

AXEL-2023

Introduction to Particle Accelerators

Lattice calculations:

- ✓ *Lattices*
- ✓ *Tune Calculations*
- ✓ *Dispersion*
- ✓ *Momentum Compaction*
- ✓ *Chromaticity*
- ✓ *Sextupoles*

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A quick recap.....

- ✓ We solved Hill's equation, which led us to the definition of transverse emittance and allowed us to describe particle motion in transverse phase space in terms of β , α , etc...
- ✓ We constructed the Transport Matrices corresponding to drift spaces and quadrupoles.
- ✓ Now we must combine these matrices with the solution of Hill's equation to evaluate β , α , etc...

Matrices & Hill's equation

- ✓ We can multiply the matrices of our drift spaces and quadrupoles together to form a transport matrix that describes a larger section of our accelerator.
- ✓ These matrices will move our particle from one point $(x(s_1), x'(s_1))$ on our phase space plot to another $(x(s_2), x'(s_2))$, as shown in the matrix equation below.

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

- ✓ The elements of this matrix are fixed by the elements through which the particles pass from point s_1 to point s_2 .
- ✓ However, we can also express (x, x') as solutions of Hill's equation.

$$x = \sqrt{\varepsilon \cdot \beta} \cos \phi \quad \text{and} \quad x' = -\alpha \sqrt{\varepsilon / \beta} \cos \phi - \sqrt{\varepsilon / \beta} \sin \phi$$

Matrices & Hill's equation (2)

$$x = \sqrt{\varepsilon \cdot \beta} \cos(\mu + \phi)$$

$$x = \sqrt{\varepsilon \cdot \beta} \cos \phi$$

$$\begin{pmatrix} x(s_2) \\ x'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x(s_1) \\ x'(s_1) \end{pmatrix}$$

$$x' = -\alpha \sqrt{\varepsilon / \beta} \cos(\mu + \phi) - \sqrt{\varepsilon / \beta} \sin(\mu + \phi)$$

$$x' = -\alpha \sqrt{\varepsilon / \beta} \cos \phi - \sqrt{\varepsilon / \beta} \sin \phi$$

- ✓ Assume that our transport matrix describes a complete turn around the machine.
- ✓ Therefore : $\beta(s_2) = \beta(s_1)$
- ✓ Let μ be the change in betatron phase over one complete turn.
- ✓ Then we get for $x(s_2)$:

$$x(s_2) = \sqrt{\varepsilon \cdot \beta} \cos(\mu + \phi) = a \sqrt{\varepsilon \cdot \beta} \cos \phi - b \alpha \sqrt{\varepsilon / \beta} \cos \phi - b \sqrt{\varepsilon / \beta} \sin \phi$$

Matrices & Hill's equation (3)

- ✓ So, for the position x at s_2 we have...

$$\sqrt{\varepsilon \cdot \beta} \cos(\mu + \phi) = a \sqrt{\varepsilon \cdot \beta} \cos \phi - b \alpha \sqrt{\varepsilon / \beta} \cos \phi - b \sqrt{\varepsilon / \beta} \sin \phi$$

$$\cos \phi \cos \mu - \sin \phi \sin \mu$$

- ✓ Equating the 'sin' terms gives: $-\sqrt{\varepsilon \cdot \beta} \sin \mu \sin \phi = -b \sqrt{\varepsilon / \beta} \sin \phi$

- ✓ Which leads to: $b = \beta \sin \mu$

- ✓ Equating the 'cos' terms gives:

$$\sqrt{\varepsilon \cdot \beta} \cos \mu \cos \phi = a \sqrt{\varepsilon \cdot \beta} \cos \phi - \alpha \sqrt{\varepsilon \cdot \beta} \sin \mu \cos \phi$$

- ✓ Which leads to: $a = \cos \mu + \alpha \sin \mu$

- ✓ We can repeat this for c and d .

Matrices & Twiss parameters

✓ Remember previously we defined:

$$\alpha = -\frac{\beta'}{2} = -\omega\omega'$$
$$\beta = \omega^2$$
$$\gamma = \frac{1 + \alpha^2}{\beta}$$

✓ These are called TWISS parameters

✓ Remember also that μ is the total betatron phase advance over one complete turn is.

$$Q = \frac{\mu}{2\pi}$$

Number of betatron oscillations per turn

✓ Our transport matrix becomes now:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}$$

Lattice parameters

$$\begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}$$

- ✓ This matrix describes one complete turn around our machine and will vary depending on the starting point (s).
- ✓ If we start at any point and multiply all of the matrices representing each element all around the machine we can calculate α , β , γ and μ for that specific point, which then will give us $\beta(s)$ and Q
- ✓ If we repeat this many times for many different initial positions (s) we can calculate our Lattice Parameters for all points around the machine.

Lattice calculations and codes

- ✓ Obviously μ (or Q) is not dependent on the initial position 's', but we can calculate the change in betatron phase, $d\mu$, from one element to the next.
- ✓ Computer codes like "MAD" or "Transport" vary lengths, positions and strengths of the individual elements to obtain the desired beam dimensions or envelope ' $\beta(s)$ ' and the desired ' Q '.
- ✓ Often a machine is made of many individual and identical sections (FODO cells). In that case we only calculate a single cell and not the whole machine, as the the functions $\beta(s)$ and $d\mu$ will repeat themselves for each identical section.
- ✓ The insertion sections have to be calculated separately.

The $\beta(s)$ and Q relation.

✓ $Q = \frac{\mu}{2\pi}$, where $\mu = \Delta\Phi$ over a complete turn

✓ But we also found: $\frac{d\phi(s)}{ds} = \frac{1}{\beta(s)}$

Over one complete turn

✓ This leads to: $Q = \frac{1}{2\pi} \int_0^s \frac{ds}{\beta(s)}$

✓ Increasing the focusing strength decreases the size of the beam envelope (β) and increases Q and vice versa.

Tune corrections

- ✓ What happens if we change the focusing strength slightly?
- ✓ The Twiss matrix for our 'FODO' cell is given by:

$$\begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}$$

- ✓ Add a small QF quadrupole, with strength dK and length ds .
- ✓ This will modify the 'FODO' lattice, and add a horizontal focusing term:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix}$$

$$dk = \frac{dK}{(B\rho)}$$

$$f = \frac{(B\rho)}{dKds}$$

- ✓ The new Twiss matrix representing the modified lattice is:

$$\begin{pmatrix} 1 & 0 \\ -dkds & 1 \end{pmatrix} \begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -\gamma \sin \mu & \cos \mu - \alpha \sin \mu \end{pmatrix}$$

Tune corrections (2)

✓ This gives

$$\begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -dkds(\cos \mu + \sin \mu) - \gamma \sin \mu & -dkds \beta \sin \mu + \cos \mu - \alpha \sin \mu \end{pmatrix}$$

- ✓ This extra quadrupole will modify the phase advance μ for the FODO cell.

New phase advance

$$\mu_1 = \mu + d\mu$$

Change in phase advance

- ✓ If $d\mu$ is small then we can ignore changes in β
- ✓ So the new Twiss matrix is just:

$$\begin{pmatrix} \cos \mu_1 + \alpha \sin \mu_1 & \beta \sin \mu_1 \\ -\gamma \sin \mu_1 & \cos \mu_1 - \alpha \sin \mu_1 \end{pmatrix}$$

Tune corrections (3)

- ✓ These two matrices represent the same FODO cell therefore:

$$\begin{pmatrix} \cos \mu + \alpha \sin \mu & \beta \sin \mu \\ -dkds(\cos \mu + \sin \mu) - \gamma \sin \mu & -dkds \beta \sin \mu + \cos \mu - \alpha \sin \mu \end{pmatrix}$$

- ✓ Which equals:

$$\begin{pmatrix} \cos \mu_1 + \alpha \sin \mu_1 & \beta \sin \mu_1 \\ -\gamma \sin \mu_1 & \cos \mu_1 - \alpha \sin \mu_1 \end{pmatrix}$$

- ✓ Combining and compare the first and the fourth terms of these two matrices gives:

$$2 \cos \mu_1 = 2 \cos \mu - dk ds \beta \sin \mu$$

Only valid for change in $b \ll$

Tune corrections (4)

$$2 \cos \mu_1 = \underline{2 \cos \mu} - dk \, ds \, \beta \sin \mu$$

Remember $\mu_1 = \mu + d\mu$
and $d\mu$ is small

$$2 \cos \mu - 2 \sin \mu d\mu$$

$$2 \sin \mu d\mu = dk \, ds \, \beta \sin \mu$$

$$d\mu = \frac{1}{2} dk \, ds \, \beta$$

,but: $dQ = d\mu / 2\pi$

In the horizontal plane this is a QF

$$dQ_h = + \frac{1}{4\pi} dk \cdot ds \cdot \beta h$$

If we follow the same reasoning for both transverse planes for both QF and QD quadrupoles

QD

$$\begin{aligned} dQ_v &= + \frac{1}{4\pi} \beta v \cdot dk_D \cdot ds_D - \frac{1}{4\pi} \beta v \cdot dk_F \cdot ds_F \\ dQ_h &= - \frac{1}{4\pi} \beta h \cdot dk_D \cdot ds_D + \frac{1}{4\pi} \beta h \cdot dk_F \cdot ds_F \end{aligned}$$

QF

Tune corrections (5)

Let $\mathbf{dk}_F = \mathbf{dk}$ for QF and $\mathbf{dk}_D = \mathbf{dk}$ for QD

$\beta_{hF}, \beta_{vF} = \beta$ at QF and $\beta_{hD}, \beta_{vD} = \beta$ at QD

Then:

$$\begin{pmatrix} dQ_v \\ dQ_h \end{pmatrix} = \begin{pmatrix} \frac{1}{4\pi} \beta_{vD} & -\frac{1}{4\pi} \beta_{vF} \\ -\frac{1}{4\pi} \beta_{hD} & \frac{1}{4\pi} \beta_{hF} \end{pmatrix} \begin{pmatrix} dk_D ds \\ dk_F ds \end{pmatrix}$$

This matrix relates the change in the tune to the change in strength of the quadrupoles.

We can invert this matrix to calculate change in quadrupole field needed for a given change in tune

Dispersion (1)

- ✓ Until now we have assumed that our beam has no energy or momentum spread:

$$\frac{\Delta E}{E} = 0 \quad \text{and} \quad \frac{\Delta p}{p} = 0$$

- ✓ Different energy or momentum particles have different radii of curvature (ρ) in the main dipoles.
- ✓ These particles no longer pass through the quadrupoles at the same radial position.
- ✓ Quadrupoles act as dipoles for different momentum particles.
- ✓ Closed orbits for different momentum particles are different.
- ✓ This horizontal displacement is expressed as the dispersion function $D(s)$
- ✓ $D(s)$ is a function of 's' exactly as $\beta(s)$ is a function of 's'

Dispersion (2)

- ✓ The displacement due to the change in momentum at any position (s) is given by:

$$\Delta x(s) = D(s) \cdot \frac{\Delta p}{p}$$

Local radial displacement due to momentum spread

Dispersion function

- ✓ $D(s)$ the dispersion function, is calculated from the lattice, and has the unit of meters.
- ✓ The beam will have a finite horizontal size due to its momentum spread.
- ✓ In the majority of the cases we have no vertical dipoles, and so $D(s)=0$ in the vertical plane.

Momentum compaction factor

- ✓ The change in orbit with the changing momentum means that the average length of the orbit will also depend on the beam momentum.
- ✓ This is expressed as the momentum compaction factor, α_p , where:

$$\frac{\Delta r}{r} = \alpha_p \frac{\Delta p}{p}$$

- ✓ α_p tells us about the change in the length of radius of the closed orbit for a change in momentum.

Chromaticity

- ✓ The focusing strength of our quadrupoles depends on the beam momentum, 'p'

$$k = \frac{dB_y}{dx} \times \frac{1}{B\rho}$$

3.3356 p

- ✓ Therefore a spread in momentum causes a spread in focusing strength

$$\frac{\Delta k}{k} = - \frac{\Delta p}{p}$$

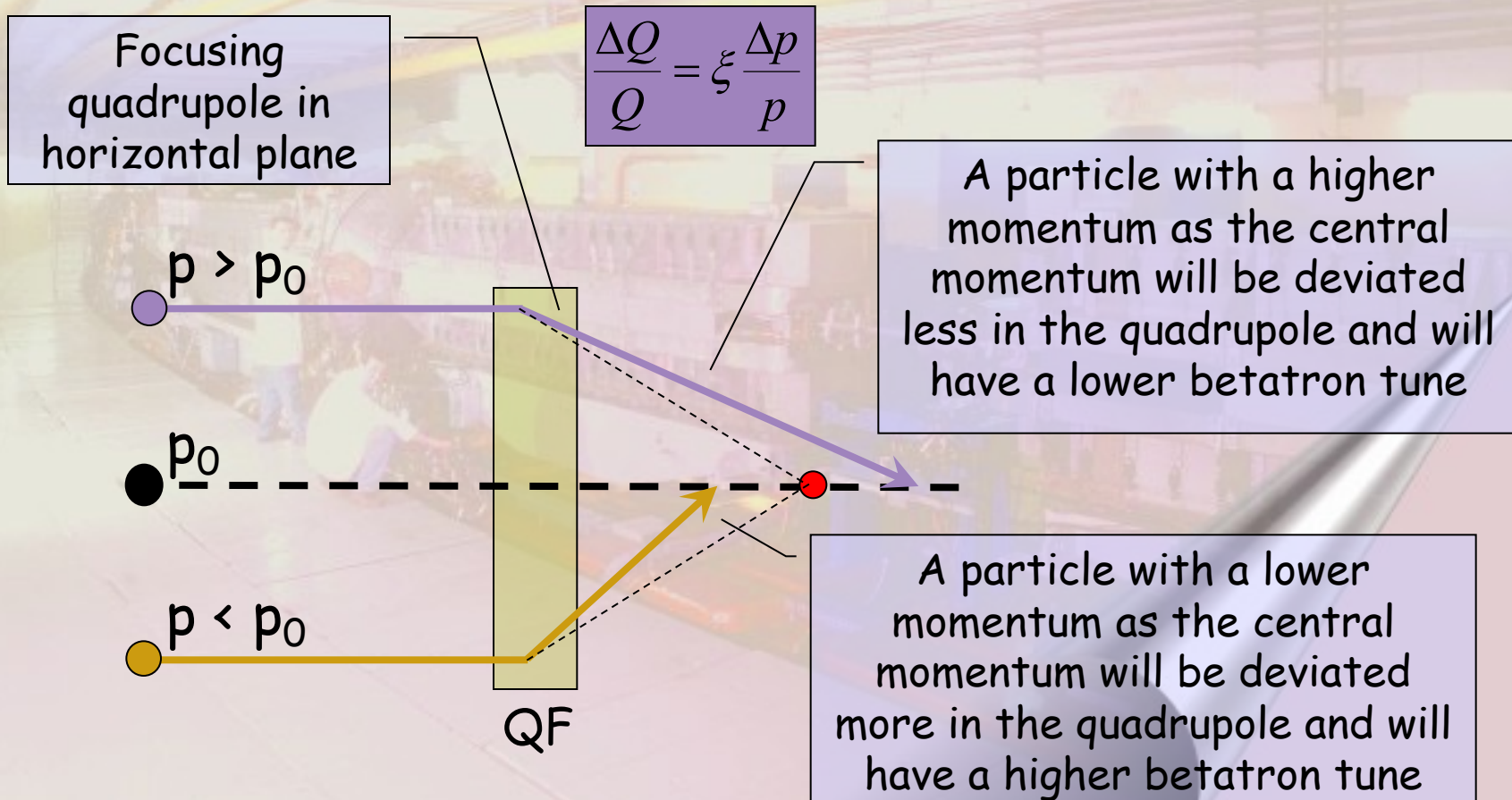
- ✓ But Q depends on the 'k' of the quadrupoles

$$\frac{\Delta Q}{Q} \propto \frac{\Delta p}{p} \longrightarrow \frac{\Delta Q}{Q} = \xi \frac{\Delta p}{p}$$

- ✓ The constant here is called : Chromaticity

Chromaticity visualized

- ✓ The chromaticity relates the tune spread of the transverse motion with the momentum spread in the beam.

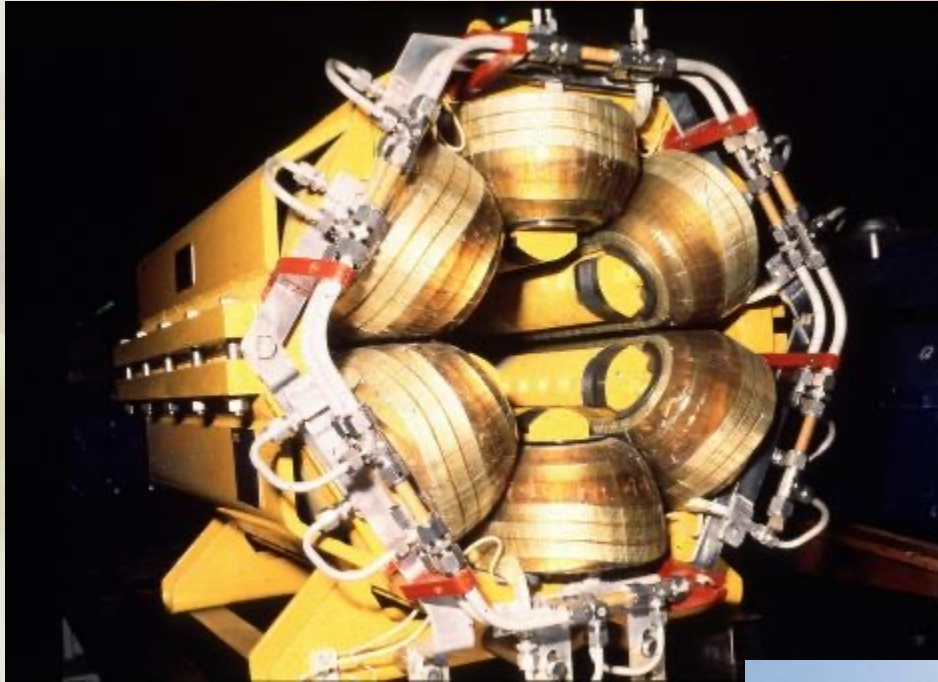


Chromaticity calculated

- ✓ Remember $\Delta Q = \frac{1}{4\pi} (\beta dk ds)$ and $\frac{\Delta k}{k} = -\frac{\Delta p}{p} \Rightarrow \Delta k = -k \frac{\Delta p}{p}$
- ✓ Therefore $\frac{\Delta Q}{Q} = -\frac{1}{4\pi} \left(\beta \frac{k}{Q} ds \right) \frac{\Delta p}{p}$

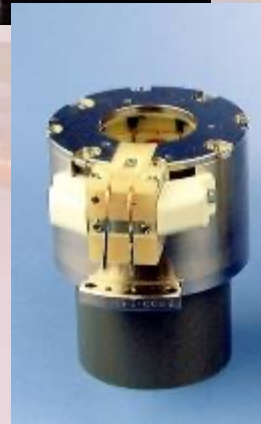
The gradient seen by the particle depends on its momentum
- ✓ This term is the Chromaticity ξ
- ✓ To correct this tune spread we need to increase the quadrupole focusing strength for higher momentum particles, and decrease it for lower momentum particles.
- ✓ This we will obtain using a Sextupole magnet

Sextupole Magnets

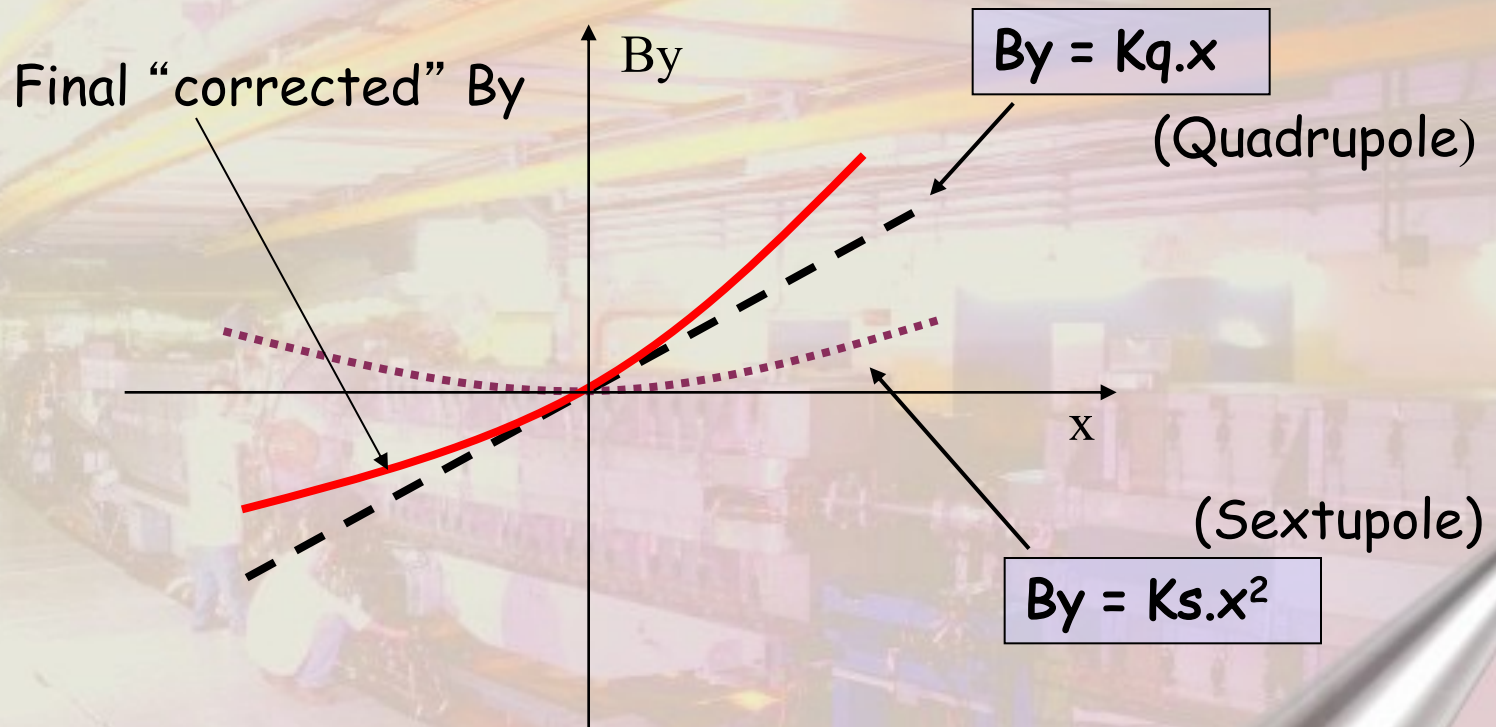


- ✓ Conventional Sextupole from LEP, but looks similar for other 'warm' machines.
- ✓ ~ 1 meter long and a few hundreds of kg.

- ✓ Correction Sextupole of the LHC
- ✓ 11cm, 10 kg, 500A at 2K for a field of 1630 T/m^2



Chromaticity correction



- ✓ Vertical magnetic field versus horizontal displacement in a quadrupole and a sextupole.

Chromaticity correction (2)

- ✓ The effect of the sextupole field is to increase the magnetic field of the quadrupoles for the positive 'x' particles and decrease the field for the negative 'x' particles.
- ✓ However, the dispersion function, $D(s)$, describes how the radial position of the particles change with momentum.
- ✓ Therefore the sextupoles will alter the focusing field seen by the particles as a function of their momentum.
- ✓ This we can use to compensate the natural chromaticity of the machine.

Sextupole & Chromaticity

- ✓ In a sextupole for $y = 0$ we have a field $B_y = C \cdot x^2$
- ✓ Now calculate 'k' the focusing gradient as we did for a quadrupole:

$$k = \frac{1}{(B\rho)} \frac{dB_y}{dx}$$

- ✓ Using $B_y = Cx^2$ which after differentiating gives $\frac{dB_y}{dx} = 2Cx$

- ✓ For k we now write $k = \frac{1}{(B\rho)} 2Cx$

- ✓ We conclude that 'k' is no longer constant, as it depends on 'x'

- ✓ So for a Δx we get $\Delta k = \frac{2C}{(B\rho)} \Delta x$ and we know that $\Delta x = D(s) \frac{\Delta p}{p}$

- ✓ Therefore $\Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$

Sextupole & Chromaticity

- ✓ We know that the tune changes with :

$$\Delta Q = \frac{1}{4\pi} \beta(s) dk ds$$

- ✓ Where: $ds = \text{sextupole length}$ and

$$dk = \Delta k = 2C \times \frac{D(s)}{(B\rho)} \times \frac{\Delta p}{p}$$

- ✓ Remember $B = C \cdot x^2$ with

$$C = \frac{1}{2} \frac{d^2 B y}{dx^2}$$

- ✓ The effect of a sextupole with length l on the particle tune Q as a function of $\Delta p/p$ is given by:

$$\frac{\Delta Q}{Q} = \frac{1}{4\pi} \ell \beta(s) \frac{d^2 B y}{dx^2} \frac{D(s)}{(B\rho)Q} \frac{\Delta p}{p}$$

- ✓ If we can make this term exactly balance the natural chromaticity then we will have solved our problem.

Sextupole & Chromaticity (2)

- ✓ There are two chromaticities:
 - ✓ horizontal $\rightarrow \xi_h$
 - ✓ vertical $\rightarrow \xi_v$
- ✓ However, the effect of a sextupole depends on $\beta(s)$, which varies around the machine
- ✓ Two types of sextupoles are used to correct the chromaticity.
 - ✓ One (SF) is placed near QF quadrupoles where β_h is large and β_v is small, this will have a large effect on ξ_h
 - ✓ Another (SD) placed near QD quadrupoles, where β_v is large and β_h is small, will correct ξ_v
- ✓ Also sextupoles should be placed where $D(s)$ is large, in order to increase their effect, since Δk is proportional to $D(s)$

Questions....,Remarks...?

Hill's equation

Lattices and tune corrections

Sextupoles

Dispersion and chromaticity

