QFT and the SM (2)

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Yesterday

- Field theory is like mechanics with 4 times
- Coupled oscillators are like fields

Today we keep diving into QFT

The quantum SHO

What is QM?

- Many ways to formulate QM
- For example, we promote $x \to \hat{x}$
- We solve QM when we know the wave function $\psi(x,t)$
- How many wave functions describe a system?

The quantum SHO

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad E_n = (n + 1/2)\hbar\omega$$

- We also like to use

$$H = (a^\dagger a + 1/2)\hbar\omega$$

with

$$a, \sim x + ip \qquad a^\dagger \sim x - ip \qquad x \sim a + a^\dagger$$

- We call $a^\dagger$ and a creation and annihilation operators

$$E = a|n\rangle \propto |n-1\rangle \qquad a^\dagger|n\rangle \propto |n+1\rangle$$

- This is abstract. What does it mean that $x \sim a + a^\dagger$?

Couple oscillators

Consider a system with 2 DOF and same mass with

$$V(x,y) = \frac{kx^2}{2} + \frac{ky^2}{2} + \alpha xy$$

The normal modes are

$$q_\pm = \frac{1}{\sqrt{2}}(x \pm y) \qquad \omega_\pm^2 = \frac{k \pm \alpha}{m}$$

What is the QM energy and spectrum of this system?

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$$|n_+, n_-\rangle$$

with

$$E_{n_+,n_-} = (n_+ + 1/2)\hbar\omega_+ + (n_- + 1/2)\hbar\omega_-$$

Couple oscillators and Fields

- With many DOFs, $a \to a_i \to a(k)$
- And the states

$$|n\rangle \to |n_i\rangle \to |n(k)\rangle$$

- And the energy

$$(n+1/2)\hbar\omega \to \sum(n_i+1/2)\hbar\omega_i \to \int [n(k)+1/2]\hbar\omega(k)dk$$

- Just like in mechanics, we expand around the minimum of the fields, and to leading order we have SHO
- In QFT fields are operators while x and t are not

SHO and photons

I have two questions:

- What is the energy that it takes to excite an harmonic oscillator by one level?
- What is the energy of the photon?

SHO and photons

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- What is the energy that it takes to excite an harmonic oscillator by one level?
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Same answer

$$\hbar\,\omega$$

- Why is the answer to both question the same? Can we learn anything from it?

What is a particle?

Excitations of SHOs are particles

More on QFT

What about masses?

- A “free” Lagrangian gives massless particle

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 \Rightarrow \omega = k \quad (\text{or } E = P)$$

- We can add “potential” terms (without derivatives)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2} m^2 \phi^2$$

- Here m is the mass of the particle. Still free particle
- (HW) Show that m is a mass of the particle by showing that $\omega^2 = k^2 + m^2$. To do it, use the E-L Eq. and “guess” a solution of the form $\phi = e^{i(kx - \omega t)}$

What about other terms?

- How do we choose what terms to add to $\mathcal{L}$?
- Must be invariant under the symmetries
- We keep some leading terms (usually, up to ϕ^4)
- Lets add $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

- We get the non-linear wave equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = m^2\phi + \lambda\phi^3$$

- We do not know how to solve it

A short summary

- Fields are a generaliztion of SHOs
- Particles are excitations of fields
- The fundamental Lagrangian is giving in terms of fields
- Our aim is to find $\mathcal{L}$
- We can only solve the linear case, that is, the equivalent of the SHO
- What can we do with higher order terms?

Perturbation theory

Perturbation theory

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In many cases, perturbation theory (PT) is a mathematical tool
- There are cases, however, that PT is a better way to describe the physics
- Many times we prefer to work with the eigenvalues of H (why?)
- Yet, at times it is better to work with the eigenvalues of H_0 (why?)

PT for 2 SHOs

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- We assume that α is small
- Classically α moves energy between the two modes
- How it goes in QM?

Drops

Fermi Golden Rule

- Recall the Fermi golden rule

$$P \propto |\mathcal{A}|^2 \times \text{P.S.} \qquad \mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

- The relevant thing to calculate is the transition amplitude, $\mathcal{A}$.

1st and 2nd order PT

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In first order we care only about the states with the same energy

$$\mathcal{A}(i \to f) \sim \langle f|H_1|i\rangle \qquad E_f = E_i$$

- 2nd order pertubation theory probe the whole spectrum

$$\mathcal{A}(i \to f) \sim \sum_n \frac{\langle f|H_1|n\rangle\langle n|H_1|i\rangle}{E_n - E_f} \qquad E_f = E_i \qquad E_n \neq E_i$$

Transitions

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- Recall

$$x \sim a_x + a_x^\dagger \qquad y \sim a_y + a_y^\dagger$$

- For a given i, for what f we have $\mathcal{A} \neq 0$?

$$\mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

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- Since $H_1 \sim x^2 y$ we see that $\Delta n_y = \pm 1$ and $\Delta n_x = 0, \pm 2$
- The amplitude is finite only for a very few fs
- What could you say if the perturbation was $x^2 y^3$?

Two SHO with small α

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y \qquad \omega_y = 2\omega_x$$

- Consider $|i\rangle = |0,1\rangle$
- Since $\omega_y = 2\omega_x$ only $f = |2,0\rangle$ is allowed by energy conservation and by the perturbation

$$\mathcal{A} \sim \langle 2,0|\alpha x^2 y|0,1\rangle \sim \alpha\langle 2,0|(a_x+a_x^\dagger)(a_x+a_x^\dagger)(a_y+a_y^\dagger)|0,1\rangle$$

- a_y in y annihilates the y “particle” and $(a_x^\dagger)^2$ in x^2 creates two x “particles”
- It is a decay of a particle y into two x particles with width $\Gamma \propto \alpha^2$ and $\tau = 1/\Gamma$

Even More PT

$$H_1 = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- Calculate $y \to 3x$ using 2nd order PT

$$\mathcal{A} \sim \langle 3,0,0|\mathcal{O}|0,1,0\rangle \qquad \mathcal{O} \sim \sum \frac{\langle 3,0,0|V'|n\rangle\langle n|V'|0,1,0\rangle}{E_n - E_{0,1,0}}$$

- Which intermediate states? $|1,0,1\rangle$ and $|2,1,1\rangle$
  - $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$
  - $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$
- The total amplitude is then

$$\mathcal{A} \propto \alpha\beta \left(\frac{\#}{\Delta E_1} + \frac{\#}{\Delta E_2}\right)$$

Closer look

$$V' = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

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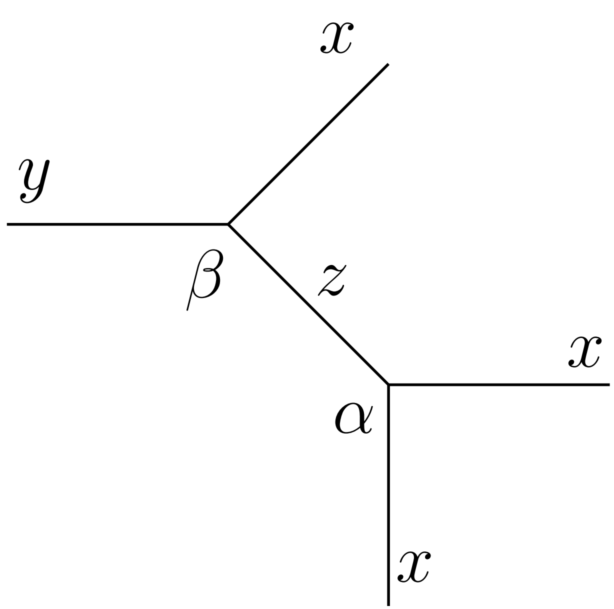

$$\mathcal{A} \propto \frac{\alpha\beta}{\Delta E}$$

Closer look: HW

- Draw the two amplitudes and show that it is the same diagram
- Show that the sum of the two diagrams give

$$\frac{1}{\omega_z^2 - q^2}$$

where q is the "energy flow" in the "z line"

General method for SHO PT

- Every x^n term with $n \geq 3$ is a vertex
- We write all the ways to get from "in" to "out"
- Each amplitude is the product of the couplings and the "off-shell" intermediate states

$$\frac{1}{\omega^2 - q^2}$$

- There are few more rules
- We add all amplitude square them and use the Fermi Golden rule

Feynman diagrams

Using PT for fields

- For one SHO we have $x \sim a + a^\dagger$
- For many SHOs we have $x_i \sim a_i + a_i^\dagger$
- For fields we then have

$$\phi \sim \int \left[a(k) + a^\dagger(k)\right] \, dk$$

Perturbation theory for fields is a generalization of that of SHO

- $\omega \to p_\mu$
- $\omega^2 \to m^2$
- We can have any energy (but one mass)

Calculations

- We usually care about $1 \to n$ or $2 \to n$ processes
- We need to make sure we have energy conservation
- External (Internal) particles are called on(off)–shell
  - On-shell: $E^2 = p^2 + m^2$
  - Off-shell: $E^2 \neq p^2 + m^2$
- $\mathcal{A} =$ the product of all the vertices and internal lines
- Each internal line with q^μ gives suppression

$$\frac{1}{m^2 - q^2}$$

- There are many more rules to get all the factors right

Some summary

- Quadratic terms describe free fields. Free particles cannot be created nor decay
- We use perturbation theory were terms with 3 or more fields in $\mathcal{L}$ are considered small
- These terms can generate and destroy particles and give dynamics
- Feynman diagrams are a tool to calculate transition amplitudes
- Many more details are needed to get calculation done
- Once calculations and experiments to check them are done, we can test our theory

Symmetries

How to “built” Lagrangians

- $\mathcal{L}$ is:
  - The most general one that is invariant under some symmetries
  - We work up to some order (usually 4)
- We need the following input:
  - What are the symmtires we impose
  - What DOFs we have and how they transform under the symmtry
- The output is
  - A Lagrangian with N parameters
  - We need to measure its parameters and test it

Symmetries and representations

Example: 3d real space in classical mechanics

- We require that $\mathcal{L}$ is invariant under rotation
- All our DOFs are assigned into vector representations $(\vec{r}_1,\ \vec{r}_2, ...)$
- We construct invariants from these DOFs. They are called singlets or scalars

$$C_{ij} \equiv \vec{r}_i \cdot \vec{r}_j$$

- We then require that V is a function of the C_{ij}s

Generalizations

- In mechanics, $\vec{r}$ lives in 3d real space and is a vector
- Fields do not live in real space. They live in some mathematical space
- They also do not have to be vectors, but can be scalars or tensors (representation)
- The idea is similar to what we did in mechanics
  - We require $\mathcal{L}$ to be invariant under rotation in that mathematical space
  - Thus $\mathcal{L}$ depends only on combinations of fields that form singlets
- All this is related to a subject called Lie groups
- We usually care about $SO(N)$, $SU(N)$ and $U(1)$

Combining representations

- It is all about generating singlets
- We all know that we can combine vectors in real space to generate singlets
- We also know how to make a spin zero from 2 spin half spinors (spin zero is a singlet!)
- There is a generalization of this procedure to any mathematical space
- As of now, all we need to know are $SU(3)$, $SU(2)$ and $U(1)$

Invariant of complex numbers

- $U(1)$ is rotation in 1d complex space
- Each complex number comes with a q that tells us how much it rotates
- When we rotate the space by an angle θ, the number rotates as

$$X \to e^{iq\theta} X$$

- Consider $q_X = 1,\ q_Y = 2,\ q_Z = 3$ and write 3rd and 4th order invariants

$$XX^*YY^* \qquad X^2Y^*$$

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$$XX^*YY^* \qquad X^2Y^* \qquad XYZ^* \qquad X^3Z^* \qquad Y^2X^*Z^*$$

$SU(2)$

- $U(2)$ is rotation in 2d complex space. We have $U(2) = SU(2) \times U(1)$
- $SU(2)$ is localy the same as rotation in 3d real space
- Rotations in this space are non-Abelian (non-commutative)
- It depends on the representation: scalar, spinor, vector
- Spin in QM is described by SU(2) rotations, so we use the same language to describe it
- For the SM all we care is that $1/2 \times 1/2 \ni 0$ so we know how to generate singlets
- How can we generate invarints from spin $1/2$ and spin $3/2$?

$SU(3)$

- $U(3)$ is rotation in 3d complex space. We have $U(3) = SU(3) \times U(1)$
- The representations we care about are singlets, triplets and octets
- Unlike $SU(2)$, in $SU(3)$ we have complex representations, 3 and $\bar{3}$
- The three quarks form a triplet (the three colors)
- To form a singlet we need to know that

$$3 \times \bar{3} \ni 1 \qquad 3 \times 3 \times 3 \ni 1$$

- This is why we have baryons and mesons

A game

A game calls “building invariants”

- Symmetry is $SU(3) \times SU(2) \times U(1)$
  - $U(1)$: Add the numbers ($\bar{X}$ has charge $-q$)
  - $SU(2)$: $2 \times 2 \ni 1$ and recall that 1 is a singlet
  - $SU(3)$: we need $3 \times \bar{3} \ni 1$ and $3 \times 3 \times 3 \ni 1$
- Fields are

$$Q(3,2)_1 \qquad U(3,1)_4 \qquad D(3,1)_{-2} \qquad H(1,2)_3$$

- What 3rd and 4th order invariants can we built?

$$(HH^*)^2 \qquad H^3 \qquad UDD \qquad QUD \qquad HQU^*$$

- HW: Find more invariants