\Theta)d\Theta = \int_0^1 \underbrace{\mathcal{L}(X)}_{iso-lkh} dX$$

where

$$\mathcal{L} = \frac{1}{\sqrt{\det(2\pi\mathbf{C})}} e^{-\frac{1}{2}(d-h(\boldsymbol{\theta}))^\dagger \mathbf{C}^{-1}(d-h(\boldsymbol{\theta}))}.$$

$$\ln \mathcal{L} = (d|h(\boldsymbol{\theta})) - \frac{1}{2}(h(\boldsymbol{\theta})|h(\boldsymbol{\theta})) - \frac{1}{2}(d|d),$$

Bayes factor:

$$\mathcal{B} = \frac{\mathcal{Z}_1}{\mathcal{Z}_2} = \frac{\int_{\Omega_\Theta} \mathcal{L}_1(\Theta)\cancel{\pi_1(\Theta)}d\Theta}{\int_{\Omega_\Theta} \mathcal{L}_2(\Theta)\cancel{\pi_2(\Theta)}d\Theta}$$

# IMR Phenomenological waveform

$$\tilde{h}_{lm}(f,\Theta) = A(f,\Theta)e^{-i\phi(f,\Theta)}$$

$$\phi_{MR} = \frac{1}{\eta}\left[\alpha_0 + \alpha_1 f - \alpha_2 f^{-1} + \frac{4}{3}\alpha_3 f^{3/4} + \alpha_4 \tan^{-1}\left(\frac{f - \alpha_5 f_{RD}}{f_{damp}}\right)\right]$$

$$\phi_{Int} = \frac{1}{\eta}\left(\beta_0 + \beta_1 f + \beta_2 \log f - \frac{\beta_3}{3} f^{-3}\right)$$

$$\phi_{Ins} = \phi_{TF2}(Mf,\Theta) + \frac{1}{\eta}\left(\sigma_0 + \sigma_1 f + \frac{3}{4}\sigma_2 f^{4/3} + \frac{3}{5} f^{5/3} + \frac{1}{2}\sigma_4 f^2\right)$$

$$\phi_{TF2} = 2\pi f\, t_c - \varphi_c - \pi/4 + \frac{3}{128}(\pi f M)^{-5/3}\sum_{i=0}^{7}\phi_i(\Theta)(\pi f M)^{i/3}$$