Introduction to open quantum systems

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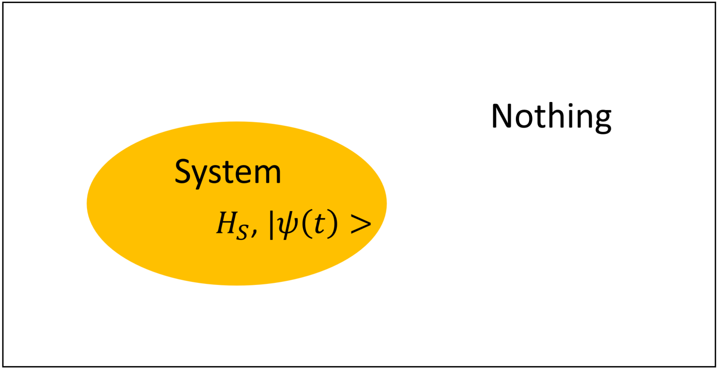
What is an open quantum system?

Closed systems are an idealization of the real ones



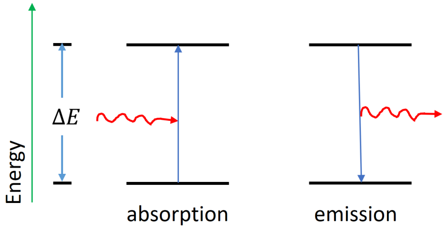
State: pure $|\psi(t)\rangle$
Evolution: Schrödinger equation

$$\frac{d|\psi(t)\rangle}{dt} = -i\, H_S|\psi(t)\rangle$$

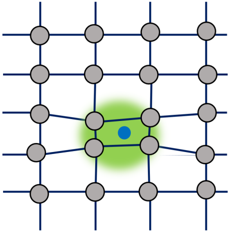
What is an open quantum system?

Real world

Atom interacts with electromagnetic fields



Electrons coupled to the phonons



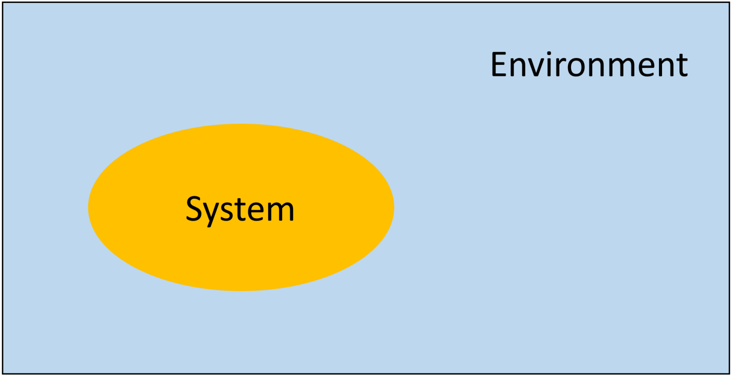
Many contexts:

Condensed Matter: Optics, Quantum Information, Circuits, . . .

High Energy Physics: AdS/CFT, Quantum gravity (BH), . . .

What is an open quantum system?

To give a more accurate description of the real world we need
Open quantum systems

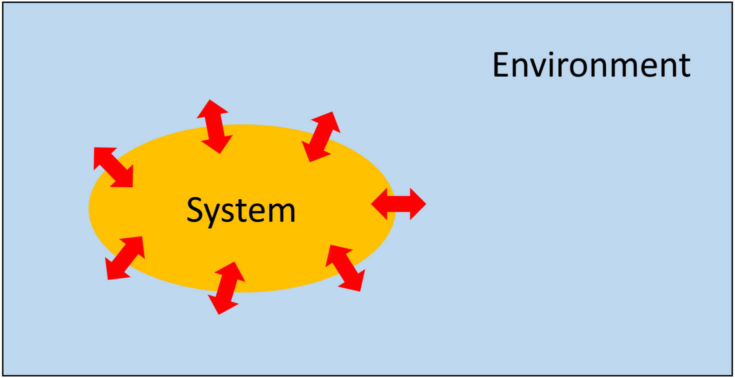


[Petruccione, Breuer, 2002; Manzano, 2020; Medvedyeva, Essler, Prosen, 2016; de Vega (lectures), 2019]

What is an open quantum system?

Real world:

Open quantum systems $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$



Problem: The total Hilbert space is "huge" due to the degree of freedom of the environment

Main goal: Understand the dynamics of the **system** tracing out the d.o.f. of the environment.

Overview:

1) Evolution of the **system**

   $\rightarrow$ Approximations $\rightarrow$ Lindblad master equation

2) Hard to solve: we look at integrable cases

   - Few words on (quantum) integrability
   - Integrable Open Quantum Systems

3) New result: Deformation of the Hubbard model

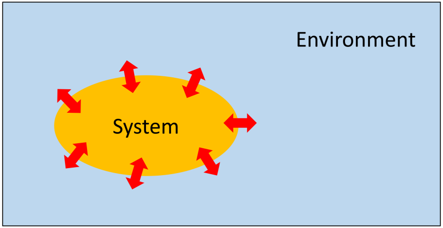
Few Basic concepts

Isolated systems: Pure states $|\psi\rangle$

$$\frac{d}{dt}|\psi(t)\rangle = -i\,H|\psi(t)\rangle \qquad \text{Schrödinger equation}$$

Open systems: Mixed states $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$\rho = \rho^\dagger, \quad \mathrm{Tr}\rho = 1, \quad \rho \geq 0$$



$$\mathcal{H}_T = \mathcal{H}_S \otimes \mathcal{H}_E, \qquad H_T = H_S \otimes 1_E + 1_S \otimes H_E + \alpha\, H_I$$

$$\dot{\rho}_T(t) = -i\,[H_T, \rho_T(t)] \qquad \text{Von-Neumann equation}$$

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Aim: Understand the dynamics of the system: $\rho = \mathrm{Tr}_E \rho_T$

Complementary Approaches:

- Rigorous methods: map that preserves the properties of the density matrix
- Microscopic derivation of dynamical evolution

$$\downarrow$$

Lindblad Master Equation

Microscopic derivation

$$\mathcal{H}_T = \mathcal{H}_S \otimes \mathcal{H}_E, \quad H_T = H_S \otimes 1_E + 1_S \otimes H_E + \alpha\, H_I$$

$$\dot{\rho}_T(t) = -i\,[H_T, \rho_T(t)] \qquad \text{Von-Neumann equation}$$

Interaction picture

$$\tilde{O} = e^{i(H_S+H_E)t} O e^{-i(H_S+H_E)t}$$

$$\dot{\tilde{\rho}}_T(t) = -i\,\alpha\,[\tilde{H}_I(t), \tilde{\rho}_T(t)]$$

Formal integration

$$\tilde{\rho}_T(t) = \tilde{\rho}_T(0) - i\,\alpha \int_0^t ds[\tilde{H}_I(s), \tilde{\rho_T}(s)]$$

$$\frac{d}{dt}\tilde{\rho}_T(t) = -i\,\alpha\,[\tilde{H}_I(t), \tilde{\rho_T}(0)] - \alpha^2 \int_0^t ds[\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_T(s)]]$$

First approximation: weak coupling

$$\frac{d}{dt}\tilde{\rho_T}(t) = -i\,\alpha\,[\tilde{H}_I(t), \tilde{\rho_T}(0)] - \alpha^2 \int_0^t ds[\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_T(t)]]$$

Notice $s \to t \to$ no memory!

Motivation

Different timescales

$$\tau_E \ll T_S$$

$$\Delta E\, T_S \sim \hbar$$

$$\tau_E \ll T_S \sim \frac{\hbar}{\alpha}$$

Main aim: Dynamic of ρ_S, take partial trace

$$\frac{d}{dt}\tilde{\rho_S}(t) = -i\,\alpha\,\mathrm{Tr}_E\,[\tilde{H}_I(t), \tilde{\rho_T}(0)] - \alpha^2 \int_0^t ds\,\mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_T(t)]]$$

Born approximation

$\tau_{corr} \ll T_S$

$$\rho_T(t) = \rho_S(t) \otimes \rho_E(t) + \rho_{correl}(t) \approx \rho_S(t) \otimes \rho_E(t)$$

Reservoir's relaxation is fast $\tau_E \ll T_S$: environment is thermal

$$\rho_E(t) = \rho_E(0) = \frac{\exp(-H_E/T)}{\mathrm{Tr}\exp(-H_E/T)}$$

We take:

$$\frac{d}{dt}\tilde{\rho_S}(t) = -i\alpha\,\mathrm{Tr}_E\,[\tilde{H}_I(t), \tilde{\rho_T}(0)] - \alpha^2 \int_0^t ds \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_T(t)]]$$

Decompose

$$\tilde{H}_I = \sum_i \tilde{S}_i \otimes \tilde{E}_i$$

$$\mathrm{Tr}_E\left[\tilde{H}_I(t), \tilde{\rho}_T(0)\right] = \sum_i \tilde{S}_i(t)\tilde{\rho}_S(0) \underbrace{\mathrm{Tr}_E[\tilde{E}_i(t)\tilde{\rho}_E(0)]}_{=0} - \tilde{\rho}_S(0)\tilde{S}_i(t)\mathrm{Tr}_E[\tilde{\rho}_E(0)\tilde{E}_i(t)] = 0$$

$$\frac{d}{dt}\tilde{\rho_S}(t) = -\alpha^2 \int_0^t ds\, \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(s), \tilde{\rho}_S(t) \otimes \tilde{\rho}_E(0)]]$$

$$\downarrow s \to t - s$$

$$\frac{d}{dt}\tilde{\rho_S}(t) = -\alpha^2 \int_0^\infty ds \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-s), \tilde{\rho}_S(t) \otimes \tilde{\rho}_E(0)]]$$

Redfield equation

Our main aim: trace out the contribution of the environment

$$H_I(t) = \sum_i S_i(t) \otimes E_i(t)$$

In the interaction picture

$$\tilde{H}_I(t) = \sum_{i,\omega} e^{-i\omega t} \tilde{S}_i(\omega) \otimes \tilde{E}_i(t) = \sum_{i,\omega} e^{i\omega t} \tilde{S}_i^\dagger(\omega) \otimes \tilde{E}_i^\dagger(t)$$

Approximation: Rotating wave

Redfield equation

$$\frac{d}{dt}\tilde{\rho_S}(t) = -\alpha^2 \int_0^\infty ds \mathrm{Tr}_E[\tilde{H}_I(t), [\tilde{H}_I(t-s), \tilde{\rho}_S(t) \otimes \tilde{\rho}_E(0)]]$$

$$\tilde{H}_I(t) \to \sum_{i,\omega} e^{i\omega t} \tilde{S}_i^\dagger(\omega) \otimes \tilde{E}_i^\dagger(t), \quad \tilde{H}_I(t-s) \to \sum_{i,\omega} e^{-i\omega(t-s)} \tilde{S}_i(\omega) \otimes \tilde{E}_i(t)$$

$$\begin{aligned}\dot{\tilde{\rho}}_S(t) = \sum_{\omega,\omega',k,l} \Big(& e^{i(\omega'-\omega)t} \Gamma_{kl}(\omega) [\tilde{S}_l(\omega)\tilde{\rho}(t), \tilde{S}_k^\dagger(\omega')] \\ & + e^{i(\omega-\omega')t} \Gamma_{lk}^*(\omega') [\tilde{S}_l(\omega), \tilde{\rho}(t)\tilde{S}_k^\dagger(\omega')] \Big)\end{aligned}$$

Γ effects of the environment

Only keep resonant terms:

$$|\omega - \omega'| \gg \alpha^2 \quad \to \quad \omega = \omega'$$

$$\dot{\tilde{\rho}}_S(t) = \sum_{\omega,k,l} \left(\Gamma_{kl}(\omega)[\tilde{S}_l(\omega)\tilde{\rho}(t), \tilde{S}_k^\dagger(\omega)] + \Gamma_{lk}^*(\omega)[\tilde{S}_l(\omega), \tilde{\rho}(t)\tilde{S}_k^\dagger(\omega)] \right)$$

Separate Hermitian and non-Hermitian parts

$$\pi_{kl} = \frac{-i}{2}(\Gamma_{kl} - \Gamma_{kl}^*)$$
$$\gamma_{kl} = \frac{1}{2}(\Gamma_{kl} + \Gamma_{kl}^*)$$

Diagonalize γ

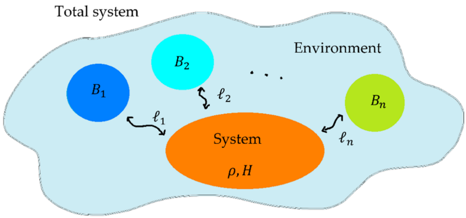
$$T\gamma T^\dagger = \mathrm{diag}(d_1, \ldots, d_n)$$

transform back to the Schrödinger picture

$$\ell_i = \sqrt{d_i} \sum_k T_{ik} S_k$$

$$\dot{\rho}(t) = -i[H + H_{LS}, \rho(t)] + \sum_i \left(\ell_i \rho(t) \ell_i^\dagger - \frac{1}{2}\{\ell_i^\dagger \ell_i, \rho(t)\} \right)$$

Summary approximation:



$$\dot{\rho}_T(t) = -i[H_T, \rho_T(t)] \to \underbrace{\dot{\rho} = i\,[\rho, H]}_{\text{Liouville equation}} + \underbrace{\sum_{a=1}^{n} \left[\ell_a \rho \ell_a^\dagger - \frac{1}{2}\{\ell_a^\dagger \ell_a, \rho\}\right]}_{\text{Dissipator}}$$

- Weak coupling $\alpha \ll 1$
- Markovian approximation (time separation)
- Rotating wave approximation

How do we solve the Lindblad master equation?

Hard to solve.

- Numerical methods
- Perturbative methods

Does an exactly solvable model exist?

Different meaning of solvability, we focus on:
Yang Baxter Integrable Lindblad systems

Reasons: The out of equilibrium dynamics can be studied:

- the Non-Equilibrium steady states can be constructed with exact methods,
- the generator of the dynamics can be diagonalized.

What do we mean by: Yang-Baxter integrable system?

Models with a high amount of symmetry and a tower of conserved charges

$$[\mathbb{H}, \mathbb{Q}_r] = [\mathbb{Q}_r, \mathbb{Q}_s] = 0, \quad r, s = 1, 2, \ldots, \infty$$

Free theories $\rightarrow$ Not a surprise!

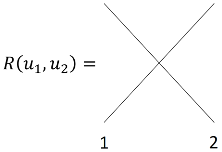
Interacting theories $\rightarrow$ May also have integrable behaviour!

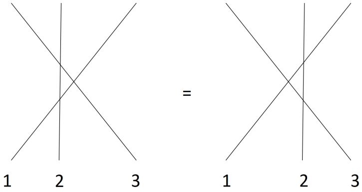
Why do we study integrable models?

Eigenvalues and eigenvectors of $\mathbb{Q}s$ can *often* be found with exact methods

What do we mean by: Yang-Baxter integrable system?

Characterized by an R-matrix solution of the Yang-Baxter equation (YBE)





$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2)$$

$$R\text{-matrix} \;\rightarrow\; [\mathbb{Q}_r, \mathbb{Q}_s] = 0$$

Why is it important to study YB Lindblad integrable models?

Isolated integrable many particle systems behave differently from non integrable models.

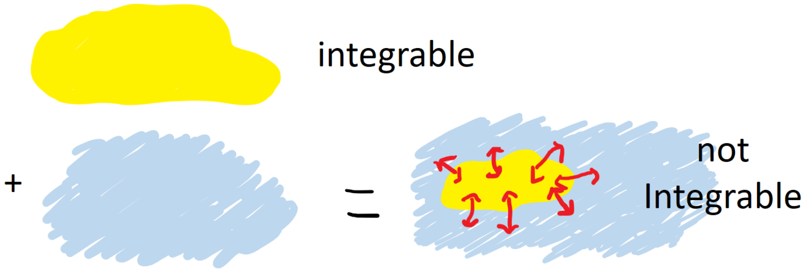
- Non integrable systems relax quickly toward an equilibrium state
- Integrable systems present an *unusual* non-equilibrium dynamics

Isolated integrable models with local charges : Generalized Gibbs Ensamble (GGE)

$$\rho^{GGE} = \frac{e^{-\sum_n \lambda_n \mathcal{Q}_n}}{\mathrm{Tr}(e^{-\sum_n \lambda_n \mathcal{Q}_n})}$$

and the dynamics is described by Generalized Hydrodynamics (GHD).

Typically



$\rightarrow$ There exist *Integrable Lindblad models*!

Open field of research: For Lindblad system, the time dependent GGE and GHD are good approximations for the time evolution.

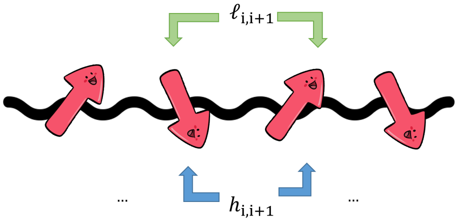
Integrable models can help to justify this statement.

Open question: Is there anything special about integrable Lindblad evolution through the NESS? What is the role of the infinite amount of conserved charges?

What do we mean by: Yang-Baxter integrable Lindblad system?

Set up: spin 1/2 chain of length L, $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$

$$\dot{\rho} = i\,[\rho, H] + \sum_{i=1}^{L} \left[\ell_{i,i+1} \rho \ell^{\dagger}_{i,i+1} - \frac{1}{2}\{\ell^{\dagger}_{i,i+1} \ell_{i,i+1}, \rho\} \right]$$



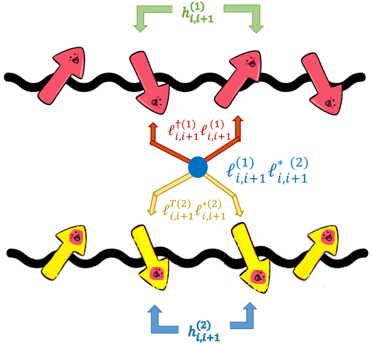
$$H = \sum_i h_{i,i+1}$$

What do we mean by: Yang-Baxter integrable Lindblad system?

$$\dot{\rho} = i\,[\rho, H] + \sum_{i=1}^{L}\left[\ell_{i,i+1}\rho\ell^{\dagger}_{i,i+1} - \frac{1}{2}\{\ell^{\dagger}_{i,i+1}\ell_{i,i+1},\rho\}\right] \;\rightarrow\; \dot{\rho} \equiv \mathcal{L}\rho$$

$$\mathcal{H} = V\otimes V, \quad V = \mathbb{C}^2 \qquad\qquad \rightarrow \quad \mathcal{H}\otimes\mathcal{H}^*$$

$$\mathcal{L}_{i,j} = -i\,h^{(1)}_{i,j} + i h^{(2)*}_{i,j} + \ell^{(1)}_{i,j}\ell^{(2)*}_{i,j} - \frac{1}{2}\ell^{(1)\dagger}_{i,j}\ell^{(1)}_{i,j} - \frac{1}{2}\ell^{(2)\,T}_{i,j}\ell^{(2)*}_{i,j}$$



Yang-Baxter integrable superoperator

Idea: Identify $\mathcal{L}$ as a (non-Hermitian) Hamiltonian
$H^{SL}_{i,j} = \mathcal{L}_{i,j} = -i\, h^{(1)}_{i,j} + i h^{(2)*}_{i,j} + \ell^{(1)}_{i,j}\ell^{(2)*}_{i,j} - \frac{1}{2}\ell^{(1)\dagger}_{i,j}\ell^{(1)}_{i,j} - \frac{1}{2}\ell^{(2)\,T}_{i,j}\ell^{(2)*}_{i,j}$

Require that $\mathcal{L}$ is one of the conserved charge of the integrable model

$$\mathbb{Q}_2 = \mathcal{L} = \sum_{i=1}^{L} \mathcal{L}_{i,i+1}$$

Construction of new integrable models:

Construction of new integrable models

$$\mathcal{L}_{i,j} = -i\, h^{(1)}_{i,j} + i h^{(2)*}_{i,j} + \ell^{(1)}_{i,j}\ell^{(2)*}_{i,j} - \frac{1}{2}\ell^{(1)\dagger}_{i,j}\ell^{(1)}_{i,j} - \frac{1}{2}\ell^{(2)T}_{i,j}\ell^{(2)*}_{i,j}$$

[de Leeuw, CP, Pozsgay, Pribytok, Retore, Ryan]

Ingredients:

- General form of h and ℓ

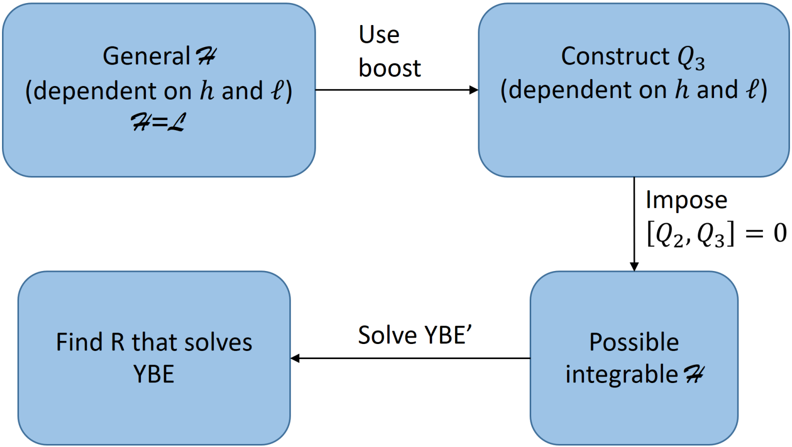
Boost automorphism mechanism

New integrable superoperators!

Boost automorphism mechanism

[de Leeuw, CP, Pozsgay, Pribytok, Retore, Ryan]

Steps:



We found many integrable models: some new one (deformation of AdS_2 and AdS_3) and a medium range deformation of Hubbard model!

New Integrable model

$$\mathcal{L} = -i\, h^{(1)} + i h^{(2)*} + \ell^{(1)} \ell^{(2)*} - \frac{1}{2} \ell^{(1)\dagger} \ell^{(1)} - \frac{1}{2} \ell^{(2)\,T} \ell^{(2)*}$$

$$h = i \left[\sigma_j^+ \sigma_{j+1}^- - \sigma_j^- \sigma_{j+1}^+ \right],$$
$$\ell = \sigma_{j+1}^z + \kappa(\sigma_j^x + \sigma_{j+2}^x)\sigma_{j+1}^x - \kappa^2 \sigma_{j+1}^z \sigma_j^x \sigma_{j+2}^x$$

What is this model?!?

Hubbard model

$$\{c_j^\alpha, c_k^\beta\} = 0 \qquad \alpha, \beta = \uparrow, \downarrow$$
$$\{c_j^\alpha, (c_k^\beta)^\dagger\} = \delta^{\alpha,\beta}\delta_{j,k}$$

Fermionic Hilbert space

States

$$|\emptyset\rangle, \qquad |\uparrow\rangle = (c^\uparrow)^\dagger|\emptyset\rangle, \qquad |\downarrow\rangle = (c^\downarrow)^\dagger|\emptyset\rangle, \qquad |\uparrow\downarrow\rangle = (c^\downarrow)^\dagger(c^\uparrow)^\dagger|\emptyset\rangle$$

Hamiltonian

$$H_{Hub} = \sum_j \left[(c_j^\uparrow)^\dagger c_{j+1}^\uparrow + (c_{j+1}^\uparrow)^\dagger c_j^\uparrow + (c_j^\downarrow)^\dagger c_{j+1}^\downarrow + (c_{j+1}^\downarrow)^\dagger c_j^\downarrow + U\, n_j^\uparrow n_j^\downarrow\right]$$

$n_j = {c_j}^\dagger c_j, \qquad U \in \mathbb{R}$

Hubbard model

$$H_{Hub} = \sum_j \left[(c_j^{\uparrow})^{\dagger} c_{j+1}^{\uparrow} + (c_{j+1}^{\uparrow})^{\dagger} c_j^{\uparrow} + (c_j^{\downarrow})^{\dagger} c_{j+1}^{\downarrow} + (c_{j+1}^{\downarrow})^{\dagger} c_j^{\downarrow} + U n_j^{\uparrow} n_j^{\downarrow} \right]$$

Particle number conservation $\sum_j n_j^{\uparrow}$ and $\sum_j n_j^{\downarrow}$

Bosonic version

→ Jordan-Wigner transformation

→ Similarity transformation

$$H'' = \sum_j \left[\underbrace{i(\sigma_j^+ \sigma_{j+1}^- - \sigma_j^- \sigma_{j+1}^+)^{(1)}}_{h_{j,j+1}{}^{(1)}} + \underbrace{i(\sigma_j^+ \sigma_{j+1}^- - \sigma_j^- \sigma_{j+1}^+)^{(2)}}_{h_{j,j+1}{}^{(2)}} + \frac{U}{4} \sigma_j^{z\,(1)} \sigma_j^{z\,(2)} \right]$$

$$\mathcal{H} = V \otimes V = \underbrace{W \otimes W}_{(1)} \otimes \underbrace{W \otimes W}_{(2)}$$

$$H'' = \sum_j \left[\underbrace{i(\sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+)^{(1)}}_{h_{j,j+1}{}^{(1)}} + \underbrace{i(\sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+)^{(2)}}_{h_{j,j+1}{}^{(2)}} + \frac{U}{4}\sigma_j^{z\,(1)}\sigma_j^{z\,(2)} \right]$$

by taking $U \to i\,U$ and renormalizing

$$\mathcal{L} = \sum_j [-i\,\underbrace{(\sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+)^{(1)}}_{h_{j,j+1}{}^{(1)}} + i\,\underbrace{[-(\sigma_j^+\sigma_{j+1}^- - \sigma_j^-\sigma_{j+1}^+)^{(2)}]}_{h_{j,j+1}{}^{(2)*}} +$$

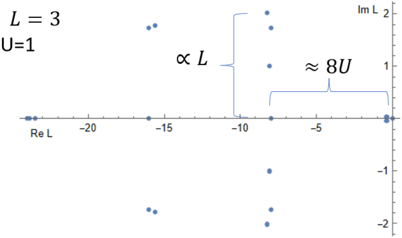
$$U\,\underbrace{\sigma_j^{z\,(1)}}_{\ell_j^(1)}\,\underbrace{\sigma_j^{z\,(2)}}_{\ell_j^{(2)*}} + \kappa 1]$$

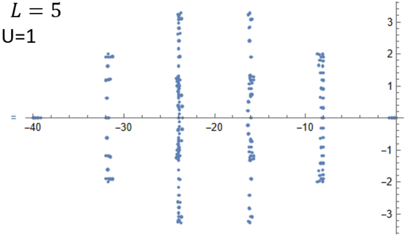
$$\mathcal{L} = -i\,h^{(1)} + i h^{(2)*} + \ell^{(1)}\ell^{(2)*} - \frac{1}{2}\ell^{(1)\dagger}\ell^{(1)} - \frac{1}{2}\ell^{(2)\,T}\ell^{(2)*}$$

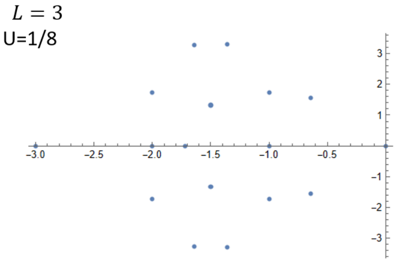
Integrability allows to compute exactly the spectrum of the model, what we should find:

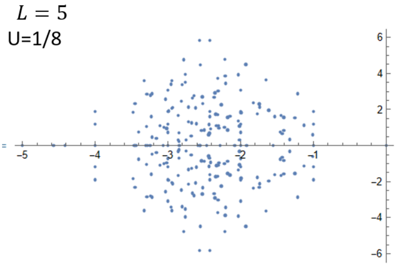
Spectrum

(studied by Medvedyeva, Essler, Prosen and also analytical solution known)









Relation with Hubbard model

$$\mathcal{L} = -i\, h^{(1)} + i h^{(2)*} + u\Big[\ell^{(1)}\ell^{(2)*} - \frac{1}{2}\ell^{(1)\dagger}\ell^{(1)} - \frac{1}{2}\ell^{(2)\,T}\ell^{(2)*}\Big]$$

$$\ell_{j,j+1,j+2} = \sigma^z_{j+1} + \kappa(\sigma^x_j + \sigma^x_{j+2})\sigma^x_{j+1} - \kappa^2\sigma^z_{j+1}\sigma^x_j\sigma^x_{j+2}$$

$u,\ \kappa \in \mathbb{R}$

- $\kappa = 0$ Hubbard model
- $\kappa \neq 0$ Range 3 deformation of the Hubbard model!
- $\kappa = \pm 1$ Two $U(1)$ conserved charges

$$\sum_j (\sigma^x_j \sigma^x_{j+1})^{(1)}, \quad \sum_j (\sigma^x_j \sigma^x_{j+1})^{(2)}$$

Main differences with Hubbard:

- Interaction spans 3 site
- Particle number conservation is broken

Previous extensions and deformations of the Hubbard model had two common properties

- Hamiltonian was always nearest neighbour interaction
- (At least) two local $U(1)$ charges

Checked that the model is not a linear combination of $\mathbb{Q}_2$ and $\mathbb{Q}_3$ of a known model!

Is this model integrable?
$\rightarrow$ YES! We found the R matrix

[arXiv:2108.02053 Gambor, Pozsgay]

Important points:

- Only known long-range deformation of the Hubbard model
- un-usual functional dependence of the R-matrix
- First long-range integrable open quantum system model

$\downarrow$

future:
Spectrum and analytical solution!

Conclusions and future work

What we saw:

- Approximations to write the Lindblad master equation
- Importance to study Lindblad integrable system
- New model: range 3 **deformation** of the Hubbard model

Possible future directions:

- Can we solve the new model by using one of the integrability techniques?
  (done for model B3 2207.14193)
- Analyze Non-Equilibrium State of the new model
- Classification of spin chain with **open boundary condition**
- Can we understand more properties of the Lindblad superoperator by using random matrix theory techniques?

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Thank you!

Why $\mathrm{Tr}_E[\tilde{E}_i(t)\tilde{\rho}_E(0)] = 0$

First, we work in Schrodinger picture

$$H_I = \sum S_i \otimes E_i$$

We define

$$H_I' = \sum_i S_i \otimes (E_i - \langle E_i \rangle_E) = \sum_i S_i \otimes E_i - \sum_i \langle E_i \rangle_E S_i \otimes 1_E$$

with $\langle E_i \rangle_E = Tr_E(\rho_E E_i)$

$$\langle H_I' \rangle = \sum_i S_i(\langle E_i \rangle_E - \langle E_i \rangle_E) = 0$$

$$H_T = H_S \otimes 1_E + 1_S \otimes H_E + \alpha H_I = H_S \otimes 1_E + 1_S \otimes H_E + \alpha(H_I' + S_i \langle E_i \rangle_E) =$$

$$(H_S + \alpha \sum_i \langle E_i \rangle_E S_i) \otimes 1_E + 1_S \otimes H_E + \alpha H_I'$$

So now $\langle H_I' \rangle_E = 0$

Symmetries

model	Hubbard	Extended
Discrete symmetry	parity invariance spin reflection Shiba $[H_1, S^\sigma S^\tau] = 0,\ L$ even	$[H_3, \otimes_{j=1}^{L} \sigma_j^z \tau_j^z] = 0$ even/odd spin conserved Shiba $[H_1, S^\sigma S^\tau] = 0,\ L$ even
Continuous symmetry	$SU(2) \otimes SU(2)$	-