AXEL-2023

Introduction to Particle Accelerators

Resonances:

- ✓ *Normalised Phase Space*
- ✓ *Dipoles, Quadrupoles, Sextupoles*
- ✓ *A more rigorous approach*
- ✓ *Coupling*
- ✓ *Tune diagram*

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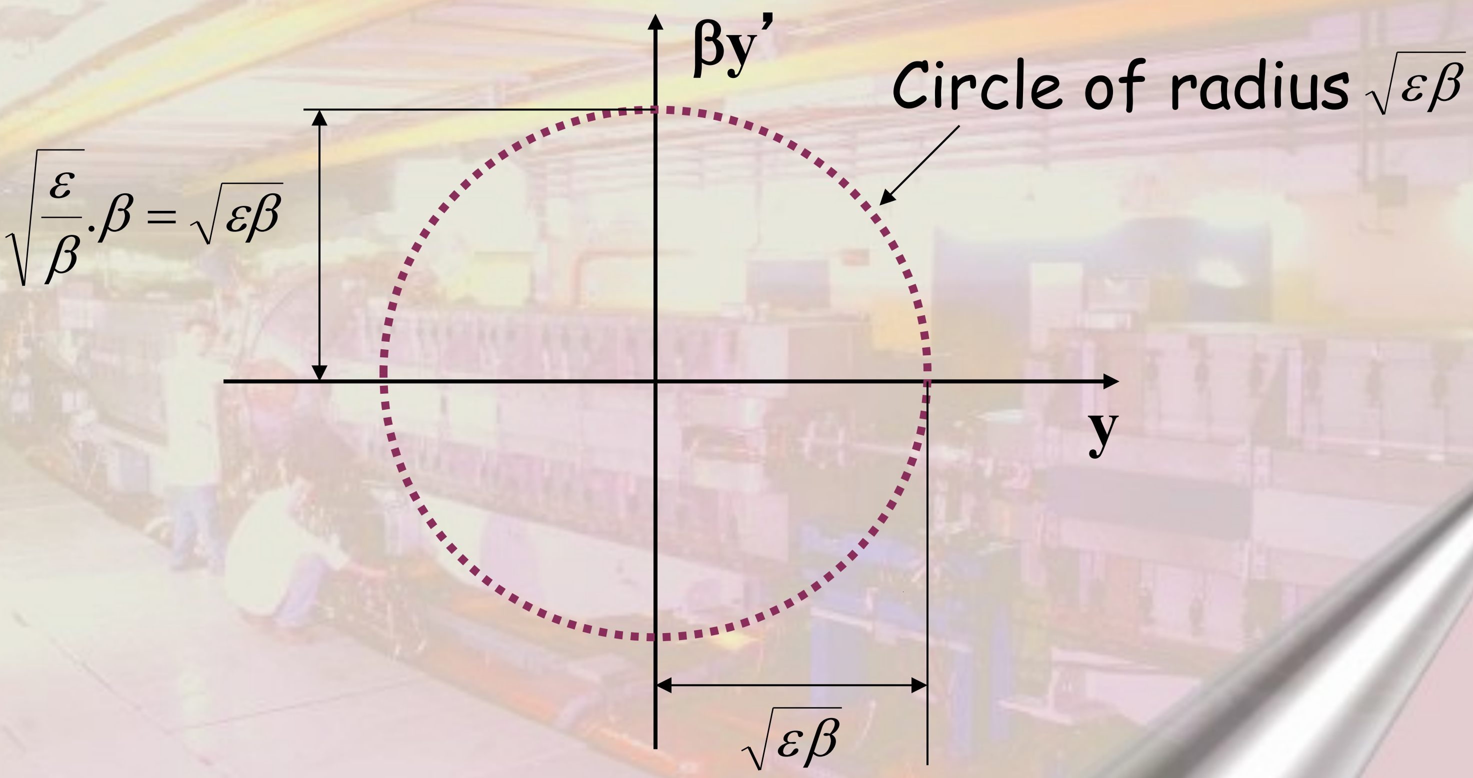
Normalised Phase Space

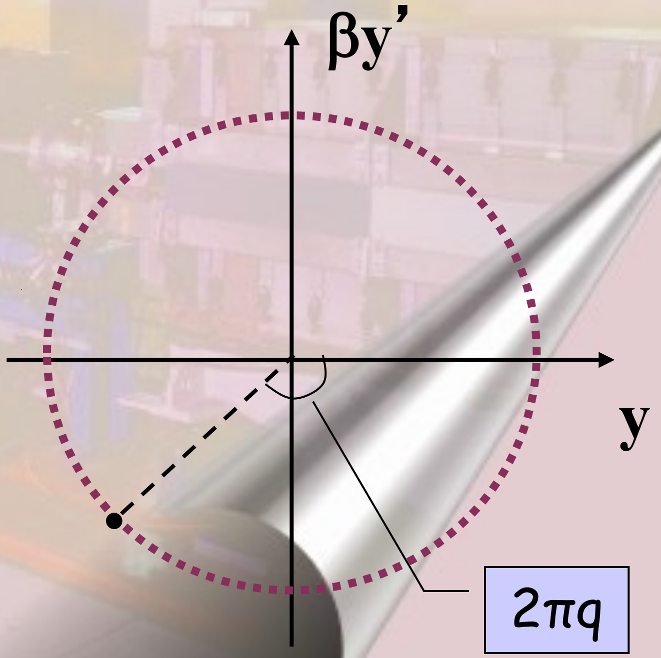


- ✓ By multiplying the y-axis by β the transverse phase space is normalised and the ellipse turns into a circle.

Phase Space & Betatron Tune

- ✓ If we unfold a trajectory of a particle that makes one turn in our machine with a tune of Q = 3.333, we get:

- ✓ This is the same as going 3.333 time around on the circle in phase space
- ✓ The net result is 0.333 times around the circular trajectory in the normalised phase space
- ✓ q is the fractional part of Q
- ✓ So here Q= 3.333 and q = 0.333

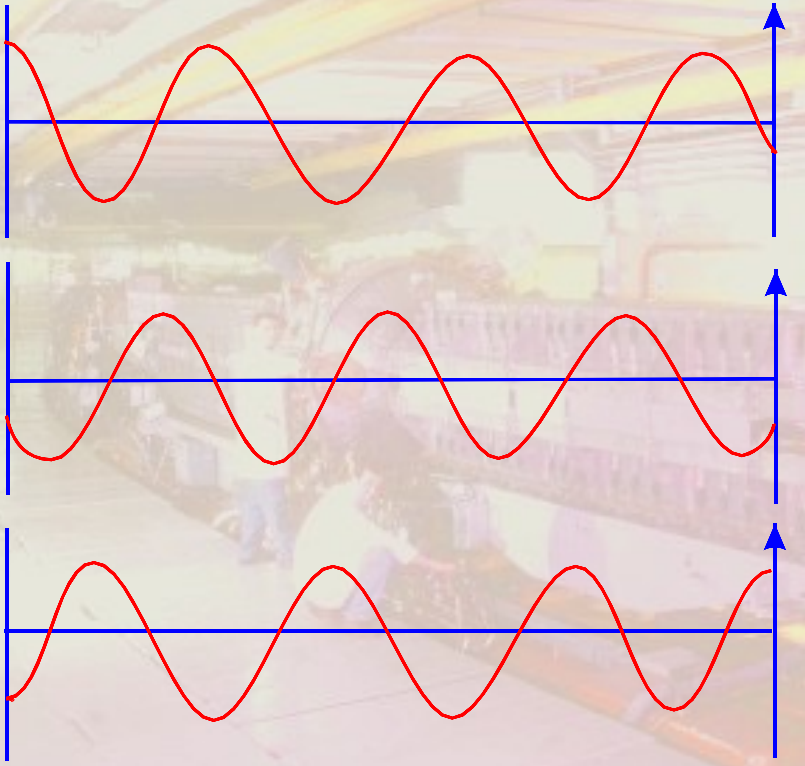


What is a resonance?

- After a certain number of turns around the machine the phase advance of the betatron oscillation is such that the oscillation repeats itself.
- For example:
  - If the phase advance per turn is 120° then the betatron oscillation will repeat itself after 3 turns.
  - This could correspond to $Q = 3.333$ or $3Q = 10$
  - But also $Q = 2.333$ or $3Q = 7$
- The order of a resonance is defined as 'n'

$n \times Q = \text{integer}$

Q = 3.333 in more detail

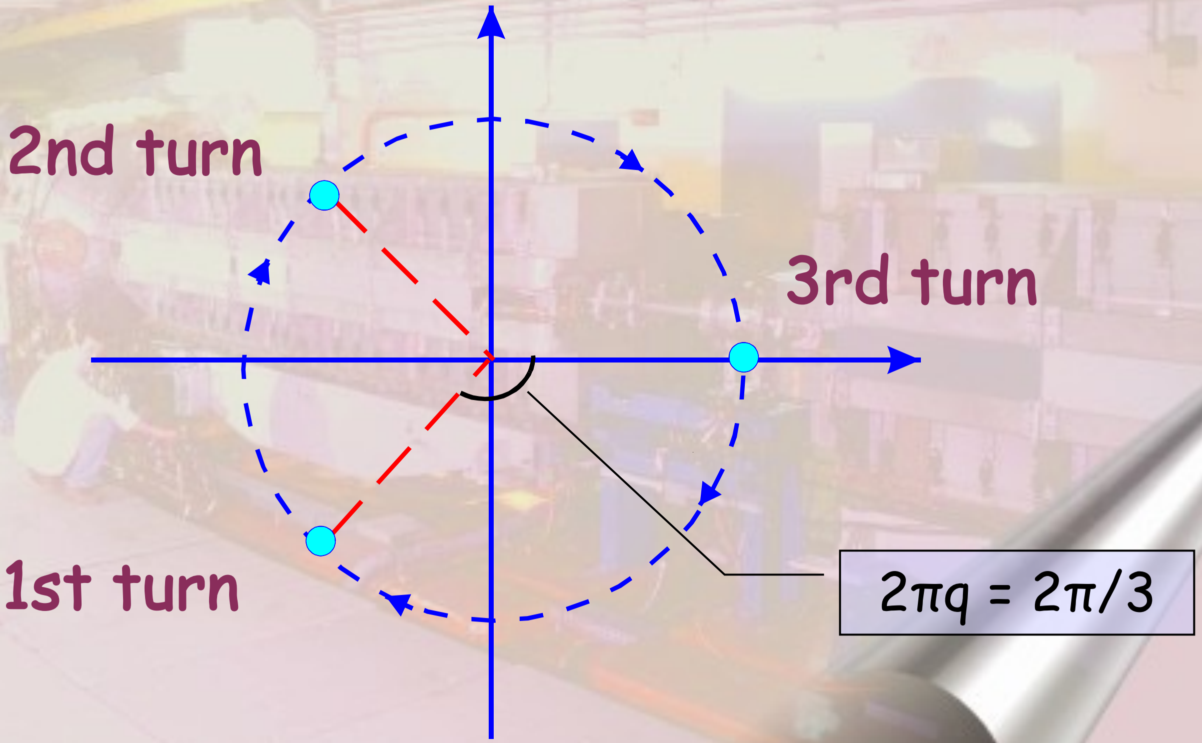


Third order resonant betatron oscillation

3Q = 10, Q = 3.333, q = 0.333

Q = 3.333 in Phase Space

- ✓ Third order resonance on a normalised phase space plot



$2\pi q = 2\pi/3$

Machine imperfections

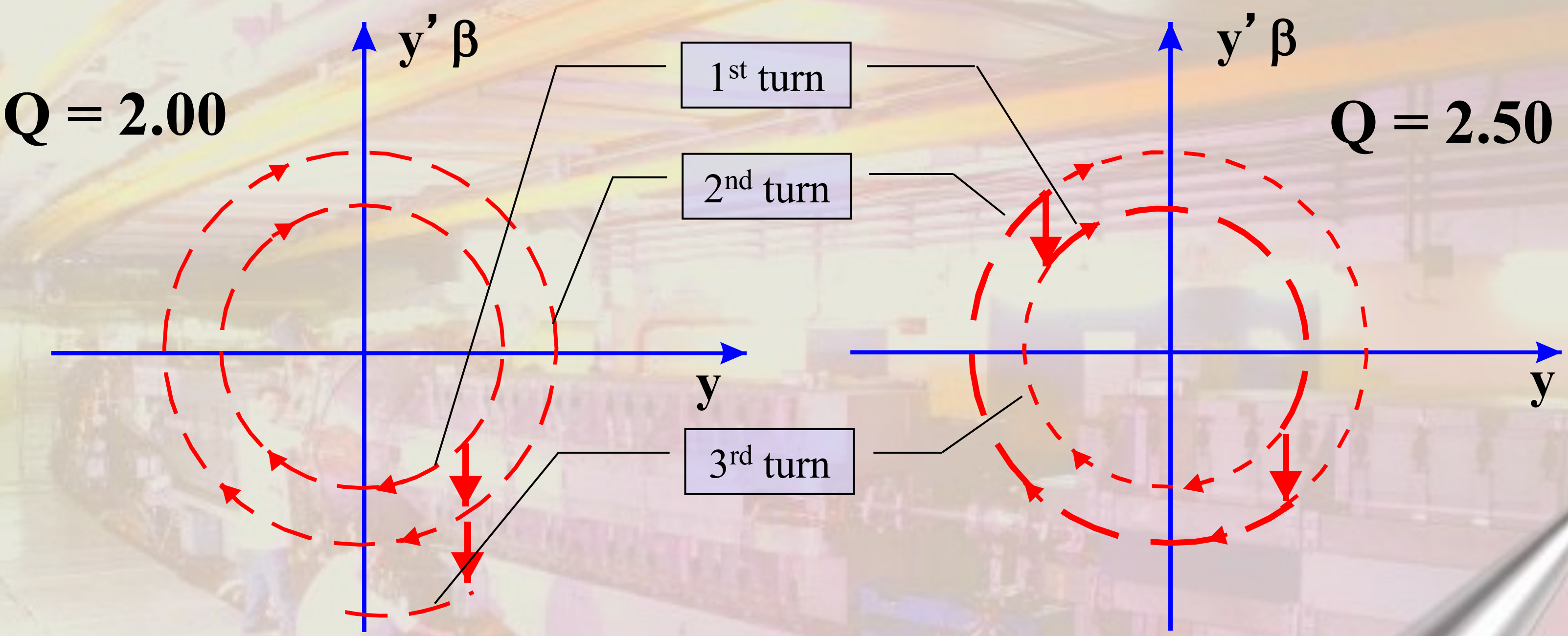
- It is not possible to construct a perfect machine.
  - Magnets can have imperfections
  - The alignment in the de machine has non zero tolerance.
  - Etc…

- So, we have to ask ourselves:
  - What will happen to the betatron oscillations due to the different field errors.
  - Therefore we need to consider errors in dipoles, quadrupoles, sextupoles, etc…

- We will have a look at the beam behaviour as a function of ‘Q’

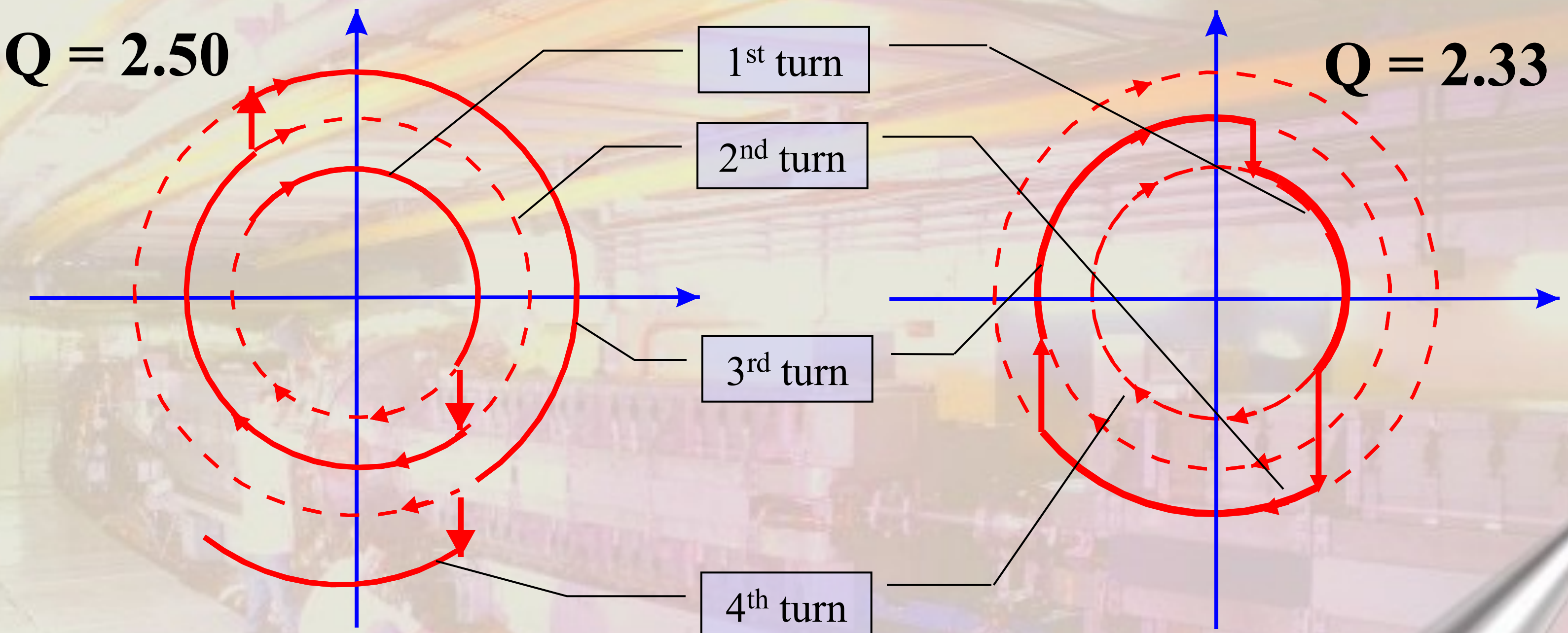
- How is it influenced by these resonant conditions?

Dipole (deflection independent of position)



- ✓ For Q = 2.00: Oscillation induced by the dipole kick grows on each turn and the particle is lost (1st order resonance Q = 2).
- ✓ For Q = 2.50: Oscillation is cancelled out every second turn, and therefore the particle motion is stable.

Quadrupole (deflection ∝ position)

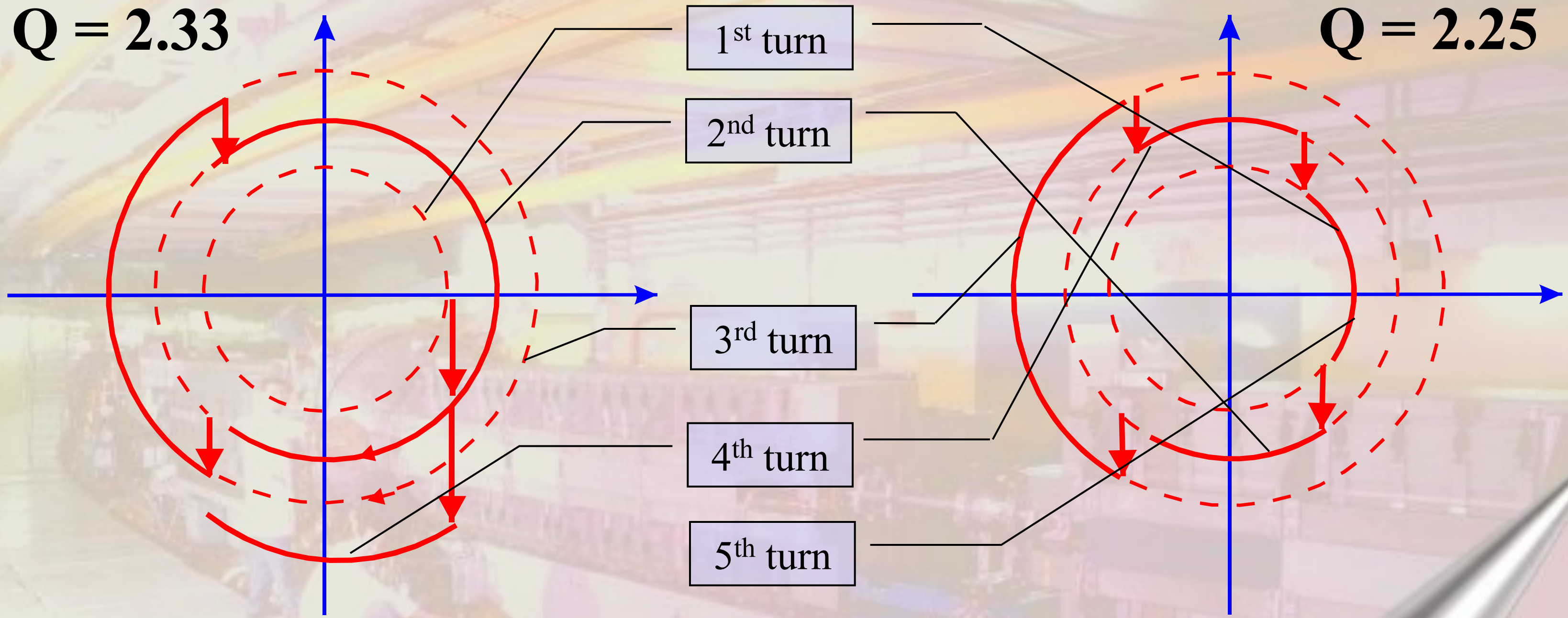


- ✓ For Q = 2.50: Oscillation induced by the quadrupole kick grows on each turn and the particle is lost

  (2nd order resonance 2Q = 5)

- ✓ For Q = 2.33: Oscillation is cancelled out every third turn, and therefore the particle motion is stable.

Sextupole (deflection ∝ position²)



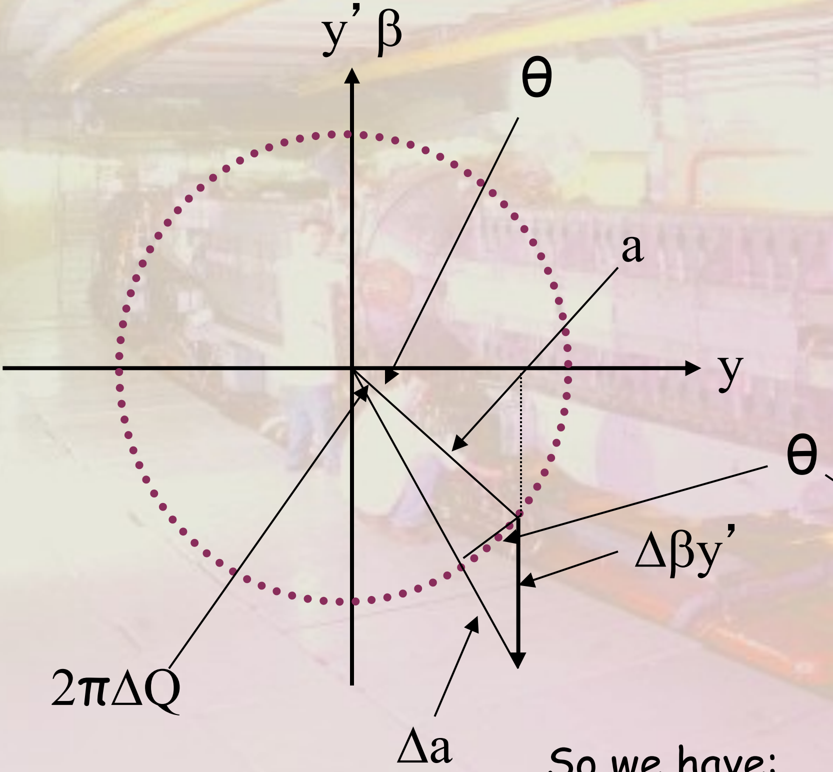
- ✓ For Q = 2.33: Oscillation induced by the sextupole kick grows on each turn and the particle is lost

  (3rd order resonance 3Q = 7)

- ✓ For Q = 2.25: Oscillation is cancelled out every fourth turn, and therefore the particle motion is stable.

More rigorous approach (1)

- ✓ Let us try to find a **mathematical expression** for the **amplitude growth** in the case of a quadrupole error:



2πQ = phase angle over 1 turn = θ

Δβy' = kick

a = old amplitude

Δa = change in amplitude

2πΔQ = change in phase

y does not change at the kick

$y = a\cos(\theta)$

In a quadrupole $\Delta y' = lky$

Only if 2πΔQ is small

So we have:

$$\Delta a = \beta \Delta y' \sin(\theta) = l\beta \sin(\theta)\, a\, k \cos(\theta)$$

More rigorous approach (2)

- ✓ So we have: $\Delta a = l \cdot \beta \cdot \sin(\theta)\ a \cdot k \cdot \cos(\theta)$ $\therefore$ $\dfrac{\Delta a}{a} = \dfrac{\ell \beta k}{2} \sin(2\theta)$

Sin(θ)Cos(θ) = 1/2 Sin (2θ)

- ✓ Each turn θ advances by 2πQ
- ✓ On the nth turn θ = θ + 2nπQ
- ✓ Over many turns: $\dfrac{\Delta a}{a} = \dfrac{\ell \beta k}{2} \sum_{n=1}^{\infty} \sin\left(2\left(\underline{\theta + 2n\pi Q}\right)\right)$

This term will be 'zero' as it decomposes in Sin and Cos terms and will give a series of + and – that cancel out in all cases where the fractional tune q ≠ 0.5

- ✓ So, for q = 0.5 the phase term, 2(θ + 2nπQ) is constant:

$$\sum_{n=1}^{\infty} \sin\left(2\left(\theta + 2n\pi Q\right)\right) = \infty$$

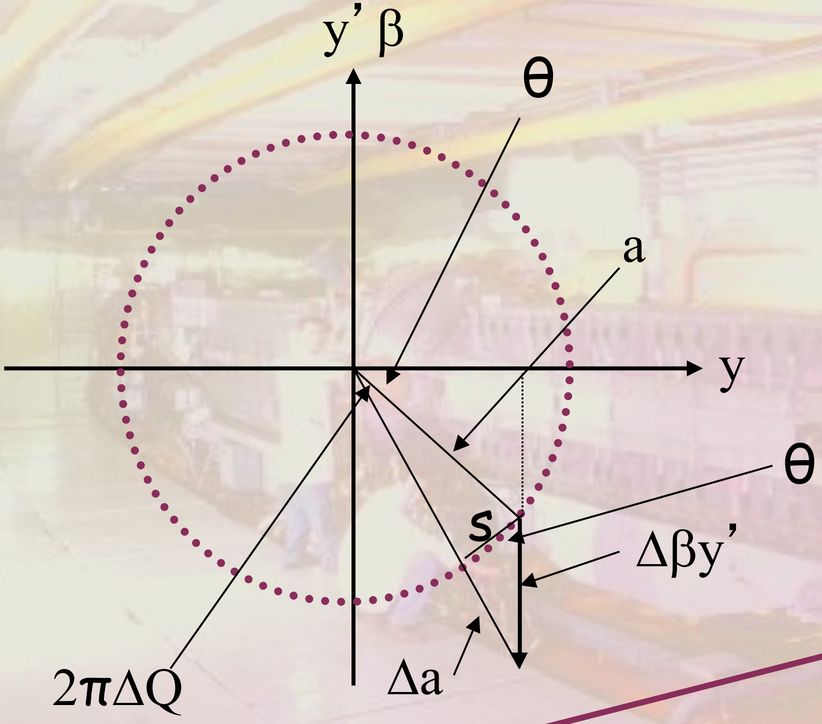
and thus: $\dfrac{\Delta a}{a} = \infty$

More rigorous approach (3)

- ✓ In this case the amplitude will grow continuously until the particles are lost.
- ✓ Therefore we conclude as before that:
  quadrupoles excite 2nd order resonances for q=0.5
- ✓ Thus for Q = 0.5, 1.5, 2.5, 3.5,…etc……

More rigorous approach (4)

- ✓ Let us now look at the phase θ for the same quadrupole error:



2πQ = phase angle over 1 turn = θ

Δβy' = kick

a = old amplitude

Δa = change in amplitude

2πΔQ = change in phase

y does not change at the kick

$y = a\cos(\theta)$

In a quadrupole **$\Delta y' = lky$**

$$s = \Delta(\beta y')\cos\theta$$

$$2\pi\Delta Q = \frac{\Delta(\beta y')\cos\theta}{a} \rightarrow \Delta Q = \frac{1}{2\pi}\cdot\frac{\beta\cdot\cos(\theta)\cdot l\cdot a\cdot k\cdot\cos(\theta)}{a}$$

2πΔQ << Therefore Sin(2πΔQ) ≈ 2πΔQ

More rigorous approach (5)

- ✓ So we have: $\Delta Q = \frac{1}{2\pi} \cdot \frac{\beta \cdot \cos(\theta) \cdot l \cdot a \cdot k \cdot \cos(\theta)}{a}$
- ✓ Since: $Cos^2(\theta) = \frac{1}{2} Cos(2\theta) + \frac{1}{2}$ we can rewrite this as:

$\Delta Q = \frac{1}{4\pi} \cdot l \cdot \beta \cdot k \cdot (\cos(2\theta) + 1)$, which is correct for the 1st turn

- ✓ Each turn θ advances by 2πQ
- ✓ On the nth turn θ = θ + 2nπQ
- ✓ Over many turns: $\Delta Q = \frac{1}{4\pi} \ell \beta k \left[\underbrace{\sum_{n=1}^{\infty} \cos\left(2(\theta + 2\pi n Q)\right)}_{\text{‘zero’}} + 1 \right]$
- ✓ Averaging over many turns: $\Delta Q = \frac{1}{4\pi} \beta.k.ds$

Stopband

- ✓ $\Delta Q = \frac{1}{4\pi}\beta.k.ds$, which is the expression for the change in Q due to a quadrupole… (fortunately !!!)

- ✓ But note that Q changes slightly on each turn

Related to Q

$$\Delta Q = \frac{1}{4\pi} l \cdot \beta \cdot k \underline{(\cos(2\theta)+1)}$$

Max variation 0 to 2

- ✓ Q has a range of values varying by: $\frac{\ell\beta k}{2\pi}$

- ✓ This width is called the **stopband** of the resonance

- ✓ So even if q is not exactly 0.5, it must not be too close, or at some point it will find itself at exactly 0.5 and 'lock on' to the resonant condition.

Sextupole kick

- ✓ We can apply the same arguments for a sextupole:
- ✓ For a sextupole $\Delta y' = \ell k y^2$ and thus $\Delta y' = \ell k a^2 \cos^2 \theta$
- ✓ We get :

$$\frac{\Delta a}{a} = \ell \beta k a \sin\theta \cos^2\theta = \frac{\ell \beta k a}{2}\left[\cos 3\theta + \cos\theta\right]$$

- ✓ Summing over many turns gives:

$$\frac{\Delta a}{a} = \frac{\ell \beta k a}{2} \sum_{n=1}^{\infty} \underline{\cos 3(\theta + 2\pi n Q)} + \underline{\cos(\theta + 2\pi n Q)}$$

3rd order resonance term

1st order resonance term

- ✓ **Sextupole** excite **1st** and **3rd** **order resonance**

q = 0

q = 0.33

Octupole kick

- ✓ We can apply the same arguments for an octupole:
- ✓ For an octupole $\Delta y' = \ell k y^3$ and thus $\Delta y' = \ell k a^3 \cos^3 \theta$
- ✓ We get : $\frac{\Delta a}{a} = \ell \beta k a^2 \sin\theta \cos^3 \theta$
- ✓ Summing over many turns gives:

4th order resonance term

2nd order resonance term

$$\frac{\Delta a}{a} \propto a^2(\overline{\cos 4(\theta+2\pi nQ)} + \overline{\cos 2(\theta+2\pi nQ)})$$

Amplitude squared

q = 0.5

q = 0.25

- ✓ Octupolar errors excite 2nd and 4th order resonance and are very important for larger amplitude particles.

Can restrict dynamic aperture

Resonance summary

- ✓ **Quadrupoles** excite **2nd** order resonances
- ✓ **Sextupoles** excite **1st** and **3rd** order resonances
- ✓ **Octupoles** excite **2nd** and **4th** order resonances
- ✓ This is true for small amplitude particles and low strength excitations
- ✓ However, for stronger excitations sextupoles will excite higher order resonance's (non-linear)

Coupling

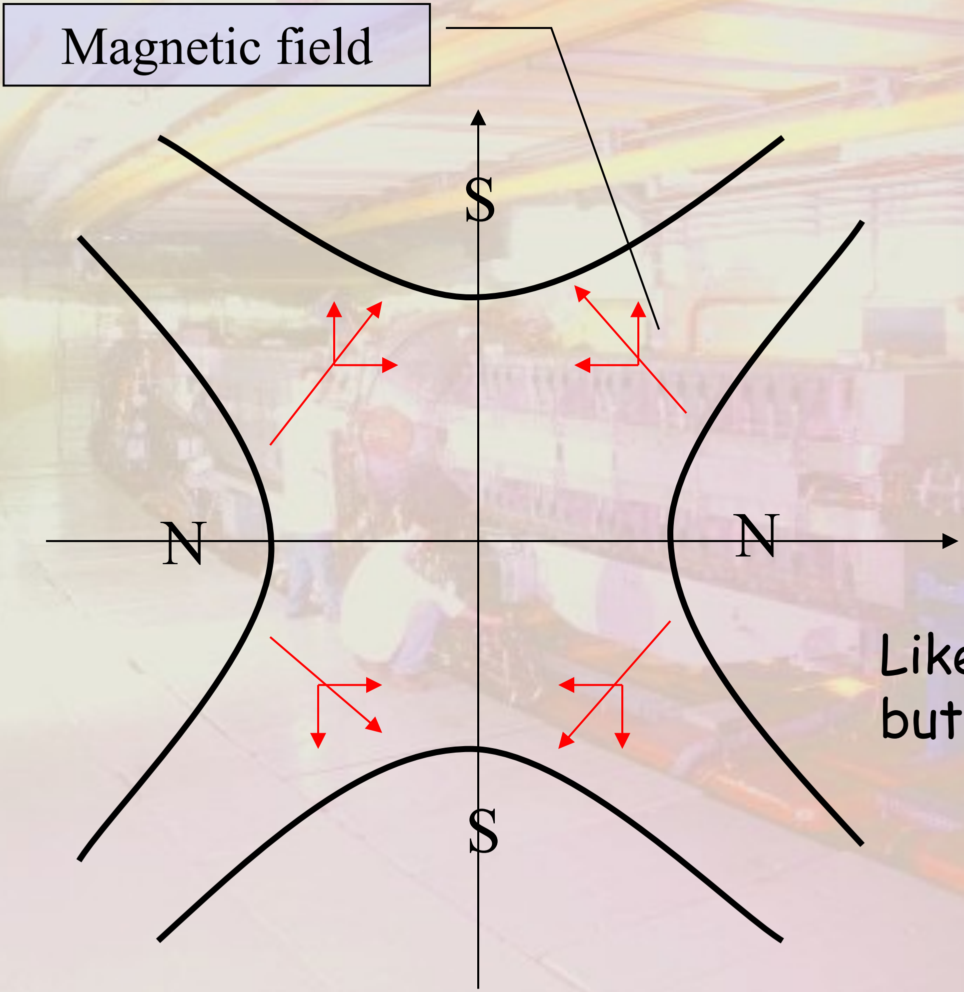
- ✓ Coupling is a phenomena, which converts betatron motion from one plane (horizontal or vertical) into motion in the other plane.

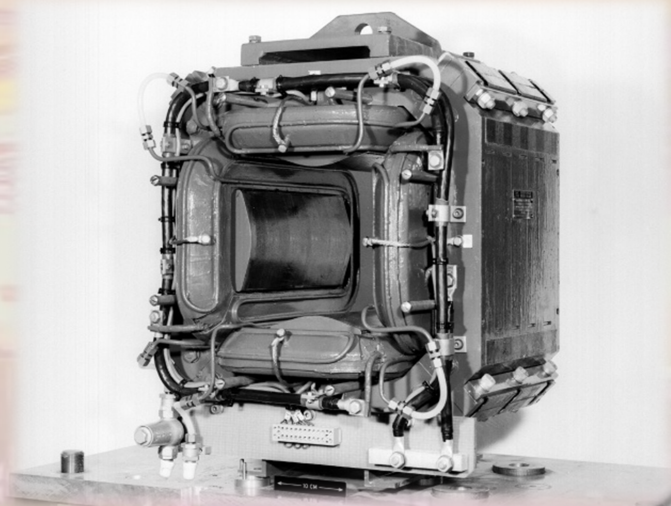
- ✓ Fields that will excite coupling are:

  - ✓ Skew quadrupoles, which are normal quadrupoles, but tilted by 45° about it's longitudinal axis.

  - ✓ Solenoidal (longitudinal magnetic field)

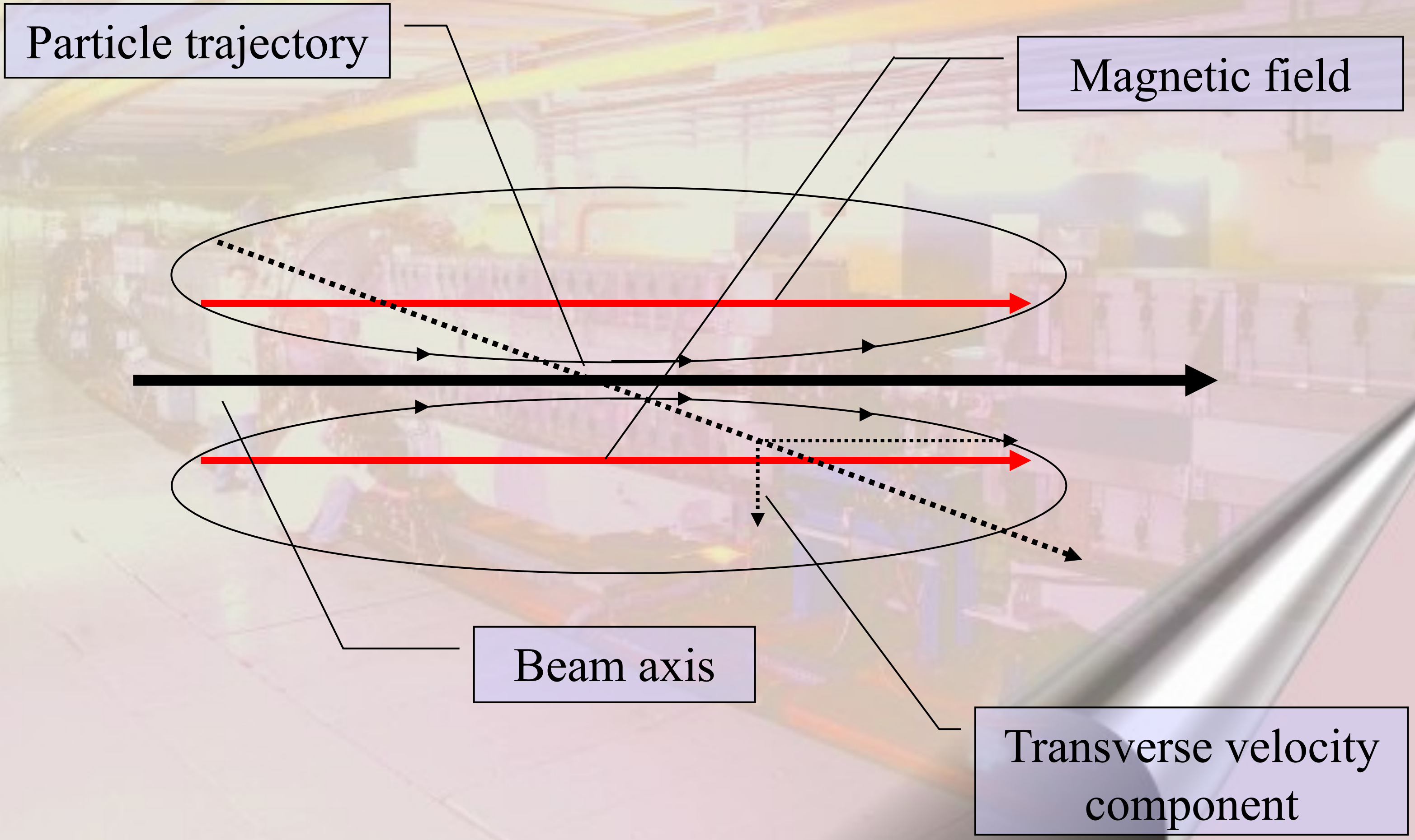
Skew Quadrupole





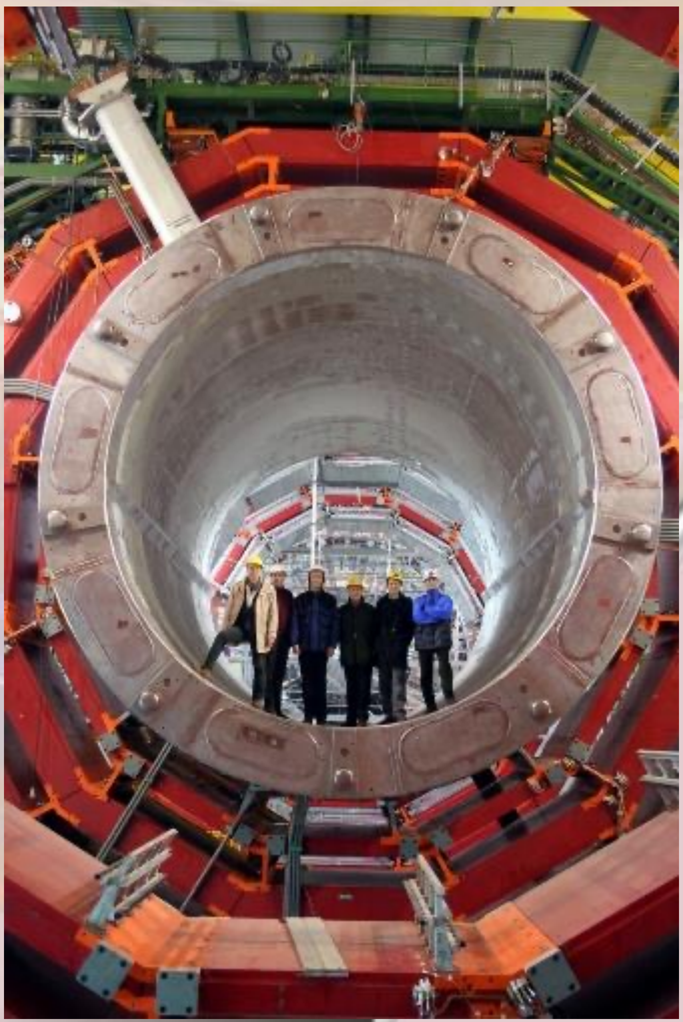
Like a normal quadrupole, but then tilted by 45°

Solenoid; longitudinal field (2)



Solenoid; longitudinal field (2)

Above:
The LPI solenoid that was used for the initial focusing of the positrons.
It was pulsed with a current of 6 kA for some 7 us, it produced a longitudinal magnetic field of 1.5 T.



At the right:
The somewhat bigger CMS solenoid

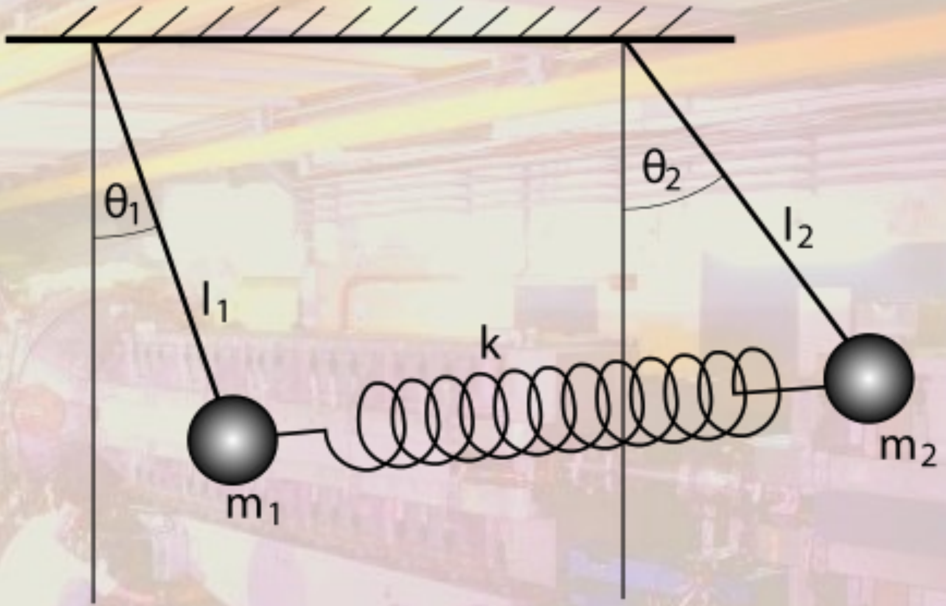
Coupling and Resonance

- ✓ This coupling means that one can transfer oscillation energy from one transverse plane to the other.
- ✓ Exactly as for linear resonances there are resonant conditions.

$$nQ_h \pm mQ_v = \text{integer}$$

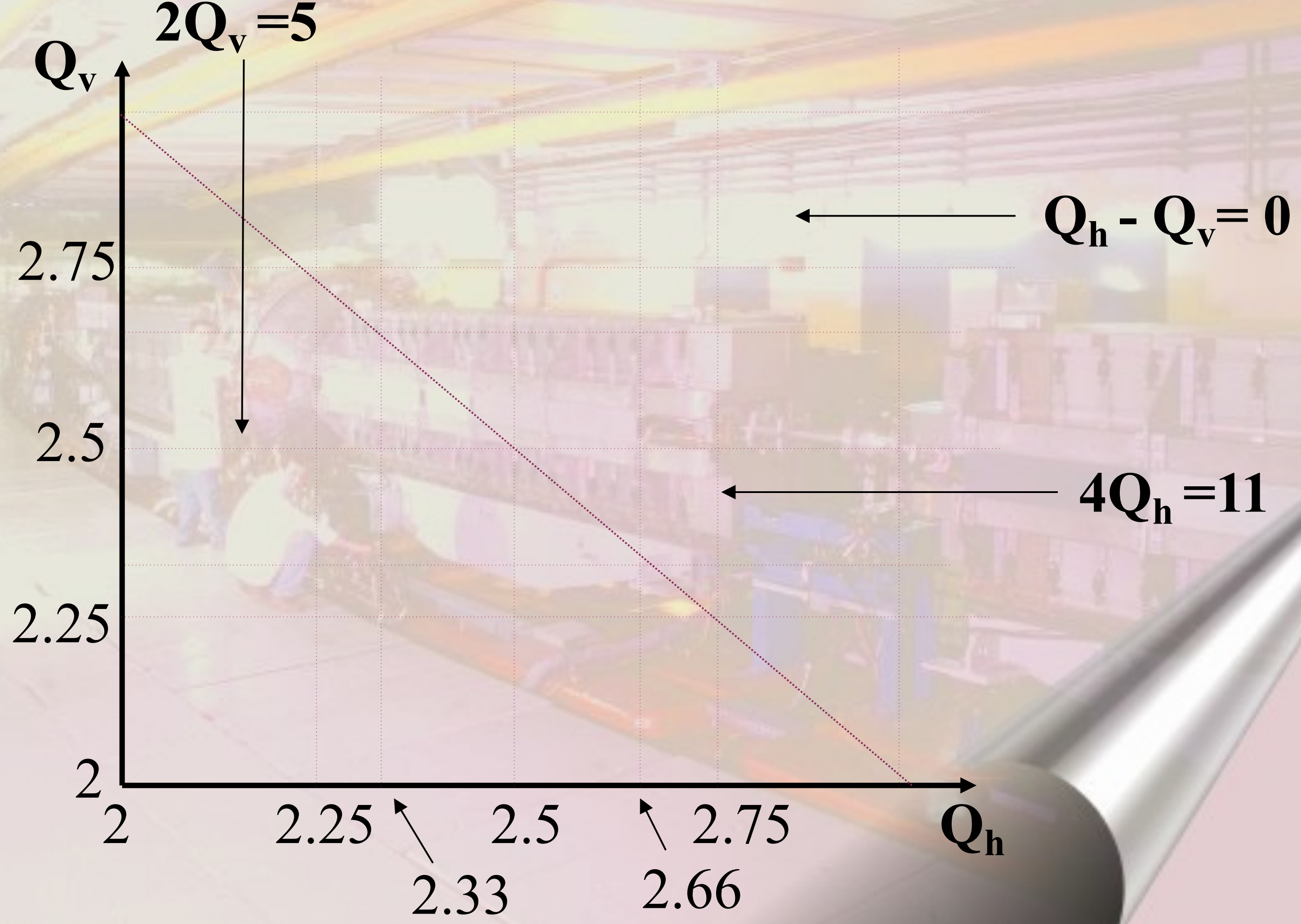
- ✓ If we meet one of these conditions the transverse oscillation amplitude will again grow in an uncontrolled way.

A mechanical equivalent

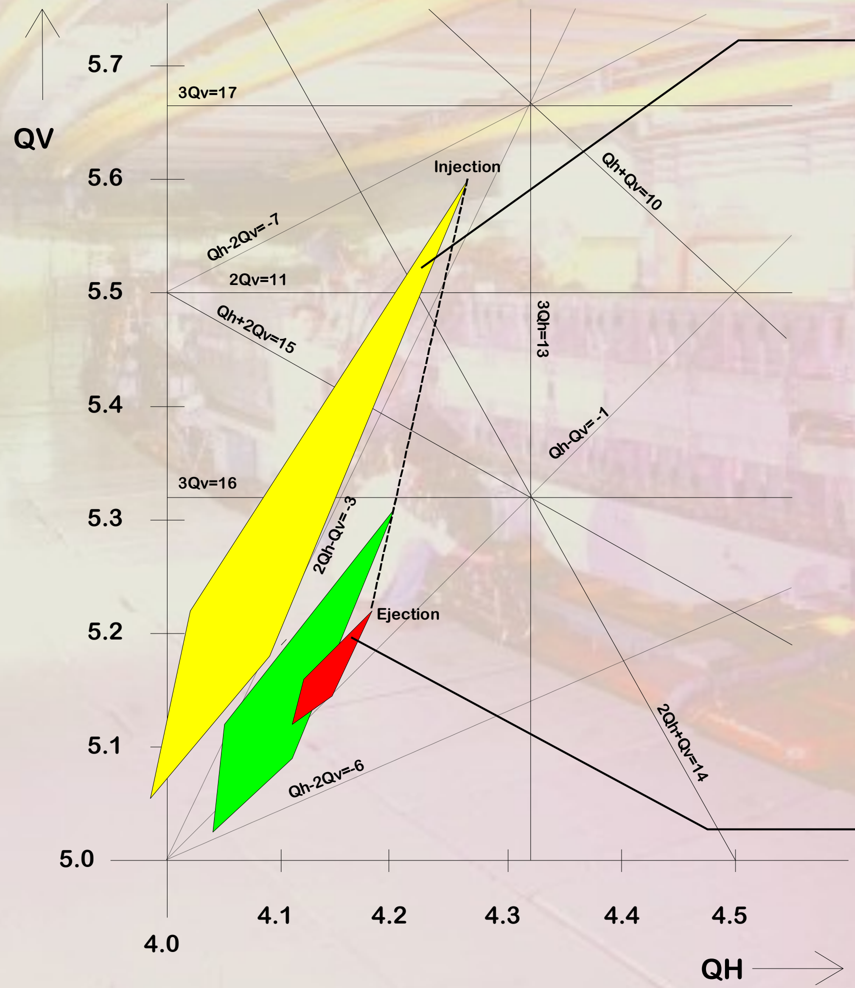


- We can transfer oscillation energy from one pendulum to the other depending on the strength 'k' of the spring

General tune diagram



Realistic tune diagram

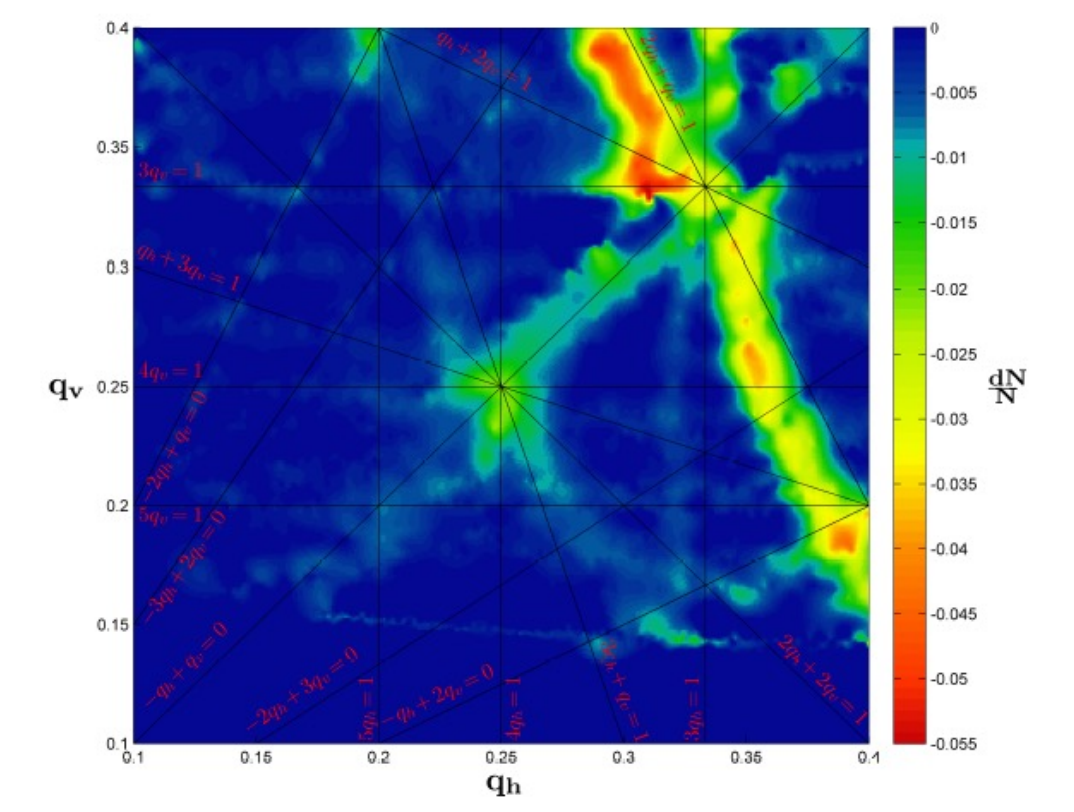


injection

ejection

During acceleration we change the horizontal and vertical tune to a place where the beam is the least influenced by resonances.

Measured tune diagram



Move a large emittance beam around in this tune diagram and measure the beam losses.

Not all resonance lines are harmful.

Conclusion

- ✓ There are many things in our machine, which will excite resonances:
  - ✓ The magnets themselves
  - ✓ Unwanted higher order field components in our magnets
  - ✓ Tilted magnets
  - ✓ Experimental solenoids (LHC experiments)

- ✓ The trick is to reduce and compensate these effects as much as possible and then find some point in the tune diagram where the beam is stable.

Questions….,Remarks…?

Phase space

Resonance

Coupling

Tune diagram