High energies. Why?

hep-th

Baltic Summer School - 3
Palanga

August 2023

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Unitarity.

Partial waves.

Resonances.

Qualitative picture of high energy scattering

unitarity

Yesterday we deduced the implications that ***causality*** imposes on interaction amplitudes.

Another basic principle of quantum physics worth exploring is called ***unitarity***.
It simply states that ***the sum of the probabilities*** of all possible outcomes of any event must ***equal one***.
The name derives from the fact that the matrix **S** that relates distant future of a process with its remote past, $\Psi_{\rm out} = \mathbf{S}\Psi_{\rm in}$,
must be ***unitary*** : from *conservation of probability*, $|\Psi_{\rm out}|^2 = |\Psi_{\rm in}|^2$, follows $|\mathbf{S}|^2 = \mathbf{S}^\dagger\mathbf{S} = \mathbf{1}$.

$$\sum_c \mathbf{S}^*_{a'c}\mathbf{S}_{ca} = \delta_{a'a}$$

In terms of the "*reaction matrix*" **T**, $\mathbf{S} = \mathbf{I} + i\mathbf{T}$ whose elements are transition amplitudes from the initial state **a** to a final state **b**,

$$\mathbf{T}_{ba} - \mathbf{T}^\dagger_{ba} \equiv \mathbf{T}_{ba} - \mathbf{T}^*_{ab} = i\left(\mathbf{T}^\dagger\mathbf{T}\right)_{ba}$$

In a time-invariant theory ($\mathbf{T}_{ab} = \mathbf{T}_{ba}$) this results in the ***unitarity relation***

$$2\mathrm{Im}\,\mathbf{T}_{ba} = \sum_c \mathbf{T}^*_{bc}\mathbf{T}_{ca}$$

$$T_{ab} = (2\pi)^4\,\delta^4\left(\sum_{i\in a} p_i - \sum_{j\in b} k_j\right)\prod_{i\in a}\frac{1}{\sqrt{2p_{0i}}}\prod_{j\in b}\frac{1}{\sqrt{2k_{0j}}}\cdot\mathcal{M}_{ab}$$

← *Lorentz invariant matrix element*

Lorentz invariant phase space of the intermediate state c

The Unitarity relation becomes

$$\frac{\mathcal{M}_{ab} - \mathcal{M}^*_{ab}}{i} = \sum_n \frac{1}{[n!]}\int \mathcal{M}_{an}(\{p\}_a;\{k\}_n)\,\mathcal{M}^*_{bn}(\{p\}_b;\{k\}_n)\,d\Pi_n$$

$$\int d\Pi_n = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\left(\sum p_f - p_A - p_B\right).$$

SIC! The **same** matrix element and the **same** phase space differential are present in the ***Differential Cross Section*** **A**+**B** -> 1,2,..n

z-boost-invariant flux factor

$$J \equiv 4|E_A\mathbf{p}_B - E_B\mathbf{p}_A| \simeq 2s$$

$$d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}|\mathcal{M}(p_A, p_B \to \{p_f\})|^2 d\Pi_n.$$

The ***flux*** and the ***phase space*** factors depend on which particles collide (initial), and which are produced (final).

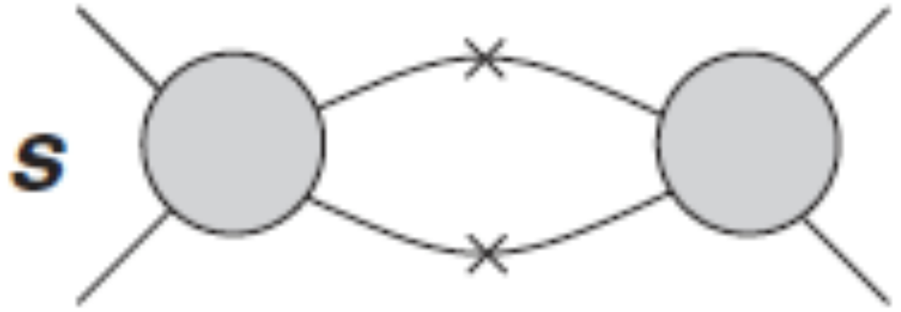
The ***matrix element*** is ***universal*** !

Choosing **b=a** - *scattering forward* - yields the "***Optical Theorem***"

$$2\,\mathrm{Im}\,\mathcal{M}_{aa} = J\cdot\sigma^a_{\rm tot}$$

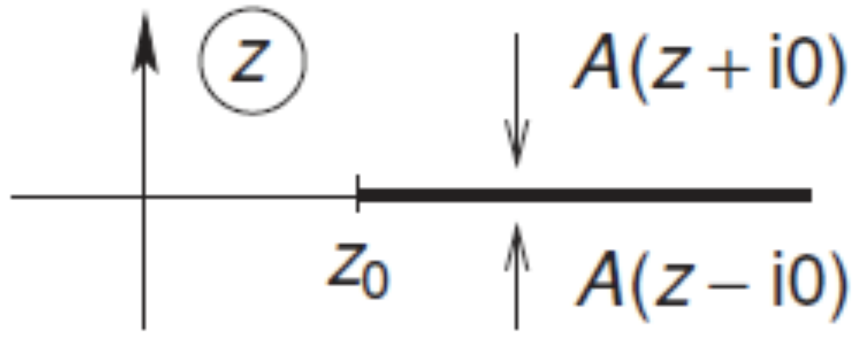
We saw that amplitudes have threshold singularities and become complex above the threshold.

For example, the first threshold singularity of the 2 -> 2 scattering amplitude appears when the colliding energy becomes large enough as to allow for production of two real (on-mass-shell) particles in the intermediate state



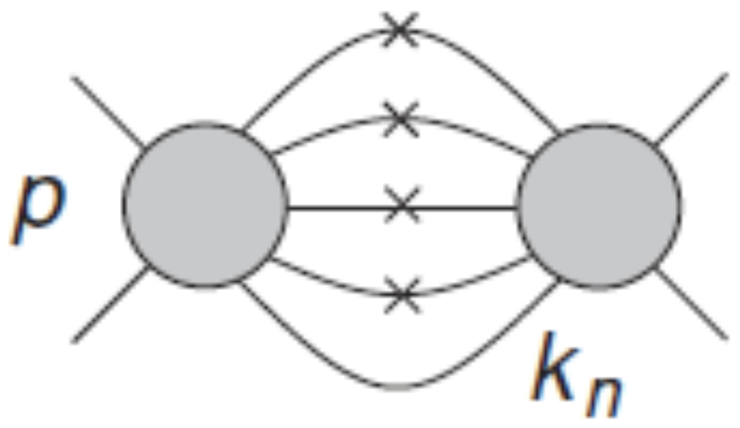
$\propto A^2$

Branch cut is characterised by ***discontinuity*** of the function around it : $\Delta A(z) = \frac{1}{2i}\left[A(z+i0) - A(z-i0)\right]$



If a function is ***real*** before the cut (as $\sqrt{z_0 - z}$ is), then its values above and below the cut become ***complex conjugate*** , and ***discontinuity*** of the function coincides with its ***imaginary part*** .

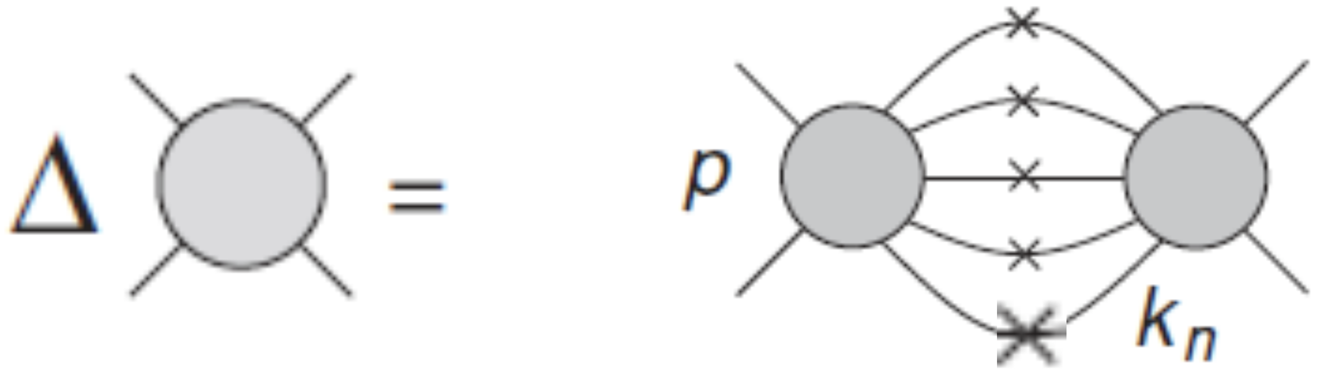
Imagine a graph with **n** (virtual) lines in the intermediate state.
Integrating over k_0 components of *n–1* loop momenta
we can put ***all but one*** propagators on-mass-shell



$\propto \frac{1}{m_n^2 - k_{0n}^2 + \mathbf{k}_n^2}.$

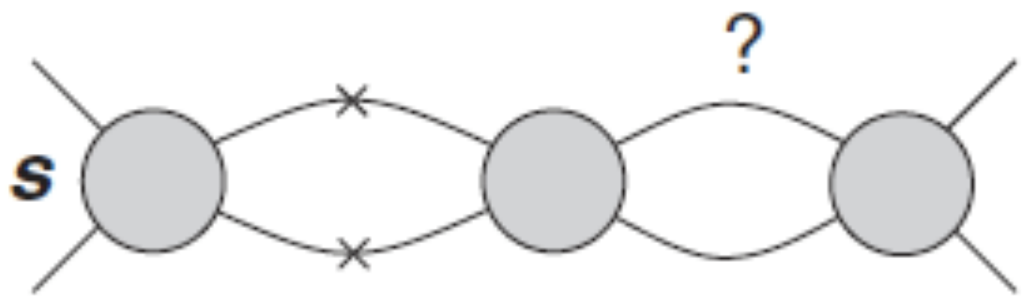
The last remaining propagator makes the amplitude singular and provides its ***discontinuity*** :

→ $\frac{1}{2i}\left[\frac{1}{m_n^2 - (p_0 + i\epsilon - \sum k_{0i})^2 + \mathbf{k}_n^2} - \frac{1}{m_n^2 - (p_0 - i\epsilon - \sum k_{0i})^2 + \mathbf{k}_n^2}\right] = \frac{2\pi i \delta_+(m_n^2 - k_n^2)}{2i}$



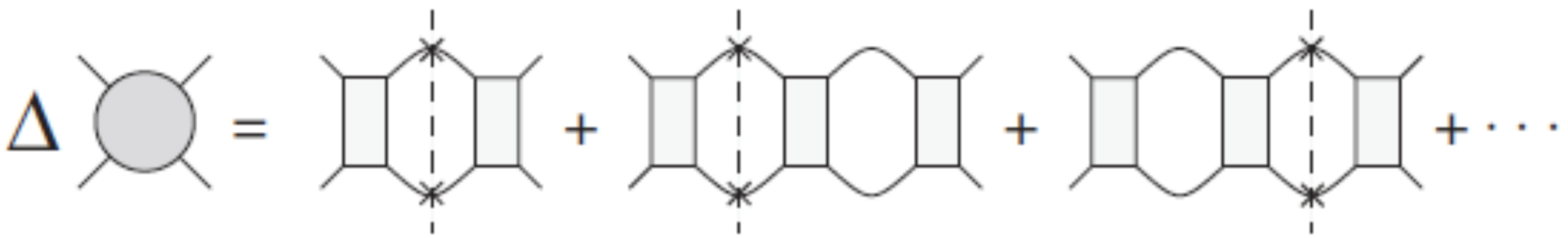
Have we solved the quest ? Almost ...

Above the threshold the grey blocks are complex themselves ...



how to deal with overlaying singularities ?

To analyse the structure of discontinuity, we separate 2-particle irreducible blocks:



unitarity
means
repetition

Then, using the algebraic relation

$$\Delta(AB) = A\cdot(\Delta B) + (\Delta A)\cdot B^*,$$

arrive, by induction, at

$$\Delta A = \sum_{i,k=1} F_1^+ \cdots F_i^+ \left[\quad\right] F_1^- \cdots F_k^- = A(s+i0)\left[\quad\right] A(s-i0).$$

Summing together the discontinuities across **n**-particle threshold branchings, we obtain

$$\Delta A_{2\to 2}(s) = \sum_n \tau_n(s) A_{2\to n}(s) A^*_{2\to n}(s)$$

This is nothing but the ***unitarity relation*** !

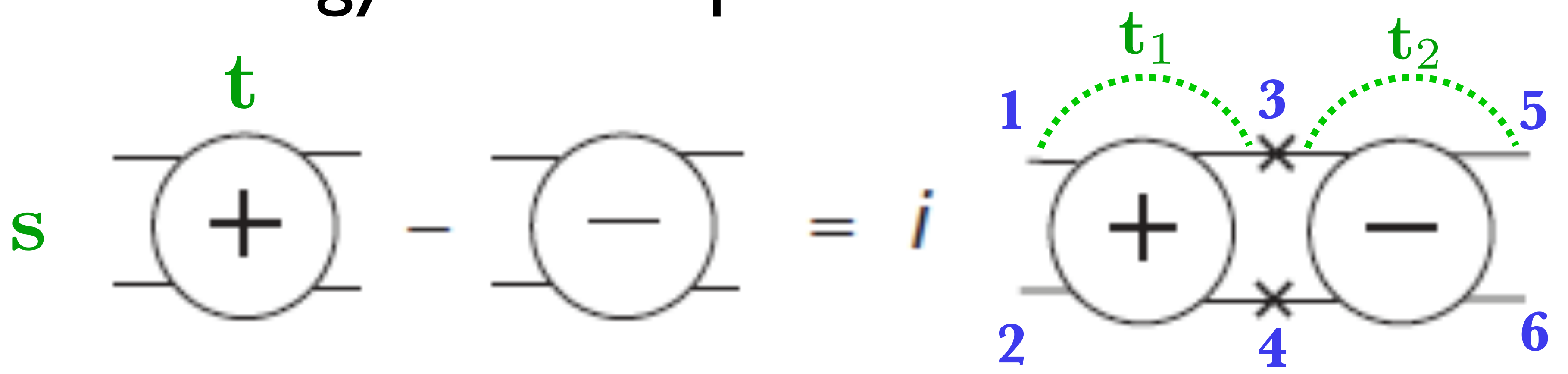
- Our (Landau) analysis of singularities automatically yields ***unitarity***.
- ***Unitarity*** assures that position and the nature of singularities is determined by the *physical spectrum* of the theory.

Have we forgotten about resonances?

Not at all. Actually, it is unitarity that will help us to understand their origin and properties.

How to examine content of unphysical sheets? Turn to the unitarity condition.

Consider, for simplicity, moderate energy below 3-particle threshold where 2-particle unitarity holds ($4m^2 < s < 9m^2$)



Written in full,

$$A(s+i\epsilon,t) - A(s-i\epsilon,t) \; = \; i\int \frac{d^4k}{(2\pi)^2}\delta(m_3^2-k^2)\delta(m_4^2-(p_1+p_2-k)^2)\cdot A(p_1,p_2,k)A^*(p_5,p_6,k).$$

$p_3 \equiv k$
$p_4 = p_1 + p_2 - k$

Or, in terms of invariants,

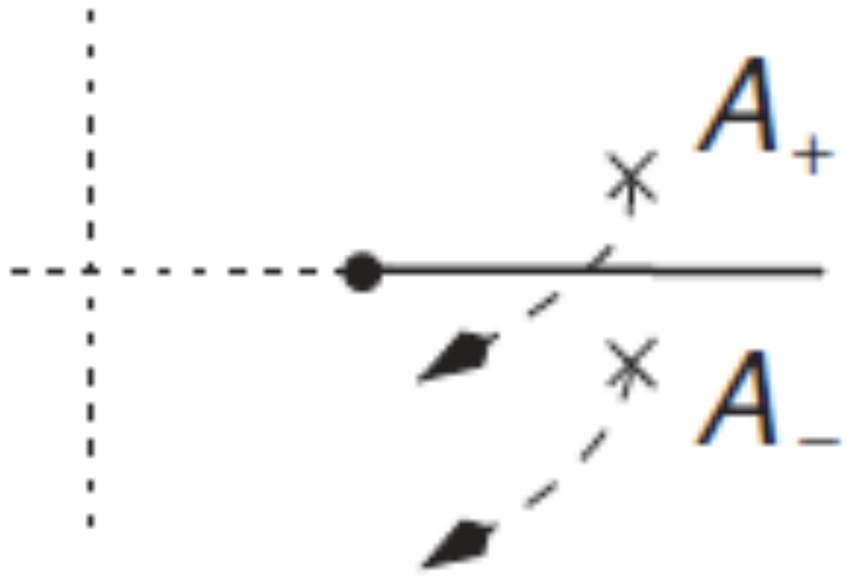
$$A(s+i\epsilon,t) - A(s-i\epsilon,t) \; = \iint dt_1\, dt_2 K(s,t_1,t_2)\cdot A(s+i\epsilon,t_1)A(s-i\epsilon,t_2)$$

We may look upon it as an inhomogeneous integral equation for A_+ .

Imagine we have solved it (which we will, shortly) and derived

$$A(s+i\epsilon) \; = \; F\left(A(s-i\epsilon)\right)$$

Then, by giving s a negative imaginary part, we will have A_- staying on the physical sheet while the argument of A_+ dives under cut and starts walking over the *first unphysical sheet* related to the 2-particle threshold !



partial waves and two-particle unitarity

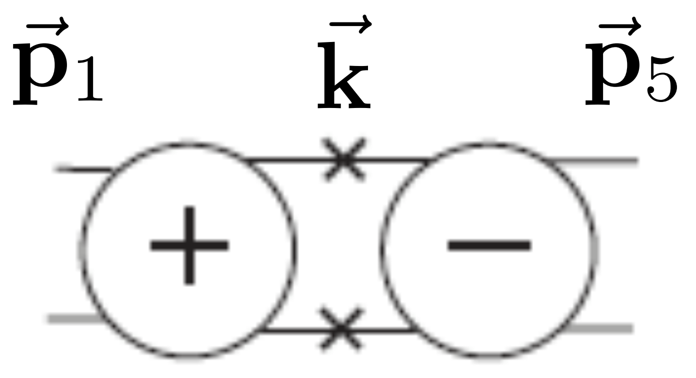
Unitarity condition is ***diagonal*** in conserved quantum numbers.

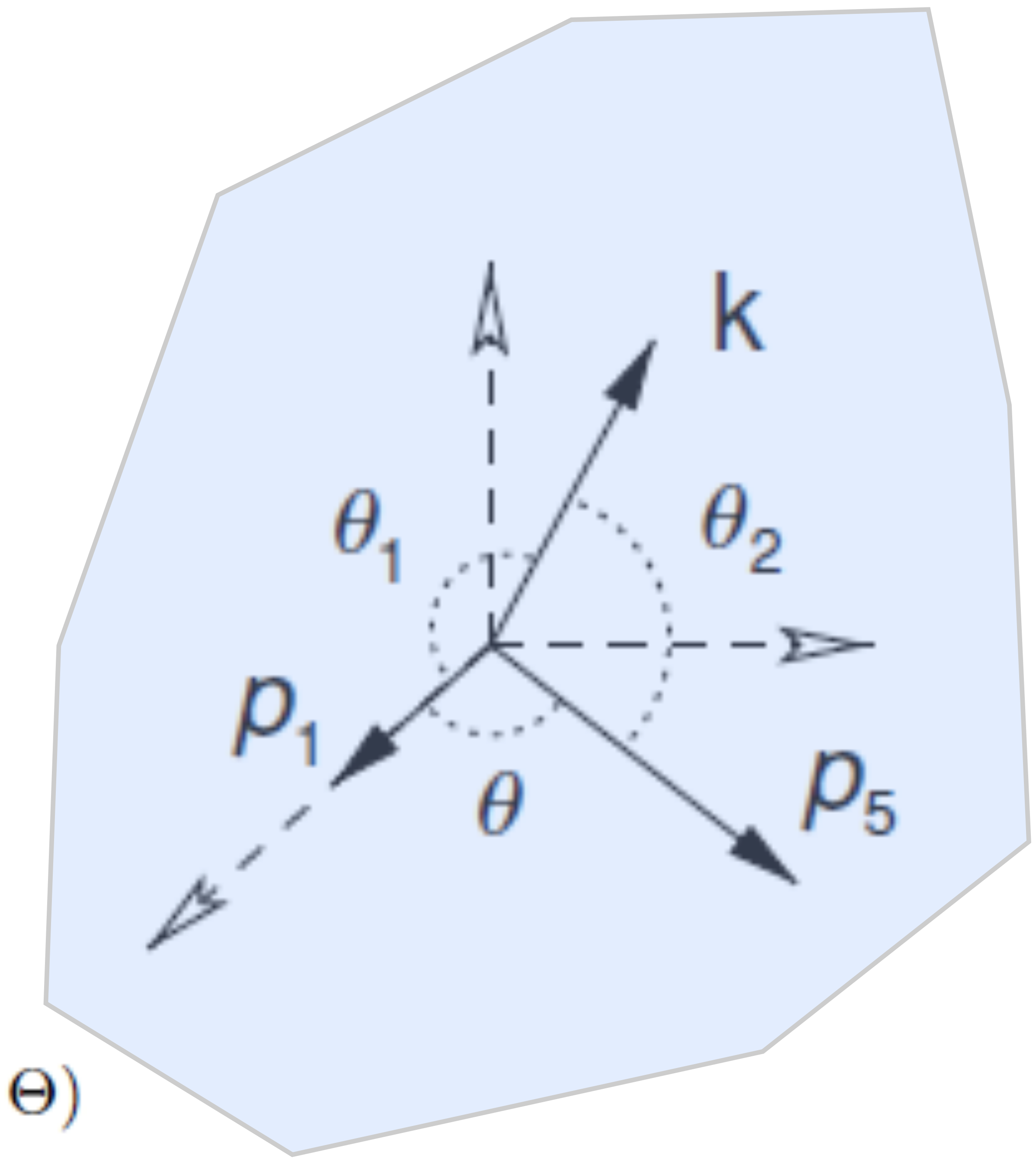
In particular, in ***Angular Momentum***.

Cast the amplitude in terms of partial wave expansion

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell(\cos\Theta)$$

and substitute it into the 2-particle unitarity condition,





$$\mathrm{Im}\, A(s,t) \;=\; \frac{|\mathbf{k}|}{8\pi\sqrt{s}} \int \frac{d\Omega}{4\pi} A(s,\cos\Theta_1) A^*(s,\cos\Theta_2)$$

Legendre polynomials are orthogonal to one another, $\int \frac{d\Omega}{4\pi} P_n(\cos\Theta_1) P_m(\cos\Theta_2) = \frac{\delta_{nm}}{2n+1} P_n(\cos\Theta)$

Therefore, the r.h.s. becomes

$$\frac{|\mathbf{k}|}{8\pi\sqrt{s}} \sum_{\ell_1,\ell_2} f_{\ell_1}(s) f^*_{\ell_2}(s) \times (2\ell_1+1)(2\ell_2+1) \int \frac{d\Omega}{4\pi} P_{\ell_1}(\cos\Theta_1) P_{\ell_2}(\cos\Theta_2) \;=\; \frac{|\mathbf{k}|}{8\pi\sqrt{s}} \sum_{\ell_1=0}^{\infty} f_{\ell_1}(s) f^*_{\ell_1}(s) (2\ell_1+1) P_{\ell_1}(\cos\Theta),$$

Comparing with the l.h.s., we arrive at the algebraic relation for partial wave amplitudes replacing the integral equation for **A**:

$$\mathrm{Im}\, f_\ell(s) \;=\; \tau f_\ell(s) f^*_\ell(s),$$

with $\tau = \tau(s) \;\equiv\; \frac{k_c}{8\pi\sqrt{s}} = \frac{1}{16\pi}\frac{k_c}{\omega_c},$

For bookkeeping sake:

c.m.s. momentum $k_c \equiv |\mathbf{k}| = \frac{\sqrt{s^2 - 2s(m_3^2+m_4^2) + (m_3^2-m_4^2)^2}}{2\sqrt{s}}, \quad \omega_c = \frac{\sqrt{s}}{2}.$

For equal masses, $k_c = \frac{\sqrt{s-4m^2}}{2}$

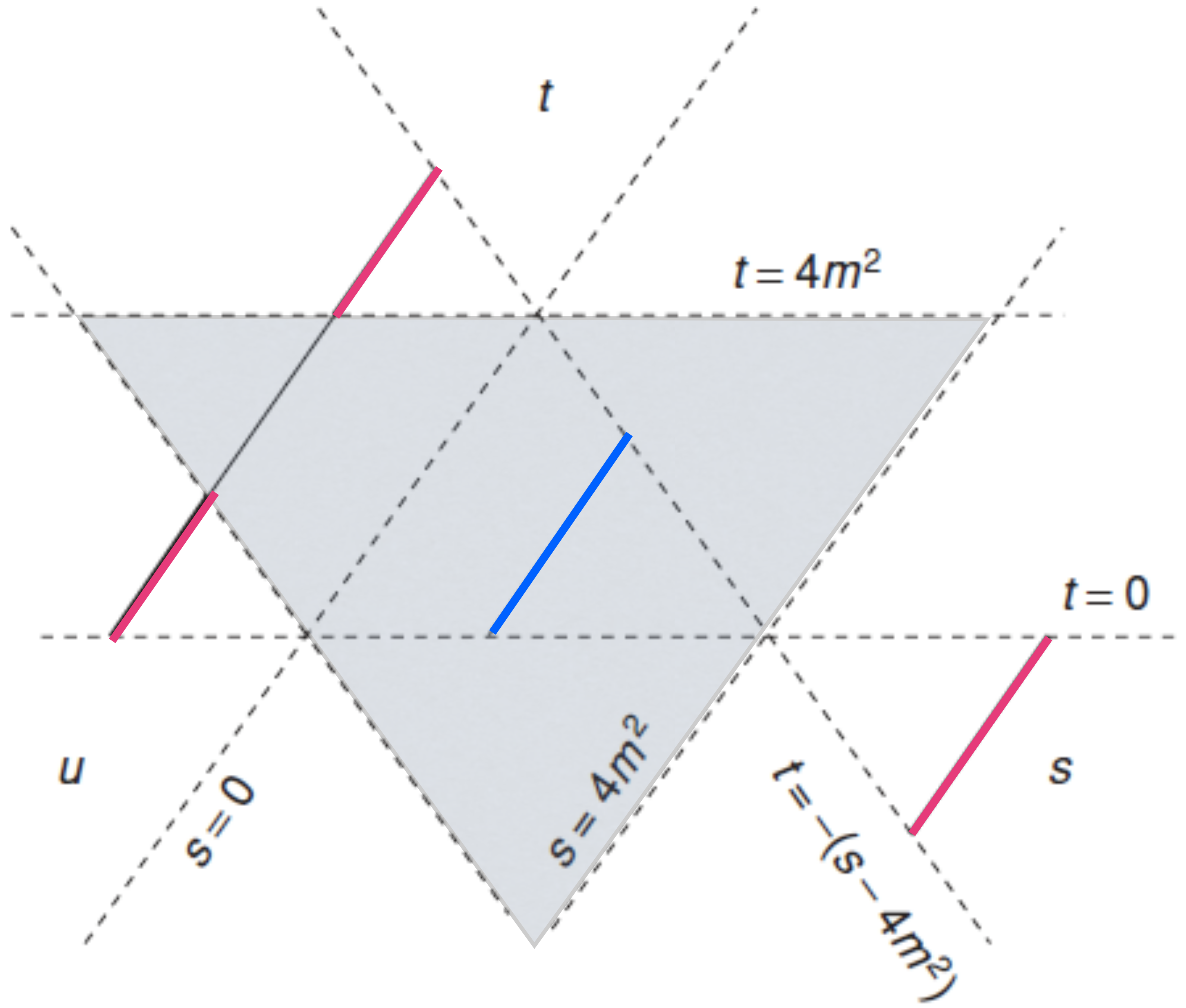
$$\mathrm{Im}\, f_\ell(s) \;=\; \tau f_\ell(s) f^*_\ell(s)$$

An inverse transformation

$$f_\ell(s) = \frac{1}{2}\int_{-1}^{1} d\cos\Theta\, P_\ell(\cos\Theta) A(s, t(\cos\Theta))$$

$$\cos\Theta = 1 + \frac{2t}{s - 4m^2}$$

On the Mandelstam plane the integral runs along a line **s** = const



The partial wave is *complex* above the threshold,
(as well as below *s=0*, due to **t**- and **u**-thresholds)

The partial wave is *real* inside the grey triangle, below *$s=4m^2$*
Therefore, its **Im** part coincides with *discontinuity*,

$$\frac{1}{2i}[f_\ell(s+i\epsilon) - f_\ell(s-i\epsilon)] \;=\; \tau(s) f_\ell(s+i\epsilon) f_\ell(s-i\epsilon)$$

and we obtain

$$f_\ell(s+i\epsilon) \;=\; \frac{f_\ell(s-i\epsilon)}{1 - 2i\tau(s) f_\ell(s-i\epsilon)}$$

When/if the partial wave amplitude on the physical sheet hit a finite value

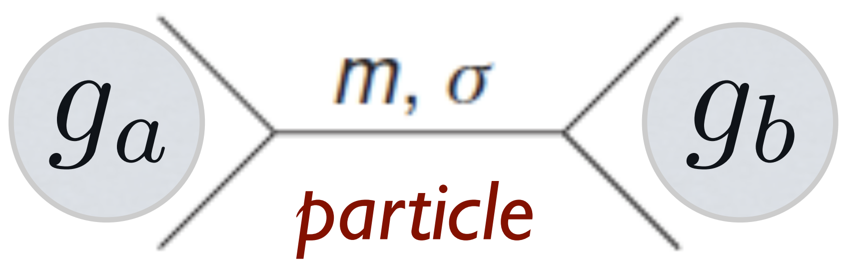
$$f_\ell(-) \;=\; \frac{1}{2i\tau(s)}$$

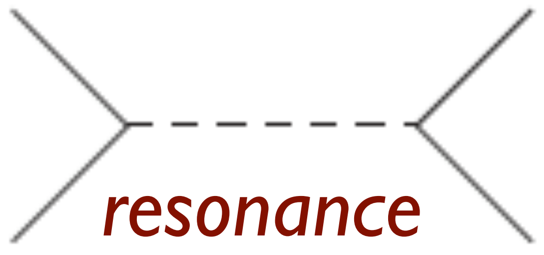
a pole on the *1st unphysical sheet* (related to 2-particle unitarity) appears, which corresponds to a ***resonance*** with spin ℓ

Threshold singularities *make hadron scattering amplitude vary,* ***Resonances*** *make it rock and roll*

Resonances are much like ordinary (stable) particles.

Contribution to the scattering amplitude :





$$f_\ell(s) = \frac{r}{m_{\rm res}^2 - s}$$

$$A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, f_\ell(s)\, P_\ell(\cos\Theta) \quad \longrightarrow \quad A^{\rm pol} = (2\ell+1)\frac{r\, P_\ell(\cos\Theta)}{m_{\rm res}^2 - s}$$

resembles the Born amplitude of the s-channel exchange of a particle with spin σ = ℓ (!)

An important property of particle exchange is ***factorisation*** $r = g_a \times g_b$

Will the residue of the resonance pole factorise into the product of "coupling constants"?

In the case of resonance exchange, it is *unitarity* that takes care of it !

Solution of the elastic unitarity : $f_\ell(s) = \dfrac{1}{2i\tau(s)}\left[e^{2i\delta_\ell} - 1\right]$

Generalisation to a multi-channel problem: $f_{ab} = \dfrac{1}{2i\sqrt{\tau_a\tau_b}}\,[\,S_{ab} - 1\,]$

Diagonalise the scattering matrix: $S_{ab} = U_{ac}S_{cd}U^\dagger_{db}$, $\quad S_{cd} = S_c I_{cd}\,, \quad S_c = \exp(2i\delta_c)$.

Let one matrix element S_1 have a resonance pole at $s = M^2 = M_1^2 - iM_2^2$.

In the pole approximation, $S_1 = \dfrac{M^{*2} - s}{M^2 - s}e^{2i\beta} = \dfrac{-2i\,{\rm Im}\,M^2}{M^2 - s}e^{2i\beta} + {\rm regular.}$

The pole contribution to the full scattering matrix becomes

$$S_{ab} \simeq U_{a1}\left[\frac{-2i\,{\rm Im}\,M^2}{M^2 - s}e^{2i\beta}\right]U^\dagger_{1b} = U_{a1}\frac{2iM_2^2}{M^2 - s}e^{2i\beta}\,U^*_{b1}$$

$$f_{ab} \simeq \frac{U_{a1}}{\sqrt{\tau_a}}\frac{M_2^2}{M^2 - s}e^{2i\beta}\frac{U^*_{b1}}{\sqrt{\tau_b}} = \frac{g_a\, g_b^*}{M^2 - s}e^{2i\beta}$$

$$f_{ab} = \frac{1}{2i\sqrt{\tau_a\tau_b}}\,[\,S_{ab} - 1\,]$$

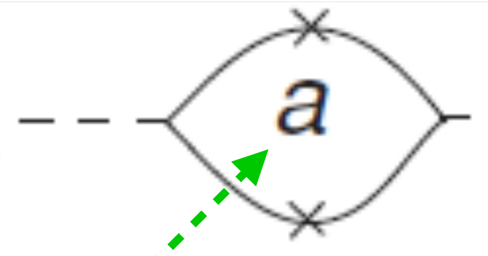
The coupling constants can be redefined as real (as long as $S_{ab} = S_{ba}$)

$$g_a \equiv U_{a1} \cdot \frac{M_2}{\sqrt{\tau_a}} \qquad \Longrightarrow |U_{a1}| \cdot \frac{M_2}{\sqrt{\tau_a}}.$$

The resonance contribution is similar to the Born diagram for spin-ℓ particle exchange: $A^{\rm res} = \frac{g_a(2\ell+1)P_\ell(z)\,g_b}{M^2 - s}\, e^{2i\beta}$.

Relation between ImM² and the coupling constants also follows from ***unitarity*** :

$$\sum_a U^*_{a1} U_{a1} = 1 \quad \Longrightarrow \quad \sum_a \tau_a g_a^2 = \sum_a$$



$$= M_2^2 \equiv M_1 \Gamma.$$

conservation of probability !

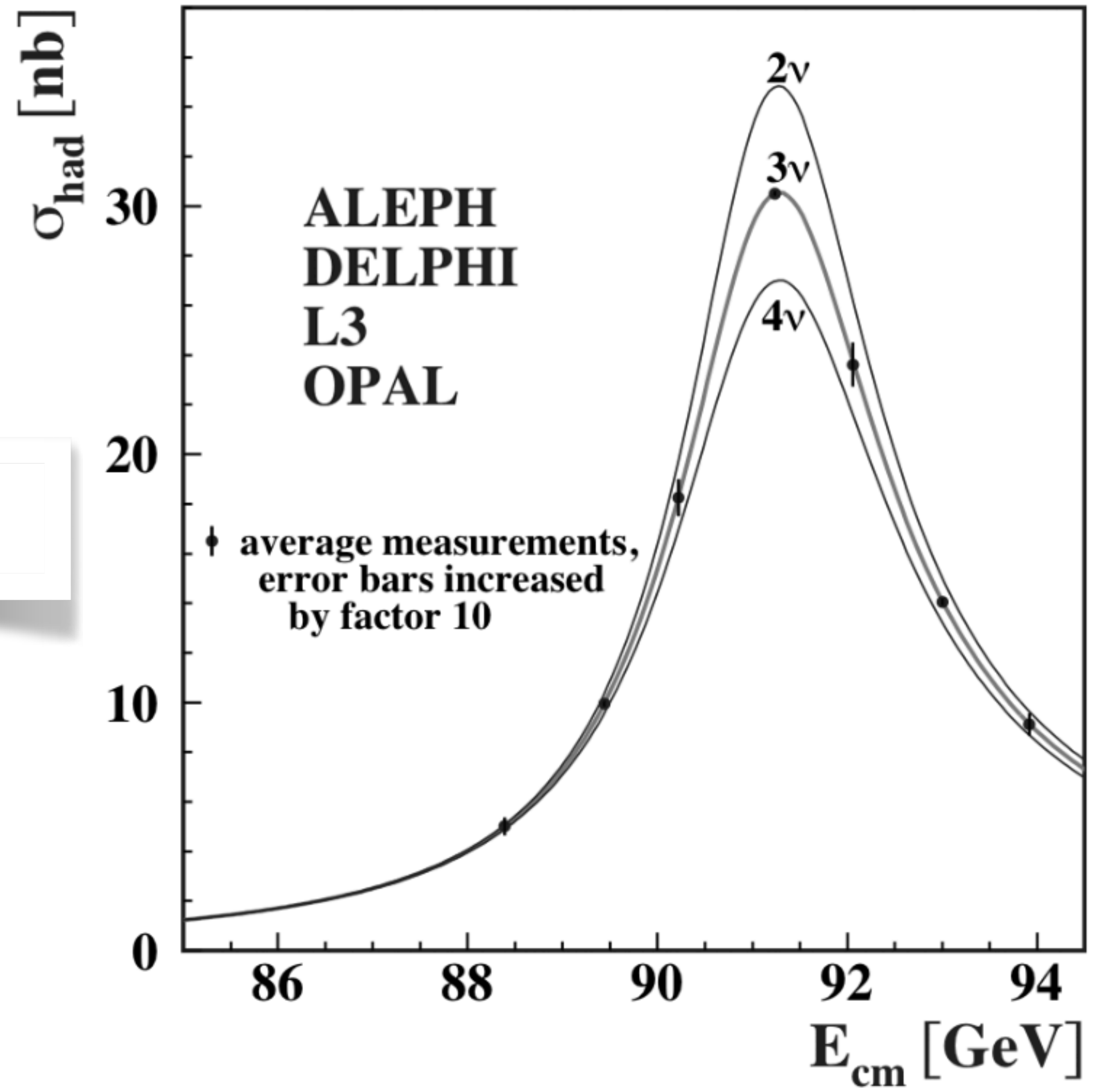
Partial decay widths sum up into the total width of the resonance.

Some of the decay modes may happen to be "*invisible*" - undetectable experimentally.

Nevertheless, they contribute to the total width
and therefore affect the production cross section - the "*line shape*".

Famous example - extraction of the number of ***light neutrinos***
from the shape of Z-boson production at LEP : $Z \to \nu\bar{\nu}$

$$N_\nu = 2.9840 \pm 0.0082.$$

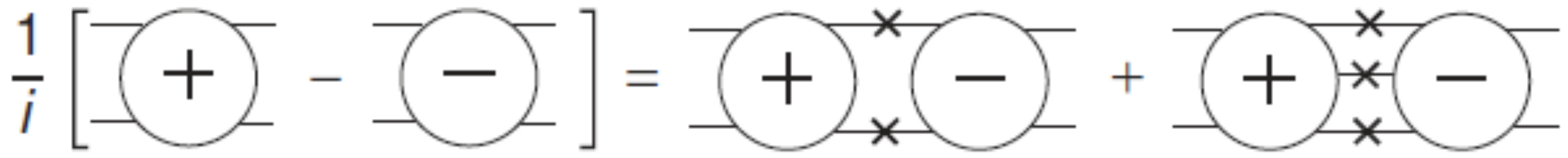


NB: we call Z (W, H), n, π (K, D, B), μ ,τ *stable particles*.
In fact, all of them are *resonances*. *Stable* w.r.t. *strong interactions*!

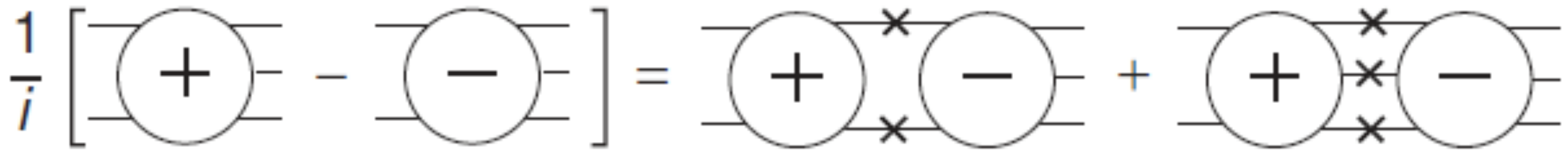
One day **p**roton may well become a resonance too. *It better does...*

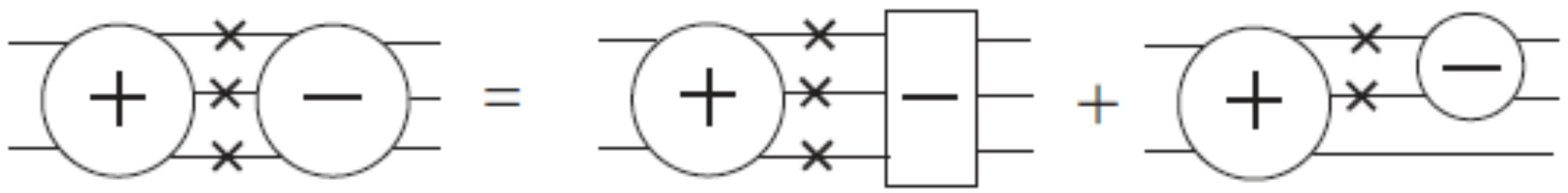
We have obtained resonances from 2-particle unitarity.
Continuation of *multi-particle* unitarity conditions is technically much more difficult.
The answer, however, turns out to be physically transparent.

Schematically,



The new ingredient, the amplitude $A_{2\to 3}$, satisfies its proper unitarity relation:





The first - *irreducible* - term gives rise to new 3-particle ***resonance poles***.

The second - *singular* - term generates ***non-pole singularities*** on the second unphysical sheet...

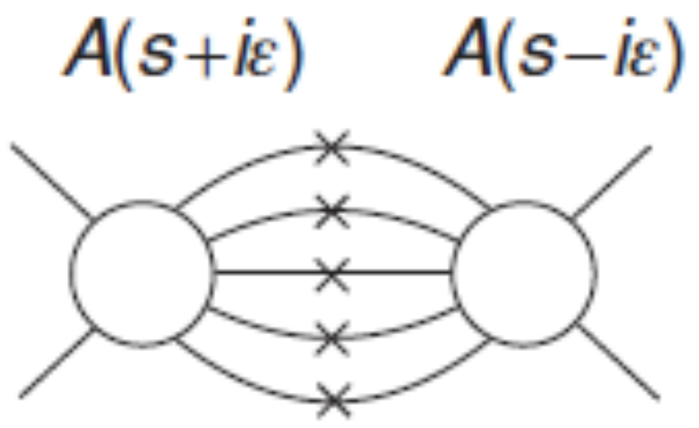
Namely, branch-points corresponding to ***resonance-particle*** and (at still higher energies) ***resonance-resonance*** threshold ***cuts***.

The full analyticity image of the amplitude :
poles - ***particles*** and ***resonances*** -
with all other singularities being generated from them via ***unitarity***.

Hunting for resonances, we have introduced partial wave expansion (*small energies*)
It turns out to be extremely useful for the analysis of ***high energy*** processes as well.

At high energies many inelastic channels open up :

$$\mathrm{Im}_s\, A(s,t) \equiv A_1(s,t) = \frac{1}{2}\sum_n$$



For t=0 we have sum of *positive* contributions, yielding $\mathrm{Im}\, A(s,0) \simeq s\sigma_{\mathrm{tot}}, \qquad s \gg \mu^2.$

How to get an amplitude increasing like **s** ?

$$\mathrm{Im}\, A(s,t) = \sum_{\ell=0}^{\infty}(2\ell+1)\ \mathrm{Im}\, f_\ell(s) P_\ell(z)$$

Partial waves are limited from above :

$$0 \le \mathrm{Im}\, f_\ell(s) \le 16\pi$$

Therefore, the source of increase - large number of "large" terms $\mathcal{O}(1)$

To estimate the characteristic number of large partial waves, $\ell < \ell_0(s)$,
we introduce the ***impact parameter*** : $\ell = k_c\rho,$ with $k_c = \frac{\sqrt{s-4\mu^2}}{2} \simeq \frac{1}{2}\sqrt{s}$ - the c.m.s. momentum,

and *define* the ***interaction radius*** through the relation $\ell_0(s) = k_c\rho_0$

$$\begin{cases} \mathrm{Im} f_\ell(s) = \mathcal{O}(1) & \text{for } 0 \le \ell < \ell_0 \\ \mathrm{Im} f_\ell(s) \ll 1 & \ell \gg \ell_0 \end{cases}$$

("*saturated*" partial waves)

classical analogy:

$\rho < \rho_0$ *projectile hits the target*

$\rho \gg \rho_0$ *projectile misses it*

elastic scattering

One can estimate the forward scattering amplitude by simply *truncating* the partial wave expansion :

$$\mathrm{Im}\, A(s,t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{Im} f_\ell(s) P_\ell(z)$$

(at t = 0 we have $P_\ell(1) = 1$)

$$\mathrm{Im} A(s,0) = \sum_{\ell=0}^{\ell_0} (2\ell+1)\, \mathrm{Im} f_\ell(s) \sim \ell_0^2 \propto s \cdot \rho_0^2$$

$$\sigma_{tot} \sim \rho_0^2$$

In relativistic theory (in marked contrast with NQM) the hadron radius ***may vary*** with energy, $\rho_0 = \rho_0(s)$
Moreover, it ***has to*** (as we shall see shortly).

But before we have to discuss qualitative picture of elastic high energy scattering

Forward scattering

Large angular momenta $\ell \sim \ell_0$ translate into small scattering angles

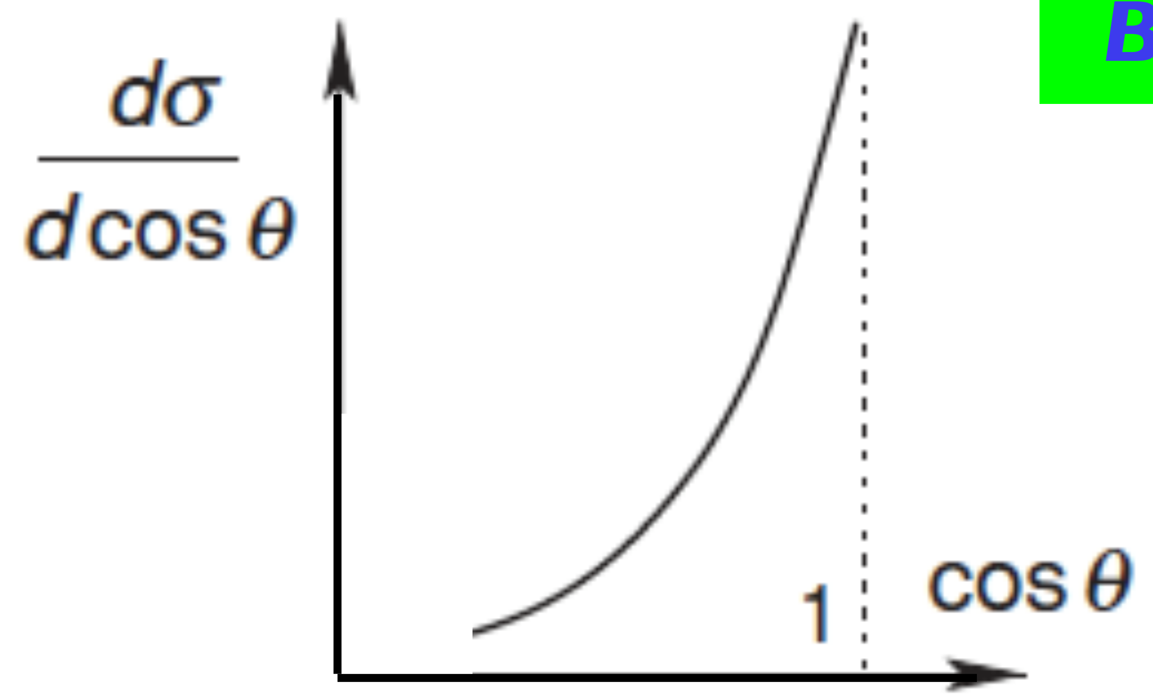
$$\Theta \sim 1/\ell_0 = 1/k_c\rho_0$$

Typical ***momentum transfer*** stays finite while energy increases :

$$-t \simeq (k_c\Theta)^2 = \rho_0^{-2} \sim \mu^2$$

Hence *forward* (*diffractive*) *cone*

Backward scattering

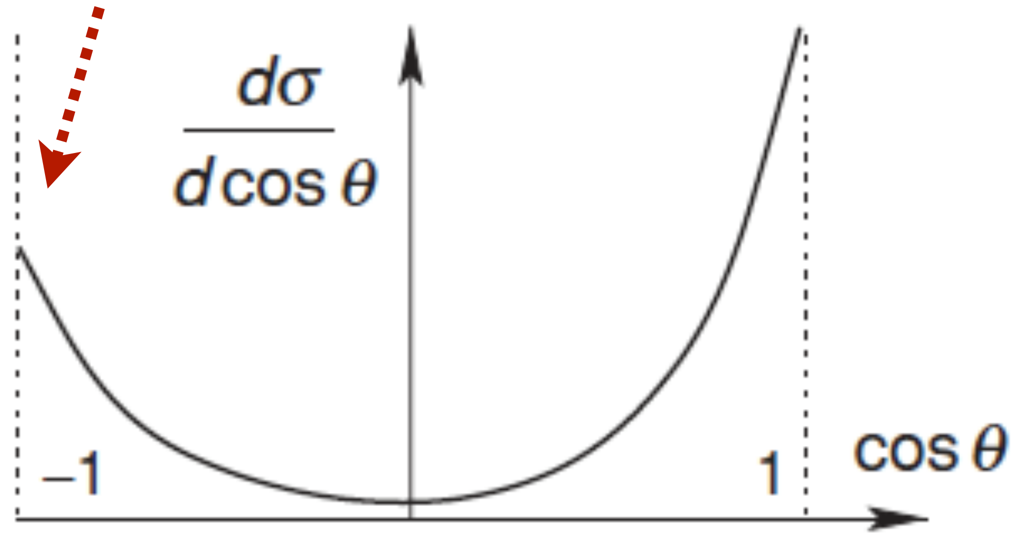


- a bizarre phenomenon specific for ***relativistic*** theory and unimaginable in NQM .

To see that such an unintuitive thing indeed happens in nature,
we have to exploit ***analytic properties*** of the interaction amplitudes.

Complex analysis in full swing. Take a deep breath.

Backward scattering



Angular momentum enters NQM Hamiltonian *analytically* via the centrifugal potential $\frac{\ell(\ell+1)}{r^2}$

As a result partial waves f_ℓ turn out to be *smooth functions* of ℓ.

In relativistic theory there is a reason for f_ℓ *to oscillate*...

To single out this oscillating behaviour, we should look into *analytic properties* of A(s, t).

Partial wave is given by the integral over $z = \cos\Theta$ from -1 to +1

$$f_\ell(s) = \frac{1}{2}\int_{-1}^{1} d\cos\Theta\, P_\ell(\cos\Theta) A(s, t(\cos\Theta))$$

The first step - to replace integration over real interval by a contour integral in z-plane.

Introduce a cousin of Legendre polynomials - "*Legendre function of the second kind*" as

$$Q_\ell(z) \equiv \frac{1}{2}\int_{-1}^{1} \frac{dz'\, P_\ell(z')}{z - z'}$$

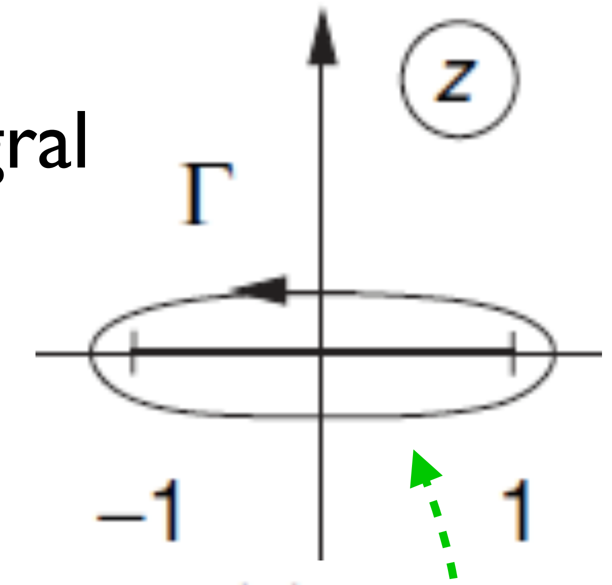
(The cousins are the two solutions of the same 2nd order differential equation.)

$Q_\ell(z)$ is regular for $|z| > 1$ and has a branch cut $-1 \le z \le 1$ with *discontinuity* across the cut

$$P_\ell = \frac{i}{\pi}\left[Q_\ell(z + i\epsilon) - Q_\ell(z - i\epsilon)\right]$$

Integration over real interval -1<z<1 gets replaced by the contour integral

$$f_\ell(s) = \int_{-1}^{1} dz\, P_\ell(z) A(s, z) = \frac{1}{2\pi i}\int_\Gamma dz\, Q_\ell(z) A(s, z).$$



since $Q_\ell(z)$ falls on a large circle, $Q_\ell(z) \overset{|z|\to\infty}{\sim} z^{-\ell-1}$

The amplitude as a function of **t** (**z**) has ***unitarity cuts***

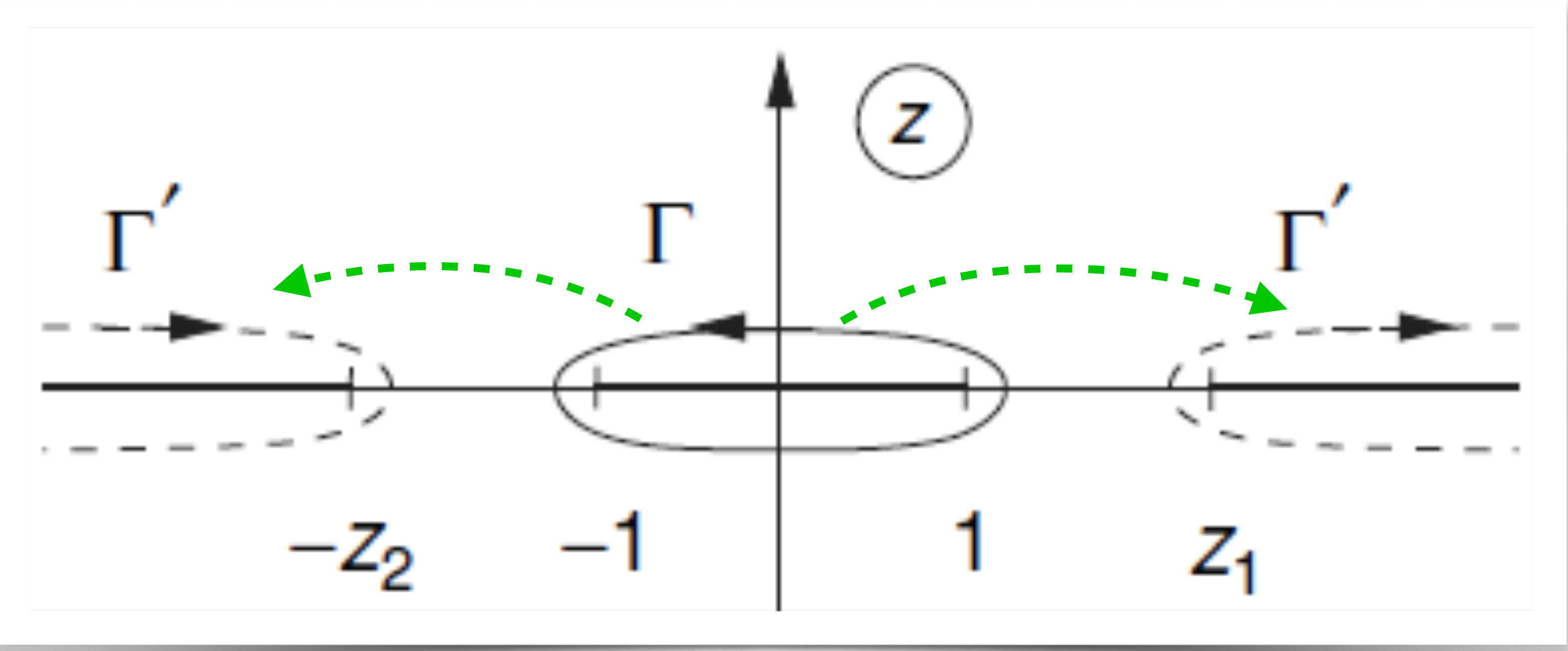
t-channel cut at **$t > t_{min}$** **&** **u-**channel cut at **$u > u_{min}$**

$$z_1 < z < +\infty; \qquad z_1 = 1 + \frac{2 \cdot t_{\min}}{s - 4\mu^2};$$

$$-\infty < z < -z_2; \qquad z_2 = 1 + \frac{2 \cdot u_{\min}}{s - 4\mu^2}.$$

in our toy model

$t_{\min} = u_{\min} = (2\mu)^2$



we can "inflate" the contour Γ and replace it by a sum of integrals around the t- and u-channel cuts Γ'

$$f_\ell(s) = \frac{1}{\pi}\int_{z_1}^{\infty} dz\, Q_\ell(z) A_3(s,z) + \frac{1}{\pi}\int_{-z_2}^{-\infty} dz\, Q_\ell(z) A_2(s,z),$$

with $A_3 = \mathrm{Im}_t A,\ A_2 = \mathrm{Im}_u A$ the *discontinuities* of the amplitude *A(s,t)* in the ***t*** and ***u*** channels.

replacing negative integration variable in the second term by $z_u = -z \geq z_2 > 0$ and using the relation $Q_\ell(-z) = (-1)^{\ell+1} Q_\ell(z)$

$$f_\ell(s) = \frac{1}{\pi}\int_{z_1}^{\infty} dz\, Q_\ell(z) A_3(s,z) + \frac{(-1)^\ell}{\pi}\int_{z_2}^{\infty} dz_u\, Q_\ell(z_u) A_2(s,-z_u).$$

We are interested in the size of the cross section. Since $\sigma_{\mathrm{tot}} \propto \mathrm{Im}_s A/s$ let's evaluate the "imaginary part" (discontinuity in s) of the p.w.:

$$\mathrm{Im}\, f_\ell = \mathrm{Im}\, f_\ell^{\mathrm{right}} + (-1)^\ell\, \mathrm{Im}\, f_\ell^{\mathrm{left}},$$

where

$$\mathrm{Im}\, f_\ell^{\mathrm{right}}(s) = \frac{1}{\pi}\int_{z_1}^{\infty} dz\, Q_\ell(z) \rho_{st}(s,t(z)),$$

$$\mathrm{Im}\, f_\ell^{\mathrm{left}}(s) = \frac{1}{\pi}\int_{z_2} dz\, Q_\ell(z) \rho_{su}(s,u(-z)).$$

forward scattering : $P_n(1) = 1$

$$A_1(s,z{=}1) = \sum_\ell (2\ell+1)\,\mathrm{Im}\, f_\ell^{\mathrm{right}} + \sum_\ell (-1)^\ell \cdot (2\ell+1)\,\mathrm{Im}\, f_\ell^{\mathrm{left}}$$

$A_1 \propto s$

large angles (90°) **:** $P_{2n+1}(0) = 0 \quad P_{2n}(0) \simeq (-1)^n \cdot 2/\sqrt{\pi n}$

$$A_1(s,z{=}0) \simeq \sum_{\ell=2n} (-1)^n \cdot \frac{2(4n+1)}{\sqrt{\pi n}} (\mathrm{Im}\, f_{2n}^{\mathrm{right}} + \mathrm{Im}\, f_{2n}^{\mathrm{left}})$$

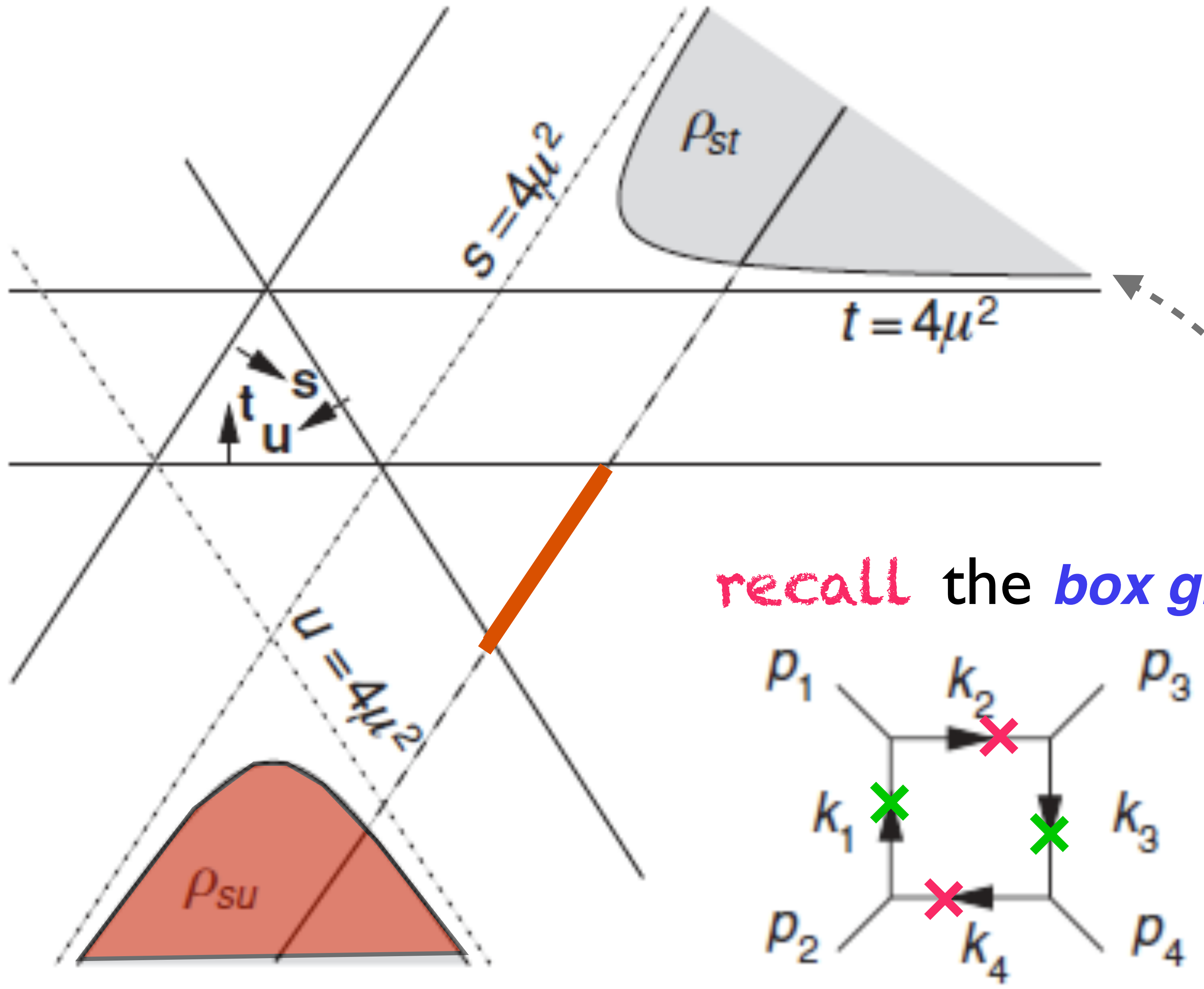
$A_1 = \mathcal{O}(1)$

backward (180°) **:** $P_n(-1) = (-1)^n$

$$A_1(s,z{=}-1) = \sum_\ell (-1)^\ell \cdot (2\ell+1)\,\mathrm{Im}\, f_\ell^{\mathrm{right}} + \sum_\ell (2\ell+1)\,\mathrm{Im}\, f_\ell^{\mathrm{left}}$$

$A_1 \propto s$ again ← ***left cut !***

recall the ***box graph***



Karplus curve

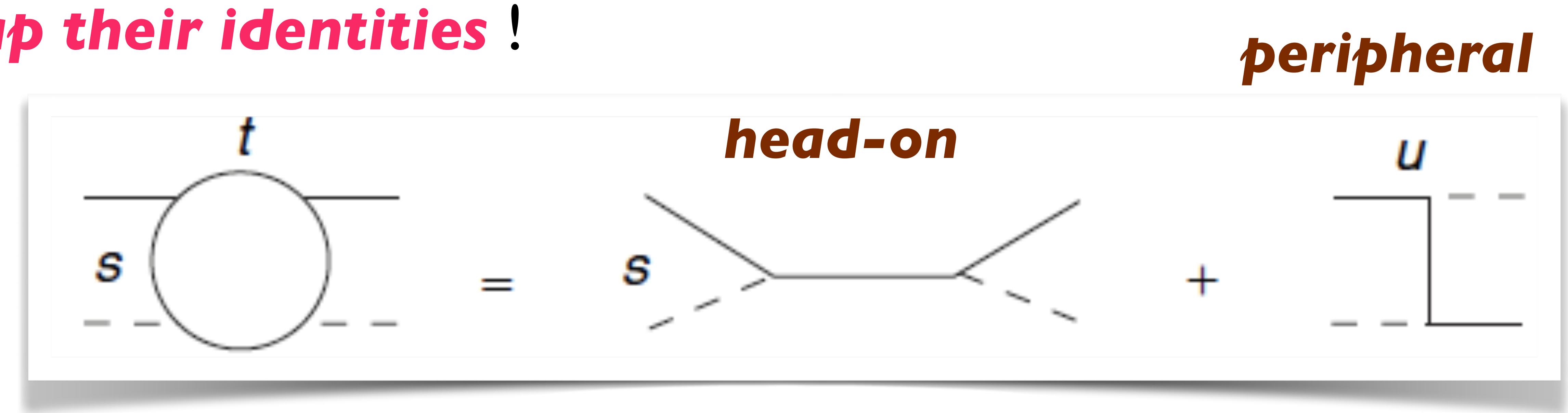
$(s - 4m^2)(t - 4m^2) = 4m^4$

Compton

In ***NQM*** backward (large momentum transfer) scattering process implies head-on collision : $\sigma \propto \lambda^2 \sim 1/s$

In ***relativistic theory*** there is an alternative to large momentum transfer :
instead of exchanging large momentum, colliding particles can ***swap their identities*** !

Example of such a phenomenon - *QED Compton scattering*.



The first amplitude in the $s \to \infty$ limit is negligible (*one partial wave*).

The second is *peripheral interaction* (*many partial waves*), $\ell_0 \sim (\sqrt{s}/2)/m_e$

Finite momentum transfer from the incident electron to the final photon, $|u| \sim m_e^2$,
means that at high energies Compton scattering occurs predominantly ***backward***.

$$\overline{|\mathcal{M}|^2} = 2e^4\left[4m^4\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right)^2 + 4m^2\left(\frac{1}{s-m^2}+\frac{1}{u-m^2}\right) - \frac{u-m^2}{s-m^2} - \frac{s-m^2}{u-m^2}\right]$$

In the high energy limit $s \simeq |t| \gg |u|$ the differential cross section peaks in a tiny angular cone of the size $\pi - \Theta_c \sim m_e/k_c$

$$\frac{d\sigma}{d\cos\Theta_c} \propto \frac{\alpha^2}{m^2-u} \simeq \frac{\alpha^2}{s(\pi-\Theta_c)^2+m^2}$$

Practical application — *By shooting energetic electrons on ionised gas, one can convert Coulomb photons into a well collimated monochromatic photon beam!*

and vice versa: *photons hitting electron gas produce collimated monochromatic electrons!*

Please mark and keep in mind the word ***peripheral*** which will become the key slogan of the discussion to follow

Return to the profile of p.w. amplitudes in the impact parameter space $(\rho = \ell/k_c)$

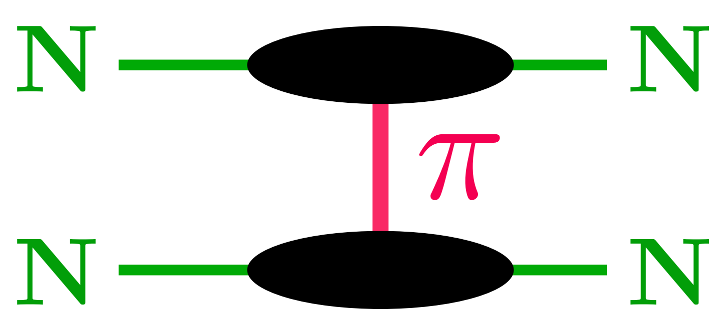
Analyticity and interaction radius

In the physical region of the **s**-channel A(s, t) has no singularities in ***t*** .
Therefore, the partial wave expansion series must be *absolutely convergent* :

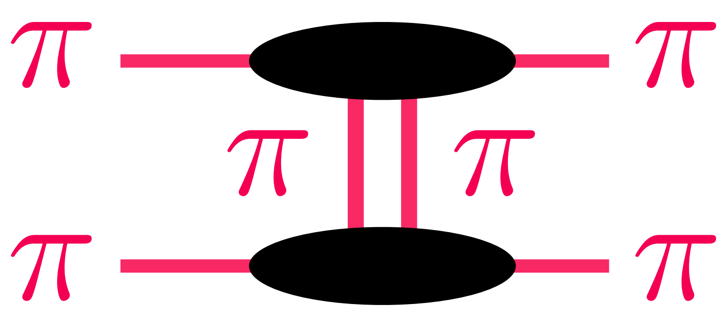
$$A(s,t) = \sum_{\ell=0}^{\infty}(2\ell+1) f_\ell(s) P_\ell(z)$$

Moreover, this should be true for ***t>0*** too , up to the *first singularity* !

For example, $t_0 = m_\pi^2$ for **nucleon** scattering;



$t_0 = 4m_\pi^2$ for scattering of **pions**:



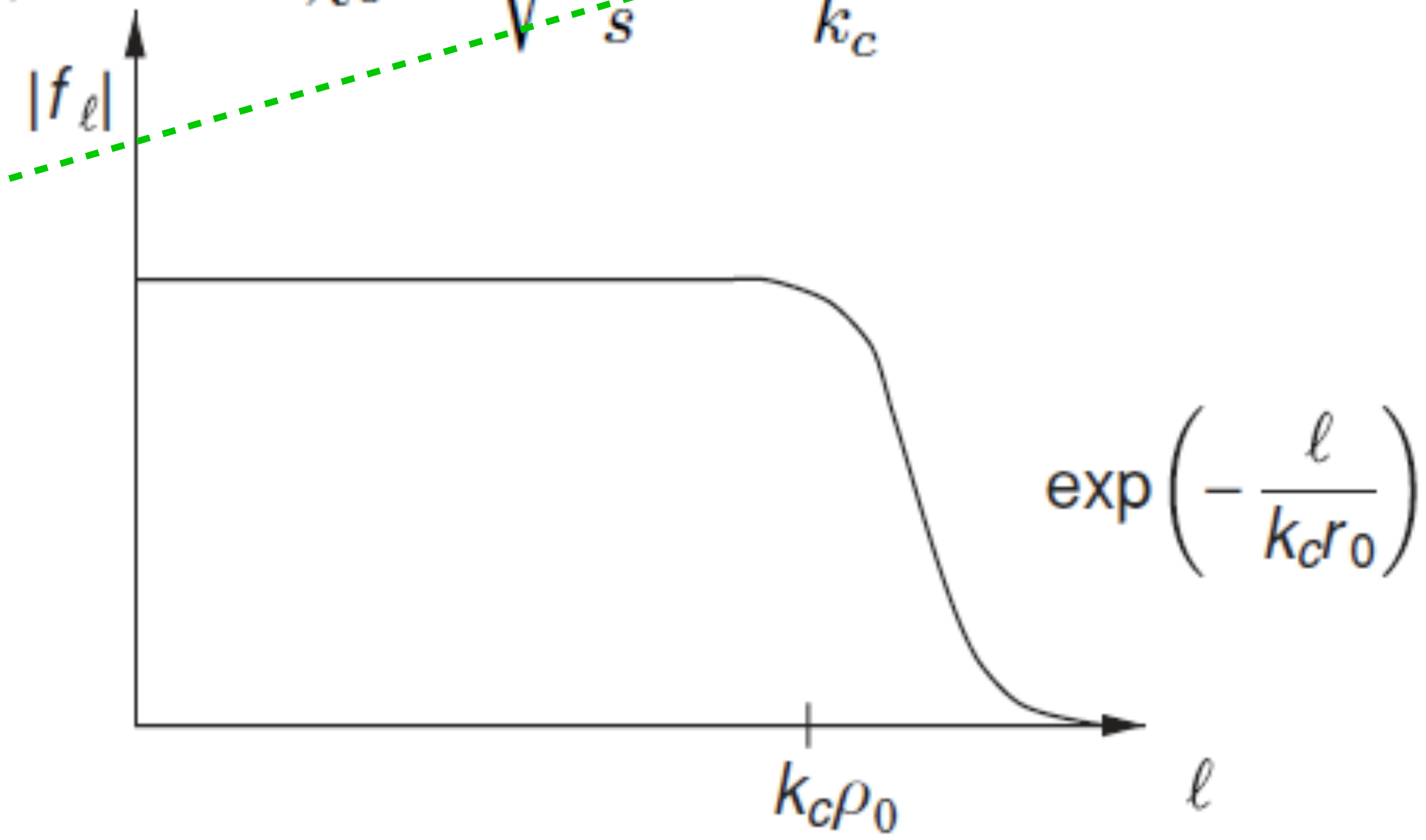
$$z = \cos\Theta = 1 + \frac{2\,t}{s-4m_\pi^2} = \cosh\chi, \quad \Theta = i\chi \quad \Longrightarrow \quad P_\ell(z) \sim e^{i\ell\Theta} + e^{-i\ell\Theta} \sim e^{\ell\,\chi(t,s)}.$$

Up to $t \le t_0$ this increase has to be damped by an *exponential fall-off* of partial waves:

$$f_\ell(s) \overset{\ell\gg1}{=} C(\ell,s)\,e^{-\ell\chi_0}, \quad \cosh\chi_0 \equiv 1 + \frac{2\cdot t_0}{s-4m_\pi^2} \simeq 1 + \frac{\chi_0^2}{2} \to 1, \qquad \chi_0 \simeq \sqrt{\frac{4t_0}{s}} \simeq \frac{\sqrt{t_0}}{k_c}.$$

position of the first ***t****-channel singularity*

$$f_\ell(s) \Longrightarrow f(\rho,s) = C(\rho,s)\,e^{-\rho/r_0}, \qquad r_0 \equiv 1/\sqrt{t_0}.$$



warning: not to confuse the hadron radius ρ_0
and the fall-off parameter r_0 !

black disk

One has little to say about partial waves corresponding to impact parameters $\rho < \rho_0(s)$

However, it is straightforward to *limit them from above*, from the first principles.
General unitarity relation for the elastic scattering partial wave amplitude reads

$$\mathrm{Im}\, f_\ell = \tau |f_\ell|^2 + \Delta_\ell$$

with $\Delta_\ell(s)$ accounting for inelastic channels.

Hadron interactions ("*soft*", "*minimum bias*") are close to this regime

Solution :

$$f_\ell(s) = \frac{1}{2i\tau(s)} \left[\eta_\ell(s)\, e^{2i\delta_\ell(s)} - 1 \right]$$

elastic scattering corresponds to $\eta_\ell = 1, \quad \Delta_\ell \equiv 0.$
"very" inelastic - to $\eta_\ell \simeq 0, \quad \Delta_\ell \simeq \Delta_{\max} \simeq 4\pi.$

Total cross section :

$$\sigma_{\mathrm{tot}} = \frac{1}{s}\sum_\ell (2\ell+1)\, \mathrm{Im}\, f_\ell \simeq \frac{1}{2\tau s}\sum_\ell^{\ell_0} (2\ell+1)\,[1 - \eta_\ell \cos(2\delta_\ell)]\,.$$

$\sigma_{\mathrm{el}} + \sigma_{\mathrm{in}} = \sigma_{\mathrm{tot}}$

elastic : $\sigma_{\mathrm{el}} \simeq \frac{1}{4\tau s}\sum_\ell^{\ell_0} (2\ell+1)\,[1 - 2\eta_\ell \cos(2\delta_\ell) + \eta_\ell^2]\,.$

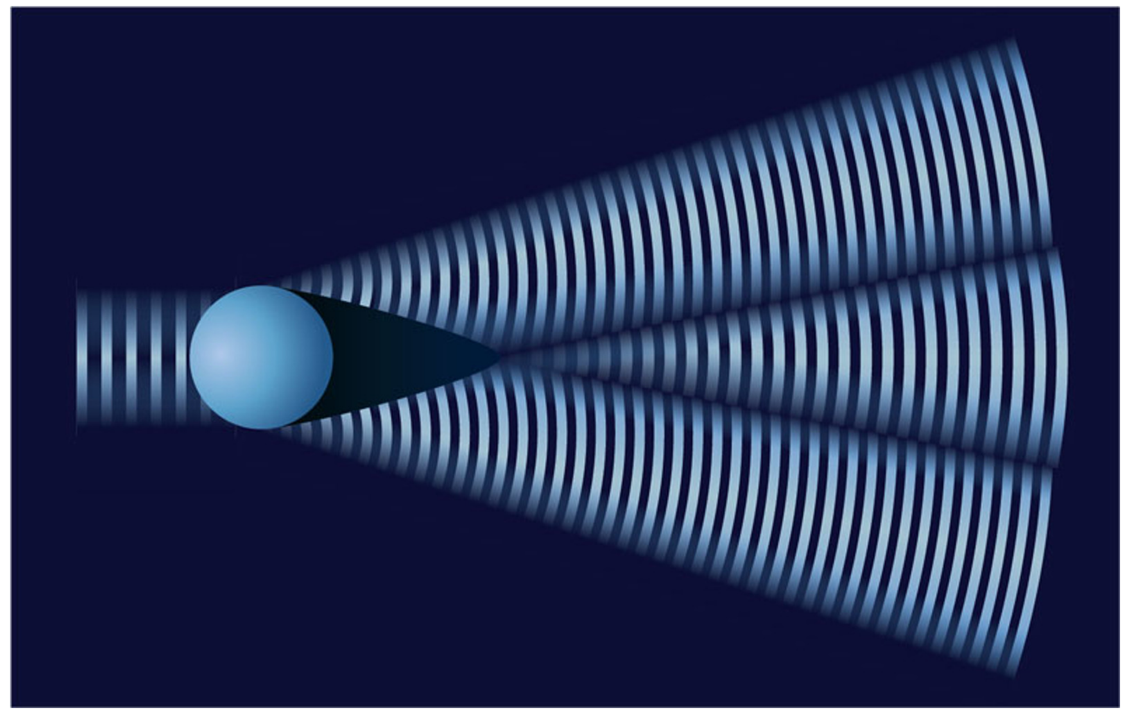
inelastic : $\sigma_{\mathrm{in}} \simeq \frac{1}{4\tau s}\sum_\ell^{\ell_0} (2\ell+1)\,[1 - \eta_\ell^2]\,.$

upper limit : $[\mathrm{Im}\, f_\ell(s)]^{\max} = \frac{1}{2\tau(s)} \simeq 8\pi \;\Rightarrow\; \sigma_{\mathrm{tot}} \le [\sigma_{\mathrm{tot}}]^{\max} \simeq \frac{8\pi}{s}\cdot \ell_0^2 \simeq 2\pi\rho_0^2\,.$

The total cross section is twice bigger than the transverse area of the target.

And for a good reason: *diffraction !*

A forward wave is necessary to form shade behind the target



Maximal inelasticity *hypothesis* $(\eta_\ell = 0)$:

$$\sigma_{\mathrm{el}} = \sigma_{\mathrm{in}} = \tfrac{1}{2}\sigma_{\mathrm{tot}} = \pi\rho_0^2\,.$$

QM scattering off a ***"black disc"***

Froissart theorem

Total cross sections *cannot grow as a power of energy*.

To guarantee ***causality***, we have to have amplitude to be *polynomially bounded* :

$$|A(s,t)| \leq s^{N(t)} \qquad \text{for } s \to \infty$$

Take a rough model with p.w. $\ell < \ell_0$ *saturated*, and $> \ell_0$ - *negligible* :

$$|A(s,t)| \leq 16\pi \sum_{\ell=0}^{\ell_0(s)} (2\ell+1)|P_\ell(z)| \sim \ell_0 \, e^{\ell_0 \chi_0(s)} \leq s^{N_1} \qquad (N_1 \equiv N(t)|_{t=4\mu^2})$$

When t > 0, Legendre polynomials increase, and the sum is dominated by the last term.

Hence the estimate

$$\rho_0(s) \leq \frac{N_1}{2\mu} \ln s,$$

$$\sigma_{\text{tot}} \leq c \ln^2 s,$$

$$c = \left(\frac{N_1}{2\mu}\right)^2 \cdot \frac{\langle \mathrm{Im}\, f \rangle}{4} \leq \left(\frac{N_1}{2\mu}\right)^2 \cdot 4\pi \leq \frac{4\pi}{m_\pi^2} \simeq 240\,\text{mb}.$$

A rigorous mathematical proof of the ***Froissart theorem*** (1961) follows from

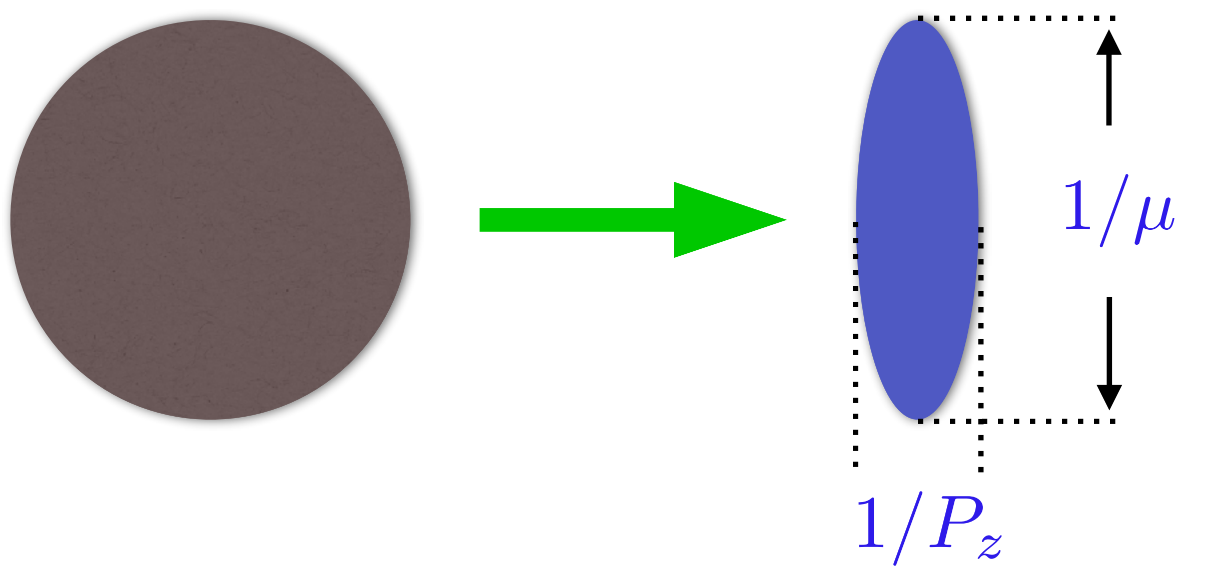
(1) singularities of $A(s,z)$ in z lie outside the physical region of the s-channel, $-1 \leq z \leq +1$

(2) for finite $|z|$ $A(s,z)$ is polynomially bounded, $|A(s,z)| < cs^N$.

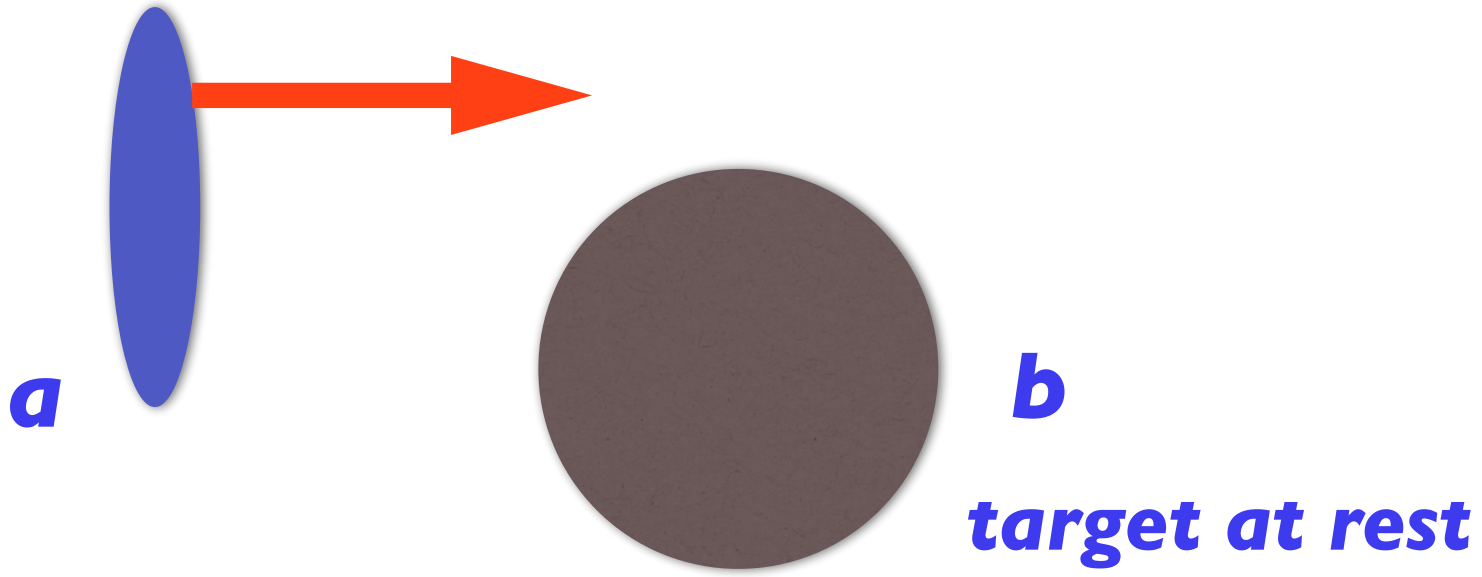
We see that the interaction radius *is allowed* to logarithmically *grow with energy*... **will it ?..**

fast *but* furious

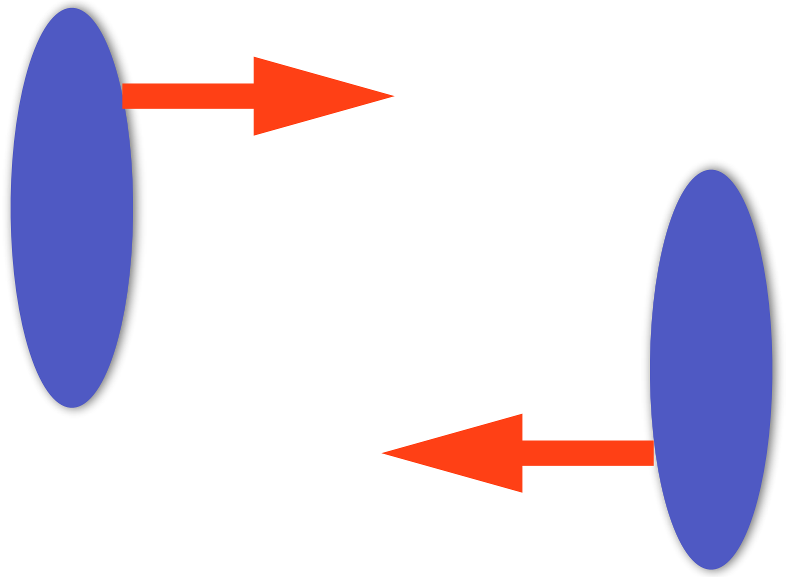
Because of the Lorentz effect, a "*ball*" gets *squeezed* into a "*pancake*" in the **z** direction (with transverse directions (**x**,**y**) unaffected)



relativistic projectile



colliding beams



Invariant "collision energy" :

$$s = (P_a + P_b)^2 = m_a^2 + m_b^2 + 2(P_a P_b) \;\simeq 2(E_a E_b - \mathbf{P}_a{\cdot}\mathbf{P}_b) \qquad \simeq 2E_a^{(\mathrm{lab})} m_b \qquad \simeq 4E^2_{(\mathrm{c.m.})} \qquad \gg (m_a + m_b)^2$$

laboratory frame: $\mathbf{P}_b = 0$ *centre-of-mass frame:* $\mathbf{P}_b = -\mathbf{P}_a$

When collision energy increases, the flight-through (*=interaction*) time goes to zero.

Interaction becomes actually instantaneous.

One can talk about ***a snapshot*** of a ***dynamically frozen incident proton***, taken by interaction with the target.

The time "*freezes*". It does. This, however, does not result in the physics of the process becoming automatically simpler.

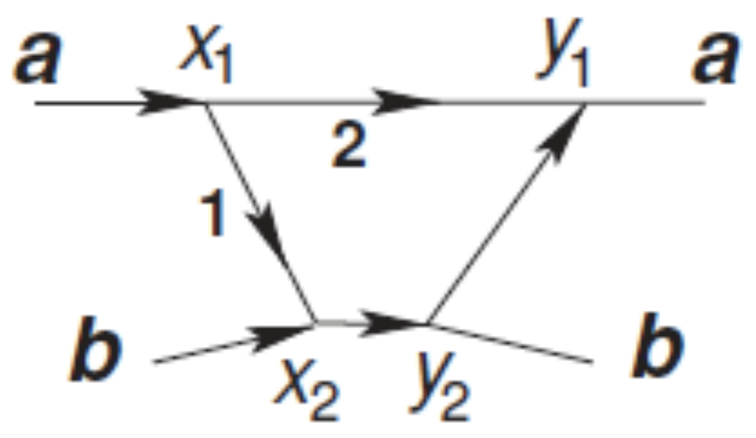
Complexity of the problem does not evaporate, but gets rerouted into internal structure of the projectile !

The point is that long before hitting the target, ***a relativistic projectile*** acquires plenty of time to ***"breathe"***
- to fluctuate into a system of virtual particles - and thus develops quite a complex ***multi-particle content.***

Let us examine why and how this happens

long live Hadron!

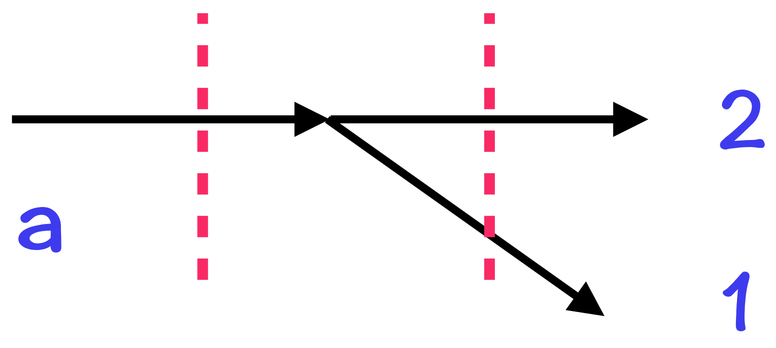
Imagine space-time picture of particle exchange between a projectile ***a*** and a target ***b***



$$x_1^0 < x_2^0 < y_2^0 < y_1^0 .$$

$$\Delta t \sim \Delta E^{-1}$$

How to estimate lifetime of a virtual state?



Look at the energy mismatch btw *intermediate* and *initial* states

$$\Delta E = E_{\mathrm{interm}} - E_{\mathrm{init}} = \sqrt{\mathbf{k}_1^2 + \mu^2} + \sqrt{\mathbf{k}_2^2 + \mu^2} - \sqrt{\mathbf{p}^2 + \mu^2}$$

When momenta are moderate, energy mismatch is of the order of the hadron scale $\Delta t \sim \mu^{-1} = \mathcal{O}(1\ \mathrm{fm/c})$

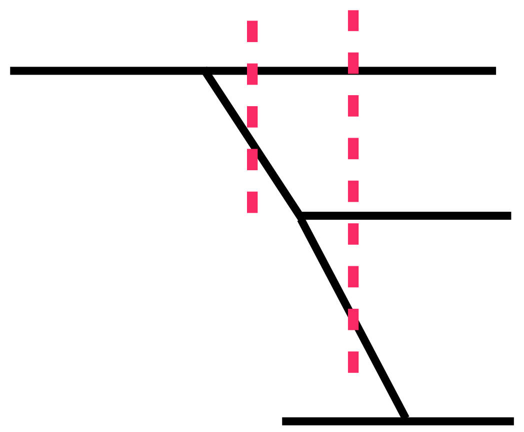
However, if momentum of the projectile ***a*** grows, $\sqrt{\mathbf{k}^2 + \mu^2} = \sqrt{k_z^2 + \mathbf{k}_\perp^2 + \mu^2} = k_z + \frac{\mathbf{k}_\perp^2 + \mu^2}{2k_z} + \ldots$

the energy defect becomes extremely small : $\simeq \frac{\mathbf{k}_{1\perp}^2 + \mu^2}{2k_{1z}} + \frac{\mathbf{k}_{2\perp}^2 + \mu^2}{2k_{2z}} - \frac{\mu^2}{2p} \simeq \frac{1}{2p}\left[\frac{\mu^2 + \mathbf{k}_\perp^2}{x(1-x)} - \mu^2\right] \sim \frac{\mu_\perp^2}{x(1-x)\,p},$

$\left(k_{1z} \equiv xp, \quad k_{2z} = (1-x)p \right)$

The *lifetime of the virtual state* is then *large* and *grows* with energy as $\Delta t \sim \frac{x(1-x)p}{\mu_\perp^2} \gg \frac{1}{\mu}$

The same consideration applies to multi-step processes *giving rise to* ***long-living*** *time-ordered cascades.*



Lifetime of a proton fluctuation into **n** pieces ("partons") with momenta $\mathbf{k}_i = (z_i P_z, \mathbf{k}_{\perp i}),\ i = 1, \ldots n$

can be estimated, analogously, as

$$\Delta t \propto \frac{1}{\Delta E} \sim \frac{E}{\mu^2}$$

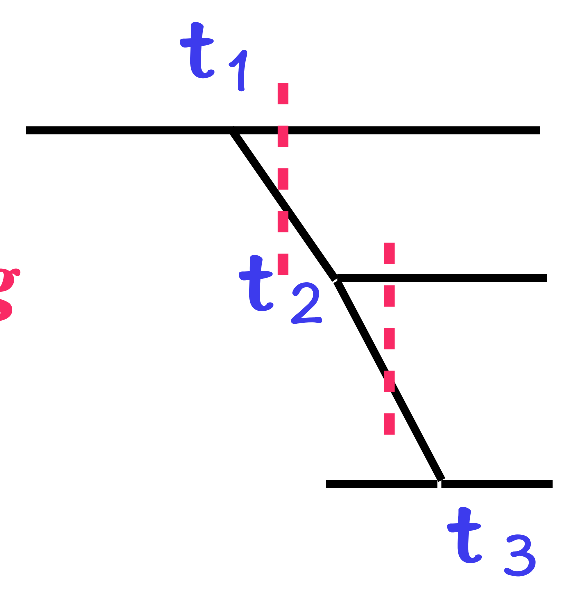
$$t \sim \frac{1}{\Delta E} \propto P_z \cdot \left(\sum_{i=1}^{n} \frac{m_{\perp i}^2}{z_i} \right)^{-1}$$

with $m_{\perp i}^2 = m_i^2 + k_{\perp i}^2$

Due to Lorentz time-dilation, a relativistic hadron acquires rich internal structure.

So drawn, the splitting process looks natural.

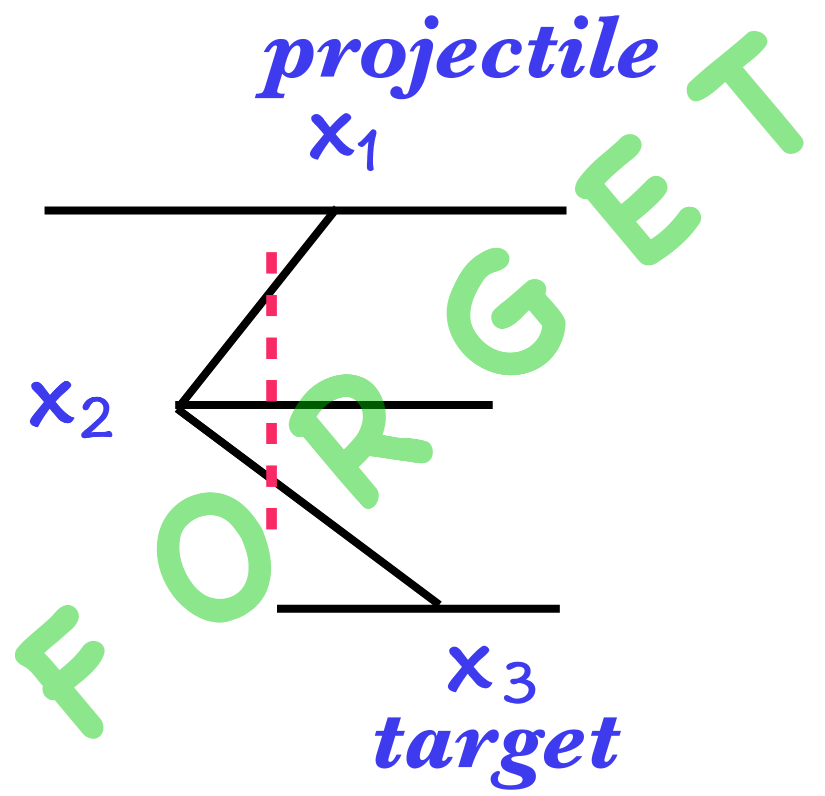
cascading



$t_1 < t_2 < t_3$

However, QFT ***does permit*** "*unnatural*" ordering of events in the configuration space (t,**r**)

vacuum fluctuation



$t_2 < t_1 , t_3$

seems *anti-causal* ?..

[no trouble: Causality of the S-matrix gets restored upon integration over coordinates.]

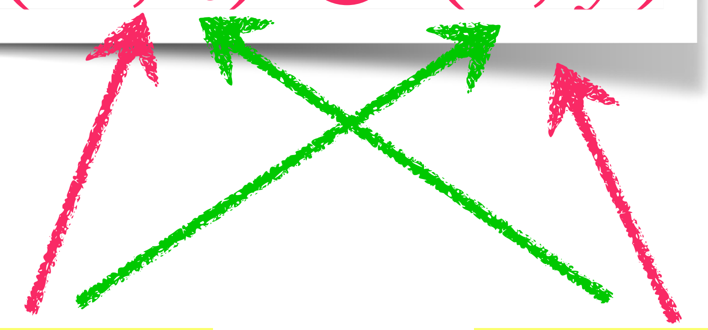
Have a look at the lifetime of such an "*anti-causal*" configuration: $\Delta t \propto \frac{1}{\Delta E} \sim \frac{1}{E}$

At very high energy vacuum fluctuations become ***vanishingly rare*** *and can be disregarded.*

The physical picture *simplifies significantly*. Effectively, **QFT** scenario reduces to a much simpler **QM** one!

Another interesting thing happening with dynamics of high energy interactions is that when particles collide along the **z** axis, our beloved 4-dimensional world effectively splits into two sub-spaces:

$$x^\mu = (ct, x, y, z) \implies (ct, z) \otimes (x, y)$$



More than just arithmetics $4 = 2 + 2$

COMPLEXITY OF DYNAMICS GETS HALVED : **SOFT** physics

curiously, it is ***the other way around*** *for the* **HARD** physics

SIMPLE SERIOUS

EXTRAS

Relativistic hadron has neither definite matter content nor definite geometrical profile in the impact parameter (transverse) plane.

Hadron is a quantum superposition of various components.

Due to the *relativistic suppression of vacuum fluctuations* with a time disorder the hadronic state can be described in terms of a multiparticle ("light-cone") wave function.
Each component has a certain number of constituents, each of which carries a certain fraction of the longitudinal momentum and has certain transverse position $(k_\perp)$.
Certain because during infinitesimal interaction time neither longitudinal momenta nor transverse coordinates of the constituents change.

High-energy interaction makes a momentary snapshot of the hadron.

What you see in this "photo" is a frozen configuration of one the virtual states belonging to the wave function of the relativistic hadron projectile.

It is clear that each state, given its specific content and configuration, interacts with the target with its proper intensity - cross section.
So what do we mean then when we refer to, say, **pp** cross section being 60 mb at ISR energy? (or about 100 mb at LHC)

- M. Good and W. Walker, Phys. Rev. D 120 (1960) 1857
- H. Miettinen and J. Pumplin, Phys. Rev. D 18 (1978) 1696

Hadron cross sections that we measure and talk about are ***average characteristics*** of strong interactions.

$$\int_0^\infty d\sigma\,\sigma\,P_h(\sigma) = \langle\sigma\rangle \equiv \sigma_{\rm tot}$$

An incident proton, or pion, interacts with ***larger*** or ***smaller*** cross section on event-by-event basis.

What one has to have in mind is a *distribution over cross sections*, specific for a given projectile hadron.

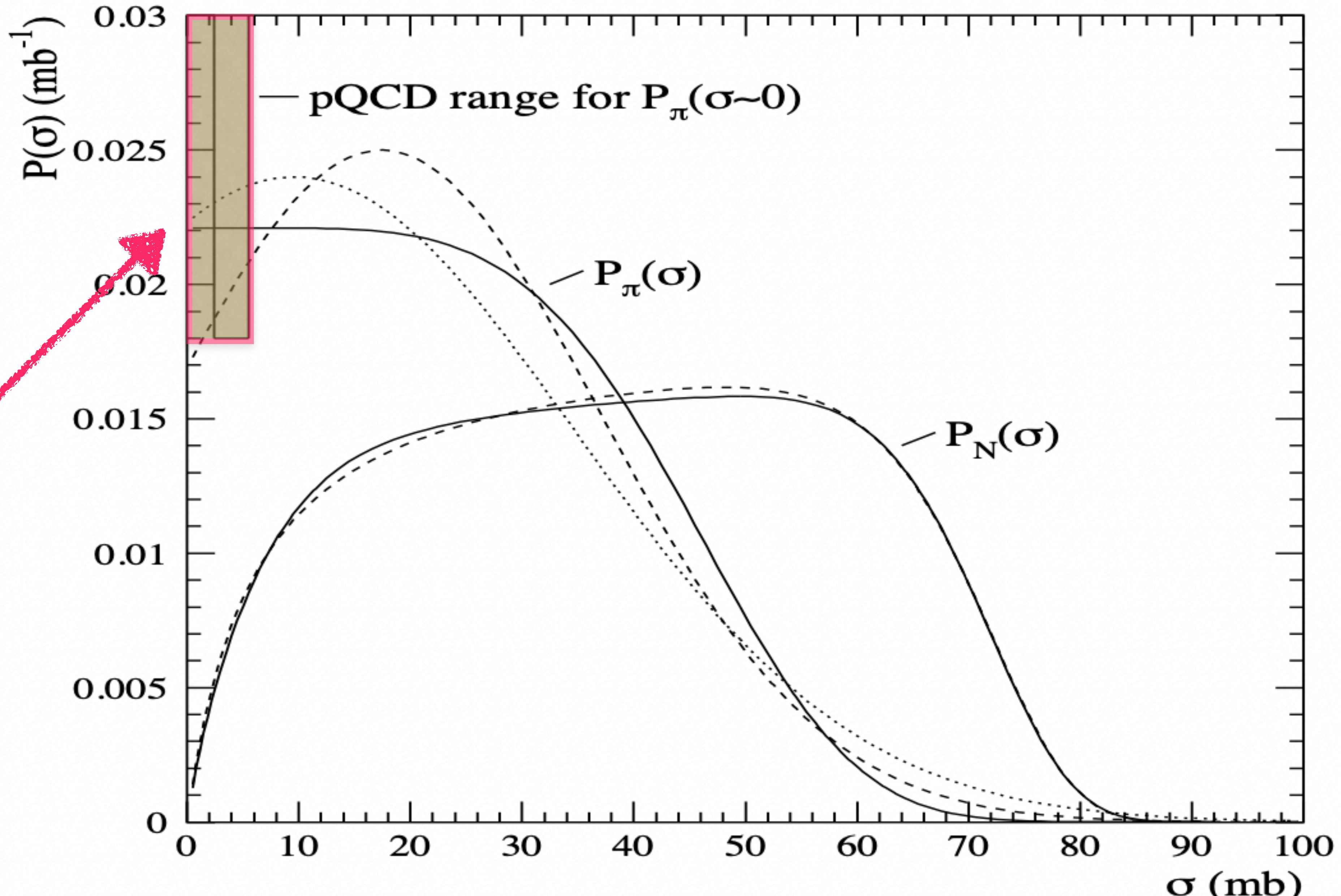
Such distributions can indeed be drawn. We do not know them with certainty.
At the same time, some global features can be firmly established.

1. dispersion of the distribution is related to *forward inelastic diffraction*

$$\sum_{h'\neq h} \left.\frac{d\sigma^{h\to h'}_{\rm diffr.}}{dt}\right|_{t=0} = \frac{\langle\sigma^2\rangle - \langle\sigma\rangle^2}{16\pi}$$

2. small-sigma limit can be estimated by means of pQCD.
   Quarks sitting close together form a colourless weakly interacting configuration.
   Hadron matter becomes transparent w.r.t. to such fluctuations.

Colour Transparency

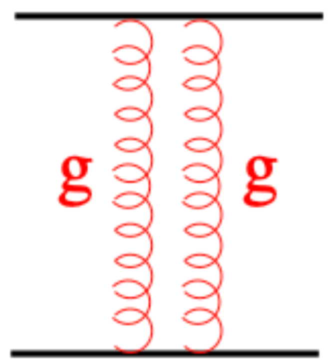


Take a pion.

(I promised not to refer to QCD but I could not possibly hide from you that hadrons do contain quarks and gluons)

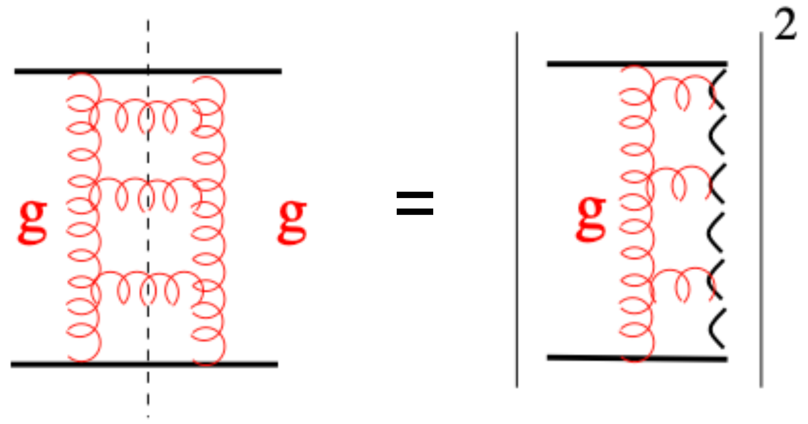
Its light-cone wave function contains various quark-gluon states $|\pi\rangle = |q\bar{q}\rangle + |q\bar{q}g\rangle + |q\bar{q}gg\rangle + \ldots$

Low-Nussinov model of the Pomeron

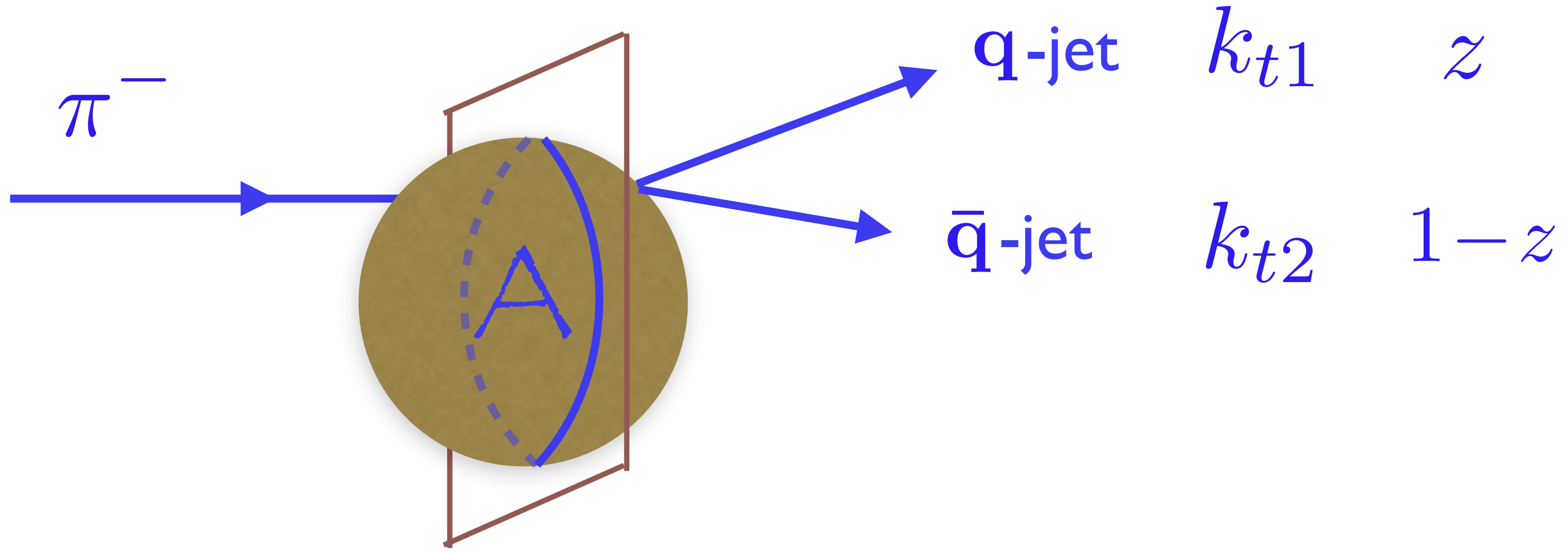


Gluon component is important: it is "internal" gluons that make hadrons interact with large (not falling with energy) cross section :

2 Im f_{el} =



(briefly, this is the picture of how colour exchange mediated by t-channel gluon results in formation of multi-hadron final states)



Let energetic pion hit a heavy nucleus.

It would normally break up the target

$$\sigma^{tot}_{\pi A} \propto R^2 \propto A^{2/3}$$

Experimental "measurement" of the pion w. f.

$$\frac{d\sigma}{dz} \simeq c\, z^2(1-z)^2 \propto |\Psi_{\pi\to q\bar{q}}(z)|^2$$

!

Should we want the target nucleus to *stay intact* (diffraction), textbooks would teach us that the probability of such eventuality decays *exponentially* with increasing nucleus size :

$$w(z) = \exp\left\{-\frac{z}{\lambda}\right\} = \exp\left\{-\sigma^{tot}_{\pi N}\,\rho_N \cdot z\right\}$$

mean free path *nucleon density in nucleus*

However we are now aware that the notion of the *total pion-nucleon Xsection* is dubious (to say the least)

There are different configurations of constituents in the pion, and some may have considerable *penetrating power*.

A quark and an antiquark sitting close have a good chance to be the one: their colour fields cancel!

All configurations that interact strongly get ***filtered out*** by the big nucleus. Only ***penetrating*** pion survives: *squeezed quark pair* = small colour dipole.

Its interaction being small, the scattering amplitude becomes coherent : $\mathcal{M}^{coh}_{\pi N} \propto A, \quad \frac{d\sigma^{coh}}{dt} \propto A^2$

Expected

$$d\sigma^{coh} = \int_{|t|\sim R_A^{-2}} dt\,\frac{d\sigma^{coh}}{dt} \propto \frac{A^2}{A^{2/3}} = A^{4/3}$$

Observed (Fermilab E791)

$$d\sigma^{coh}_{diffr.} \propto A^{1.6}$$

Will one see a pion on the backside of the target?

Hardly. Admixture of a small-dipole component in a normal pion is tiny.

The most probable outcome - production of two hadron jets emanating from the quark and the antiquark.

Size of the dipole $\simeq 1/k_t$. Selection of final state jets with $k_t > 1.5\,\mathrm{GeV}$ guaranteed that the squeezed pion propagated as a compact dipole all the way through the nucleus.

One of many a bright manifestation of Colour Transparency