# High energies. Why?

## hep-math

**Baltic Summer School - 3**

**Palanga**

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**disclamer**

At Saaremaa summer school-2022 I gave a sketchy overview of the basic ideas and principles of Quantum Field Theory, of the history of hadrons and quarks and, finally, of Deep Inelastic Scattering processes that allow us to probe them.

That was a ***sitcom*** *. Continuing within entertainment vocabulary, this year I decided to choose ***thriller*** **.

I am well aware that some of you are not prepared to go through such an ordeal unharmed.
I envisage three (*not mutually exclusive*) possible outcomes: SBC. You're likely to be **S**cared, sooner or later you might get **B**ored, or you might start feeling **C**urious. My **C** bets are modest, but I still think it's worth a try.

***I felt a strong urge to show you how theoretical physics works.***

**motivation**

1973 marked a major breakthrough in particle physics: ***Quantum Chromodynamics*** has been invented.
**QCD** is routinely referred to as the true ***microscopic theory of hadrons and their interactions***.

To be honest, nowadays we are rather far from awarding QCD this honourable mention.

Unlike electromagnetic and weak sectors of the **Standard Model** where QFT methods provide us with amazingly accurate predictions, ascension of QCD to the status of a legitimate QFT is impeded by the problem of ***colour confinement***.

We **are** confident that strongly interacting particles do consist of ***quarks*** and ***gluons*** that carry "**colour**".
But we do not understand why (and how) the **only** objects that fly and hit detectors - ***mesons*** and ***baryons*** - happen to be ***colourless***.

We will celebrate the 50th anniversary of the theory-of-hadrons-to-be by "anti-QCD lectures".

The reason for this choice was twofold.

1 Curiously, it is the most important characteristics of strong interactions such as total cross sections, elastic and inelastic scattering processes, the pattern of multi-particle production and multiplicity fluctuations, etc lie beyond the grasp of QCD as we master it today.

2 Under the magic spell of QCD, these important topics disappeared from the particle physics curriculum (for 5 decades!).

In this mini-course we will discuss qualitative picture of high-energy hadron interactions that we will derive from the basic principles of relativity, causality and conservation of probability.

* *situation comedy*
** *a novel, play, or film with an exciting plot, typically involving crime or espionage*

SBC : **S**cared (and/or) **B**ored (and/or) **C**urious

*The history of Particle Physics is that of never ending rush for higher and higher accelerator energies.*

# What do we need High Energies for?

- To be able to produce **new particles**, ever more massive — **heavier**
- To probe smaller and **smaller distances** in search of substructure — **deeper**
- To translate a cumbersome **QFT** picture into an easier to grasp QM - and sometimes even Classical - language, in order to visualise the structure of particles and the physics of their interaction — **simpler**

***Turning energy into matter***

***Turning energy into transferred momentum***

***Turning energy into clarity***

# Turning energy into matter

heavier

The first obvious motivation for higher energy race is search for new massive particles.

Have you seen this relation? $E = mc^2$ Is it correct? $nop$

?

$E = mc^2$

$E_0 = mc^2$

$E = m_0 c^2$

$E_0 = m_0 c^2$

true: $E = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$

$E_0 = mc^2$ internal energy of a massive particle = energy of a particle at rest

A couple of historic examples

$J/\psi$ Frascati, Italy — ADONE e+e- accelerator was 100 MeV short of charm discovery in November 1974

$H$ LEP (1989-2000) -> LHC

Nobel noble searches : SUSY particles (photino, gluino, zino, wino, squarks, sleptons, ...)

H', Z', lepto-quarks, heavy neutrinos,...

**High energy is a necessary (though not sufficient) condition for new discoveries.**

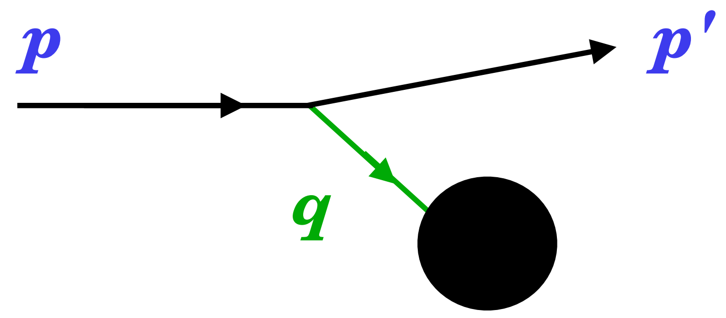
deeper

# *Turning energy into transferred momentum*

*Large **collision energy** allows for large **momentum transfer.***

*When we speak about " **large momentum**" we do not mean large values of energy and/or 3-momentum components.*
*Those depend on the reference frame.* *What we always have in mind is an **invariant quantity** - the Lorentz square*

$$(p_\mu - p'_\mu)^2 \equiv (q_\mu)^2 = q_0^2 - \mathbf{q}^2 \equiv q^2$$




***Large momentum transfer***

$$|q^2| \gg m^2$$

It is important to keep in mind that in **QFT** interacting objects ***do not like to exchange large momentum.***
Small-distance-dominated processes are ***rare***.

$$\frac{d\sigma}{dq^2} \propto \frac{1}{q^4}$$

It is ***large momentum transfer processes*** that serve as microscope, allowing us to study internal structure of matter, with strongly interacting particles - ***hadrons*** - being the primary subject of interest.

Classical example of such a microscope is ***Deep Inelastic*** lepton-hadron ***Scattering*** (DIS)
which reveals quark-gluon content of a proton and how it evolves with the "resolution" $Q^2 = |q^2|$.

Thanks to ***Asymptotic Freedom***, the quark-gluon and gluon-gluon interactions at small distances (space-time intervals) stay under quantitative control by means of perturbative QCD (expansion in small coupling $\alpha_s$ ).

This is the realm of ***rare*** **"Hard Processes"**.

Meanwhile, the ***bulk*** of hadron interactions are ***peripheral*** and do not penetrate the depth of matter.
Essential distances here are large, pQCD fails, the language of quarks and gluons is hardly justifiable.

(**mark**: I did not say *inapplicable*) simpler

**The nature of Relativistic Quantum Field Theory is paradoxical.**

**On one hand, its dynamics is undoubtedly complicated.**

**On the other hand, it turns out to be extremely restricted.**

**Quantum Physics** is inherently **complex.** But this is only *the* "*first level*" of complexity.

In **Non-relativistic Quantum Mechanics** we at least know what objects we dealing with, and how many they are.

While in the context of **Relativistic QFT** we don't have this luxury. Here the number of the objects involved **fluctuates**.

Any problem becomes not simply a **multi-body problem** but the one with **indefinite number of "bodies"!**

**This is a direct consequence of Relativity, and it significantly complicates the picture.**

**Meantime, the same Relativity provides powerful tools for simplifying the description of certain problems.**

**And this applies, in the first place, to high energy interaction phenomena**

***Physical Observables*** *are* ***frame-independent*** *(relativistically invariant).* ***Physical interpretation is not !***

"The Physical Picture" is revealed in all its glory *in high energy collisions.*

The aim of our discourse - ***Turning energy into clarity***

***We will dive into various aspects of this strange but true declaration:***

1. ***simplifying the high energy interactions picture*** (resurrecting the notion of classical ***impact parameter***)
2. ***reducing 4-dimensional dynamics to a product of complementary (longitudinal and transversal) 2-dimensional worlds***
3. ***replacing Quantum Field Theory battlefield by Quantum Mechanical terrain***

BUT FIRST ...

no joy without suffering...

# Analytic functions

***Quantum physics - from the QM wave function all the way to QFT - is the world of complex numbers and functions.***

Functions we are dealing with in physics, and in QFT in particular, are analytic functions

**Maths = the art of cheating by the rules** *(cons, codified)*

An example of a perfectly legitimate *double stunt* : $(1+3+9+27+\cdots)! = \sqrt{\pi}$

Looks crazy but can be justified, *unambiguously*, using the **analytic continuation** trick.

Analytic continuation ***is a trick of making sense of meaningless math constructions.***

Generalise : **3** -> **z** $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}\bigg|_{z=3} = \frac{1}{1-3} = -\frac{1}{2}$

The r.h.s. make sense everywhere but **z=1.**

Similar story with the function

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t} \qquad n! = \Gamma(n+1)$$

which is well defined by this integral for any complex value of **z** as long as **Re z** >0

How to make sense now of $(-1/2)! = \sqrt{\pi}$ ?

$$n! = \int_0^\infty dt\, t^n e^{-t}\bigg|_{n=-1/2} = \int_0^\infty \frac{dt}{\sqrt{t}} e^{-t} = 2\int_0^\infty dx\, e^{-x^2} = \int_{-\infty}^{+\infty} dx\, e^{-x^2}$$

$$= \sqrt{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, e^{-x^2-y^2}} = \sqrt{\int_0^{2\pi} d\phi \int_0^\infty \frac{d(r^2)}{2}\, e^{-r^2}} = \sqrt{\pi}$$

We justified our double-con having analytically continued

1. series beyond its convergence radius |z|<1
2. a function n! from positive integers to any z

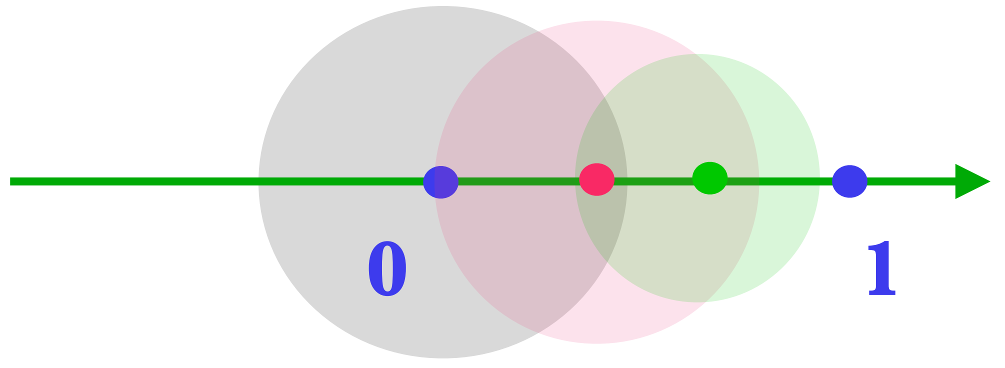
# walking the plane

Geometric series is too simple an example. Replace it with general Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \; a_n = \frac{f^{[n]}(z_0)}{n!}$$

Knowing the value of the function and of all its derivatives at some point $\mathbf{z_0}$ we expand our knowledge to all points inside the convergence circle $|\mathbf{z} - \mathbf{z_0}| < R$.

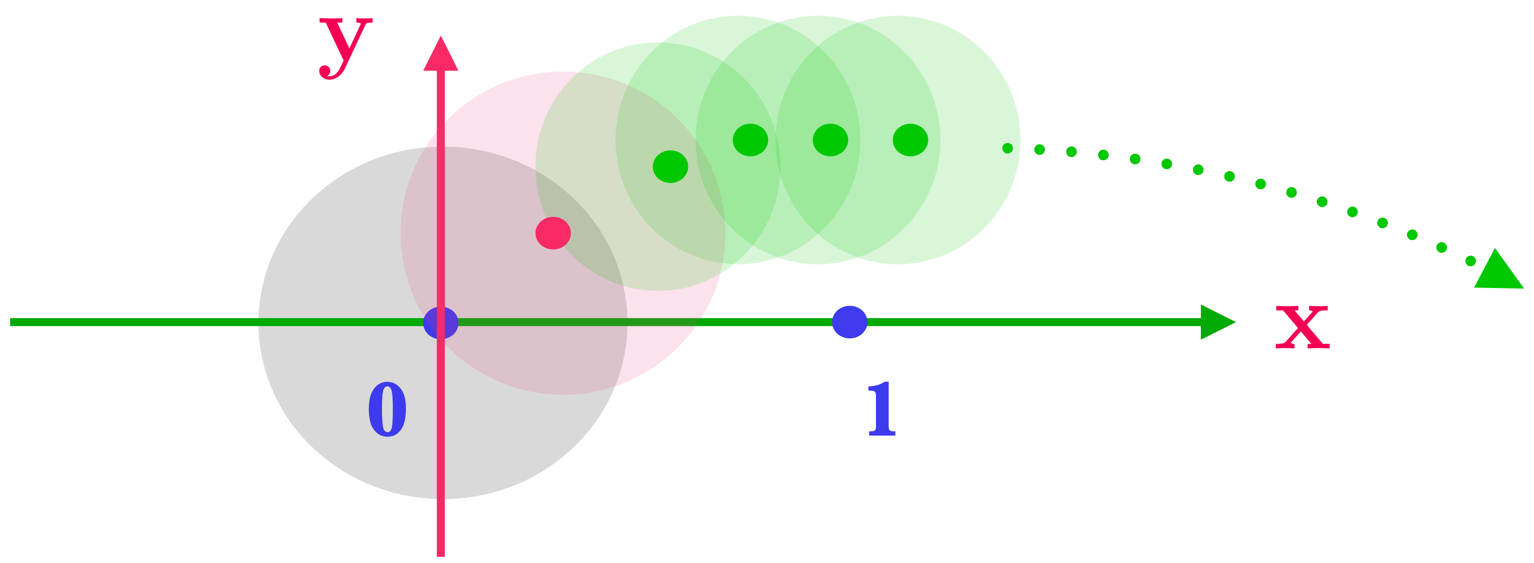
Then, constructing Taylor series around a new point inside this circle, we may explore the function further from the initial point.



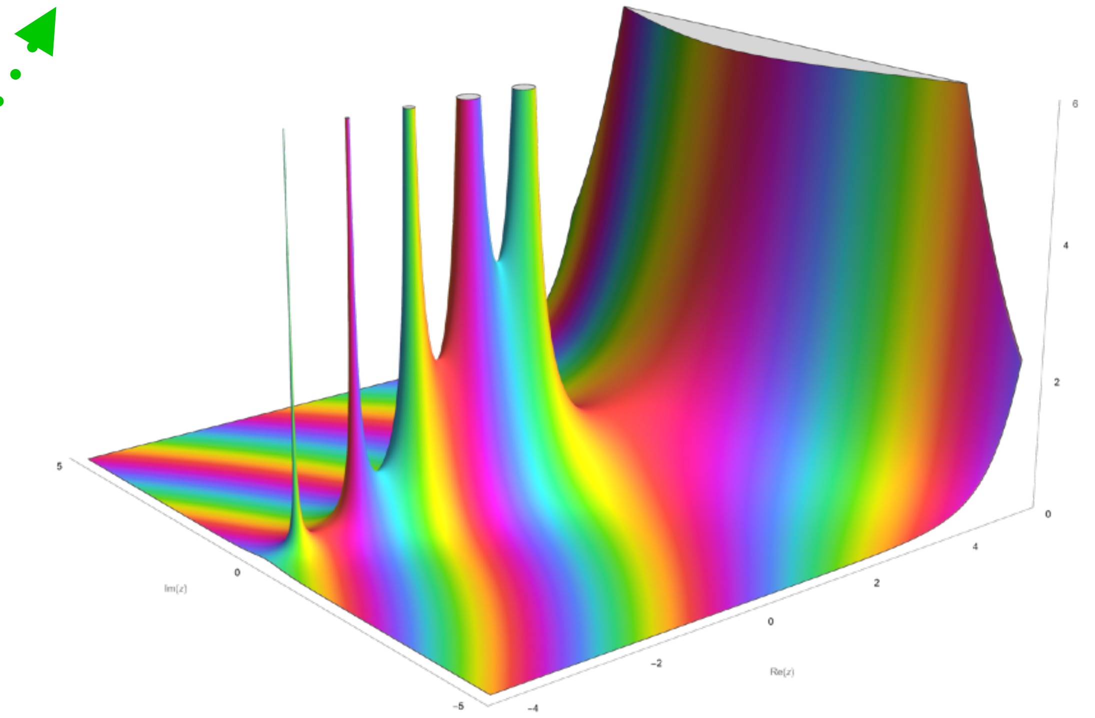

In our example, we won't be able to pass to z>1 : since the radius of convergence of the series shrinks when we approach the point z=1 where the function we examine is singular

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

The head-on attack fails? Make a flanking manoeuvre by diving into the **complex plane** $z = x + iy$




$$\Gamma(z) \longleftarrow n! = \Gamma(n+1)$$



This way one attempts to *analytically continue* the function from a limited domain (vicinity of one point) to the values where the original expression did not make sense.

One succeeds in uniquely expanding all over the complex plane when the function happens to have <u>isolated singularities</u> like simple poles in z, which is the case in both our examples:

**success not guaranteed**

Look, e.g. at $f(z) = (1-z)^{1/2}$

Point z=1 looks harmless: $f(1)=0$ Still, this is a singular point! (No Taylor series around z=1)

How to continue this function?

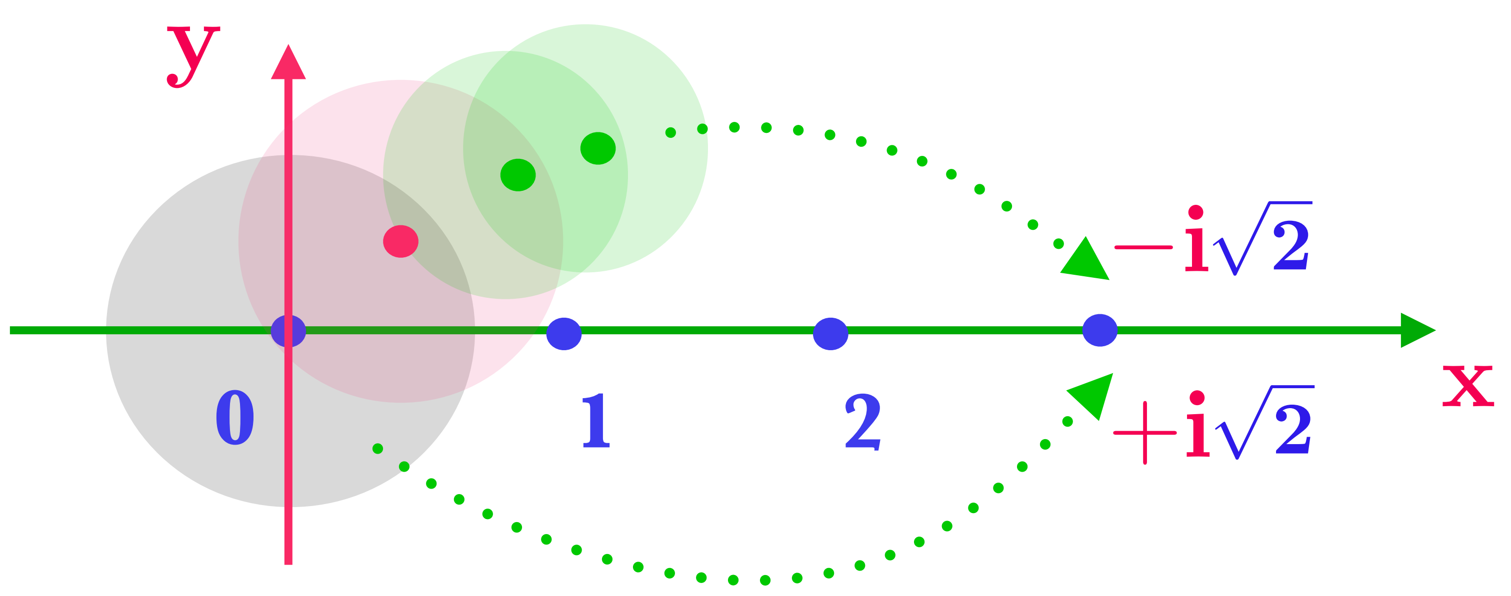
Recall the Newton binomial formula: $(1-x)^n = 1 - nx + \frac{n(n-1)}{2}x^2 + \ldots + (-x)^n = \sum_{k=0}^{\infty}(-x)^k C_n^k$

*combinatorial coefficients* $C_n^k = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{k!\Gamma(n+1-k)}$

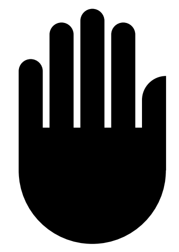
*converge for* $|x| \leq 1$

To generalise to non-integer **n,** replace factorials by the Gamma-function, resulting in series $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256} - \ldots$

*What will happen if we attempt to pass by the singular point* ***z=1*** *in the complex plane as we did before ?*




the value depends on the continuation path !

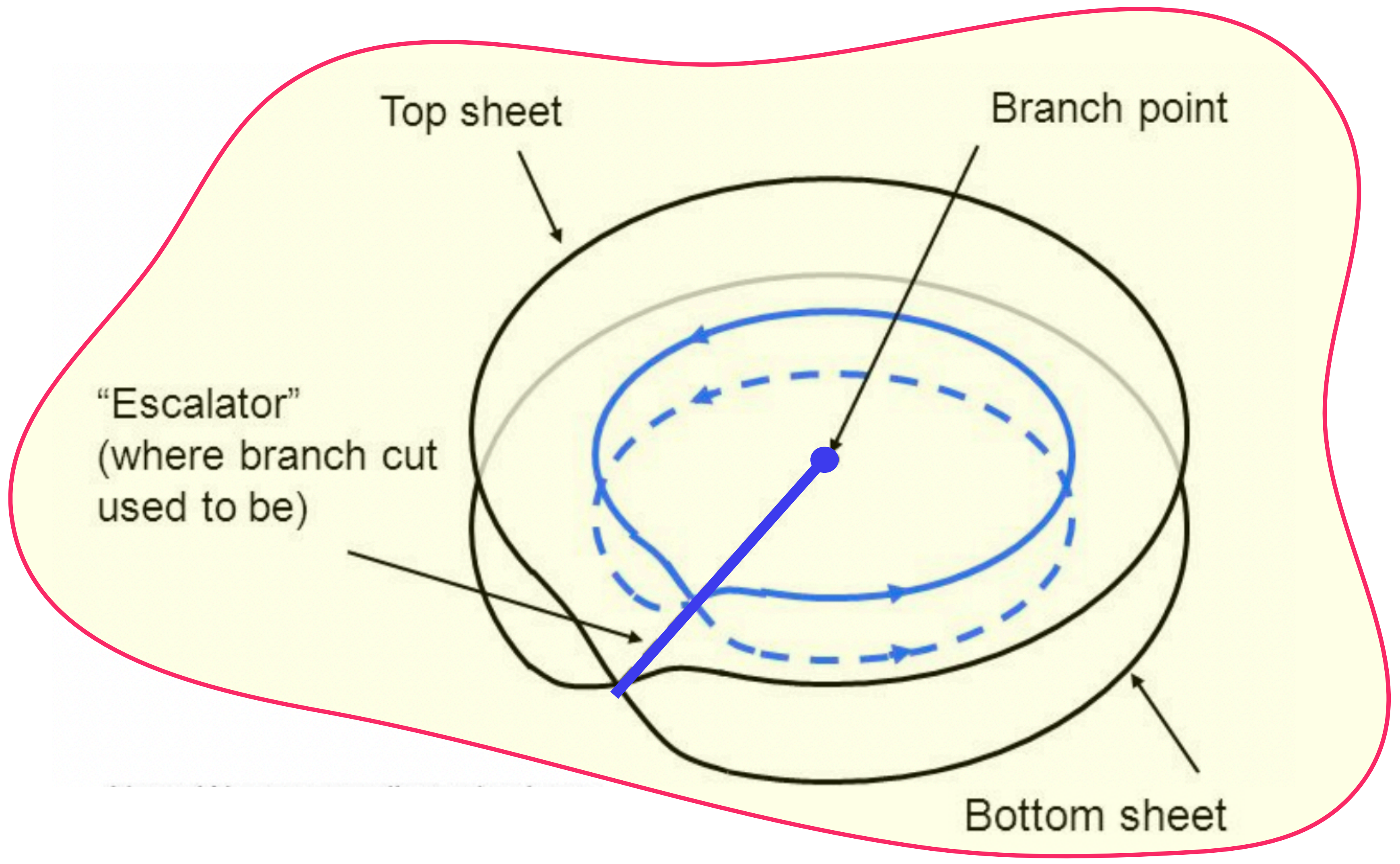


ain't a function
oops ...

Another way to treat this "confusion" :

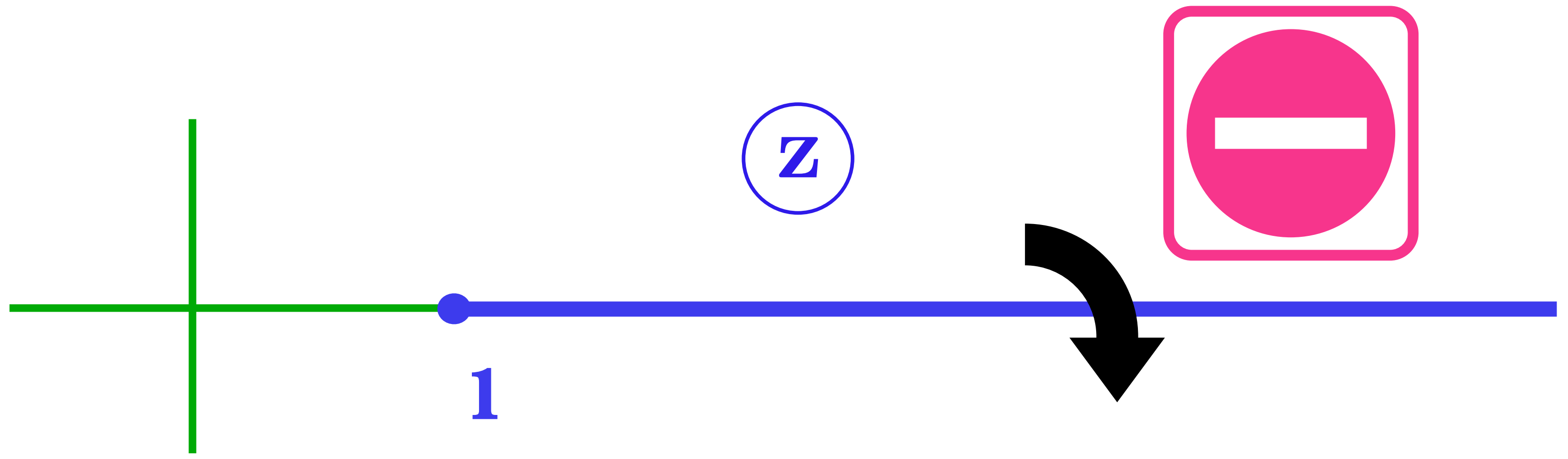
upgrade the habitat of our function from the **complex plane** to Riemann surface

- two complex planes glued together along the branch cut :




Our function ***branches*** above z=1.

To **define** it, we are forced to ***cut*** the plane along the ray $1 \leq z < +\infty$



math summary

***Analytic functions*** is a small subset of functions in the complex plane : $f(x,y) \;= f(x+iy, x-iy) \equiv f(z, \bar{z}) \longrightarrow f(\mathbf{z})$

***This seemingly harmless "detail" has truly dramatic consequences :***

★ vectors $\nabla\,\mathbf{Re}f, \nabla\,\mathbf{Im}f$ with $\nabla \equiv \left(\frac{d}{dx}, \frac{d}{dy}\right)$ correspond to two-dimensional flow(s) of incompressible liquid $\mathbf{div}\nabla f = \Delta f = 0$

★ mapping $\mathbf{z} \to \mathbf{f}(\mathbf{z})$ is ***conformal*** - *preserves small shapes* (angles)

★ an analytic function ***must have*** singularity(ies) somewhere in the complex plane (*with the only exception* $f(z) \equiv \mathrm{const}$ )

★ knowing the function on the **border** $z \in C$, you get it everywhere **inside**: $f(w) = \oint_C \frac{dz}{2\pi i}\frac{f(z)}{z-w}$ (Cauchy)

★ an **integral** btw two points ***does not depend on the integration path*** (*provided no singularities are crossed while deforming it*)

★ an **analytic function** is fully determined by its ***singularities***

***Time to ask :*** ***What was the purpose of a bungee dive into complex analysis ?***

**Scattering amplitude** is an ***analytic function*** of its variables: **energy**, **momentum transfer**

***causality*** (no effect before cause)

***relativity*** (crossing)

***unitarity*** (conservation of probability)

Its ***singularities*** are determined by the **physical spectrum** of the theory

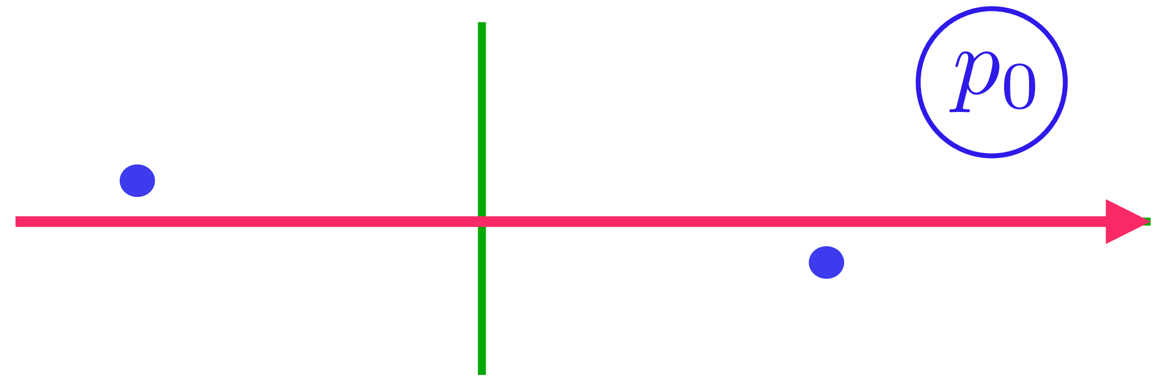
# Analyticity and Causality

Look at two-particle scattering in the *coordinate space*

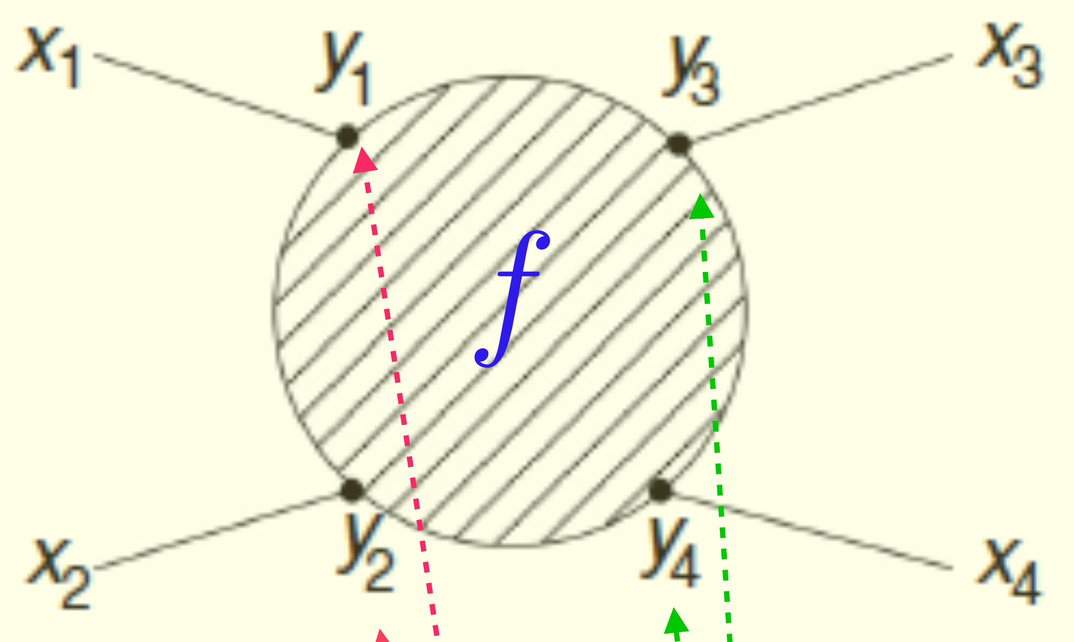
$e^{-i\mathbf{p_0}(y_0-x_0)+i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}$

Free particle propagator $$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\int_{-\infty}^{+\infty}\frac{dp_0}{2\pi i}\frac{\exp\{-ip^\mu(y-x)_\mu\}}{m^2-p^2-i\epsilon}$$

$(m^2+\mathbf{p}^2)-p_0^2$




$$A(x_1,x_2;x_3,x_4) =$$




$$= \int f(y_1,y_2,y_3,y_4)\left\{\prod_{i=1}^{4} D(y_i - x_i)\, d^4y_i\right\}.$$

depending on time ordering of x and y, reduces to the product of *free wave functions*

$$\psi_{\mathbf{p}}(x) = \frac{e^{-iE\,x_0 + i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{2E}} \qquad E \equiv \sqrt{m^2+\mathbf{p}^2}$$

$$D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{\exp\{-ip^\mu(y-x)_\mu\}}{2p_0} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\psi_{\mathbf{p}}(y)\cdot\psi^*_{\mathbf{p}}(x), \qquad y_0 > x_0.$$

And for the outgoing lines, $D(y_\mu - x_\mu) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\psi_{\mathbf{p}}(x)\cdot\psi^*_{\mathbf{p}}(y), \qquad x_0 > y_0.$

In the momentum space, the interaction amplitude takes the form

$$\mathcal{M}(p_i) = \int f(y_1,y_2,y_3,y_4)\, e^{-i(p_1y_1+p_2y_2)+i(p_3y_3+p_4y_4)}\prod d^4y_i.$$

For nearly forward scattering ( $p_1 \approx p_3,\ p_2 \approx p_4$ )

$$\mathcal{M} \Longrightarrow (2\pi)^4\delta(p_1+p_2-p_3-p_4)\int e^{ip_1(y_3-y_1)} f(y_{13};p_2)d^4y_{13}$$

For **causality** to hold, we must have

$$f(y) = \vartheta(y_0)\vartheta(y_\mu^2)\cdot f_1(y) + f_0(y),$$

$$\mathcal{M}(E_1) = \int d^4y\, f_1(y)\cdot\vartheta(y_0)\vartheta(y_\mu^2)\, e^{ip_1y} = \int d^3\mathbf{y}\int_{\sqrt{\mathbf{y}^2}}^{\infty} dt\, e^{iE_1(t-v_1z)} f_1(y)$$

where $\mathbf{f_0}$ should not contribute: $\int d^4y\, f_0(y)\, \exp\{ip_1y\} = 0$

for a *physical* momentum $p_1 = \left(\sqrt{m^2+\mathbf{p}^2},\ \mathbf{p}\right)$

*The theta-functions ensure that the phase of the exponent is positively definite:*

$$t>0, \quad t > \sqrt{z^2+\rho_\perp^2} \geq |z| > |v_1 z| \Longrightarrow (t - v_1 z) > 0\,.$$

As a consequence, **M(E₁)≡M(s)** is a ***regular analytic function*** in the upper half-plane of complex energies $\mathbf{E_1}$.

*Reversing the logic:*

$$f(t) = \int_{-\infty}^{+\infty} dE \, e^{-iEt} \mathcal{M}(E).$$

We need to have $|\mathcal{M}(\mathrm{Im}\, E \to +\infty)| < \exp(\gamma \, \mathrm{Im}\, E)$ for arbitrary $\gamma > 0.$

Otherwise, the response would not vanish at small but finite negative times $-\gamma \leq t < 0.$

***Conclusion***: for causality to hold, amplitude should have NO singularities in the upper half-plane **Im E >0** ( ***be regular*** )

This is not the end of the story though.

In *real world* physical amplitudes cannot have singularities ***anywhere*** in the complex plane of energy, either **Im E >0** or **Im E <0 .**

Moreover, the same applies to *all variables* a physical amplitude depends on, be it *energy* or *momentum transfer*.

The reason for that is ***Special Relativity*** that physical phenomena must respect. Interaction amplitudes being no exception.

**Relativity swift recap**

***4-vectors***

$x^\mu = (ct; x, y, z) = (ct, \mathbf{r})$ — space-time position vector

$p^\mu = (E/c; p_x, p_y, p_z) = (E/c, \mathbf{p})$ — 4-momentum vector

***Lorentz invariants***

$(x^\mu)^2 \equiv t^2 - \mathbf{r}^2$ — space-time interval (**c=1**)

$(p^\mu)^2 \equiv E^2 - \mathbf{p}^2 = m^2$ — squared mass

***scalar product of 4-vectors***

$p^\mu x_\mu \equiv E\, t - (\mathbf{p} \cdot \mathbf{r})$ — phase of relativistic wave function

$e^{-iEt + i(\mathbf{p} \cdot \mathbf{x})}$

***"invariant energy"***

$$s \equiv (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 = (E_1^{(c.m.)} + E_2^{(c.m.)})^2$$ *centre-of-mass frame* $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{0}$

$$= m_1^2 + m_2^2 + 2(p_1 p_2) = m_1^2 + m_2^2 + 2\mathbf{E_1} m_2$$ *laboratory frame* $\mathbf{p}_2 = \mathbf{0}$

Let us calculate the number of independent variables that characterise a **2** -> (**n-2**) process (**n**-leg amplitude):

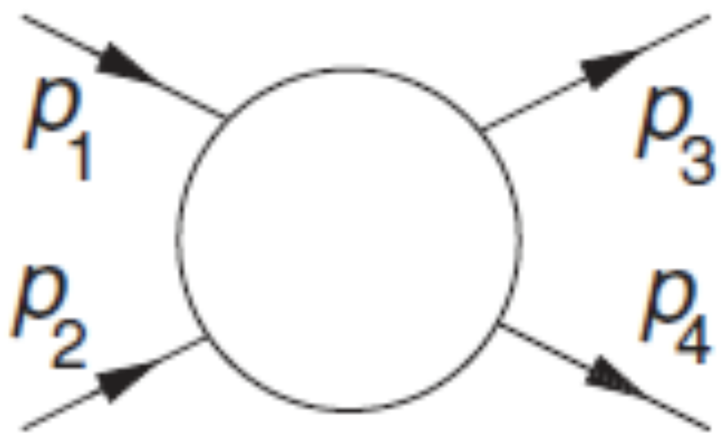
**crossing**

momentum components
- overall energy-momentum conservation
  - on-mass-shell conditions
    - subtract 3 rotations and 3 boosts

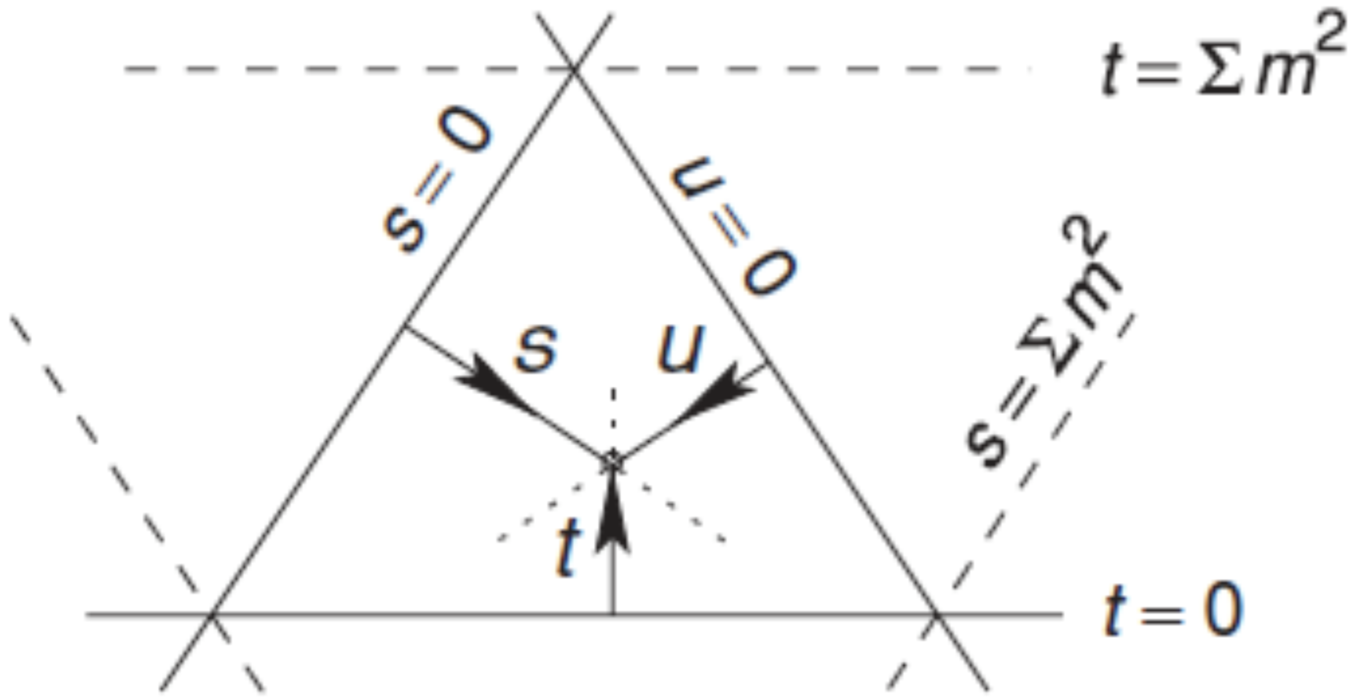
$$4(n-1)-n-6 = 3n-10$$

For example, 2->2 = 2 invariants, 2->3 = 5 invariants, etc.

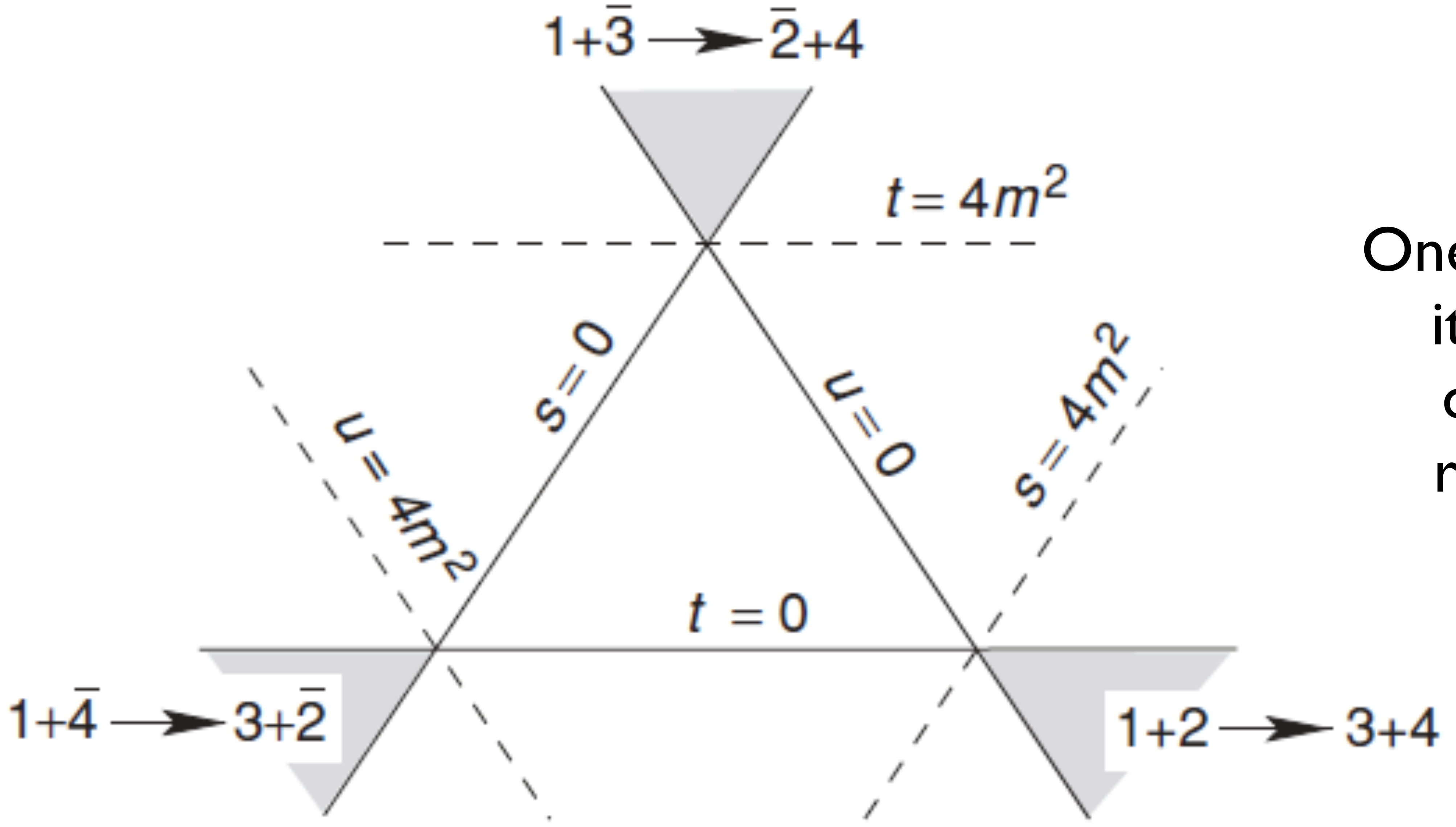
***Mandelstam variables*** for **2->2** scattering




$$s = (p_1+p_2)^2;$$
$$t = (p_1-p_3)^2; \quad s+t+u=4m^2,$$
$$u = (p_1-p_4)^2.$$




***Mandelstam plane***



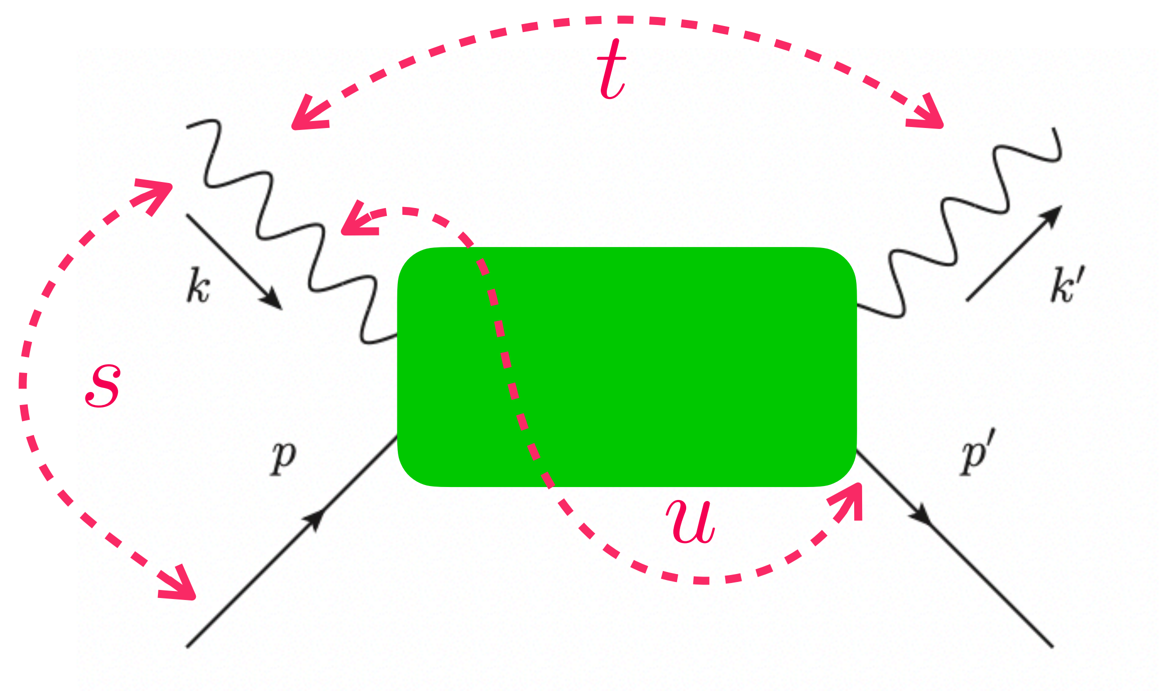

So, each of the Mandelstam variables **s**, **t**, **u** plays a role of **energy** in one of the cross-channels, and therefore is the subject of consideration for **causality**.

One amplitude in different regions of its Mandelstam invariants **s**, **t**, **u** describes 3 different processes related by relativistic **crossing**

***Crossing reactions on the Mandelstam plane***

**Compton scattering**

Consider electron-photon scattering process as an example




Mandelstam variables

in the lab. system (electron at rest)

$$s = (p+k)^2 = 2p\cdot k + m^2 = 2mE_\gamma + m^2$$
$$t = (k'-k)^2 = -2k\cdot k' = -2E_\gamma E'_\gamma(1-\cos\Theta)$$
$$u = (p'-k)^2 = -2k\cdot p' + m^2$$

(check)

**s** translates into the incident photon energy.

So our causality consideration directly applies to it.

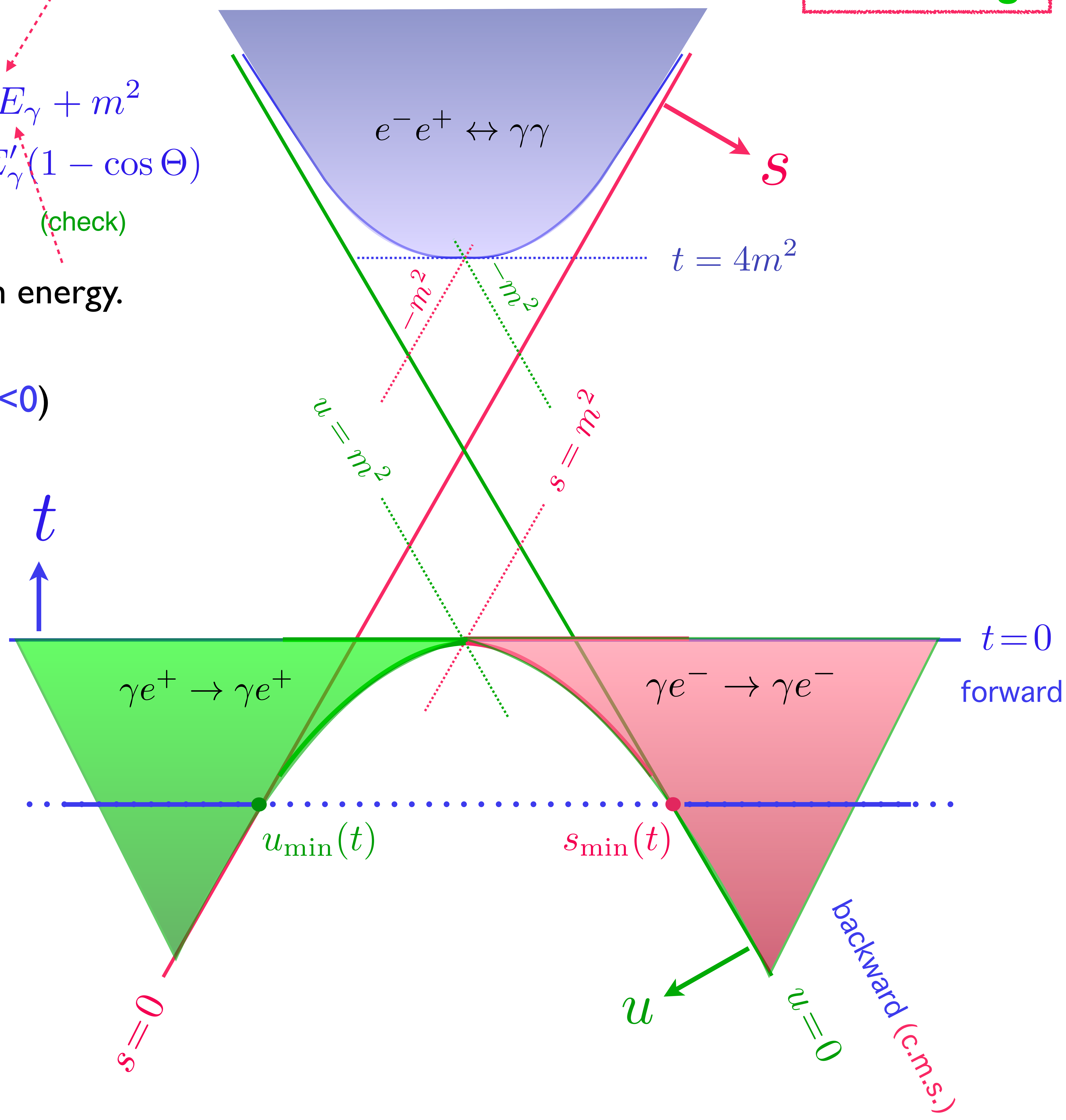
Fix the momentum transfer variable at some physical value (**t<0**) and study the amplitude $A(s,t)$ as a function of **s** .

Physical region of the s-channel starts at $s = s_{\min} \sim 2m^2 - t$ and extends to $+\infty$ .

Recall that in the causality check, to make sure that the *effect* follows the *cause*, we had to integrate over energy (**s**) from minus to plus infinity.

$$u = -s - t + \sum m^2$$

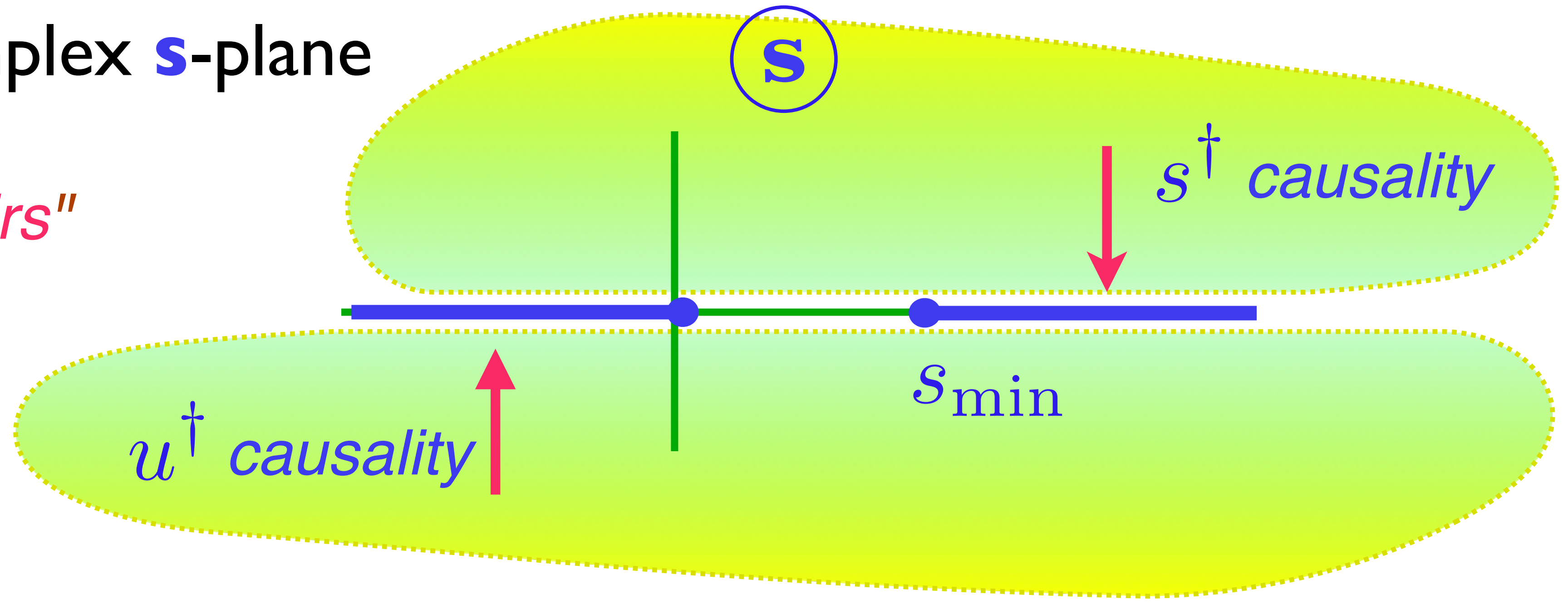
At negative **s** we hit the physical region of the crossing channel $u \geq u_{\min}$ where our amplitude describes *photon-positron* scattering with **u** as energy.




Look now at the complex **s**-plane

No singularities *"upstairs"*

Neither *"downstairs"*

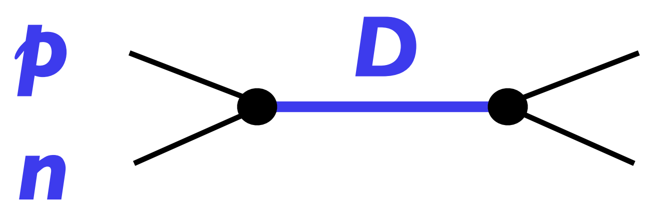


We conclude that interaction amplitudes may not have singularities anywhere in the complex plane of any of its Mandelstam variables.

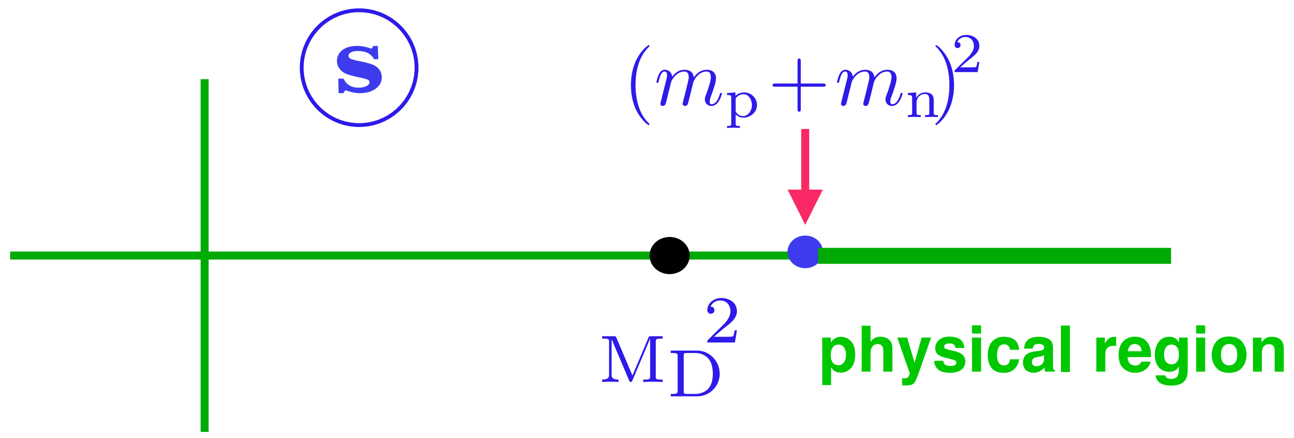
Singularities may lie only on the real axis. What are these singularities?

First of all, the amplitude has a pole at $s = M^2$ with ***M*** the mass of the **s**-channel exchange particle.

For example, a **pn** scattering amplitude has a pole at the deuteron mass:

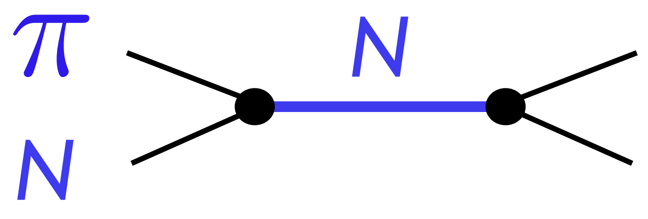


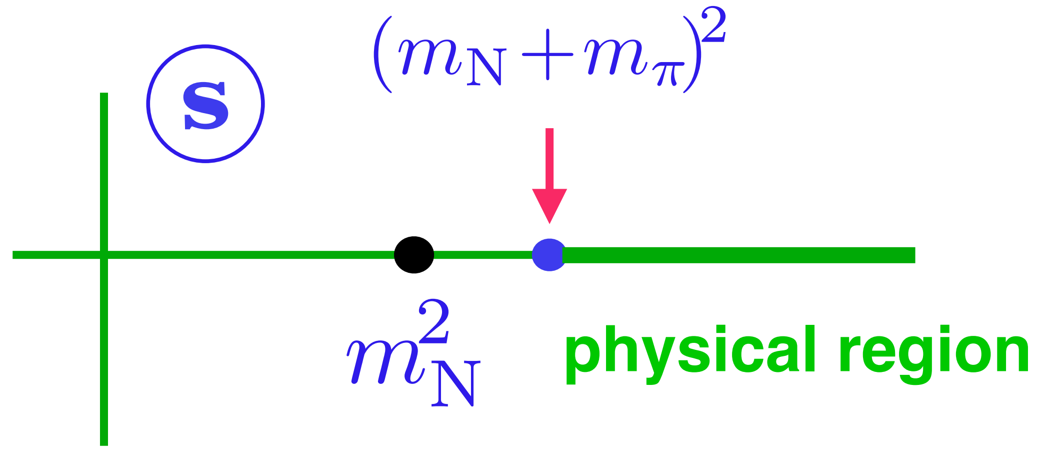

$$A_{pn}(s) = \frac{\lambda^2}{M_D^2 - s} + A^{[\mathrm{reg}]}(s)$$

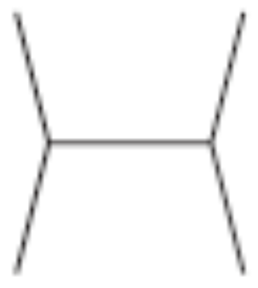
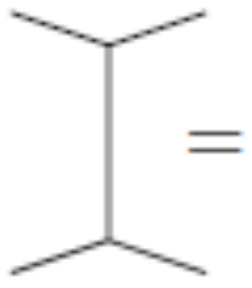



This pole singularity lies ***outside the physical scattering region*** since deuteron is a stable particle, being *lighter* than **p+n**.

The same can be said, e.g., for pion-nucleon scattering:




 $= \dfrac{\lambda^2}{m^2 - s}$,  $= \dfrac{\lambda^2}{m^2 - t}$,  $= \dfrac{\lambda^2}{m^2 - u}$

Obviously, poles may appear in any crossing channel

What is the meaning of a pole in momentum transfer **t** from the **s**-channel point of view?

the NQM Born scattering amplitude $f_B(\mathbf{q}) = -\dfrac{2m}{4\pi}\displaystyle\int d^3\mathbf{r}\, e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r})$ develops *singularity* when the integral diverges for some complex value of $\mathbf{q}$

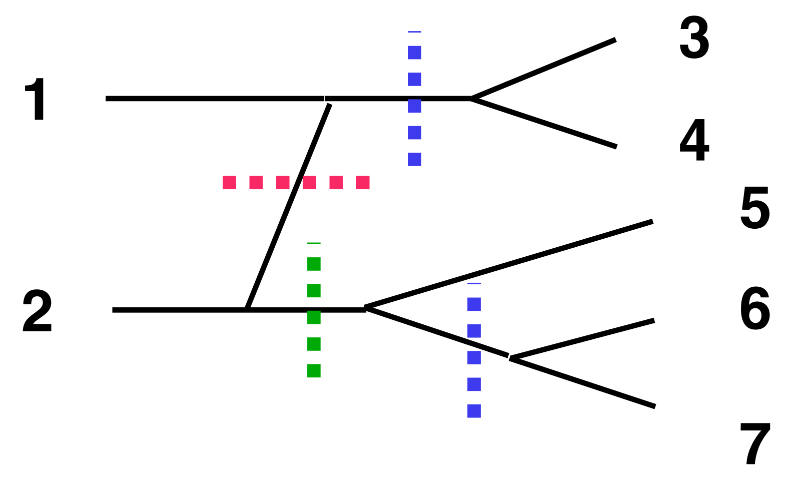
$V(r) \propto r^{-n}, \quad r \to \infty,$ power tail ⇒ $t = -q^2_{\mathrm{sing}} = 0$

$V(r) = \dfrac{A}{r} e^{-mr},$ *Yukawa* potential ⇒ $f_B \propto \dfrac{A}{m^2 + \mathbf{q}^2} = \dfrac{\lambda^2}{m^2 - t}$ → 

**no Van-der-Waals strong forces,** since all the hadrons are ***massive***

## Any singularities other than poles around?

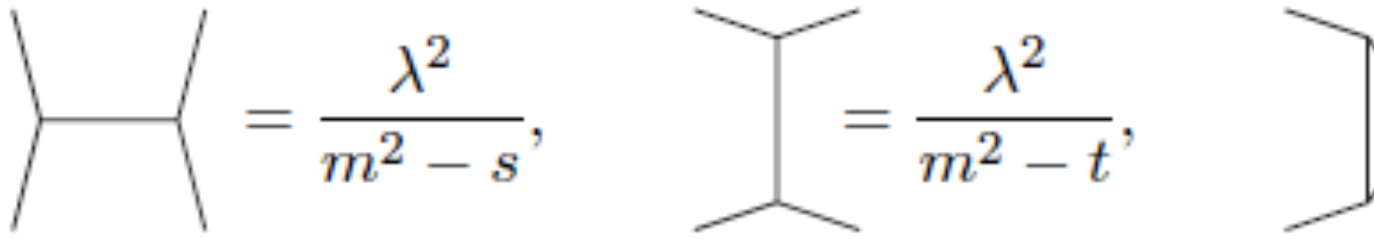
**Tree graph** can be separated in two pieces by cutting through **one line**




pole in the Mandelstam "invariant mass"

$s_{34}$
$s_{134}$
$s_{567}$
$s_{67}$

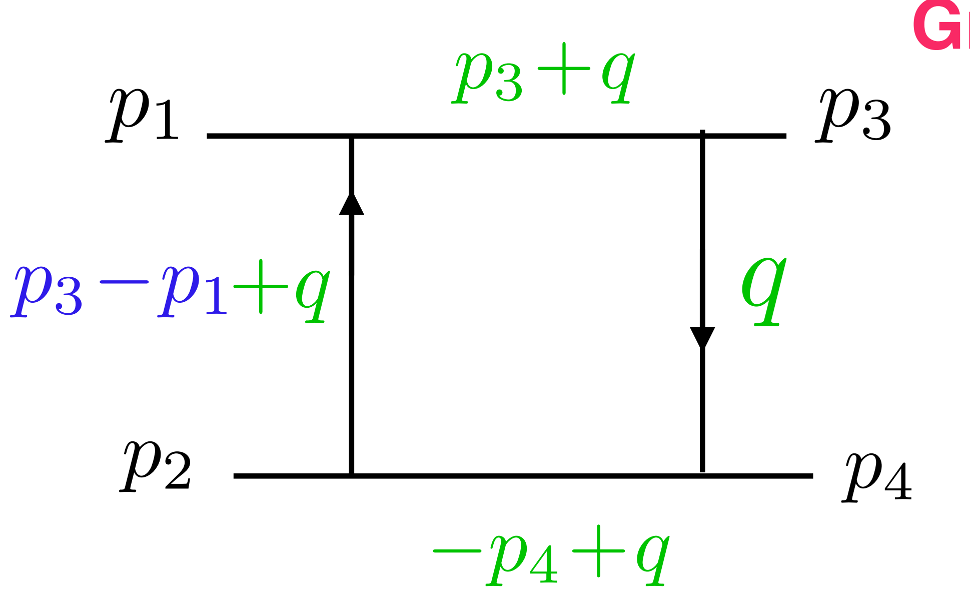
Poles are characteristic for the so-called **Tree Graphs**, the simples of which we have drawn above



$$= \frac{\lambda^2}{m^2 - s}, \qquad = \frac{\lambda^2}{m^2 - t}, \qquad = \frac{\lambda^2}{m^2 - u}$$

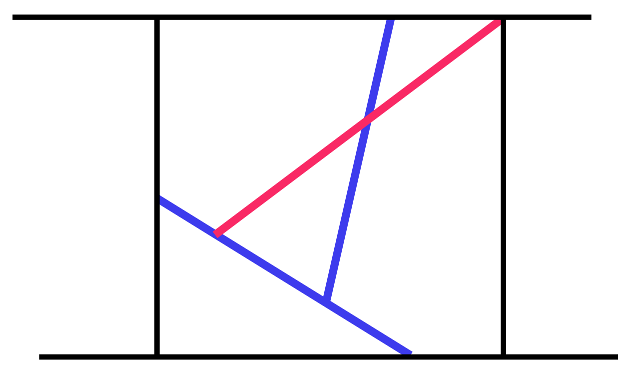
## What happens when more than one particle is exchanged?

### Graphs with Loops




4-integral over $q$

**= one** loop



**three** loops

**four** loops

A Feynman diagram with ***n*** internal lines has a general structure with $\ell$ the number of independent integrations - *loops*.

$$A_{n\ell} = \int \frac{d^4q_1\, d^4q_2 \cdots d^4q_\ell}{[(2\pi)^4 i]^\ell} \frac{1}{(m_1^2 - k_1^2)(m_2^2 - k_2^2)\cdots(m_n^2 - k_n^2)}$$

The analysis of such multi-integral expression is rather complicated.
The answer, however, turns out to be quite compact and transparent.

### *Landau rules*

In each loop, ascribe to each internal line ***i*** a real number $\alpha_i \geq 0$.

$$\sum_i \alpha_i k_i^\mu = 0 \qquad \text{along each loop}$$

resembles ***Kirchhoff current law*** *equations* for electric circuits, with momentum $\mathbf{k_i}$ playing the role of the current, and $\boldsymbol{\alpha_i}$ that of resistance.

$$k_i^2 = m_i^2 \quad \text{or} \quad \alpha_i = 0$$

showing that each internal line either has to have the **on-mass-shell momentum** or should be dropped from consideration (*short-circuited*).

# singularities (character)

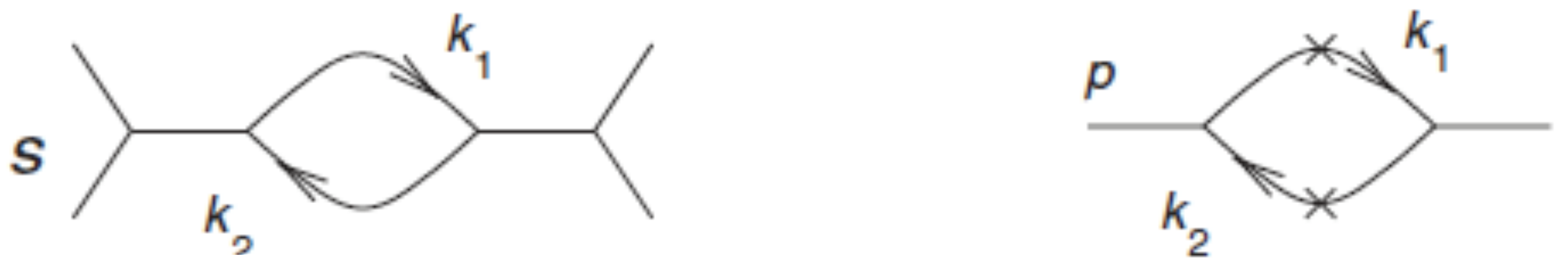
The **character** of the singularity is determined by the critical number $\mathcal{E} = 4\ell - n - 1$

$$A(s) \propto (s_0 - s)^{\mathcal{E}/2}$$ for $\mathcal{E}$ ***odd*** or ***negative***

$$\propto (s_0 - s)^{\mathcal{E}/2} \ln(s_0 - s)$$ when $\mathcal{E}$ is ***even*** and ***positive***

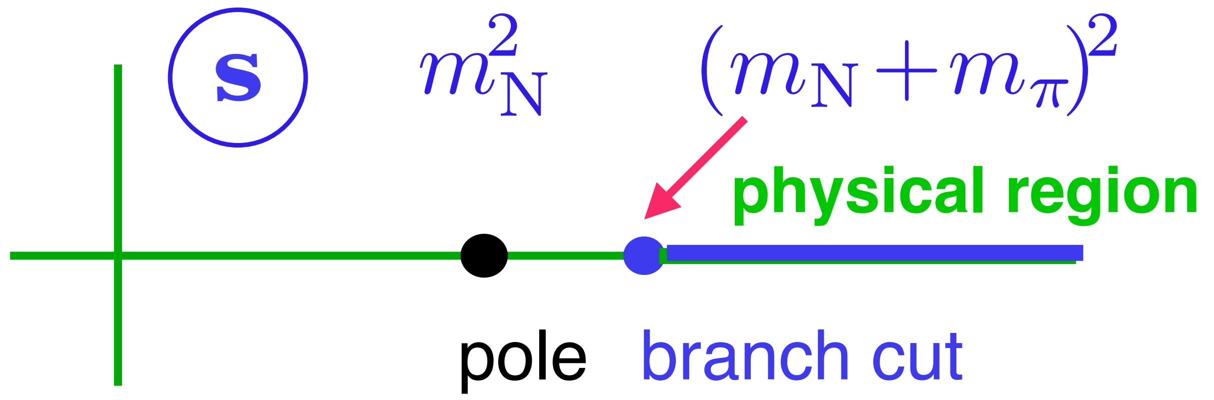
According to Landau rules, this applies to **any** amplitude that has a 2-particle division, that is, can have 2 real particles in the intermediate state (Feynman diagrams be damned)

Simplest example: one loop, two lines




$$s_0 = p^2 = (m_1 + m_2)^2$$ - **two-particle threshold**

$$\mathcal{E} = 4 \cdot 1 - 2 - 1 = 1 \longrightarrow A(s) \propto \sqrt{s_0 - s}$$



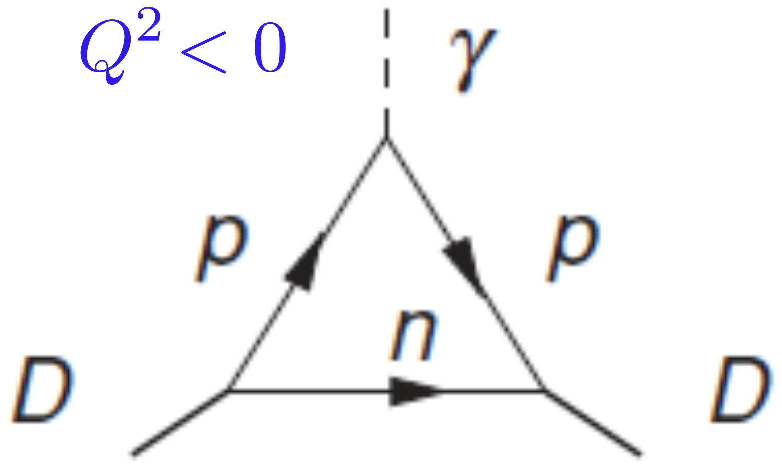

We see that the interaction amplitude ***in the physical region*** is always **complex** and is actually sitting on the **branch cut**

So, the answer to the question *"what sort of singularities the amplitude may have in the complex energy plane"* is as follows:

**Poles** (at particle/bound state masses) and **branch cuts** (starting at n-particle thresholds) **on the real axis** (!)

*Landau rules produce more sophisticated singularities too.*

So-called "***anomalous singularities***" appear near the physical region and are typical for loosely bound systems (see, e.g., deuteron scattering).

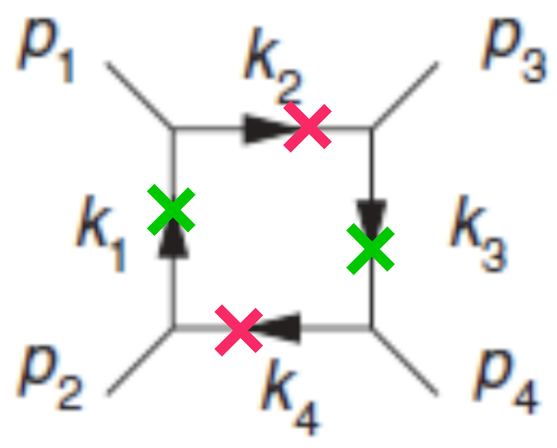



$$m_p + m_n - M_D = \epsilon \ll m \; (m = m_p \simeq m_n)$$

$$Q_0^2 \simeq 16\, m \cdot \epsilon \ll m^2,$$

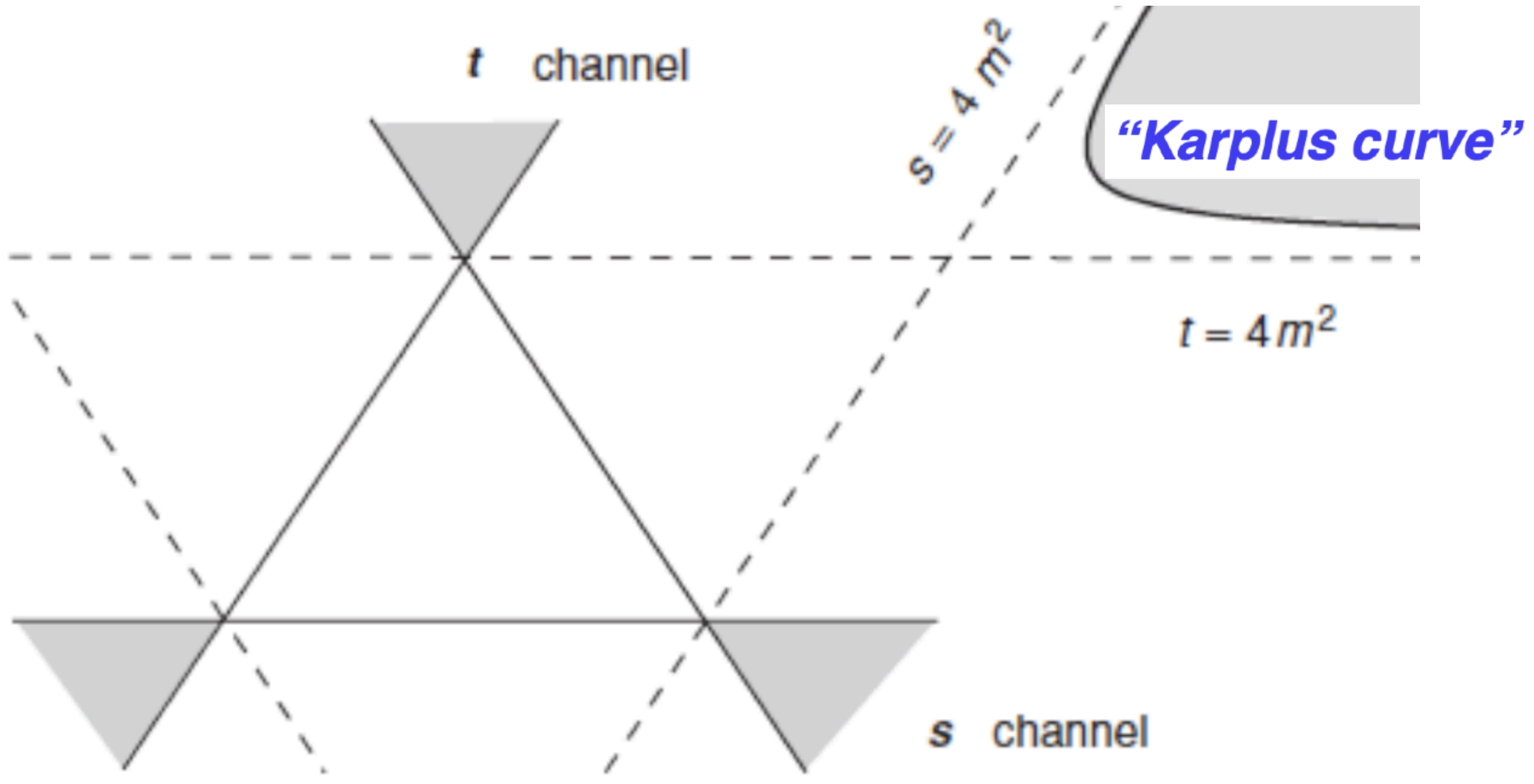
***Anomalous singularities*** are plentiful in the physics of nuclei in general.

the ***box graph***




**Equation for the Landau surface**

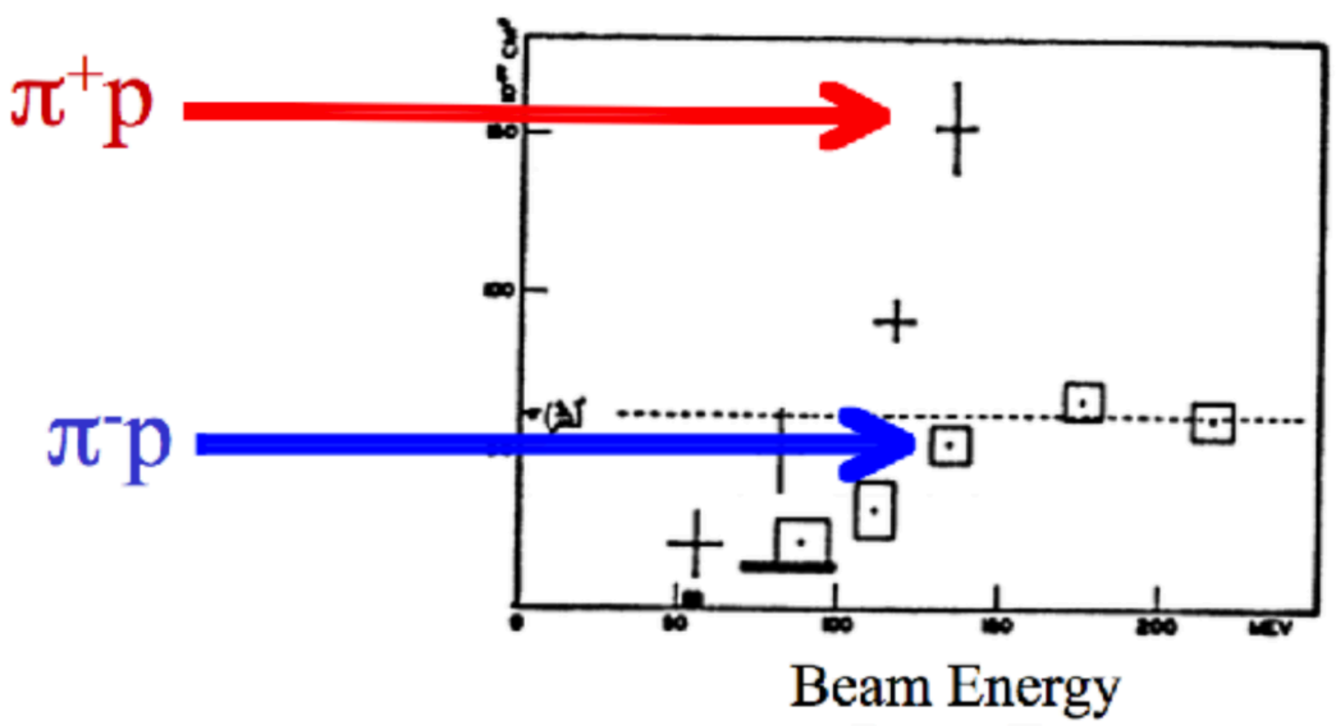
$$(s - 4m^2)(t - 4m^2) = 4m^4$$



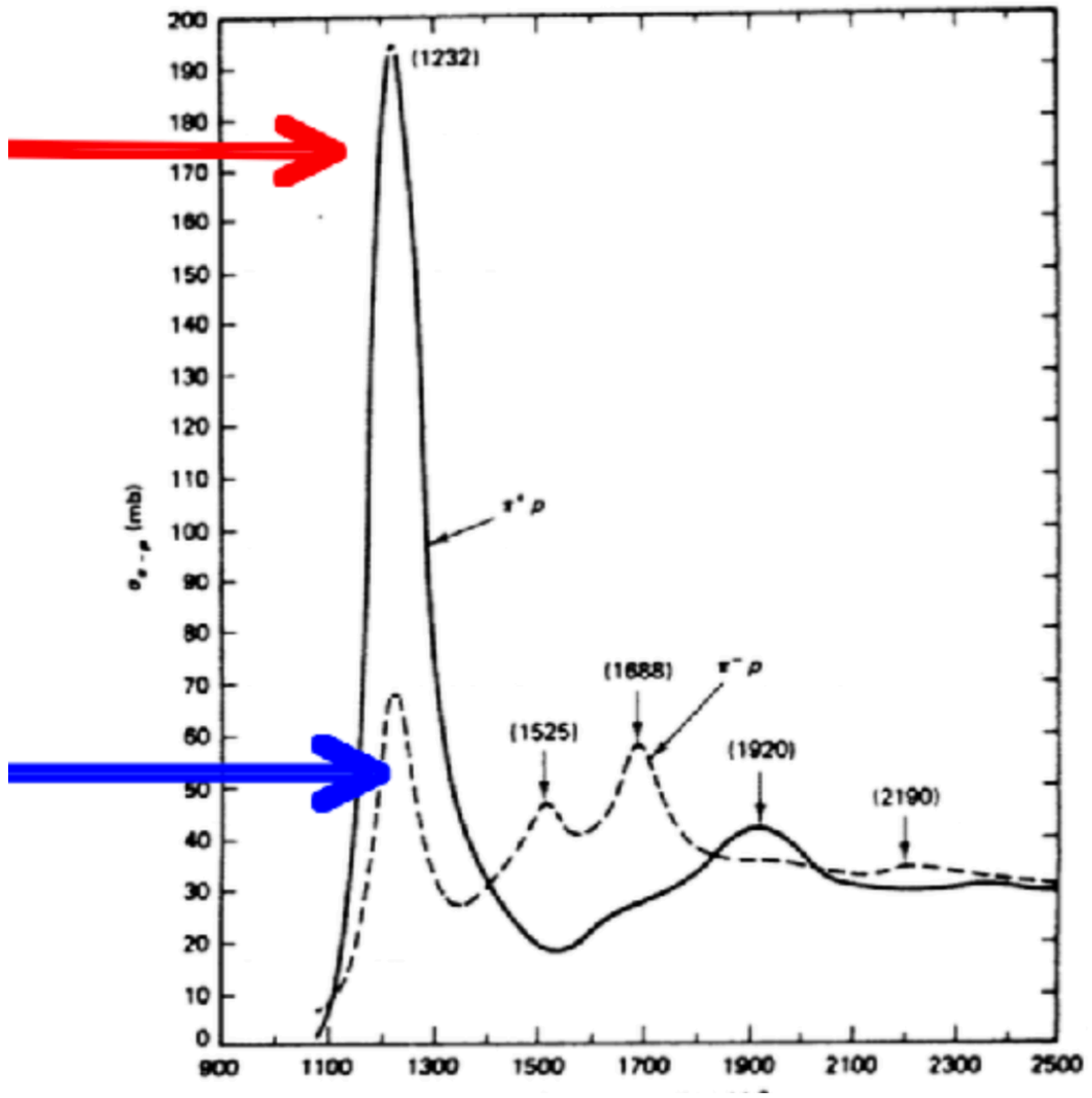
In 1952 the first pion beam experiment by the Enrico Fermi group demonstrated a rapid growth of the pion-proton cross section with pion beam energy.

Discovery of the first ***baryon resonance*** $\Delta$ (1230).

$$\Delta^-, \Delta^0, \Delta^+, \Delta^{++}$$




Modern compilation of the $\pi p$ cross section shows multiple bumps and wiggles.




This was not expected by Fermi & Co (neither by anyone else).

$\pi p$ interaction amplitude being an ***analytic function*** in the complex energy plane, it was expected to ***change smoothly*** at the scales of the order of masses that determine ***position of singularities*** on the real axis of the energy variable.

Threshold singularities are pretty mild.

If they were the ***only driver*** of the behaviour of strong interaction amplitudes, the physics of hadrons would have been boring ...

Apparently, we are missing something.

There has to be a ***hidden reason*** for interaction amplitudes to behave furiously.

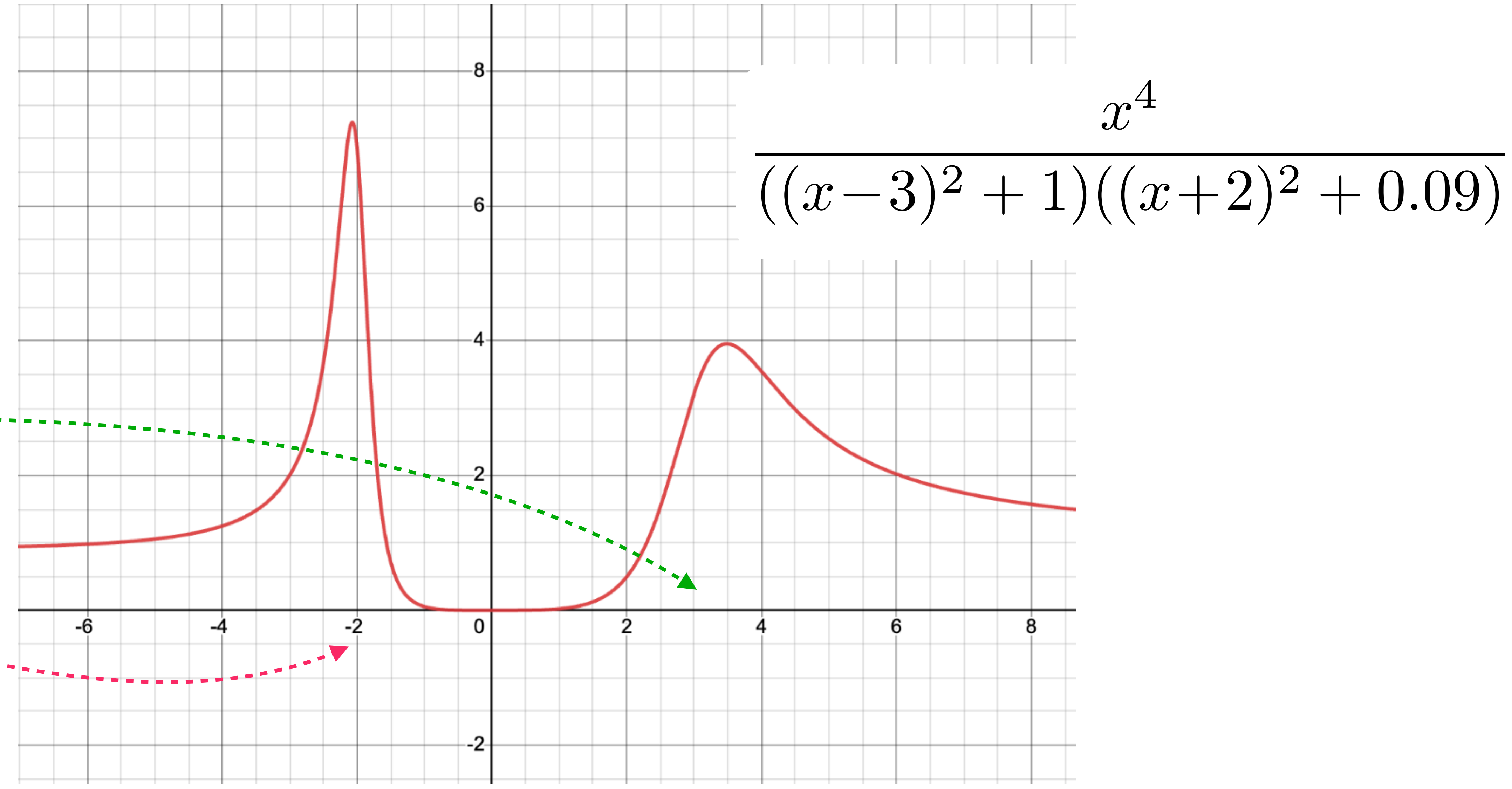
Status as seen in

| Particle | $J^P$ | overall | $N\gamma$ | $N\pi$ | $N\eta$ | $N\sigma$ | $N\omega$ | $\Lambda K$ | $\Sigma K$ | $N\rho$ | $\Delta\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Delta(1232)$ | $3/2^+$ | **** | **** | **** | F | | | | | | |
| $\Delta(1600)$ | $3/2^+$ | *** | *** | *** | o | | | | | * | *** |
| $\Delta(1620)$ | $1/2^-$ | **** | *** | **** | | r | | | | *** | *** |
| $\Delta(1700)$ | $3/2^-$ | **** | **** | **** | | b | | | | ** | *** |
| $\Delta(1750)$ | $1/2^+$ | * | | * | | i | | | | | |
| $\Delta(1900)$ | $1/2^-$ | ** | ** | ** | | | d | | ** | ** | ** |
| $\Delta(1905)$ | $5/2^+$ | **** | **** | **** | | | d | | *** | ** | ** |
| $\Delta(1910)$ | $1/2^+$ | **** | ** | **** | | | e | | * | * | ** |
| $\Delta(1920)$ | $3/2^+$ | *** | ** | *** | | | n | | *** | | ** |
| $\Delta(1930)$ | $5/2^-$ | *** | | *** | | | | | | | |
| $\Delta(1940)$ | $3/2^-$ | ** | ** | * | F | | | | | | |
| $\Delta(1950)$ | $7/2^+$ | **** | **** | **** | o | | | | *** | * | *** |
| $\Delta(2000)$ | $5/2^+$ | ** | | | | r | | | | | ** |
| $\Delta(2150)$ | $1/2^-$ | * | | * | | b | | | | | |
| $\Delta(2200)$ | $7/2^-$ | * | | * | | i | | | | | |
| $\Delta(2300)$ | $9/2^+$ | ** | | ** | | | d | | | | |
| $\Delta(2350)$ | $5/2^-$ | * | | * | | | d | | | | |
| $\Delta(2390)$ | $7/2^+$ | * | | * | | | e | | | | |
| $\Delta(2400)$ | $9/2^-$ | ** | | ** | | | n | | | | |
| $\Delta(2420)$ | $11/2^+$ | **** | * | **** | | | | | | | |
| $\Delta(2750)$ | $13/2^-$ | ** | | ** | | | | | | | |
| $\Delta(2950)$ | $15/2^+$ | ** | | ** | | | | | | | |

And here our dive into analytic functions will start paying back

## How to force a function to change fast?

Take as an example of a respectable function of a real argument **x**




What is the origin of the "**bumps**" near **x=-2** and **x=3** ?

Our function is regular on the real axis, but has **poles** in the complex points

$x = -2 \pm 0.3i$ and $x = 3 \pm i$

These ***complex poles*** affect the behaviour of the function the more, the closer to the real axis they are located.

Looks quite similar to "$\pi p$ wiggles" and tempting.

However, it is clear that such a construction does not solve our trouble :

> the *fundamental principle* of causality forbids interaction amplitudes to have any singularities (poles included) at complex energy values.

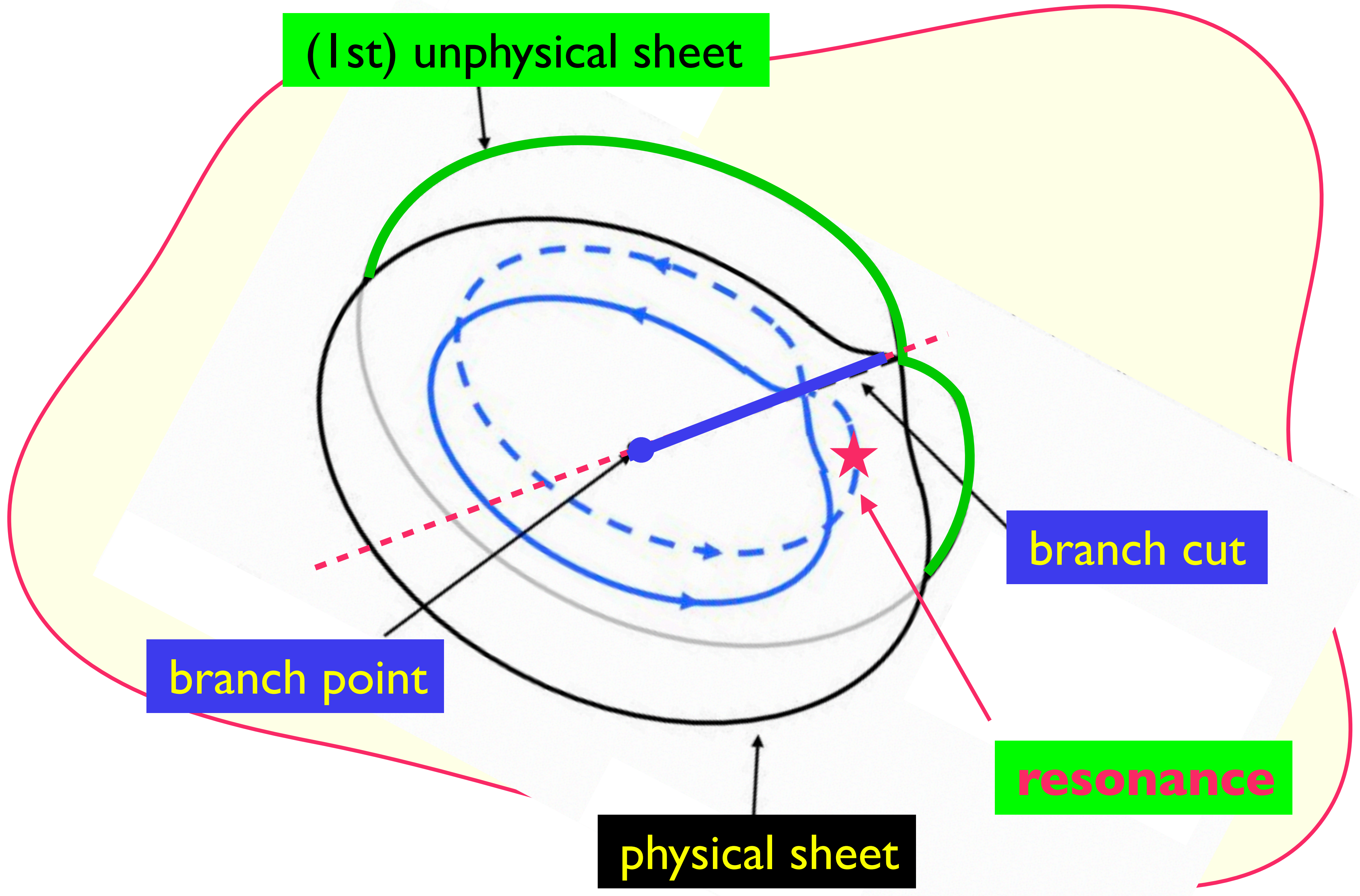
Recall that, having branch cuts, the amplitude lives not on the *complex plane* but instead on a Riemann surface which consists of more than one "*sheet*".

A pole at complex energy will not violate causality if it sits on the unphysical sheet !

This is exactly what **resonances** do.

Resonance is an unstable particle that corresponds to a pole in the interaction amplitude having "complex mass" and sitting on the unphysical sheet (*under the cut*).

Difference between resonances and stable particles is rather elusive.




> It is another *fundamental principle* – unitarity – that takes care of it.