Precise QCD+QED resummed predictions

For the Drell-Yan process





Leandro Cieri

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MWDays23 Workshop



Precise resummed QCD+QED predictions

- Motivation
- Brief TH intro
- QCD resummation at N4LLa [DYTurbo]

[2303.12781, 2202.10343, 2111.14509, 2103.04974, 1910.07049]

• QED+QCD resummation at NLL [DYqT]

[2302.05403, 1805.11948]

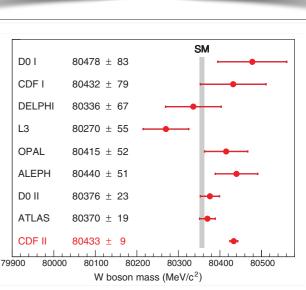
Outlook

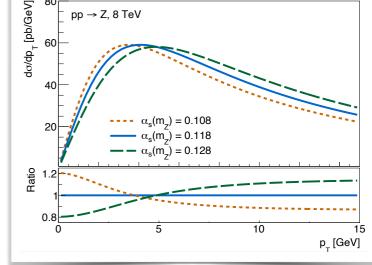
Motivation

Drell-Yan process

- Standard candle for precision measurements and theory at the LHC
- Detector calibration
- Extraction of PDFs
- Precise measurement of the strong coupling







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Data at small transverse momentum is very relevant

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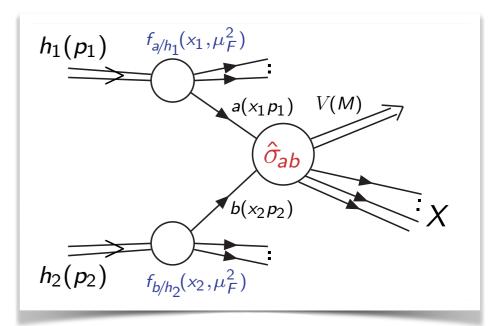
Transverse momentum resummation

Data at small transverse momentum is very relevant

QCD qT resummation at N4LLa [DYTurbo]

Transverse momentum resummation

Camarda, LC, Ferrera [2023]



$$h_1 + h_2 \to V + X \to l_3 + l_4 + X$$

up to N4LL+N4LO accuracy

 $\frac{d\sigma_{h_1h_2\to l_3l_4}}{d^2\mathbf{q_T}dM^2dyd\Omega}(\mathbf{q_T}, M^2, y, s, \Omega) = \sum_{a_1, a_2} \int_0^1 dx_1 \int_0^1 dx_2 f_{a_1/h_1}(x_1, \mu_F^2) f_{a_2/h_2}(x_2, \mu_F^2) \frac{d\hat{\sigma}_{a_1a_2\to l_3l_4}}{d^2\mathbf{q_T}\,dM^2\,d\hat{y}\,d\Omega}$

• $f_{a/h_1}(x_1, \mu_F^2)$: Non perturbative universal parton densities (PDFs), $\mu_F \sim M$.

 σ_{ab} : Hard scattering cross section. Process dependent, calculable with a perturbative expansion in the strong coupling $\alpha_s(M)$ (M $\gg \Lambda_{QCD} \sim 1$ GeV).

This framework relies in the QCD factorization property of the cross sections
 Collins, Soper, Sterman [1988]
 Aybat, Sterman [2008]

Transverse momentum resummation

Camarda, LC, Ferrera [2023]

$h_{1}(p_{1}) \xrightarrow{f_{a/h_{1}}(x_{1},\mu_{F}^{2})} \xrightarrow{a(x_{1}p_{1}) V(M)} \xrightarrow{a(x_{1}p_{1}) V(M)} \xrightarrow{f_{a/h_{1}}(x_{2},\mu_{F}^{2})} \xrightarrow{f_{b/h_{2}}(x_{2},\mu_{F}^{2})} \xrightarrow{f$

$$h_1 + h_2 \to V + X \to l_3 + l_4 + X$$

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• $f_{a/h_1}(x_1, \mu_F^2)$: Non perturbative universal parton densities (PDFs), $\mu_F \sim M$.

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Beware! violation of strict collinear factorization beyond N3LO (two loop amplitudes with 5 external legs and $n \ge 4$ QCD partons) Catani, de Florian, Rodrigo [2011] Forshaw, Seymour, Siodmok [2012]

Transverse momentum resummation

up to N4LL+N4LO accuracy

The cross section can be decomposed as

$$[d\hat{\sigma}_{a_1a_2 \to l_3l_4}] = [d\hat{\sigma}_{a_1a_2 \to l_3l_4}^{(\text{res.})}] + [d\hat{\sigma}_{a_1a_2 \to l_3l_4}^{(\text{fin.})}]$$

$$\left[d\hat{\sigma}_{a_{1}a_{2}\to l_{3}l_{4}}^{(\text{res.})}\right] = \sum_{b_{1},b_{2}=q,\bar{q}} \frac{d\hat{\sigma}_{b_{1}b_{2}\to l_{3}l_{4}}^{(0)}}{d\mathbf{\Omega}} \frac{1}{\hat{s}} \int_{0}^{\infty} \frac{db}{2\pi} b J_{0}(bq_{T}) \mathcal{W}_{a_{1}a_{2},b_{1}b_{2}\to V}(b,M,\hat{y},\hat{s};\alpha_{S},\mu_{R}^{2},\mu_{F}^{2})$$

Bozzi, Catani, de Florian, Grazzini [2005] Bozzi, Catani, de Florian, Ferrera, Grazzini [2011] Catani, de Florian, Ferrera, Grazzini [2015] $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/Q) \times \exp{\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}}$

$$\mathcal{H}_V(\alpha_S) = H_V(\alpha_S) \, C(\alpha_S) \, C(\alpha_S)$$

$$\mathcal{H}_{V}(\alpha_{S}) = 1 + \sum_{n=1}^{\infty} \left(\frac{\alpha_{S}}{\pi}\right)^{n} \mathcal{H}_{V}^{(n)}$$
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Camarda, LC, Ferrera [2023]

 $h_1(p_1) \Longrightarrow$

 $h_2(p_2)$

= 0

Transverse momentum resummation

Camarda, LC, Ferrera [2023]

 $\lim_{Q_T \to 0} \int_0^{Q_T^2} dq_T^2 \left[\frac{d\hat{\sigma}_{Fab}^{(\text{fin.})}}{dq_T^2} \right]$

 $h_1(p_1) =$

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up to N4LL+N4LO accuracy

$$\mathcal{W}_{V}(b, M; \alpha_{S}, \mu_{R}^{2}, \mu_{F}^{2}) = \mathcal{H}_{V}(\alpha_{S}; M/\mu_{R}, M/\mu_{F}, M/Q) \times \exp\{\mathcal{G}(\alpha_{S}, L; M/\mu_{R}, M/Q)\}$$

Hard-virtual factor

$$\mathcal{H}_{V}(\alpha_{S}) = H_{V}(\alpha_{S}) C(\alpha_{S}) C(\alpha_{S})$$
Sudakov

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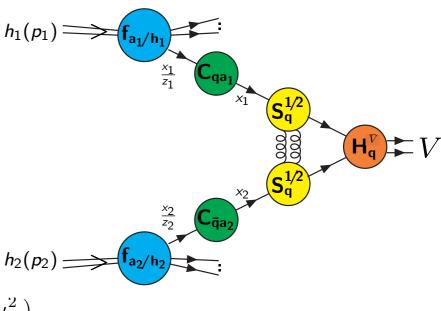
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Camarda, LC, Ferrera [2023]



We are interested in the impact of the resummation

 $L \equiv \ln(Q^2 b^2 / b_0^2)$

Missing N4LO f.o contribution

Without matching: artificial estimation of Q variation with +1 prescription

For a discussion about the f.o, resummation and the matching please see Alex and Tobias talks

Transverse momentum resummation

up to N4LL+N4LO accuracy

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$$\begin{aligned} \mathcal{H}_{V}^{(1)} &= H_{V}^{(1)} + C^{(1)} + C^{(1)}, \\ \mathcal{H}_{V}^{(2)} &= H_{V}^{(2)} + C^{(2)} + C^{(2)} + H_{V}^{(1)}(C^{(1)} + C^{(1)}) + C^{(1)}C^{(1)}, \\ \mathcal{H}_{V}^{(3)} &= H_{V}^{(3)} + C^{(3)} + C^{(3)} + H_{V}^{(2)}(C^{(1)} + C^{(1)}) + H_{V}^{(1)}(C^{(2)} + C^{(2)} + C^{(1)}C^{(1)}) \\ &+ C^{(2)}C^{(1)} + C^{(2)}C^{(1)}, \\ \mathcal{H}_{V}^{(4)} &= H_{V}^{(4)} + C^{(4)} + C^{(4)} + H_{V}^{(3)}(C^{(1)} + C^{(1)}) + H_{V}^{(2)}(C^{(2)} + C^{(2)} + C^{(1)}C^{(1)}) \\ &+ H_{V}^{(1)}(C^{(3)} + C^{(3)} + C^{(2)}C^{(1)} + C^{(2)}C^{(1)}) + C^{(3)}C^{(1)} + C^{(3)}C^{(1)} + C^{(2)}C^{(2)} \end{aligned}$$

Camarda, LC, Ferrera [2023] $h_1(p_1)$ f_{a_1/h_1} r_1 r_2 r_2

 $\left(\mathbf{H}_{\mathbf{q}}^{\overline{V}} \right)$

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Resummation scheme

Resummation scheme dependent statement!!!!

Partially known

$$\begin{aligned}
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&+ C^{(2)}C^{(1)} + C^{(2)}C^{(1)}, \\
\mathcal{H}_{V}^{(4)} &= H_{V}^{(4)} + C^{(4)} + C^{(4)} + H_{V}^{(3)}(C^{(1)} + C^{(1)}) + H_{V}^{(2)}(C^{(2)} + C^{(2)} + C^{(1)}C^{(1)}) \\
&+ H_{V}^{(4)}(C^{(3)} + C^{(3)} + C^{(2)}C^{(1)} + C^{(2)}C^{(1)}) + C^{(3)}C^{(1)} + C^{(3)}C^{(1)} + C^{(2)}C^{(2)}
\end{aligned}$$
Unknown

Transverse momentum resummation Camarda, LC, Ferrera [2023]

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up to N4LL+N4LO accuracy

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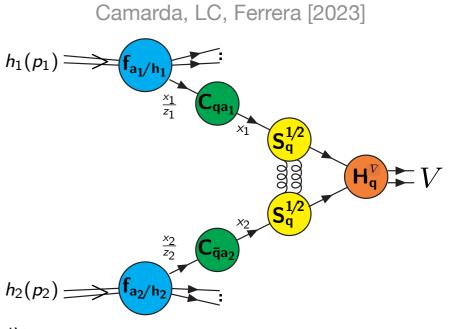
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$$\mathcal{H}_V(\alpha_S) = H_V(\alpha_S) \, C(\alpha_S) \, C(\alpha_S)$$

The $\delta(1-z)$ contribution can be computed from the four-loop quark form factor

$$\begin{split} \mathcal{H}_{V}^{(1)} &= H_{V}^{(1)} + C^{(1)} + C^{(1)}, \\ \mathcal{H}_{V}^{(2)} &= H_{V}^{(2)} + C^{(2)} + C^{(2)} + H_{V}^{(1)}(C^{(1)} + C^{(1)}) + C^{(1)}C^{(1)}, \\ \mathcal{H}_{V}^{(3)} &= H_{V}^{(3)} + C^{(3)} + C^{(3)} + H_{V}^{(2)}(C^{(1)} + C^{(1)}) + H_{V}^{(1)}(C^{(2)} + C^{(2)} + C^{(1)}C^{(1)}) \\ &+ C^{(2)}C^{(1)} + C^{(2)}C^{(1)}, \\ \mathcal{H}_{V}^{(4)} &= H_{V}^{(4)} + C^{(4)} + C^{(4)} + H_{V}^{(3)}(C^{(1)} + C^{(1)}) + H_{V}^{(2)}(C^{(2)} + C^{(2)} + C^{(1)}C^{(1)}) \\ &+ H_{V}^{(1)}(C^{(3)} + C^{(3)} + C^{(2)}C^{(1)} + C^{(2)}C^{(1)}) + C^{(3)}C^{(1)} + C^{(3)}C^{(1)} + C^{(2)}C^{(2)} \end{split}$$



Transverse momentum resummation

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 $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$

$$\mathcal{G}(\alpha_{S}, L) = -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left[A(\alpha_{S}(q^{2})) \ln \frac{M^{2}}{q^{2}} + \widetilde{B}(\alpha_{S}(q^{2})) \right]$$

$$= L g^{(1)}(\alpha_{S}L) + g^{(2)}(\alpha_{S}L) + \sum_{n=1}^{\infty} \left(\frac{\alpha_{S}}{\pi}\right)^{n} g^{(n+2)}(\alpha_{S}L)$$

g(n) controls and resums the $\alpha_{S}L^{k}$ (k $\geq 1)$ logarithmic terms

$$\widetilde{B}(\alpha_S) = B(\alpha_S) + 2\beta(\alpha_S) \frac{d\ln C(\alpha_S)}{d\ln \alpha_S} + 2\gamma(\alpha_S)$$

$$\lambda = \frac{1}{\pi} \beta_0 \,\alpha_S(\mu_R^2) \,L \ , \ \overline{B}^{(n)} = \widetilde{B}^{(n)} + A^{(n)} \ln \frac{M^2}{Q^2}$$

- At N4LL we need the resummation coefficients
- A5 : 1–3·10⁻³ relative uncertainty
- B4 : negligible uncertainty
- C4 : 1–2·10⁻³ relative uncertainty
- γ4 singlet : 1–3·10⁻³ relative uncertainty (non-singlet negligible)

$+\frac{A^{(4)}}{3\beta_0} \left(\frac{\beta_1}{\beta_0^2} \left[\frac{\lambda(-12+42\lambda-28\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right]\right]$
$+\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln\frac{Q^2}{\mu_R^2}\right)+\overline{B}^{(3)}\left(\frac{\beta_1}{\beta_0^2}\left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3}+\frac{\ln(1-\lambda)}{(1-\lambda)^3}\right]$
$+\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3}\ln\frac{Q^2}{\mu_R^2}\right) + A^{(3)}\left(-\frac{\beta_2}{4\beta_0^3}\frac{\lambda^3(4-\lambda)}{(1-\lambda)^4}\right)$
$+\frac{\beta_1^2}{\beta_0^4} \left[\frac{\lambda(12-24\lambda+52\lambda^2-13\lambda^3)}{36(1-\lambda)^4} + \frac{\ln(1-\lambda)}{3(1-\lambda)^3} + \frac{1-4\lambda}{2(1-\lambda)^4} \ln^2(1-\lambda) \right]$
$+\frac{\beta_1}{\beta_0^2} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln \frac{Q^2}{\mu_R^2}$
$-\frac{\lambda^2(6-4\lambda+\lambda^2)}{2(1-\lambda)^4}\ln^2\frac{Q^2}{\mu_R^2}\right) + \overline{B}^{(2)}\left(-\frac{\beta_2}{3\beta_0^2}\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^3}\frac{(3-\lambda)\lambda^2}{(3(1-\lambda)^3)^3}\right) + \frac{\beta_1^2}{\beta_0^3}\frac{(3-\lambda)\lambda^2}{(3(1-\lambda)^3)^3}$
$-\frac{\ln^2(1-\lambda)}{(1-\lambda)^3}\right) - \frac{2\beta_1}{\beta_0}\frac{\ln(1-\lambda)}{(1-\lambda)^3}\ln\frac{Q^2}{\mu_R^2} - \beta_0\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3}\ln^2\frac{Q^2}{\mu_R^2}\right)$
$+ A^{(2)} \left(-\frac{\beta_3}{12\beta_0^3} \frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3\beta_0^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{(1-\lambda)^4} \right)^{-1} + \frac{\beta_1\beta_2}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{(1-\lambda$
$+\frac{1-4\lambda+9\lambda^2}{(1-\lambda)^4}\ln(1-\lambda)\right)+\frac{\beta_1^3}{\beta_0^5}\left(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{36(1-\lambda)^4}\right)$
$-\frac{\left(1-4\lambda+9\lambda^{2}\right)}{3\left(1-\lambda\right)^{4}}\ln(1-\lambda)-\frac{\lambda}{\left(1-\lambda\right)^{4}}\ln^{2}(1-\lambda)-\frac{1-4\lambda}{3\left(1-\lambda\right)^{4}}\ln^{3}(1-\lambda)\right)$
$+\left[\frac{\beta_2}{3\beta_0^2}\frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4}+\frac{\beta_1^2}{\beta_0^3}\left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4}-\frac{2\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right.\right.$
$-\frac{1-4\lambda}{(1-\lambda)^4}\ln^2(1-\lambda)\bigg)\bigg]\ln\frac{Q^2}{\mu_R^2} + \frac{\beta_1}{\beta_0}\bigg[-\frac{\lambda}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda)\bigg]\ln^2\frac{Q^2}{\mu_R^2}$
$+ \frac{\beta_0}{3} \frac{\lambda^2 (6 - 4\lambda + \lambda^2)}{(1 - \lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) + \overline{B}^{(1)} \Biggl(- \frac{\beta_3}{6\beta_0^2} \frac{(3 - 2\lambda)\lambda^2}{(1 - \lambda)^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{(3 - 2\lambda)\lambda^2}{3(1 - \lambda)^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{(1 - \lambda)^3} \Bigr) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{(3 - 2\lambda)\lambda^2}{3(1 - \lambda)^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{(3 - 2\lambda)\lambda^2}{3(1 - \lambda)^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{3(1 - \lambda)^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{\beta_0^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{\beta_0^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigr) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{\beta_0^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{\beta_0^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigr) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl(\frac{\beta_1 \beta_2}{\beta_0^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigr) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigl) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigg) + \frac{\beta_1 \beta_2}{\beta_0^3} \Bigg$
$+\frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)\right)+\frac{\beta_1^3}{\beta_0^4}\bigg(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3}-\frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)-\frac{\ln^2(1-\lambda)}{2(1-\lambda)^3}\bigg)$
$+\frac{\ln^{3}(1-\lambda)}{3(1-\lambda)^{3}}\right) + \left\lfloor \frac{\beta_{2}}{\beta_{0}} \frac{\lambda}{(1-\lambda)^{3}} + \frac{\beta_{1}^{2}}{\beta_{0}^{2}} \left(-\frac{\lambda}{(1-\lambda)^{3}} - \frac{\ln(1-\lambda)}{(1-\lambda)^{3}} + \frac{\ln^{2}(1-\lambda)}{(1-\lambda)^{3}} \right) \right\rfloor \ln \frac{Q^{2}}{\mu_{R}^{2}}$
$+\beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right)$
$+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12 + 42\lambda - 52\lambda^2 + 7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \right) \right)$
$+\frac{\beta_4}{3\beta_0^3} \left(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda)\right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4}\ln(1-\lambda)\right) + \frac{\beta_1^2\beta_2}{\beta_0^5} \left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{36(1-\lambda)^4}\right)$
$-\frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) + $
$+\frac{\lambda^2(-3+2\lambda-2\lambda^2)}{3(1-\lambda)^4}\ln(1-\lambda)-\frac{(1-3\lambda)\lambda}{(1-\lambda)^4}\ln^2(1-\lambda)-\frac{1-6\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)$
$+\frac{1-4\lambda}{6(1-\lambda)^4}\ln^4(1-\lambda)\right) + \left[-\frac{\beta_3}{6\beta_0^2}\frac{\lambda^2(-3-2\lambda+2\lambda^2)}{(1-\lambda)^4} - \frac{\beta_1\beta_2}{\beta_0^3}\left(\frac{2\lambda^3}{3(1-\lambda)^3} + \frac{3\lambda^2}{(1-\lambda)^4}\ln(1-\lambda)\right)\right]$
$+\frac{\beta_1^3}{\beta_0^4}\bigg(-\frac{\lambda^2(3-2\lambda+2\lambda^2)}{6(1-\lambda)^4}-\frac{(1-3\lambda)\lambda}{(1-\lambda)^4}\ln(1-\lambda)-\frac{1-6\lambda}{2(1-\lambda)^4}\ln^2(1-\lambda)$
$+\frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\bigg)\bigg[\ln\frac{Q^2}{\mu_R^2}+\bigg[-\frac{3\beta_2}{2\beta_0}\frac{\lambda^2}{(1-\lambda)^4}+\frac{\beta_1^2}{2\beta_0^2}\bigg(-\frac{(1-3\lambda)\lambda}{(1-\lambda)^4}-\frac{(1-6\lambda)}{(1-\lambda)^4}\ln(1-\lambda)\bigg)\bigg]$
$+ \frac{(1-4\lambda)}{(1-\lambda)^4} \ln^2(1-\lambda) \bigg) \bigg] \ln^2 \frac{Q^2}{\mu_R^2} + \frac{\beta_1}{3} \bigg[\frac{\lambda(2+6\lambda-4\lambda^2+\lambda^3)}{2(1-\lambda)^4} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \bigg] \ln^3 \frac{Q^2}{\mu_R^2}$
$-\frac{\beta_0^2}{12}\frac{(6-4\lambda+\lambda^2)\lambda^2}{(1-\lambda)^4}\ln^4\frac{Q^2}{\mu_R^2}\right),\qquad\qquad g(5) \ still \ fits \ in \ a \ slide!$

Transverse momentum resummation

up to N4LL+N4LO accuracy

 $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$

$$\mathcal{G}(\alpha_{S}, L) = -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left[A(\alpha_{S}(q^{2})) \ln \frac{M^{2}}{q^{2}} + \widetilde{B}(\alpha_{S}(q^{2})) \right]$$

$$= L g^{(1)}(\alpha_{S}L) + g^{(2)}(\alpha_{S}L) + \sum_{n=1}^{\infty} \left(\frac{\alpha_{S}}{\pi}\right)^{n} g^{(n+2)}(\alpha_{S}L)$$

g(n) controls and resums the $\alpha_{S}L^{k}$ (k $\geq 1)$ logarithmic terms

$$\widetilde{B}(\alpha_S) = B(\alpha_S) + 2\beta(\alpha_S) \frac{d\ln C(\alpha_S)}{d\ln \alpha_S} + 2\gamma(\alpha_S)$$

$$\lambda = \frac{1}{\pi} \beta_0 \, \alpha_S(\mu_R^2) \, L \ , \ \overline{B}^{(n)} = \widetilde{B}^{(n)} + A^{(n)} \ln \frac{M^2}{Q^2}$$

- At N4LL we need the resummation coefficients
- (A5) 1–3.10⁻³ relative uncertainty
- B4 : negligible uncertainty
- C4: 1–2·10⁻³ relative uncertainty
- γ4 singlet : 1–3·10⁻³ relative uncertainty (non-singlet negligible)

$\alpha_S L) = -\frac{A^{(5)}}{12\beta_0^2} \frac{\lambda}{(1-\lambda)^4} - \frac{\overline{B}^{(4)}}{3\beta_0} \frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3}$
$+\frac{A^{(4)}}{3\beta_0} \left(\frac{\beta_1}{\beta_0^2} \left[\frac{\lambda(-12+42\lambda-28\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right]\right)$
$+\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln\frac{Q^2}{\mu_R^2}\right)+\overline{B}^{(3)}\left(\frac{\beta_1}{\beta_0^2}\left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3}+\frac{\ln(1-\lambda)}{(1-\lambda)^3}\right]\right)$
$+\frac{\lambda(3-3\lambda+\lambda^{2})}{(1-\lambda)^{3}}\ln\frac{Q^{2}}{\mu_{R}^{2}}\right)+A^{(3)}\left(-\frac{\beta_{2}}{4\beta_{0}^{3}}\frac{\lambda^{3}(4-\lambda)}{(1-\lambda)^{4}}\right)$
$+\frac{\beta_1^2}{\beta_0^4} \left[\frac{\lambda(12 - 24\lambda + 52\lambda^2 - 13\lambda^3)}{36(1-\lambda)^4} + \frac{\ln(1-\lambda)}{3(1-\lambda)^3} + \frac{1 - 4\lambda}{2(1-\lambda)^4} \ln^2(1-\lambda) \right]$
$+\frac{\beta_1}{\beta_0^2} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln \frac{Q^2}{\mu_R^2}$
$-\frac{\lambda^2(6-4\lambda+\lambda^2)}{2(1-\lambda)^4}\ln^2\frac{Q^2}{\mu_R^2}\right) + \overline{B}^{(2)}\left(-\frac{\beta_2}{3\beta_0^2}\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^3}\left(\frac{(3-\lambda)\lambda^2}{3(1-\lambda)^3}\right)\right)$
$-\frac{\ln^2(1-\lambda)}{(1-\lambda)^3}\bigg) - \frac{2\beta_1}{\beta_0}\frac{\ln(1-\lambda)}{(1-\lambda)^3}\ln\frac{Q^2}{\mu_R^2} - \beta_0\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3}\ln^2\frac{Q^2}{\mu_R^2}\bigg)$
$+ A^{(2)} \Bigg(- \frac{\beta_3}{12\beta_0^3} \frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3\beta_0^4} \bigg(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \bigg) + \frac{\beta_1\beta_2}{6(1-\lambda)^4} \bigg) \bigg) + \frac{\beta_1\beta_2}{6(1-\lambda)^4} \bigg) $
$+\frac{1-4\lambda+9\lambda^{2}}{(1-\lambda)^{4}}\ln(1-\lambda)\right)+\frac{\beta_{1}^{3}}{\beta_{0}^{5}}\left(\frac{\lambda(-12+42\lambda-64\lambda^{2}+25\lambda^{3})}{36(1-\lambda)^{4}}\right)$
$-\frac{(1-4\lambda+9\lambda^2)}{3(1-\lambda)^4}\ln(1-\lambda) - \frac{\lambda}{(1-\lambda)^4}\ln^2(1-\lambda) - \frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\bigg) \\ \left[\beta_{2\lambda}\left(3+4\lambda-\lambda^2\right)\lambda^2 - \beta_{2\lambda}^2\left(-(3+4\lambda-\lambda^2)\lambda^2\right) - 2\lambda\right] \\ + \beta_{2\lambda}\left(-(3+4\lambda-\lambda^2)\lambda^2\right) + \beta_$
$+ \left[\frac{\beta_2}{3\beta_0^2} \frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^3} \left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{2\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right)\right] = 0^2$
$-\frac{1-4\lambda}{(1-\lambda)^4}\ln^2(1-\lambda)\bigg) \left[\ln\frac{Q^2}{\mu_R^2} + \frac{\beta_1}{\beta_0} \left[-\frac{\lambda}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda) \right] \ln^2\frac{Q^2}{\mu_R^2} \right]$
$+\frac{\beta_0}{3}\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^3\frac{Q^2}{\mu_R^2}\right)+\overline{B}^{(1)}\left(-\frac{\beta_3}{6\beta_0^2}\frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3}+\frac{\beta_1\beta_2}{\beta_0^3}\left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3}\right)\right)$
$+\frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)\right) + \frac{\beta_1^3}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3}\right)$
$+\frac{\ln^{3}(1-\lambda)}{3(1-\lambda)^{3}}\right) + \left\lfloor\frac{\beta_{2}}{\beta_{0}}\frac{\lambda}{(1-\lambda)^{3}} + \frac{\beta_{1}^{2}}{\beta_{0}^{2}}\left(-\frac{\lambda}{(1-\lambda)^{3}} - \frac{\ln(1-\lambda)}{(1-\lambda)^{3}} + \frac{\ln^{2}(1-\lambda)}{(1-\lambda)^{3}}\right)\right\rfloor \ln\frac{Q^{2}}{\mu_{R}^{2}}$
$+\beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right)$
$+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12 + 42\lambda - 52\lambda^2 + 7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \right) \right)$
$+\frac{\beta_4}{3\beta_0^3} \left(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda)\right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4}\right)$
$-\frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4}\ln(1-\lambda)\right)+\frac{\beta_1^2\beta_2}{\beta_0^5}\left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{36(1-\lambda)^4}\right)$
$-\frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right)$
$+\frac{\lambda^{2}(-3+2\lambda-2\lambda^{2})}{3(1-\lambda)^{4}}\ln(1-\lambda) - \frac{(1-3\lambda)\lambda}{(1-\lambda)^{4}}\ln^{2}(1-\lambda) - \frac{1-6\lambda}{3(1-\lambda)^{4}}\ln^{3}(1-\lambda)$
$+\frac{1-4\lambda}{6(1-\lambda)^4}\ln^4(1-\lambda)\bigg) + \left[-\frac{\beta_3}{6\beta_0^2}\frac{\lambda^2(-3-2\lambda+2\lambda^2)}{(1-\lambda)^4} - \frac{\beta_1\beta_2}{\beta_0^3}\bigg(\frac{2\lambda^3}{3(1-\lambda)^3} + \frac{3\lambda^2}{(1-\lambda)^4}\ln(1-\lambda)\bigg)\right]$
$+\frac{\beta_1^3}{\beta_0^4} \bigg(-\frac{\lambda^2 (3-2\lambda+2\lambda^2)}{6(1-\lambda)^4} - \frac{(1-3\lambda)\lambda}{(1-\lambda)^4} \ln(1-\lambda) - \frac{1-6\lambda}{2(1-\lambda)^4} \ln^2(1-\lambda)$
$+\frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\bigg)\bigg]\ln\frac{Q^2}{\mu_R^2}+\bigg[-\frac{3\beta_2}{2\beta_0}\frac{\lambda^2}{(1-\lambda)^4}+\frac{\beta_1^2}{2\beta_0^2}\bigg(-\frac{(1-3\lambda)\lambda}{(1-\lambda)^4}-\frac{(1-6\lambda)}{(1-\lambda)^4}\ln(1-\lambda)\bigg)\bigg]$
$+ \frac{(1-4\lambda)}{(1-\lambda)^4} \ln^2(1-\lambda) \bigg) \bigg] \ln^2 \frac{Q^2}{\mu_R^2} + \frac{\beta_1}{3} \bigg[\frac{\lambda(2+6\lambda-4\lambda^2+\lambda^3)}{2(1-\lambda)^4} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \bigg] \ln^3 \frac{Q^2}{\mu_R^2}$
$-\frac{\beta_0^2 \left(6-4\lambda+\lambda^2\right)\lambda^2}{(1-\lambda)^4} \ln^4 \frac{Q^2}{\mu_R^2}\right), \qquad \qquad g(5) \ still \ fits \ in \ a \ slide!$

Transverse momentum resummation

up to N4LL+N4LO accuracy

 $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$

$$\mathcal{G}(\alpha_{S}, L) = -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left[A(\alpha_{S}(q^{2})) \ln \frac{M^{2}}{q^{2}} + \widetilde{B}(\alpha_{S}(q^{2})) \right]$$

$$= L g^{(1)}(\alpha_{S}L) + g^{(2)}(\alpha_{S}L) + \sum_{n=1}^{\infty} \left(\frac{\alpha_{S}}{\pi}\right)^{n} g^{(n+2)}(\alpha_{S}L)$$

g(n) controls and resums the $\alpha_{S}L^{k}$ (k $\geq 1)$ logarithmic terms

$$\widetilde{B}(\alpha_S) = B(\alpha_S) + 2\beta(\alpha_S) \frac{d\ln C(\alpha_S)}{d\ln \alpha_S} + 2\gamma(\alpha_S)$$

$$\lambda = \frac{1}{\pi} \,\beta_0 \,\alpha_S(\mu_R^2) \,L \ , \ \overline{B}^{(n)} = \widetilde{B}^{(n)} + A^{(n)} \ln \frac{M^2}{Q^2}$$

- At N4LL we need the resummation coefficients
- (A5) 1–3.10⁻³ relative uncertainty
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- C4: 1–2·10⁻³ relative uncertainty
- γ4 singlet : 1–3·10⁻³ relative uncertainty (non-singlet negligible)

$\begin{split} sL_1 &= \left(\frac{4^{(0)}}{12k_3^2} \frac{\lambda}{(1-\lambda)^4} (1-\lambda)^4}{3k_3^2} (1-\lambda)^4 (1-\lambda)^3\right) \\ &+ \frac{4^{(0)}}{3k_5} \left(\frac{\beta_1}{k_5^2} \left[\frac{\lambda(-12+42\lambda-28\lambda^2+7\lambda^3)}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \right] \\ &+ \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln \frac{Q^2}{\mu_R^2} \right) + \overline{B}^{(3)} \left(\frac{\beta_1}{3k_5^2} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \\ &+ \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln \frac{Q^2}{\mu_R^2} \right) + A^{(3)} \left(-\frac{\beta_2}{4k_5^2} \frac{\lambda^3(4-\lambda)}{(1-\lambda)^4} \right) \\ &+ \frac{\beta_1^2}{(1-\lambda)^4} \left[\frac{\lambda(12-24\lambda+22\lambda^2-13\lambda^3)}{3(1-\lambda)^4} + \frac{\ln(1-\lambda)}{3(1-\lambda)^3} + \frac{1-4\lambda}{2(1-\lambda)^4} \ln^2(1-\lambda) \right] \\ &+ \frac{\beta_1^2}{\beta_1^2} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln \frac{Q^2}{\mu_R^2} \\ &- \frac{\lambda^2(6-4\lambda+\lambda^2)}{2(1-\lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{\beta_1^2}{(1-\lambda)^3} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{\mu_R^2} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln \frac{Q^2}{\mu_R^2} \\ &- \frac{\lambda^2(6-4\lambda+\lambda^2)}{2(1-\lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{B^{(2)}}{(1-\lambda)^3} \left(-\frac{\beta_2}{\mu_R^2} \frac{\lambda^3(4-\lambda)}{\mu_R^2} \right) \\ &- \frac{\lambda^2(6-4\lambda+\lambda^2)}{2(1-\lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{B^{(2)}}{(1-\lambda)^3} \left(-\frac{\beta_2}{\mu_R^2} \frac{\lambda^3(4-\lambda)}{\mu_R^2} \right) \\ &- \frac{\beta_1^2}{(1-\lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{B^{(2)}}{(1-\lambda)^4} \left(-\frac{\beta_2}{\mu_R^2} \frac{\lambda^3(4-\lambda)}{\mu_R^2} \right) \\ &- \frac{B^{(2)}}{(1-\lambda)^4} \ln^2 \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{A^{(2)}}{(1-\lambda)^4} \left(-\frac{\beta_2}{\mu_R^2} \frac{\lambda^3(4-\lambda)}{\mu_R^2} \right) \\ &+ \frac{A^{(2)}}{(1-\lambda)^4} \left(-\frac{\beta_2}{\mu_R^2} \frac{\lambda^3(4-\lambda)}{\mu_R^2} \right) \\ &+ \frac{A^{(2)}}{(1-\lambda)^4} \ln^2 \ln(1-\lambda) \right) \\ &+ \left[\frac{\beta_2}{k_5^2} \frac{\lambda^3(4-\lambda)}{(1-\lambda)^4} \ln^2 \ln^2 \frac{\lambda^3}{\mu_R^2} \right] \\ &+ \frac{B^{(3)}}{(1-\lambda)^4} \ln^2 \ln^2 \frac{\lambda^3}{\mu_R^2} \right] \\ &+ \frac{B^{(3)}}{3(1-\lambda)^4} \ln^2 \ln^2 \frac{\lambda^3}{\mu_R^2} \right) \\ &+ \frac{B^{(3)}}{3(1-\lambda)^4} \ln^2 \ln^2 \frac{\lambda^3}{\mu_R^2} \right) \\ &+ \frac{B^{(3)}}{k_5^3} \left(-\frac{\lambda^3}{(1-\lambda)^3} + \frac{B^{(3)}}{\mu_R^2} \right) \\ &+ \frac{B^{(3)}}{k_5^3} \left(-\frac{\lambda^3}{(1-\lambda)^3} + \frac{B^{(3)}}{\mu_R^2} \right) \\ &+ \frac{B^{(3)}}{(1-\lambda)^4} \ln^2 \ln^2 \frac{\lambda^3}{\mu_R^2} \right] \\ \\ &+ \frac{B^{(3)}}{3(1-\lambda)^3} \ln(1-\lambda) \right) \\ &+ \frac{B^{(3)}}{\mu_R^2} \left(-\frac{\lambda^3}{(1-\lambda)^3} + \frac{B^{(3)}}{\mu_R^2} \right) \\ \\ &+ \frac{B^{(3)}}{k_5^3} \left(-\frac{\lambda^3}{(1-\lambda)^3} + \frac{B^{(3)}}{\mu_R^2} \right) \\ \\ &+ \frac{B^{(3)}}{k_5^3} \left(-\frac{\lambda^3}{(1-\lambda)^3} + \frac{B^{(3)}}{\mu_R^2} \right) \\ \\ &+ \frac{B^{(3)}}{k_5^3} \left(-\frac{\lambda^3}{(1-\lambda)^3} + \frac{B^{(3)}}{$
$\begin{split} &+\frac{\lambda^{2}(6-4\lambda+\lambda^{2})}{(1-\lambda)^{4}}\ln\frac{Q^{2}}{\mu_{R}^{2}}\right) +\overline{B}^{(3)}\left(\frac{\beta_{L}}{\beta_{0}^{2}}\left[\frac{\lambda(3-3\lambda+\lambda^{2})}{3(1-\lambda)^{3}}+\frac{\ln(1-\lambda)}{(1-\lambda)^{3}}\right] \\ &+\frac{\lambda(3-3\lambda+\lambda^{2})}{(1-\lambda)^{3}}\ln\frac{Q^{2}}{\mu_{R}^{2}}\right) +A^{(3)}\left(-\frac{\beta_{2}}{4\beta_{0}^{3}}\frac{\lambda^{3}(4-\lambda)}{(1-\lambda)^{4}}\right) \\ &+\frac{\beta_{L}^{2}}{\beta_{0}^{2}}\left[\frac{\lambda(12-24\lambda+52\lambda^{2}-13\lambda^{3})}{3(1-\lambda)^{4}}+\frac{\ln(1-\lambda)}{3(1-\lambda)^{4}}+\frac{1-4\lambda}{2(1-\lambda)^{4}}\ln^{2}(1-\lambda)\right] \\ &+\frac{\beta_{L}^{2}}{\beta_{0}^{2}}\left[\frac{\lambda(3-3\lambda+\lambda^{2})}{3(1-\lambda)^{4}}+\frac{1-4\lambda}{(1-\lambda)^{4}}\ln(1-\lambda)\right]\ln\frac{Q^{2}}{\mu_{R}^{2}} \\ &-\frac{\lambda^{2}(6-4\lambda+\lambda^{2})}{2(1-\lambda)^{4}}\ln^{2}\frac{Q^{2}}{\mu_{R}^{2}}\right) +\overline{B}^{(2)}\left(-\frac{\beta_{2}}{\beta_{0}^{2}}\frac{(3-\lambda)\lambda^{2}}{(1-\lambda)^{3}}+\frac{\beta_{1}^{2}}{\beta_{0}^{2}}\left(\frac{(3-\lambda)\lambda^{2}}{(1-\lambda)^{3}}\right) \\ &-\frac{\hbar^{2}(1-\lambda)}{(1-\lambda)^{4}}\ln^{2}\frac{Q^{2}}{\mu_{R}^{2}}\right) +\overline{B}^{(2)}\left(-\frac{\beta_{2}}{\beta_{0}^{2}}\frac{(3-\lambda)\lambda^{2}}{(1-\lambda)^{3}}+\frac{\beta_{1}^{2}}{\beta_{0}^{2}}\left(\frac{(3-\lambda)\lambda^{2}}{(1-\lambda)^{3}}\right) \\ &+\frac{\hbar^{2}(1-\lambda)}{(1-\lambda)^{4}}\ln^{2}\frac{Q^{2}}{\mu_{R}^{2}}\right) \\ &+\frac{\hbar^{2}(1-\lambda)}{(1-\lambda)^{4}}\ln^{2}\frac{\beta_{1}^{2}}{\mu_{R}^{2}}\right) +\overline{B}^{(2)}\left(-\frac{\beta_{2}}{\beta_{0}^{2}}\frac{(3-\lambda)\lambda^{2}}{(1-\lambda)^{3}}\ln^{2}\left(\frac{Q^{2}}{\mu_{R}^{2}}\right) \\ &+\frac{\hbar^{2}(1-\lambda)}{(1-\lambda)^{4}}\ln(1-\lambda)\right) +\frac{\beta_{1}^{3}}{\beta_{0}^{2}}\left(\frac{\lambda(6-21\lambda+44\lambda^{2}-20\lambda^{3})}{6(1-\lambda)^{4}}\right) \\ &+\frac{1-4\lambda+9\lambda^{2}}{(1-\lambda)^{4}}\ln(1-\lambda)\right) +\frac{\beta_{1}^{3}}{\beta_{0}^{2}}\left(\frac{\lambda(1-2+42\lambda-64\lambda^{2}+25\lambda^{3})}{6(1-\lambda)^{4}}\right) \\ &+\frac{1-4\lambda+9\lambda^{2}}{3(1-\lambda)^{4}}\ln(1-\lambda)\right) +\frac{\beta_{1}^{3}}{\beta_{0}^{2}}\left(\frac{-(3+4\lambda-\lambda^{2})\lambda^{2}}{3(1-\lambda)^{4}}-\frac{2\lambda}{(1-\lambda)^{4}}\ln^{2}(1-\lambda)\right) \\ &+\left[\frac{\beta_{2}}{\beta_{2}^{3}}\frac{(3+4\lambda-\lambda^{2})\lambda^{2}}{(1-\lambda)^{4}}+\frac{\beta_{1}^{2}}{\beta_{0}^{2}}\left(-\frac{(3+4\lambda-\lambda^{2})\lambda^{2}}{3(1-\lambda)^{4}}-\frac{2\lambda}{(1-\lambda)^{4}}\ln(1-\lambda)\right)\ln^{2}\frac{Q^{2}}{\mu_{R}^{2}}\right] \\ &+\frac{\beta_{1}}(-\lambda)^{2}(6-4\lambda+\lambda^{2})\ln^{3}}{1(1-\lambda)}\ln^{3}\frac{Q^{2}}{\mu_{R}^{2}}\right) +\overline{B}^{(1)}\left(-\frac{\beta_{3}}{\beta_{0}^{2}}\frac{(3-2\lambda)\lambda^{2}}{(1-\lambda)^{4}}\ln(1-\lambda)\right)\ln^{2}\frac{Q^{2}}{\mu_{R}^{2}}} \\ &+\frac{\beta_{1}^{3}}(6-4\lambda+\lambda^{2})\ln^{3}}{1(1-\lambda)^{4}}\ln^{3}\frac{Q^{2}}{\mu_{R}^{2}}\right) +\overline{B}^{(1)}\left(-\frac{\beta_{1}^{3}}{(1-\lambda)^{4}}\ln(1-\lambda)\right)\ln^{3}\frac{Q^{2}}{\mu_{R}^{2}}} \\ &+\frac{\beta_{1}^{3}}(6-4\lambda+\lambda^{2})h^{2}(1-\lambda)^{4}}h^{2}(1-\lambda)^{4}}{h^{2}}h^{2}\left(-\frac{\lambda^{2}}{(1-\lambda)^{4}}h^{2}(1-\lambda)\right)\ln^{3}\frac{Q^{2}}{\mu_{R}^{2}}} \\ &+\frac{\beta_{1}^{3}}(6-4\lambda+\lambda^{2})h^{2}}{h^{3}}\left(-\frac{\beta_{1}^{3}}{h^{3}}\left(-\frac{\beta_{1}^$
$\begin{aligned} + \frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3} \ln \frac{Q^2}{\mu_R^2} + A^{(3)} \left(-\frac{\beta_2}{4k_0^3} \frac{\lambda^3(4-\lambda)}{(1-\lambda)^4} + \frac{\beta_1^2}{3(1-\lambda)^4} \ln^2(1-\lambda) \right] \\ + \frac{\beta_1^2}{\beta_0^4} \left[\frac{\lambda(12-24\lambda+52\lambda^2-13\lambda^3)}{3(1-\lambda)^4} + \frac{\ln(1-\lambda)}{3(1-\lambda)^3} + \frac{1-4\lambda}{2(1-\lambda)^4} \ln^2(1-\lambda) \right] \\ + \frac{\beta_1^2}{\beta_0^4} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln \frac{Q^2}{\mu_R^2} \\ - \frac{\lambda^2(6-4\lambda+\lambda^2)}{3(1-\lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right] + \overline{B}^{(2)} \left(-\frac{\beta_2}{3k_0^2} \frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3-\lambda)\lambda^2}{3(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \right) \right) \\ + A^{(2)} \left(-\frac{\beta_2}{12k_0^3} \frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3k_0^2} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} + \frac{1-(4\lambda+9\lambda^2)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3k_0^2} \left(\frac{\lambda(1-21+44\lambda^2-20\lambda^3)}{3(1-\lambda)^4} + \frac{1-(4\lambda+9\lambda^2)}{(1-\lambda)^4} \ln(1-\lambda) - \frac{1-4\lambda}{3(1-\lambda)^4} \ln^3(1-\lambda) \right) \\ + \left[\frac{\beta_2}{3k_0^2} \left(\frac{3+4\lambda-\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4} \ln^3(1-\lambda) \right) \right] \ln^2 \frac{Q^2}{\mu_R^2} \right] \\ + \frac{\beta_0}{3} \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3} - \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln^2 \frac{Q^2}{\mu_R^2} \\ + \frac{\beta_0}{3} \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ + \frac{\beta_1(1-\lambda)}{3(1-\lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) + \overline{B}^{(1)} \left(-\frac{\beta_3}{63} \frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^2} \left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3} \right) \\ + \frac{\lambda}{3(1-\lambda)^3} \ln(1-\lambda) \right) + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \right) \\ + \frac{\beta_1}{3(1-\lambda)^3} \left(\frac{\lambda}{12} + \frac{\lambda}{2\lambda} + \frac{\lambda}{\beta_0^2} \right) \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ + \lambda^{(1)} \left(\frac{\beta_2}{\beta_0^2} \left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^3} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_2}{\beta_0^2} \left(\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{12(1-\lambda)^4} \right) \\ + \frac{\beta_1\beta_0^2}{3\beta_0^2} \left(\frac{\lambda(12-2\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_2}{\beta_0^2} \left(\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{12(1-\lambda)^4} \right) \\ - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1\beta_0^2}{\beta_0^2} \left(\frac{\lambda(2-5\lambda)}{3(1-\lambda)^3} + \frac{\lambda}{\beta_0^2} \right) \\ + \frac{\lambda}{3\beta_0^2} \left(\frac{\lambda}{12-2\lambda+40\lambda^2-13\lambda$
$\begin{split} &+ \frac{\beta_1^2}{\beta_0^2} \left[\frac{\lambda(12 - 24\lambda + 52\lambda^2 - 13\lambda^3)}{3(1 - \lambda)^4} + \frac{\ln(1 - \lambda)}{3(1 - \lambda)^3} + \frac{1 - 4\lambda}{2(1 - \lambda)^4} \ln^2(1 - \lambda) \right] \\ &+ \frac{\beta_1}{\beta_0^2} \left[\frac{\lambda(3 - 3\lambda + \lambda^2)}{3(1 - \lambda)^3} + \frac{1 - 4\lambda}{(1 - \lambda)^4} \ln(1 - \lambda) \right] \ln \frac{Q^2}{\mu_R^2} \\ &- \frac{\lambda^2(6 - 4\lambda + \lambda^2)}{2(1 - \lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right] \\ &+ \overline{B^2}(2) \left(- \frac{\beta_2}{3\beta_0^2} \frac{(3 - \lambda)\lambda^2}{(1 - \lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3 - \lambda)\lambda^2}{(1 - \lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3 - \lambda)\lambda^2}{3(1 - \lambda)^3} \right) \right] \\ &- \frac{\ln^2(1 - \lambda)}{(1 - \lambda)^3} - \frac{2\beta_1}{\beta_0} \frac{\ln(1 - \lambda)}{(1 - \lambda)^3} \ln \frac{Q^2}{\mu_R^2} - \beta_0 \frac{\lambda(3 - 3\lambda + \lambda^2)}{(1 - \lambda)^3} \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ A^{(2)} \left(- \frac{\beta_2}{12\beta_0^2} \frac{\lambda^3(8 - 5\lambda)}{(1 - \lambda)^4} + \frac{\beta_1\beta_2}{\beta_0^2} \left(\frac{\lambda(-21\lambda + 44\lambda^2 - 20\lambda^3)}{6(1 - \lambda)^4} \right) \\ &+ \frac{1 - 4\lambda + 9\lambda^2}{(1 - \lambda)^4} \ln(1 - \lambda) \right) + \frac{\beta_1^2}{\beta_0^2} \left(\frac{\lambda(-12 + 42\lambda - 64\lambda^2 + 25\lambda^3)}{3(1 - \lambda)^4} \right) \\ &- \left(\frac{1 - 4\lambda + 9\lambda^2}{(1 - \lambda)^4} \ln(1 - \lambda) \right) - \frac{\lambda}{\beta_0^2} \left(\frac{\lambda(-12 + 42\lambda - 64\lambda^2 + 25\lambda^3)}{3(1 - \lambda)^4} \right) \\ &- \left(\frac{1 - 4\lambda + 9\lambda^2}{(1 - \lambda)^4} \ln(1 - \lambda) \right) + \frac{\beta_1^2}{\beta_0^2} \left(- \frac{(3 + 4\lambda - \lambda^2)\lambda^2}{3(1 - \lambda)^4} - \frac{2\lambda}{(1 - \lambda)^4} \ln^3(1 - \lambda) \right) \\ &+ \left[\frac{\beta_2}{3\beta_0^2} \frac{(3 + 4\lambda - \lambda^2)\lambda^2}{(1 - \lambda)^4} + \frac{\beta_1^2}{\beta_0^2} \left(- \frac{(3 - 4\lambda)\lambda^2}{3(1 - \lambda)^4} - \frac{1 - 4\lambda}{(1 - \lambda)^4} \ln(1 - \lambda) \right) \right] \ln^2 \frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3} \frac{\lambda^2(6 - 4\lambda + \lambda^2)}{(1 - \lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{\beta_0}{3} \frac{\lambda^2(6 - 4\lambda + \lambda^2)}{(1 - \lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{\beta_1}{(1 - \lambda)^3} \ln(1 - \lambda) + \frac{\beta_1^3}{\beta_0^2} \left(- \frac{(3 - 2\lambda)\lambda^2}{(1 - \lambda)^3} - \frac{\lambda}{(1 - \lambda)^3} \ln(1 - \lambda) - \frac{12(1 - \lambda)}{3(1 - \lambda)^3} \right) \\ &+ \frac{\beta_1}{3(1 - \lambda)^3} \right) + \left[\frac{\beta_2}{\beta_0} \frac{\lambda}{(1 - \lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(- \frac{\lambda}{(1 - \lambda)^3} - \frac{\ln(1 - \lambda)}{(1 - \lambda)^3} + \frac{\ln^2(1 - \lambda)}{(1 - \lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[- \frac{\lambda(3 - 3\lambda + \lambda^2)}{\beta_0^2} + \frac{\ln(1 - \lambda)}{1(\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3 - 3\lambda + \lambda^2)}{3(1 - \lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ \lambda(1 \left(\frac{\beta_2}{\beta_0^2} \left(\frac{\lambda(12 - 42\lambda + 40\lambda^2 - 13\lambda^3)}{1(\lambda)^3} + \ln(1 - \lambda) \right) + \frac{\beta_1\beta_0^3}{\beta_0^2} \left(- \frac{\lambda(2 - 5\lambda)}{3(1 - \lambda)^3} \frac{(3 - \lambda)^4}{\mu_R^2} \right) \\ &+ \frac{\lambda(1 - \lambda)^3}{\lambda(1 - \lambda)^3} \frac{\lambda}{\lambda(1 - \lambda)^3} + \lambda(1 -$
$\begin{split} &+ \frac{\beta_1}{\beta_0^2} \left[\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right] \ln \frac{Q^2}{\mu_R^2} \\ &- \frac{\lambda^2(6-4\lambda+\lambda^2)}{2(1-\lambda)^4} \ln^2 \frac{Q^2}{\mu_R^2} \right] + \overline{B}^{(2)} \left(-\frac{\beta_2}{3\beta_0^2} \frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3-\lambda)\lambda^2}{3(1-\lambda)^3} \right) \\ &- \frac{\ln^2(1-\lambda)}{\beta_0} - \frac{2\beta_1}{\beta_0} \frac{\ln(1-\lambda)}{(1-\lambda)^3} \ln \frac{Q^2}{\mu_R^2} - \beta_0 \frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3} \ln^2 \frac{Q^2}{\mu_R^2} \right) \\ &+ A^{(2)} \left(-\frac{\beta_2}{12\beta_0^3} \frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{\beta_0^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right) \\ &+ \frac{1-4\lambda+9\lambda^2}{(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1^3}{\beta_0^6} \left(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{36(1-\lambda)^4} \right) \\ &- \left(\frac{1-4\lambda+9\lambda^2}{3(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1^3}{\beta_0^6} \left(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{3(1-\lambda)^4} - \frac{1-4\lambda}{3(1-\lambda)^4} \ln^3(1-\lambda) \right) \\ &+ \left[\frac{\beta_2}{3\beta_0^2} \frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4} \ln^3(1-\lambda) \right) \right] \ln^2 \frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3\beta_0^2} \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{\beta_1(1-\lambda)}{3(1-\lambda)^4} \ln^3(1-\lambda) \right) + \frac{\beta_1^3}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \right) \\ &+ \frac{\lambda^3(1-\lambda)}{3(1-\lambda)^3} \right) \\ &+ \left[\frac{\beta_2}{\beta_0^2} \left(\frac{\lambda(1-2+42\lambda+6\lambda^2-13\lambda^3)}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(\frac{(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\beta_1^2}{(1-\lambda)^3} - \frac{\lambda(1-\lambda)}{(1-\lambda)^3} - \frac{\ln(1-\lambda)}{(1-\lambda)^3} + \frac{\ln^2(1-\lambda)}{(1-\lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ A^{(1)} \left(\frac{\beta_2}{\beta_0^2} \left(\frac{\lambda(1-2+42\lambda+40\lambda^2-13\lambda^3)}{1(1-\lambda)^3} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{\beta_0^2} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1\beta_2}{\beta_0^2} \left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{\beta_0^2} \left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{10(1-\lambda)} - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1\beta_3}{\beta_0^2} \left(-\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^4} + \frac{\lambda(3-3\lambda^2+3\lambda^3)}{(1-\lambda)^4} - \frac{\lambda(3-3\lambda+\lambda^2)}$
$\begin{split} &-\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^2\frac{Q^2}{\mu_R^2}\right) + \overline{B}^{(2)}\left(-\frac{\beta_2}{3\beta_0^2}\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2}\left(\frac{(3-\lambda)\lambda^2}{(1-\lambda)^3}\right) \\ &-\frac{\ln^2(1-\lambda)}{(1-\lambda)^3}\right) - \frac{2\beta_1}{\beta_0}\frac{\ln(1-\lambda)}{(1-\lambda)^3}\ln\frac{Q^2}{\mu_R^2} - \beta_0\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3}\ln^2\frac{Q^2}{\mu_R^2}\right) \\ &+ A^{(2)}\left(-\frac{\beta_3}{12\beta_0^3}\frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3\beta_0^4}\left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4}\right) \\ &+\frac{1-4\lambda+9\lambda^2}{(1-\lambda)^4}\ln(1-\lambda)\right) + \frac{\beta_1^3}{\beta_0^5}\left(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{36(1-\lambda)^4}\right) \\ &-\frac{(1-4\lambda+9\lambda^2)}{(1-\lambda)^4}\ln(1-\lambda) - \frac{\lambda}{(1-\lambda)^4}\ln^2(1-\lambda) - \frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\right) \\ &+ \left[\frac{\beta_2}{3\beta_0^2}\frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^2}\left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{2\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right)\ln^2\frac{Q^2}{\mu_R^2}\right) \\ &+ \frac{\beta_0}{3\beta_0^2}\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^3\frac{Q^2}{\mu_R^2}\right) + \overline{B}^{(1)}\left(-\frac{\beta_3}{6\beta_0^2}\frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^2}\left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3}\right) \\ &+ \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)\right) + \frac{\beta_1^3}{\beta_0^2}\left(-\frac{(3-2\lambda)\lambda^2}{(6(1-\lambda)^3)} - \frac{\lambda(1-\lambda)}{(1-\lambda)^3}\ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3}\right) \\ &+ \frac{\lambda}{n^3(1-\lambda)^3}\right) + \left[\frac{\beta_2}{\beta_0}\frac{\lambda}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2}\left(-\frac{\lambda}{(1-\lambda)^3} - \frac{\ln(1-\lambda)}{3(1-\lambda)^3} + \frac{n^2(1-\lambda)}{2(1-\lambda)^3}\right)\right]\ln\frac{Q^2}{\mu_R^2} \\ &+ \beta_1\left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{1(1-\lambda)^3}\right]\ln^2\frac{Q^2}{\mu_R^2} + \beta_0^2\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3}\ln^3\frac{Q^2}{\mu_R^2}\right) \\ &+ A^{(1)}\left(\frac{\beta_2^2}{3\beta_0^4}\left(\frac{\lambda(-12+42\lambda+52\lambda^2+1\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda)\right) + \frac{\beta_1\beta_3}{6\beta_0^4}\left(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4}\right) \\ &- \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4}\ln(1-\lambda) + \frac{\beta_1\beta_3}{\beta_0^2}\left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{3(1-\lambda)^4}\right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^2}{2\beta_0^6}\left(-\frac{\lambda^3(2+3\lambda)}{3(1-\lambda)^4}\right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^2}{2\beta_0^6}\left(-\frac{\lambda^3(2+3\lambda)}{3(1-\lambda)^4}\right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^2}{2\beta_0^6}\left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^2}{2\beta_0^6}\left(-\frac{\lambda^3(2+3\lambda)}{3(1-\lambda)^4}\right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^2}{$
$\begin{split} &-\frac{\ln^2(1-\lambda)}{(1-\lambda)^3}\right) - \frac{2\beta_1}{\beta_0}\frac{\ln(1-\lambda)}{(1-\lambda)^3}\ln\frac{Q^2}{\mu_R^2} - \beta_0\frac{\lambda(3-3\lambda+\lambda^2)}{(1-\lambda)^3}\ln^2\frac{Q^2}{\mu_R^2}\right) \\ &+ A^{(2)}\bigg(-\frac{\beta_3}{12\beta_0^3}\frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3\beta_0^3}\bigg(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \\ &+ \frac{1-4\lambda+9\lambda^2}{(1-\lambda)^4}\ln(1-\lambda)\bigg) + \frac{\beta_1^3}{\beta_0^3}\bigg(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{36(1-\lambda)^4} \\ &- \frac{(1-4\lambda+9\lambda^2)}{(1-\lambda)^4}\ln(1-\lambda) - \frac{\lambda}{(1-\lambda)^4}\ln^2(1-\lambda) - \frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\bigg) \\ &+ \bigg[\frac{\beta_2}{3\beta_0^2}\frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^3}\bigg(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{2\lambda}{(1-\lambda)^4}\ln(1-\lambda)\bigg]\ln^2\frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3\beta_0^2}\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^3\frac{Q^2}{\mu_R^2}\bigg) + \overline{B}^{(1)}\bigg(-\frac{\beta_3}{6\beta_0^2}\frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^2}\bigg(\frac{(3-2\lambda)\lambda^2}{(3(1-\lambda)^3)} \\ &+ \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)\bigg) + \frac{\beta_1^2}{\beta_0^4}\bigg(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3}\bigg)\bigg]\ln\frac{Q^2}{\mu_R^2} \\ &+ \beta_1\bigg[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3}\bigg]\ln^2\frac{Q^2}{\mu_R^2} + \beta_0^2\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3}\ln^3\frac{Q^2}{\mu_R^2}\bigg) \\ &+ A^{(1)}\bigg(\frac{\beta_2^2}{3\beta_0^4}\bigg(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda)\bigg)\bigg) \\ &+ \frac{\beta_4}{3\beta_0^3}\bigg(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda)\bigg) + \frac{\beta_1\beta_3}{\beta_0^2}\bigg(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4}\bigg) \\ &- \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4}\ln(1-\lambda)\bigg) + \frac{\beta_1^2\beta_2}{2(1-\lambda)^4}\ln^2(1-\lambda)\bigg) + \frac{\beta_1^2\beta_0}{\beta_0^2}\bigg(-\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{3(1-\lambda)^4}\bigg) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^4}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\bigg) + \frac{\beta_1^4}{\beta_0^2}\bigg(-\frac{\lambda^3(2+3\lambda)}{3(1-\lambda)^4}\bigg) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\bigg) + \frac{\beta_1^4}{\beta_0^2}\bigg(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\bigg) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\bigg) + \frac{\beta_1^4}{\beta_0^2}\bigg(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\bigg) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\bigg) + \frac{\beta_1^4}{2\beta_0^6}\bigg(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\bigg) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\bigg) \\ &+ \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\bigg$
$\begin{aligned} &+A^{(2)} \left(-\frac{\beta_3}{12\beta_0^3} \frac{\lambda^3(8-5\lambda)}{(1-\lambda)^4} + \frac{\beta_1\beta_2}{3\beta_0^4} \left(\frac{\lambda(6-21\lambda+44\lambda^2-20\lambda^3)}{6(1-\lambda)^4} \right. \\ &+ \frac{1-4\lambda+9\lambda^2}{(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1^3}{\beta_0^5} \left(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{36(1-\lambda)^4} \right. \\ &- \left(\frac{1-4\lambda+9\lambda^2}{3(1-\lambda)^4} \ln(1-\lambda) - \frac{\lambda}{(1-\lambda)^4} \ln^2(1-\lambda) - \frac{1-4\lambda}{3(1-\lambda)^4} \ln^3(1-\lambda) \right) \\ &+ \left[\frac{\beta_2}{3\beta_0^2} \frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^3} \left(- \frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{2\lambda}{(1-\lambda)^4} \ln(1-\lambda) \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3\beta_0^2} \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) + \overline{B}^{(1)} \left(-\frac{\beta_3}{6\beta_0^2} \frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^3} \left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3} \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ \frac{\beta_1}{3(1-\lambda)^3} \ln(1-\lambda) \right) + \frac{\beta_1^3}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \right) \\ &+ \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) \right) + \frac{\beta_1^3}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \right) \\ &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} \\ &+ \beta_0 \frac{\lambda^2}{3\beta_0^4} \left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \right) \\ &+ \frac{\beta_4}{3\beta_0^3} \left(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{\beta_0^2} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} \right) \\ &- \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4} \ln(1-\lambda) \\ &+ \frac{\beta_1^2\beta_2}{\beta_0^2} \left(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{\beta_0^2} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} \right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^4}{\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{3(1-\lambda)^4} \right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^2\beta_0^2}{\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} \right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^4}{\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} \right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^2\beta_0^2}{3(1-\lambda)^4} \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln^2(1-\lambda) + \frac{\beta_1^2\beta_0^2}{3(1-\lambda)^4} \left(-\frac{\lambda^3(2+3\lambda)}{3(1-\lambda)^4} \right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^4} \ln^2(1$
$\begin{split} &+ \frac{1-(4)+9\lambda^2}{(1-\lambda)^4}\ln(1-\lambda) + \frac{\beta_0^3}{\beta_0^5} \left(\frac{\lambda(-12+42\lambda-64\lambda^2+25\lambda^3)}{36(1-\lambda)^4}\right) \\ &- \frac{(1-4\lambda+9\lambda^2)}{3(1-\lambda)^4}\ln(1-\lambda) - \frac{\lambda}{(1-\lambda)^4}\ln^2(1-\lambda) - \frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda) \\ &+ \left[\frac{\beta_2}{3\beta_0^2}\frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^3} \left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{2\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right)\right] \ln^2\frac{Q^2}{\mu_R^2} \\ &- \frac{1-4\lambda}{(1-\lambda)^4}\ln^2(1-\lambda) \right) \left] \ln\frac{Q^2}{\mu_R^2} + \frac{\beta_1}{\beta_0} \left[-\frac{\lambda}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right] \ln^2\frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3}\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^3\frac{Q^2}{\mu_R^2} + \overline{B}^{(1)} \left(-\frac{\beta_3}{6\beta_0^2}\frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^3} \left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3} + \frac{\lambda}{\beta_0^4}\left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3}\right) \\ &+ \frac{\lambda}{3(1-\lambda)}\ln(1-\lambda) + \frac{\beta_1^3}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{(1-\lambda)^3}\right) \right] \ln\frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3}\right] \ln^2\frac{Q^2}{\mu_R^2} + \beta_0^2\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3}\ln^3\frac{Q^2}{\mu_R^2}\right) \\ &+ A^{(1)} \left(\frac{\beta_2^3}{\beta\delta_0^4} \left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda)\right) + \frac{\beta_1\beta_3}{\delta\delta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)^4}\ln(1-\lambda)\right) + \frac{\beta_1\beta_3}{\beta\delta_0^5} \left(-\frac{\lambda(2-5\lambda)}{3(1-\lambda)^4}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) \end{split}$
$\begin{split} &-\frac{(1-4\lambda+9\lambda^2)}{3(1-\lambda)^4}\ln(1-\lambda)-\frac{\lambda}{(1-\lambda)^4}\ln^2(1-\lambda)-\frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\right)\\ &+\left[\frac{\beta_2}{3\beta_0^2}\frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4}+\frac{\beta_1^2}{\beta_0^3}\left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4}-\frac{2\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right)\right]\ln^2\frac{Q^2}{\mu_R^2}\\ &-\frac{1-4\lambda}{(1-\lambda)^4}\ln^2(1-\lambda)\right)\right]\ln\frac{Q^2}{\mu_R^2}+\frac{\beta_1}{\beta_0}\left[-\frac{\lambda}{(1-\lambda)^4}-\frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda)\right]\ln^2\frac{Q^2}{\mu_R^2}\\ &+\frac{\beta_0}{3}\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^3\frac{Q^2}{\mu_R^2}\right)+\overline{B}^{(1)}\left(-\frac{\beta_3}{6\beta_0^2}\frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3}+\frac{\beta_1\beta_2}{\beta_0^3}\left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3}\right)\\ &+\frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)\right)+\frac{\beta_1^3}{\beta_0^4}\left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3}-\frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda)-\frac{\ln^2(1-\lambda)}{2(1-\lambda)^3}\right)\\ &+\frac{\ln^3(1-\lambda)}{3(1-\lambda)^3}\right)+\left[\frac{\beta_2}{\beta_0}\frac{\lambda}{(1-\lambda)^3}+\frac{\beta_1^2}{\beta_0^2}\left(-\frac{\lambda}{(1-\lambda)^3}-\frac{\ln(1-\lambda)}{(1-\lambda)^3}+\frac{\ln^2(1-\lambda)}{(1-\lambda)^3}\right)\right]\ln\frac{Q^2}{\mu_R^2}\\ &+\beta_1\left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3}+\frac{\ln(1-\lambda)}{(1-\lambda)^3}\right]\ln^2\frac{Q^2}{\mu_R^2}+\beta_0^2\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3}\ln^3\frac{Q^2}{\mu_R^2}\right)\\ &+A^{(1)}\left(\frac{\beta_2^2}{3\beta_0^4}\left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4}-\ln(1-\lambda)\right)+\frac{\beta_1\beta_3}{6\beta_0^4}\left(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4}-\frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{12(1-\lambda)}\ln(1-\lambda)\right)+\frac{\beta_1\beta_3}{\beta_0^2}\left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{3(1-\lambda)^4}-\frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda)-\frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right)+\frac{\beta_1^4}{2\beta_0^6}\left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) \end{split}$
$\begin{split} &+ \left[\frac{\beta_2}{3\beta_0^2} \frac{(3+4\lambda-\lambda^2)\lambda^2}{(1-\lambda)^4} + \frac{\beta_1^2}{\beta_0^3} \left(-\frac{(3+4\lambda-\lambda^2)\lambda^2}{3(1-\lambda)^4} - \frac{2\lambda}{(1-\lambda)^4} \ln(1-\lambda)\right) - \frac{1-4\lambda}{(1-\lambda)^4} \ln^2(1-\lambda)\right] \ln^2 \frac{Q^2}{\mu_R^2} \\ &- \frac{1-4\lambda}{(1-\lambda)^4} \ln^2(1-\lambda) \right) \ln^2 \frac{Q^2}{\mu_R^2} + \frac{\beta_1}{\beta_0} \left[-\frac{\lambda}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda)\right] \ln^2 \frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3} \frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) + \overline{B}^{(1)} \left(-\frac{\beta_3}{6\beta_0^2} \frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^3} \left(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3} + \frac{\lambda}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \right) \right] \\ &+ \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) + \frac{\beta_1^2}{\beta_0^4} \left(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} - \frac{\ln(1-\lambda)}{(1-\lambda)^3} + \frac{\ln^2(1-\lambda)}{(1-\lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3}\right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2}\right) \\ &+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda)\right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2 - 10\lambda^3 + 4\lambda^4}{12(1-\lambda)^4} \ln(1-\lambda)\right) + \frac{\beta_1\beta_2}{\beta_0^5} \left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{3(1-\lambda)^4} - \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^4} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda)\right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda)\right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) \\ &+ \frac{\beta_1}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} + \frac{\beta_1}{2\beta_0^6} \right) \right) \\ &+ \frac{\beta_1}{\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} + \frac{\beta_1}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} + \frac{\beta_1}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} + \frac{\beta_1}{2\beta_0^6} \right) \right) \\ &+ \frac{\beta_1}{\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} + \frac{\beta_1}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} + \frac{\beta_1}{2\beta_0^6} + \frac{\beta_1}{2\beta_0^6} + \frac{\beta_1}{2\beta$
$\begin{split} &-\frac{1-4\lambda}{(1-\lambda)^4}\ln^2(1-\lambda)\Big) \Bigg] \ln\frac{Q^2}{\mu_R^2} + \frac{\beta_1}{\beta_0} \Bigg[-\frac{\lambda}{(1-\lambda)^4} - \frac{1-4\lambda}{(1-\lambda)^4}\ln(1-\lambda) \Bigg] \ln^2\frac{Q^2}{\mu_R^2} \\ &+ \frac{\beta_0}{3}\frac{\lambda^2(6-4\lambda+\lambda^2)}{(1-\lambda)^4}\ln^3\frac{Q^2}{\mu_R^2} \Bigg) + \overline{B}^{(1)} \Bigg(-\frac{\beta_3}{6\beta_0^2}\frac{(3-2\lambda)\lambda^2}{(1-\lambda)^3} + \frac{\beta_1\beta_2}{\beta_0^2} \Big(\frac{(3-2\lambda)\lambda^2}{3(1-\lambda)^3} \\ &+ \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda) \Big) + \frac{\beta_1^3}{\beta_0^4} \Big(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3}\ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \\ &+ \frac{\ln^3(1-\lambda)}{3(1-\lambda)^3} \Big) + \Bigg[\frac{\beta_2}{\beta_0}\frac{\lambda}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \Big(-\frac{\lambda}{(1-\lambda)^3} - \frac{\ln(1-\lambda)}{(1-\lambda)^3} + \frac{\ln^2(1-\lambda)}{(1-\lambda)^3} \Big) \Bigg] \ln\frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \Bigg[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \Bigg] \ln^2\frac{Q^2}{\mu_R^2} + \beta_0^2\frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3\frac{Q^2}{\mu_R^2} \Bigg) \\ &+ A^{(1)} \Bigg(\frac{\beta_2}{3\beta_0^4} \Big(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \Big) \\ &+ \frac{\beta_4}{3\beta_0^3} \Big(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \Big) + \frac{\beta_1\beta_3}{6\beta_0^4} \Big(-\frac{\lambda(2-5\lambda)}{3}\frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} \\ &- \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4} \ln(1-\lambda) \Bigg) + \frac{\beta_1^2\beta_2}{\beta_0^5} \Big(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{3(1-\lambda)^4} \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \Bigg) + \frac{\beta_1^4}{2\beta_0^6} \Big(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} \Bigg) \end{aligned}$
$\begin{aligned} &+ \frac{\beta_0}{3} \frac{\lambda^2 (6 - 4\lambda + \lambda^2)}{(1 - \lambda)^4} \ln^3 \frac{Q^2}{\mu_R^2} \right) + \overline{B}^{(1)} \left(-\frac{\beta_3}{6\beta_0^2} \frac{(3 - 2\lambda)\lambda^2}{(1 - \lambda)^3} + \frac{\beta_1 \beta_2}{\beta_0^3} \left(\frac{(3 - 2\lambda)\lambda^2}{3(1 - \lambda)^3} \right) \\ &+ \frac{\lambda}{(1 - \lambda)^3} \ln(1 - \lambda) \right) + \frac{\beta_1^3}{\beta_0^4} \left(-\frac{(3 - 2\lambda)\lambda^2}{6(1 - \lambda)^3} - \frac{\lambda}{(1 - \lambda)^3} \ln(1 - \lambda) - \frac{\ln^2(1 - \lambda)}{2(1 - \lambda)^3} \right) \\ &+ \frac{\ln^3(1 - \lambda)}{3(1 - \lambda)^3} \right) + \left[\frac{\beta_2}{\beta_0} \frac{\lambda}{(1 - \lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{\lambda}{(1 - \lambda)^3} - \frac{\ln(1 - \lambda)}{(1 - \lambda)^3} + \frac{\ln^2(1 - \lambda)}{(1 - \lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[-\frac{\lambda(3 - 3\lambda + \lambda^2)}{2(1 - \lambda)^3} + \frac{\ln(1 - \lambda)}{(1 - \lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3 - 3\lambda + \lambda^2)}{3(1 - \lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12 + 42\lambda - 52\lambda^2 + 7\lambda^3)}{12(1 - \lambda)^4} - \ln(1 - \lambda) \right) + \frac{\beta_1 \beta_3}{6\beta_0^4} \left(-\frac{\lambda(2 - 5\lambda)}{3} \frac{(3 - 3\lambda + \lambda^2)}{(1 - \lambda)^4} \right) \\ &- \frac{2 - 8\lambda + 9\lambda^2 - 10\lambda^3 + 4\lambda^4}{(1 - \lambda)^4} \ln(1 - \lambda) \right) + \frac{\beta_1^2 \beta_2}{\beta_0^5} \left(\frac{\lambda(12 - 42\lambda + 52\lambda^2 + 5\lambda^3)}{3(1 - \lambda)^4} \right) \\ &- \frac{(-1 + 3\lambda - 3\lambda^2 + 3\lambda^3)}{3(1 - \lambda)^3} \ln(1 - \lambda) - \frac{3\lambda^2}{2(1 - \lambda)^4} \ln^2(1 - \lambda) \right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2 + 3\lambda)}{6(1 - \lambda)^4} \right) \end{aligned}$
$\begin{aligned} &+ \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) \bigg) + \frac{\beta_1^3}{\beta_0^4} \bigg(-\frac{(3-2\lambda)\lambda^2}{6(1-\lambda)^3} - \frac{\lambda}{(1-\lambda)^3} \ln(1-\lambda) - \frac{\ln^2(1-\lambda)}{2(1-\lambda)^3} \\ &+ \frac{\ln^3(1-\lambda)}{3(1-\lambda)^3} \bigg) + \bigg[\frac{\beta_2}{\beta_0} \frac{\lambda}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \bigg(-\frac{\lambda}{(1-\lambda)^3} - \frac{\ln(1-\lambda)}{(1-\lambda)^3} + \frac{\ln^2(1-\lambda)}{(1-\lambda)^3} \bigg) \bigg] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \bigg[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \bigg] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \bigg) \\ &+ A^{(1)} \bigg(\frac{\beta_2^2}{3\beta_0^4} \bigg(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \bigg) \\ &+ \frac{\beta_4}{3\beta_0^3} \bigg(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \bigg) + \frac{\beta_1\beta_3}{6\beta_0^4} \bigg(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} \\ &- \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4} \ln(1-\lambda) \bigg) + \frac{\beta_1^2\beta_2}{\beta_0^5} \bigg(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{3(1-\lambda)^4} \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \bigg) + \frac{\beta_1^4}{2\beta_0^6} \bigg(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} \bigg) \end{aligned}$
$\begin{aligned} &+ \frac{\ln^3(1-\lambda)}{3(1-\lambda)^3} + \left[\frac{\beta_2}{\beta_0} \frac{\lambda}{(1-\lambda)^3} + \frac{\beta_1^2}{\beta_0^2} \left(-\frac{\lambda}{(1-\lambda)^3} - \frac{\ln(1-\lambda)}{(1-\lambda)^3} + \frac{\ln^2(1-\lambda)}{(1-\lambda)^3} \right) \right] \ln \frac{Q^2}{\mu_R^2} \\ &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \right) \right) \\ &+ \frac{\beta_4}{3\beta_0^3} \left(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} \right) \\ &- \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1^2\beta_2}{\beta_0^5} \left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{3(1-\lambda)^4} \right) \\ &- \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} \right) \end{aligned}$
$\begin{aligned} &+ \beta_1 \left[-\frac{\lambda(3-3\lambda+\lambda^2)}{2(1-\lambda)^3} + \frac{\ln(1-\lambda)}{(1-\lambda)^3} \right] \ln^2 \frac{Q^2}{\mu_R^2} + \beta_0^2 \frac{\lambda(3-3\lambda+\lambda^2)}{3(1-\lambda)^3} \ln^3 \frac{Q^2}{\mu_R^2} \right) \\ &+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12+42\lambda-52\lambda^2+7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \right) \right) \\ &+ \frac{\beta_4}{3\beta_0^3} \left(\frac{\lambda(12-42\lambda+40\lambda^2-13\lambda^3)}{12(1-\lambda)^4} + \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1^2\beta_2}{\beta_0^5} \left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{36(1-\lambda)^4} - \frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \frac{\lambda^3(2+3\lambda)}{2(1-\lambda)^4} + \frac{\lambda^4}{2\beta_0^6} - \frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \frac{\lambda^4}{2\beta_0^6} - \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{2\beta_0^6} + \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{2\beta_0^6} + \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{2\beta_0^6} + \frac{\lambda^4}{6(1-\lambda)^4} - \frac{\lambda^4}{6(1-\lambda$
$+ A^{(1)} \left(\frac{\beta_2^2}{3\beta_0^4} \left(\frac{\lambda(-12 + 42\lambda - 52\lambda^2 + 7\lambda^3)}{12(1-\lambda)^4} - \ln(1-\lambda) \right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2-5\lambda)}{3} \frac{(3-3\lambda+\lambda^2)}{(1-\lambda)^4} - \frac{2-8\lambda + 9\lambda^2 - 10\lambda^3 + 4\lambda^4}{(1-\lambda)^4} \ln(1-\lambda) \right) + \frac{\beta_1^2\beta_2}{\beta_0^5} \left(\frac{\lambda(12 - 42\lambda + 52\lambda^2 + 5\lambda^3)}{36(1-\lambda)^4} - \frac{(-1+3\lambda - 3\lambda^2 + 3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \frac{(-1+3\lambda - 3\lambda^2 + 3\lambda^3)}{3(1-\lambda)^3} \ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4} \ln^2(1-\lambda) \right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4} - \lambda^3(2+3$
$ + \frac{\beta_4}{3\beta_0^3} \left(\frac{\lambda(12 - 42\lambda + 40\lambda^2 - 13\lambda^3)}{12(1 - \lambda)^4} + \ln(1 - \lambda) \right) + \frac{\beta_1\beta_3}{6\beta_0^4} \left(-\frac{\lambda(2 - 5\lambda)}{3} \frac{(3 - 3\lambda + \lambda^2)}{(1 - \lambda)^4} - \frac{2 - 8\lambda + 9\lambda^2 - 10\lambda^3 + 4\lambda^4}{(1 - \lambda)^4} \ln(1 - \lambda) \right) + \frac{\beta_1^2\beta_2}{\beta_0^5} \left(\frac{\lambda(12 - 42\lambda + 52\lambda^2 + 5\lambda^3)}{36(1 - \lambda)^4} - \frac{(-1 + 3\lambda - 3\lambda^2 + 3\lambda^3)}{3(1 - \lambda)^3} \ln(1 - \lambda) - \frac{3\lambda^2}{2(1 - \lambda)^4} \ln^2(1 - \lambda) \right) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2 + 3\lambda)}{6(1 - \lambda)^4} - \frac{\lambda^3(2 - 3\lambda)}{6(1 - \lambda)^4} + \frac{\lambda^3(2 - 3\lambda)}{6(1 - \lambda)^4} - \frac{\lambda^3(2 - 3\lambda)}{2(1 - \lambda)^4} + \frac{\lambda^3(2 - 3\lambda)}{2(1 - \lambda)^4} + \frac{\lambda^3(2 - 3\lambda)}{6(1 - \lambda)^4} + \lambda^3(2 - $
$-\frac{2-8\lambda+9\lambda^2-10\lambda^3+4\lambda^4}{(1-\lambda)^4}\ln(1-\lambda)\right)+\frac{\beta_1^2\beta_2}{\beta_0^5}\left(\frac{\lambda(12-42\lambda+52\lambda^2+5\lambda^3)}{36(1-\lambda)^4}-\frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda)-\frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda)\right)+\frac{\beta_1^4}{2\beta_0^6}\left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}+\frac{\beta_1^4}{6(1-\lambda)^4}\right)$
$-\frac{(-1+3\lambda-3\lambda^2+3\lambda^3)}{3(1-\lambda)^3}\ln(1-\lambda) - \frac{3\lambda^2}{2(1-\lambda)^4}\ln^2(1-\lambda) + \frac{\beta_1^4}{2\beta_0^6} \left(-\frac{\lambda^3(2+3\lambda)}{6(1-\lambda)^4}\right) + \frac{\beta_1^4}{6(1-\lambda)^4} + \frac{\beta_1^4}{6(1-\lambda)^4}$
$+\frac{1-4\lambda}{6(1-\lambda)^4}\ln^4(1-\lambda)\bigg) + \Bigg[-\frac{\beta_3}{6\beta_0^2}\frac{\lambda^2(-3-2\lambda+2\lambda^2)}{(1-\lambda)^4} - \frac{\beta_1\beta_2}{\beta_0^3}\bigg(\frac{2\lambda^3}{3(1-\lambda)^3} + \frac{3\lambda^2}{(1-\lambda)^4}\ln(1-\lambda)\bigg)$
$+\frac{\beta_1^3}{\beta_0^4}\bigg(-\frac{\lambda^2(3-2\lambda+2\lambda^2)}{6(1-\lambda)^4}-\frac{(1-3\lambda)\lambda}{(1-\lambda)^4}\ln(1-\lambda)-\frac{1-6\lambda}{2(1-\lambda)^4}\ln^2(1-\lambda)$
$+\frac{1-4\lambda}{3(1-\lambda)^4}\ln^3(1-\lambda)\bigg)\bigg]\ln\frac{Q^2}{\mu_R^2}+\bigg[-\frac{3\beta_2}{2\beta_0}\frac{\lambda^2}{(1-\lambda)^4}+\frac{\beta_1^2}{2\beta_0^2}\bigg(-\frac{(1-3\lambda)\lambda}{(1-\lambda)^4}-\frac{(1-6\lambda)}{(1-\lambda)^4}\ln(1-\lambda)\bigg)\bigg]$
$+ \frac{(1-4\lambda)}{(1-\lambda)^4} \ln^2(1-\lambda) \bigg) \Bigg] \ln^2 \frac{Q^2}{\mu_R^2} + \frac{\beta_1}{3} \Bigg[\frac{\lambda(2+6\lambda-4\lambda^2+\lambda^3)}{2(1-\lambda)^4} + \frac{1-4\lambda}{(1-\lambda)^4} \ln(1-\lambda) \Bigg] \ln^3 \frac{Q^2}{\mu_R^2} + \frac{1-4\lambda}{2(1-\lambda)^4} + \frac{1-4\lambda}{2(1-\lambda)^4} + \frac{1-4\lambda}{2(1-\lambda)^4} + \frac{1-4\lambda}{2(1-\lambda)^4} + \frac{1-4\lambda}{2(1-\lambda)^4} + \frac{1-4\lambda}{2(1-\lambda)^4} + $
$-\frac{\beta_0^2}{12}\frac{(6-4\lambda+\lambda^2)\lambda^2}{(1-\lambda)^4}\ln^4\frac{Q^2}{\mu_R^2}\right),\qquad \qquad g(5) \text{ still fits in a slide!}$

Transverse momentum resummation Camarda, LC, Ferrera [2023] up to N4LL+N4LO accuracy

 $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$

$$\mathcal{G}(\alpha_{S}, L) = -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left[A(\alpha_{S}(q^{2})) \ln \frac{M^{2}}{q^{2}} + \widetilde{B}(\alpha_{S}(q^{2})) \right]$$

$$= L g^{(1)}(\alpha_{S}L) + g^{(2)}(\alpha_{S}L) + \sum_{n=1}^{\infty} \left(\frac{\alpha_{S}}{\pi}\right)^{n} g^{(n+2)}(\alpha_{S}L)$$

g(n) controls and resums the $\alpha_{S}L^{k}$ (k $\geq 1)$ logarithmic terms

$$\widetilde{B}(\alpha_S) = B(\alpha_S) + 2\beta(\alpha_S) \frac{d\ln C(\alpha_S)}{d\ln \alpha_S} + 2\gamma(\alpha_S)$$

$$\lambda = \frac{1}{\pi} \beta_0 \, \alpha_S(\mu_R^2) \, L \ , \ \overline{B}^{(n)} = \widetilde{B}^{(n)} + A^{(n)} \ln \frac{M^2}{Q^2}$$

- At N4LL we need the resummation coefficients
- (A5) $1-3\cdot10^{-3}$ relative uncertainty
- (B4) negligible uncertainty
- C4: 1–2·10⁻³ relative uncertainty
- γ (γ 4) singlet : 1–3·10⁻³ relative uncertainty (non-singlet negligible)

Herzog, Moch, Ruijl, Ueda, Vermaseren, Vogt [2019] Henn, Korchemsky, Mistlberger [2020] von Manteuffel, Panzer, Schabinger [2020] Moult, Xing Zhu, Jiao Zhu [2022]

[DYTurbo]

ncertainty (non-singlet negligible) Falcioni, Herzog, Moch, Vogt [2023] Moch, Ruijl, Ueda, Vermaseren, Vogt [2022]

Transverse momentum resummation Camarda, LC, Ferrera [2023] up to N4LL+N4LO accuracy

[DYTurbo]

 $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$

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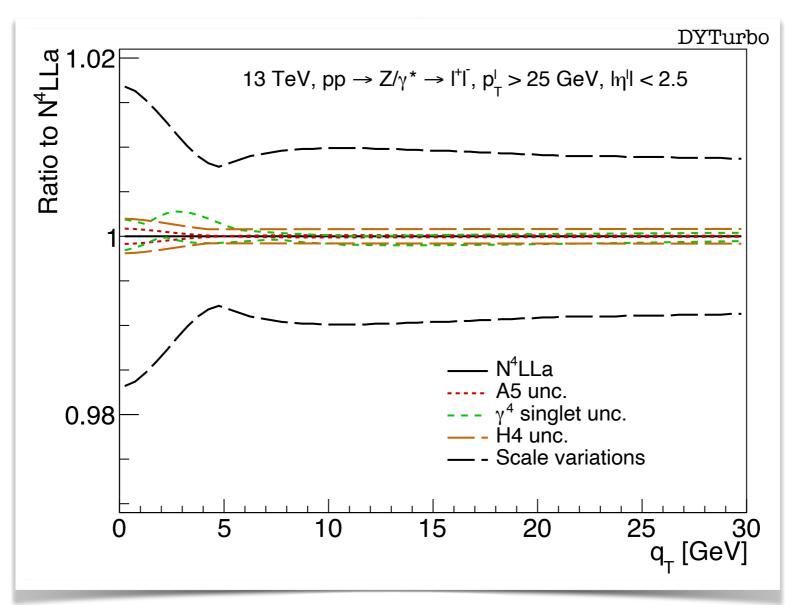
We rely on the Levin transform assigning 100% uncertainty → We assume that the Levin transform estimates the correct sign and order of magnitude.

Transverse momentum resummation

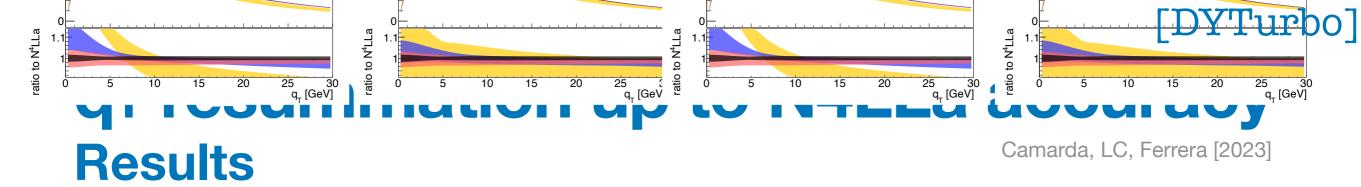
Camarda, LC, Ferrera [2023]

up to N4LL+N4LO accuracy

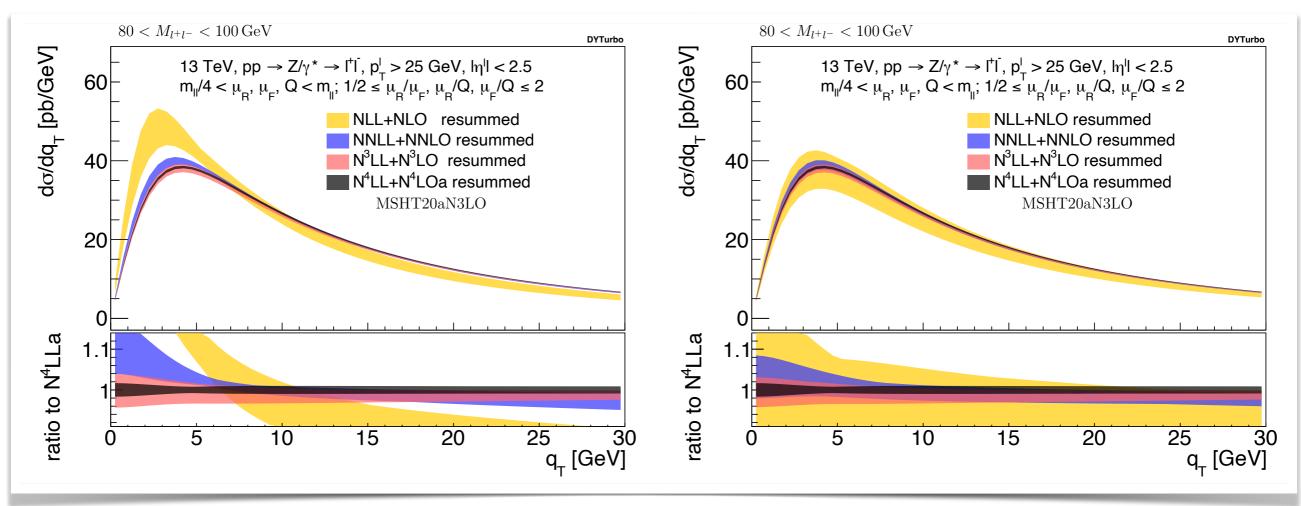
Anticipating our results



- The uncertainties in the N4LL+N4LO approximation are found to be 5 to 10 times smaller compared to the missing higher order uncertainties estimated through scale variations.
- All "main" channels already present at NNLO : qqbar, qg, gg.
- N4LL is the first order at which all the combination of the channels are opened: {q,qb,qp,qbp,g} x {q,qb,qp,qbp,g} (all combinations)

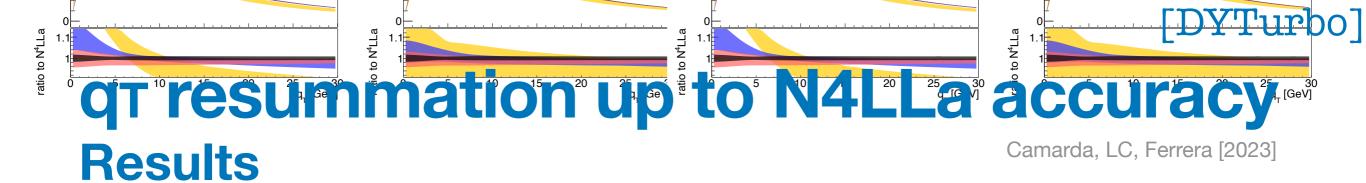


The qT spectrum of $Z/\gamma *$ bosons with lepton selection cuts at the LHC ($\sqrt{s} = 13$ TeV) at various perturbative orders

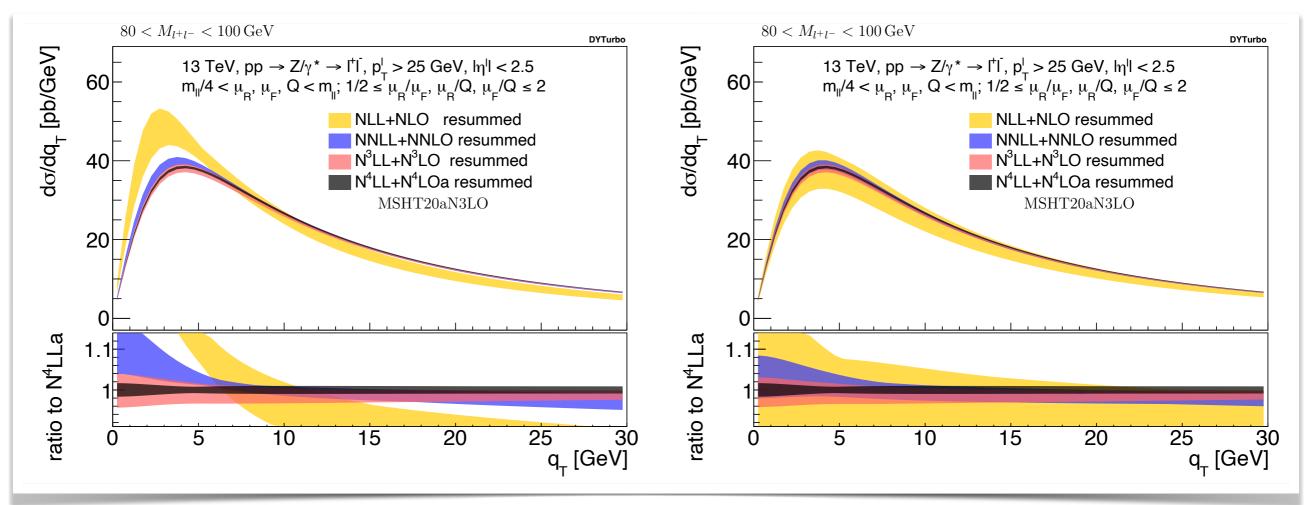


Formal mismatch between the N³LO Altarelli-Parisi evolution as encoded in the N³LO parton densities functions and the corresponding N^kLO evolution included in the N^(k+1)LL partonic resummed formula. The order of Altarelli-Parisi evolution in the resummed prediction is equal to the order of the parton densities

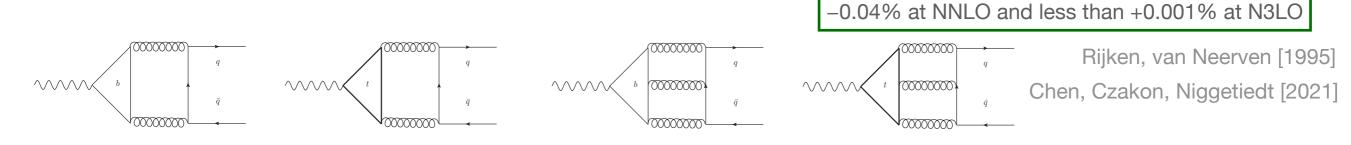
- Negligible impact at N3LL and N4LL on the choice → we apply this strategy in the next slides
- Scale dependence reduced a factor 2 from N3LL to N4LL → N4LL accuracy is at the 1%-1.5%



The qT spectrum of $Z/\gamma *$ bosons with lepton selection cuts at the LHC ($\sqrt{s} = 13$ TeV) at various perturbative orders

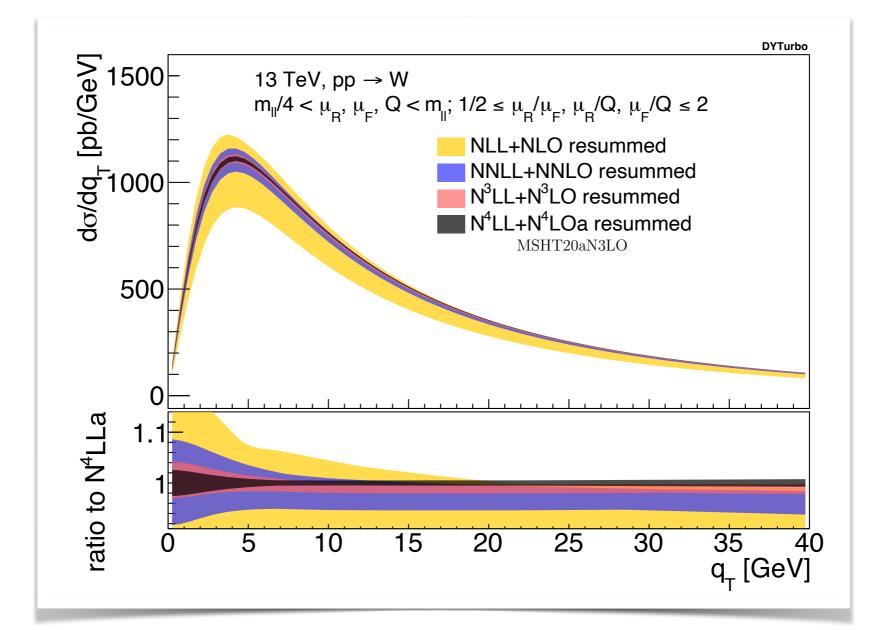


- Negligible impact at N3LL and N4LL on the choice → we apply this strategy in the next slides
- Scale dependence reduced a factor 2 from N3LL to N4LL → N4LL accuracy is at the 1%-1.5%
- Effect of a finite top-quark mass including the singlet contributions mediated by heavy-quark loops at NNLO and N³LO included

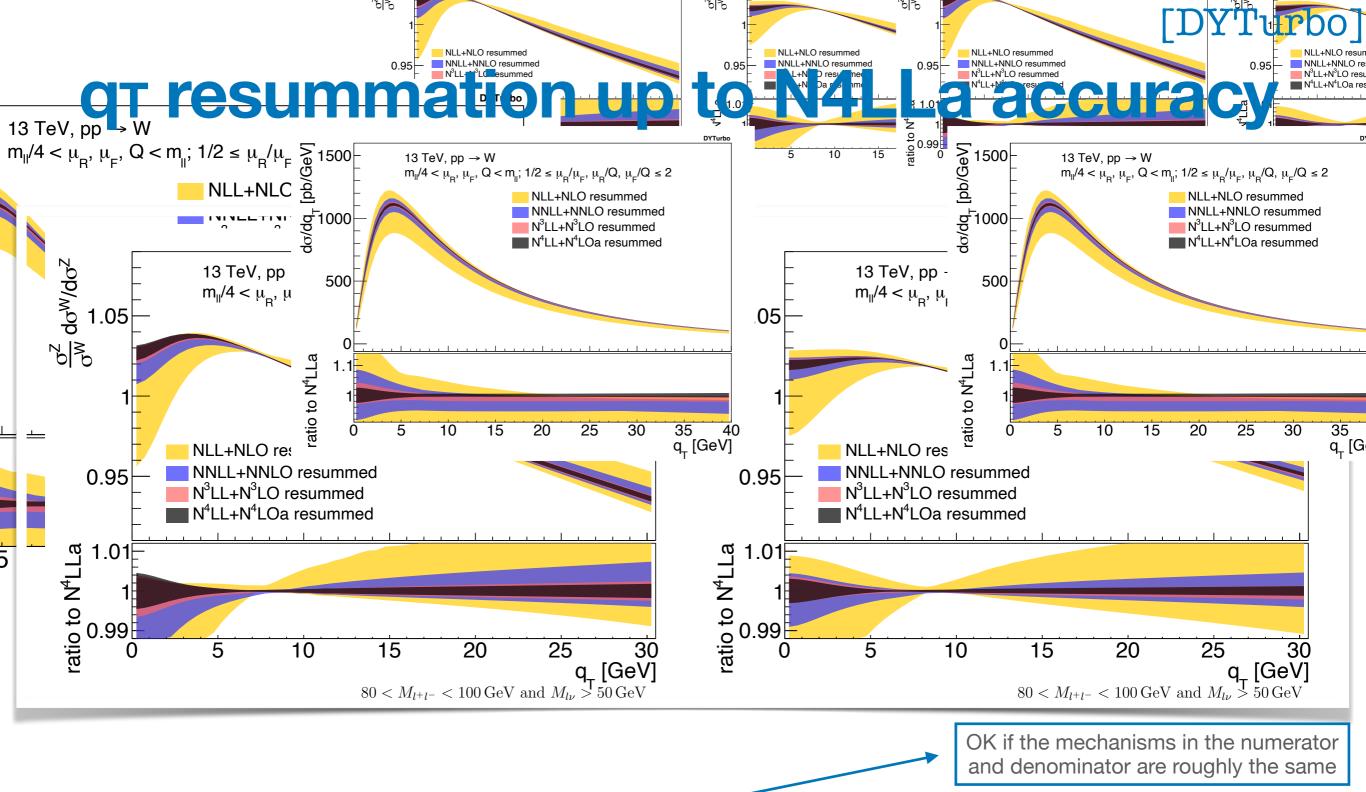


qT resummation up to N4LLa accuracy Camarda, LC, Ferrera [2023]

The qT spectrum of W+ and W– bosons with inclusive leptonic decay at the LHC ($\sqrt{s} = 13$ TeV) at various perturbative orders



The scale variation at N4LLa accuracy is around $\pm 2\%$ at qT ~ 1GeV, then it reduces at $\pm 1\%$ level at the peak (qT ~ 4GeV), it decreases further to $\pm 0.5\%$ for qT ~ 7 GeV and remains below $\pm 1\%$ level up to qT ~ 30 GeV.



Correlated scale variation

$$R(q_T) = \frac{\sigma_Z}{\sigma_W} \frac{d\sigma_W}{dq_T} \Big/ \frac{d\sigma_Z}{dq_T}$$

- N4LL very relevant removing uncertainties in the W/Z pT distribution ratio
- However, analysis is not complete : flavour-dependent intrinsic kT, processdependent EW effects

Next part of the talk

Main differences respect to pure QCD case On-shell Z and W production

The cross section can be decomposed as Catani, Grazzini, Torre [2014] $\begin{bmatrix} d\hat{\sigma}_{a_1a_2 \rightarrow l_3l_4} \end{bmatrix} = \begin{bmatrix} d\hat{\sigma}_{a_1a_2 \rightarrow l_3l_4}^{(\text{rcs.})} \end{bmatrix} + \begin{bmatrix} d\hat{\sigma}_{a_1a_2 \rightarrow l_3l_4}^{(\text{fin.})} \end{bmatrix} + \begin{bmatrix} d\hat{\sigma}_{a_1a_2 \rightarrow l_3l_4}^{(\text{fin.})} \end{bmatrix}$ $\begin{bmatrix} d\hat{\sigma}_{a_1a_2 \rightarrow l_3l_4} \end{bmatrix} = \sum_{b_1, b_2 = q, \bar{q}} \frac{d\hat{\sigma}_{b_1b_2 \rightarrow l_3l_4}^{(0)}}{d\Omega} \frac{1}{\hat{s}} \int_0^\infty \frac{db}{2\pi} b J_0(bq_T) \mathcal{W}_{a_1a_2, b_1b_2 \rightarrow V}(b, M, \hat{y}, \hat{s}; \alpha_S, \mu_R^2, \mu_F^2)$ $\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$ $L = \log\left(\frac{b^2 Q^2}{b_0^2} + 1\right)$ Here we include the f.o predictions at NLO (QED and QCD) Unitary constraint $L = \log\left(\frac{b^2 Q^2}{b_0^2} + 1\right)$ Here we include the f.o predictions at NLO (QED and QCD) Unitary constraint $L = \log\left(\frac{b^2 Q^2}{b_0^2} + 1\right)$ Here we include the f.o predictions at NLO (QED and QCD) Unitary constraint $L = \log\left(\frac{b^2 Q^2}{b_0^2} + 1\right)$ Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD) Here we include the f.o predictions at NLO (QED and QCD)

• W on-shell at NLL+NLO: colourless and charged final state → New Autieri, LC, Ferrera, Sborlini [2023]

The resummation formalism can be obtained with plain abelianization of the QCD results and it will be not presented in detail here

[DYqT]

Autieri, LC, Ferrera, Sborlini [2023]

Naive abelianization of the QCD results does not work. Apart from this fact, the (more involved) abelizanization procedure has to be applied to the QCD resummation for ttbar final state

Main differences respect to pure QCD case **On-shell Z and W production**

The cross section can be decomposed as

$$[d\hat{\sigma}_{a_1a_2 \to l_3l_4}] = [d\hat{\sigma}_{a_1a_2 \to l_3l_4}^{(\text{res.})}] + [d\hat{\sigma}_{a_1a_2 \to l_3l_4}^{(\text{fin.})}]$$

$$\left[d\hat{\sigma}_{a_{1}a_{2}\to l_{3}l_{4}}^{(\text{res.})}\right] = \sum_{b_{1},b_{2}=q,\bar{q}} \frac{d\hat{\sigma}_{b_{1}b_{2}\to l_{3}l_{4}}^{(0)}}{d\mathbf{\Omega}} \frac{1}{\hat{s}} \int_{0}^{\infty} \frac{db}{2\pi} b J_{0}(bq_{T}) \mathcal{W}_{a_{1}a_{2},b_{1}b_{2}\to V}(b,M,\hat{y},\hat{s};\alpha_{S},\mu_{R}^{2},\mu_{F}^{2})$$

$$\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$$

 $L = \log\left(\frac{b^2 Q^2}{b_0^2} + 1\right)$ Unitary constraint

W on-shell at NLL+NLO: colourless and charged final state \rightarrow New Autieri, LC, Ferrera, Sborlini [2023]

$$\mathcal{G}_{N}'(\alpha,L) = -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left(A'(\alpha(q^{2})) \log\left(\frac{M^{2}}{q^{2}}\right) + \widetilde{B}_{N}'(\alpha(q^{2})) + D'(\alpha(q^{2})) \right) \qquad D'(\alpha) = \frac{\alpha}{\pi} D'^{(1)} + \sum_{n=2}^{+\infty} \left(\frac{\alpha}{\pi}\right)^{n} D'^{(n)}$$

$$\mathcal{G}_{N}'(\alpha_{\mathrm{S}},\alpha,L) = \mathcal{G}_{N}(\alpha_{\mathrm{S}},L) + L g'^{(1)}(\alpha L) + g'^{(2)}_{N}(\alpha L) + \sum_{n=3}^{+\infty} \left(\frac{\alpha}{\pi}\right)^{n-2} g'^{(n)}_{N}(\alpha L)$$

$$+ g'^{(1,1)}(\alpha_{\mathrm{S}}L,\alpha L) + \sum_{\substack{n,m=1\\n+m\neq 2}}^{+\infty} \left(\frac{\alpha_{\mathrm{S}}}{\pi}\right)^{n-1} \left(\frac{\alpha}{\pi}\right)^{m-1} g'^{(n,m)}_{N}(\alpha_{\mathrm{S}}L,\alpha L)$$

$$\mathrm{New \ linear \ logarithmic \ term \ lit \ is \ specific \ of \ charged \ highmass \ system \ production \ and \ it \ is \ due \ to \ QED \ soft \ non-collinear \ (wide \ angle) \ radiation \ from \ the \ underlying \ subprocess \ due \ to \ delta \ due \ to \ delta \ due \ to \ delta \ subprocess \ due \ to \ delta \ subprocess \ due \ to \ delta \ due \ to \ delta \ subprocess \ due \ to \ delta \ to \ subprocess \ due \ to \ delta \ subprocess \ due \ subprocess \ due$$

$$D'^{(1)} = -\frac{e_V^2}{2}$$

Autieri, LC, Ferrera, Sborlini [2023]

$$\begin{array}{c} h_1(p_1) & \overbrace{\mathbf{f_{a_1/h_1}}}^{\mathbf{f_{a_1/h_1}}} & \overbrace{\mathbf{C_{qa_1}}}^{\mathbf{r_1}} & \overbrace{\mathbf{S_q}}^{\mathbf{l/2}} \\ \mu_F^2) & \overbrace{\mathbf{S_q}}^{\mathbf{l/2}} & \overbrace{\mathbf{H_q}}^{\mathbf{H_q}} & V \\ M/Q) \\ h_2(p_2) & \overbrace{\mathbf{f_{a_2/h_2}}}^{\mathbf{r_2}} & \overbrace{\mathbf{C_{qa_2}}}^{\mathbf{r_2}} & \overbrace{\mathbf{S_q}}^{\mathbf{l/2}} \\ \end{array}$$

Main differences respect to pure QCD case On-shell Z and W production

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$$\left[d\hat{\sigma}_{a_{1}a_{2}\to l_{3}l_{4}}^{(\text{res.})}\right] = \sum_{b_{1},b_{2}=q,\bar{q}} \frac{d\hat{\sigma}_{b_{1}b_{2}\to l_{3}l_{4}}^{(0)}}{d\mathbf{\Omega}} \frac{1}{\hat{s}} \int_{0}^{\infty} \frac{db}{2\pi} b J_{0}(bq_{T}) \mathcal{W}_{a_{1}a_{2},b_{1}b_{2}\to V}(b,M,\hat{y},\hat{s};\alpha_{S},\mu_{R}^{2},\mu_{F}^{2})$$

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$$\mathcal{G}'_{N}(\alpha,L) = -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left(A'(\alpha(q^{2})) \log\left(\frac{M^{2}}{q^{2}}\right) + \widetilde{B}'_{N}(\alpha(q^{2})) + D'(\alpha(q^{2})) \right) \qquad D'(\alpha) = \frac{\alpha}{\pi} D'^{(1)} + \sum_{n=2}^{+\infty} \left(\frac{\alpha}{\pi}\right)^{n} D'^{(n)}$$

$$g^{\prime(1)}(\alpha L) = \frac{A_q^{\prime(1)}}{\beta_0'} \frac{\lambda' + \ln(1 - \lambda')}{\lambda'} ,$$

$$g_N^{\prime(2)}(\alpha L) = \frac{\widetilde{B}_{q,N}^{\prime(1)}}{\beta_0'} \ln(1 - \lambda') - \frac{A_q^{\prime(2)}}{\beta_0'^2} \left(\frac{\lambda'}{1 - \lambda'} + \ln(1 - \lambda')\right) + \frac{A_q^{\prime(1)}\beta_1'}{\beta_0'^3} \left(\frac{1}{2}\ln^2(1 - \lambda') + \frac{\ln(1 - \lambda')}{1 - \lambda'} + \frac{\lambda'}{1 - \lambda'}\right)$$

$$g^{\prime(1,1)}(\alpha_S L, \alpha L) = \frac{A_q^{(1)} \beta_{0,1}}{\beta_0^2 \beta_0^{\prime}} h(\lambda, \lambda^{\prime}) + \frac{A_q^{\prime(1)} \beta_{0,1}^{\prime}}{\beta_0^{\prime 2} \beta_0} h(\lambda^{\prime}, \lambda)$$

 $h_2(p_2)$

$$h(\lambda,\lambda') = -\frac{\lambda'}{\lambda-\lambda'}\ln(1-\lambda) + \ln(1-\lambda')\left[\frac{\lambda(1-\lambda')}{(1-\lambda)(\lambda-\lambda')} + \ln\left(\frac{-\lambda'(1-\lambda)}{\lambda-\lambda'}\right)\right] - \operatorname{Li}_2\left(\frac{\lambda}{\lambda-\lambda'}\right) + \operatorname{Li}_2\left(\frac{\lambda(1-\lambda')}{\lambda-\lambda'}\right),$$

[DYqT]

Autieri, LC, Ferrera, Sborlini [2023]

"New" mixed contribution not present by trivial abelianization of QCD results

Main differences respect to pure QCD case **On-shell Z and W production**

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$$\mathcal{W}_V(b, M; \alpha_S, \mu_R^2, \mu_F^2) = \mathcal{H}_V(\alpha_S; M/\mu_R, M/\mu_F, M/Q) \times \exp\{\mathcal{G}(\alpha_S, L; M/\mu_R, M/Q)\}$$

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W on-shell at NLL+NLO: colourless and charged final state \rightarrow New Autieri, LC, Ferrera, Sborlini [2023]

$$\begin{aligned} \mathcal{G}'_{N}(\alpha,L) &= -\int_{b_{0}^{2}/b^{2}}^{Q^{2}} \frac{dq^{2}}{q^{2}} \left(A'(\alpha(q^{2})) \log\left(\frac{M^{2}}{q^{2}}\right) + \widetilde{B}'_{N}(\alpha(q^{2})) + D'(\alpha(q^{2})) \right) \\ A'^{(1)} &= \frac{e_{qf}^{2} + e_{\bar{q}f'}^{2}}{2}, \\ A'^{(2)} &= -\frac{5}{9} \frac{e_{qf}^{2} + e_{\bar{q}f'}^{2}}{2} N^{(2)}, \\ \widetilde{B}'^{(1)} &= B'^{(1)} + \gamma'^{(1)}_{qfqf,N} + \gamma'^{(1)}_{\bar{q}f'\bar{q}f',N}, \end{aligned} \qquad \begin{aligned} N^{(2)} &= 3\sum_{q=1}^{n_{f}} e_{q}^{2} + \sum_{l=1}^{n_{l}} e_{l}^{2}, \\ B'^{(1)} &= -\frac{3}{2} \frac{e_{qf}^{2} + e_{\bar{q}f'}^{2}}{2}, \\ \gamma'^{(1)}_{qq,N} &= e_{q}^{2} \left(\frac{3}{4} + \frac{1}{2N(N+1)} - \gamma_{E} - \psi_{0}(N+1)\right) \\ \gamma'^{(1)}_{qr,N} &= \frac{3}{2} e_{q}^{2} \frac{N^{2} + N + 2}{N(N+1)(N+2)}, \end{aligned} \qquad \begin{aligned} N^{(2)} &= 3\sum_{q=1}^{n_{f}} e_{q}^{2} + \sum_{l=1}^{n_{l}} e_{l}^{2}, \\ New \text{ linear logarithmic term} \end{aligned}$$

$$\begin{aligned} \text{It is specific of charged high-mass system production and it is due to QED soft non-collinear (wide angle) radiation from the underlying subprocess \end{aligned}$$

$$\begin{aligned} D'(\alpha) &= \frac{\alpha}{\pi} D'^{(1)} + \sum_{n=2}^{\infty} \left(\frac{\alpha}{\pi}\right)^{n} D'^{(n)} \\ New \text{ linear logarithmic term} \end{aligned}$$

$$h_{1}(p_{1}) \xrightarrow{f_{a_{1}/h_{1}}} \xrightarrow{r_{1}} C_{qa_{1}}$$

$$\mu_{R}^{2}, \mu_{F}^{2})$$

$$\mu_{R}, M/Q) \} \xrightarrow{x_{2}}{z_{2}} C_{\bar{q}a_{2}}$$

$$h_{2}(p_{2}) \xrightarrow{f_{a_{2}/h_{2}}} \xrightarrow{r_{2}} C_{\bar{q}a_{2}}$$

Autieri, LC, Ferrera, Sborlini [2023]

[DYqT]

е

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Main differences respect to pure QCD case On-shell Z and W production

The cross section can be decomposed as

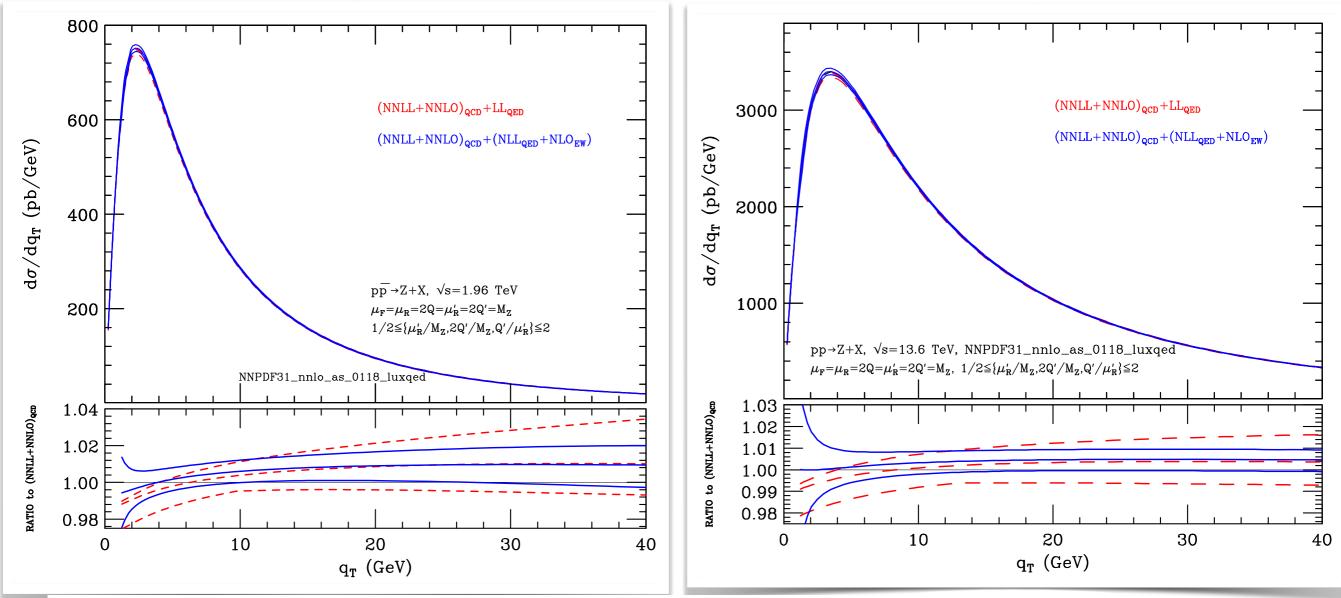
$$\begin{split} \left[d\hat{\sigma}_{a_{1}a_{2} \rightarrow l_{3}l_{4}} \right] &= \left[d\hat{\sigma}_{a_{1}a_{2} \rightarrow l_{3}l_{4}}^{(\text{fres.})} \right] + \left[d\hat{\sigma}_{a_{1}a_{2} \rightarrow l_{3}l_{4}}^{(\text{fin.})} \right] & \stackrel{h_{1}(p_{1})}{\longrightarrow} \left[f_{n/h} f_$$

- The hard virtual factor H'^V requires the definition of subtraction operators I, suitable to treat massive and charged final states → we left this topic to the discussion session
- The expansion of the f.o contribution served as a check for the involved abelianization procedure → we left this topic to the discussion session (also the linear power corrections)

Autieri, LC, Ferrera, Sborlini [2023]



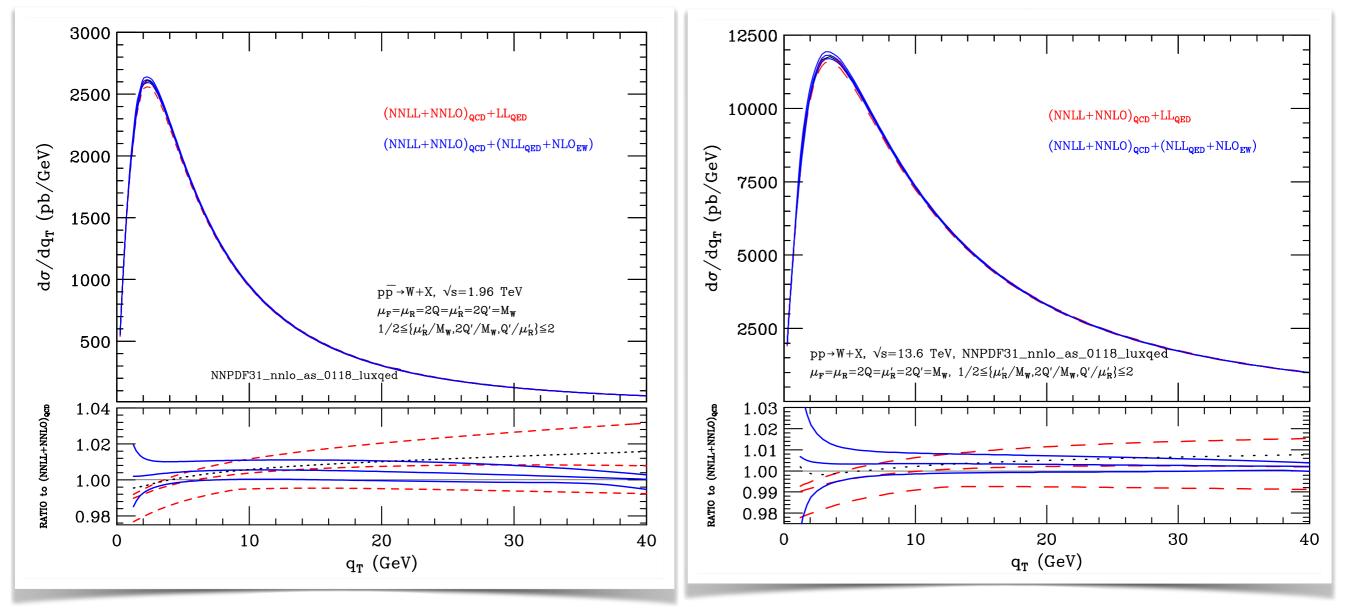
QED+QCD qT resummation at NLL+NLO Autieri, LC, Ferrera, Sborlini [2023]



• The scale variation band is reduced by roughly a factor 2 with the inclusion of the NLL+NLO corrections

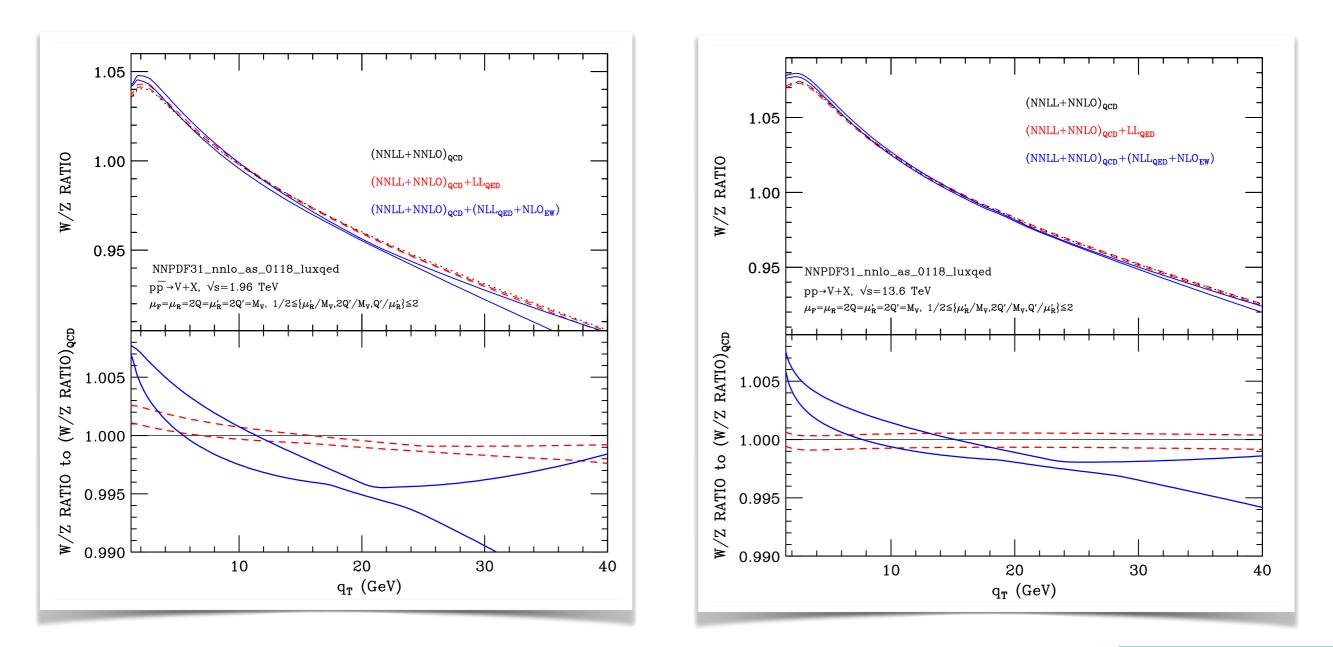
- At the Tevatron and at the LHC, QED uncertainty is dominated by the renormalization scale at LL accuracy and resummation scale at NLL+NLO LC, Ferrera, Sborlini [2018]
- The effect of EW loop corrections is extremely small (per-mille level effect)
- Overall order 0.5% at the LHC at NLL

QED+QCD qT resummation at NLL+NLO Autieri, LC, Ferrera, Sborlini [2023]



- The NLL+NLO prediction without the effect of soft wide-angle QED radiation (black dotted curve)
- NLL+NLO scale variation band reduction factor 1.5-2 for qT \lesssim 20GeV and up to a factor 3 for qT \gtrsim 30GeV
- Overall order 0.5% at the LHC at NLL

QED+QCD qT resummation at NLL+NLO Autieri, LC, Ferrera, Sborlini [2023]



- Correlated scale variation → use the difference between the prediction at NLL+NLO and the LL?
- The impact of NLL+NLO QED corrections is to make the distribution softer at O(0.5 1%) level
- This is the combined effect of the W distribution slightly softer and the Z distribution harder

Example of what happen with different mechanisms in numerator and denominator

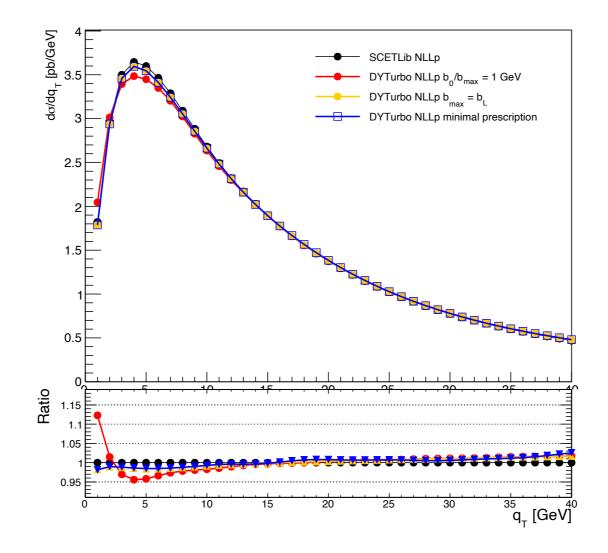
Outlook

- N4LL QCD plays a relevant role removing uncertainties in the W/Z pT distribution ratio
- NLL+NLO QCD+QED corrections to on shell Z and W boson production introduce non negligible effects for the W/Z pT distribution ratio
- QCD resummation at N4LL is implemented in the public code DYTurbo
- NLL+NLO QCD+QED corrections to on shell Z and W boson production are encoded in DYqT. (very soon in DYTurbo)
- Full NLL+NLO QCD+QED corrections to Z and W boson production with decays → very soon in DYTurbo

Thank you!!!

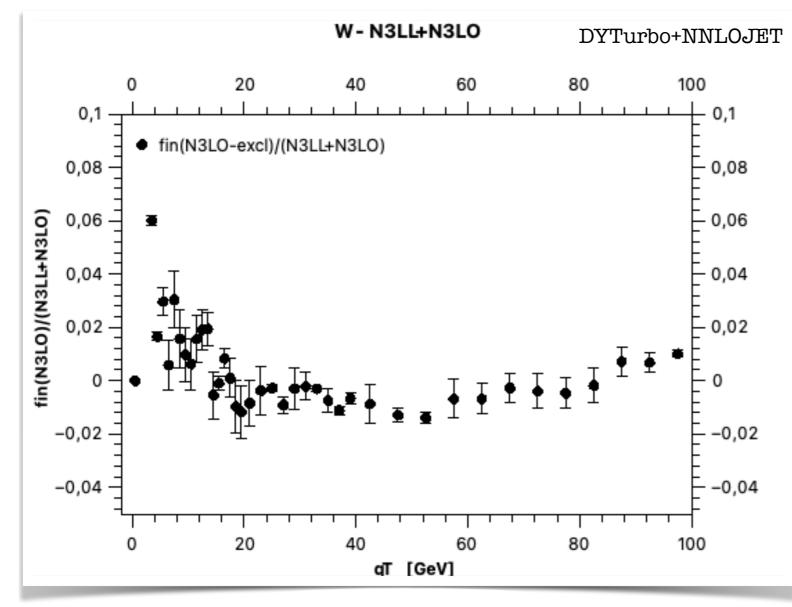
Backup slides

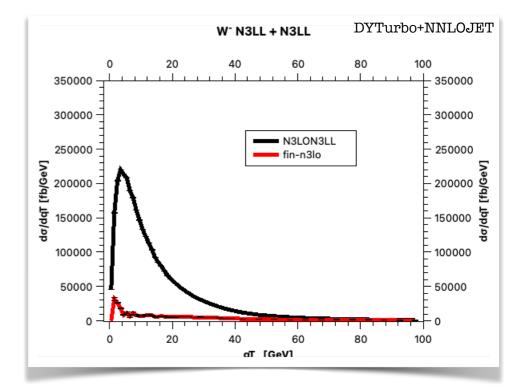
Comparison of b* and minimal prescription



- keep bstar with bmax = b0/1GeV to evaluate PDFs, but integrate up to or beyond the Landau pole in the Sudakov
- In one prescription bmax = bL with $bL = b0 \cdot exp(1/(2\alpha s \beta 0))$
- In the other prescription the path of integration is deformed in the complex plane (minimal prescription)

Size of the finite part W-





Non perturbative model used in the N4LLa

For the non-perturbative (NP) effects at very small transverse momenta we introduced, in the conjugated bspace, a NP form factor of the form

Collins, Rogers [2015]

$$S_{NP}(b) = \exp\{-g_1 b^2 - g_K(b) \ln(M^2/Q_0^2)\}$$

$$g_K(b) = g_0 \left(1 - \exp\left[-\frac{C_F \alpha_S ((b_0/b_\star)^2) b^2}{\pi g_0 b_{\lim}^2} \right] \right)$$

$$g_1 = 0.5 \text{ GeV}^2, Q_0 = 1 \text{ GeV}, g_0 = 0.3, b_{\text{lim}} = 1.5 \text{ GeV}^{-1}$$

$$b_{\star}^2 = b^2 b_{\rm lim}^2 / (b^2 + b_{\rm lim}^2)$$

Other choices available in the code

$$S_{NP}(b) = \exp\left\{-\left(g_1 + g_2 \log\left(\frac{m}{Q_0}\right) + g_3 \log\left(\frac{100m}{\sqrt{s}}\right)\right)b^2\right\}$$