

Complex ~~Difficult~~ times

Pavel Cejnar

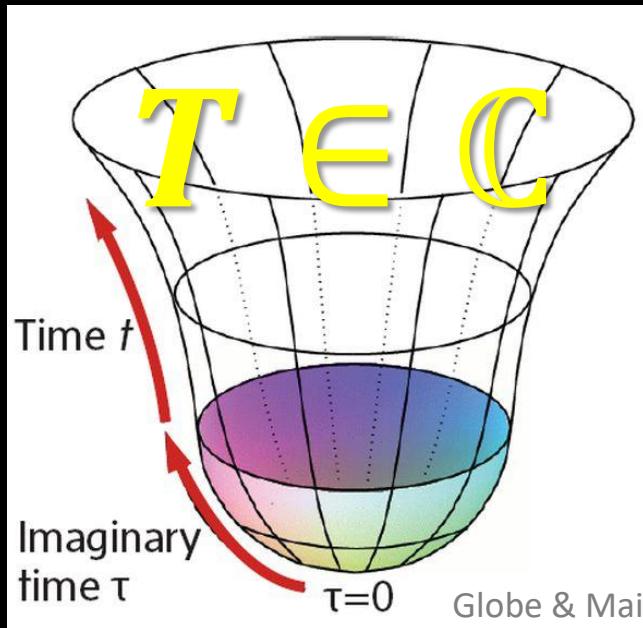


Malá Skála 2023

Complex

times

$$T_I \equiv \text{Im } T$$



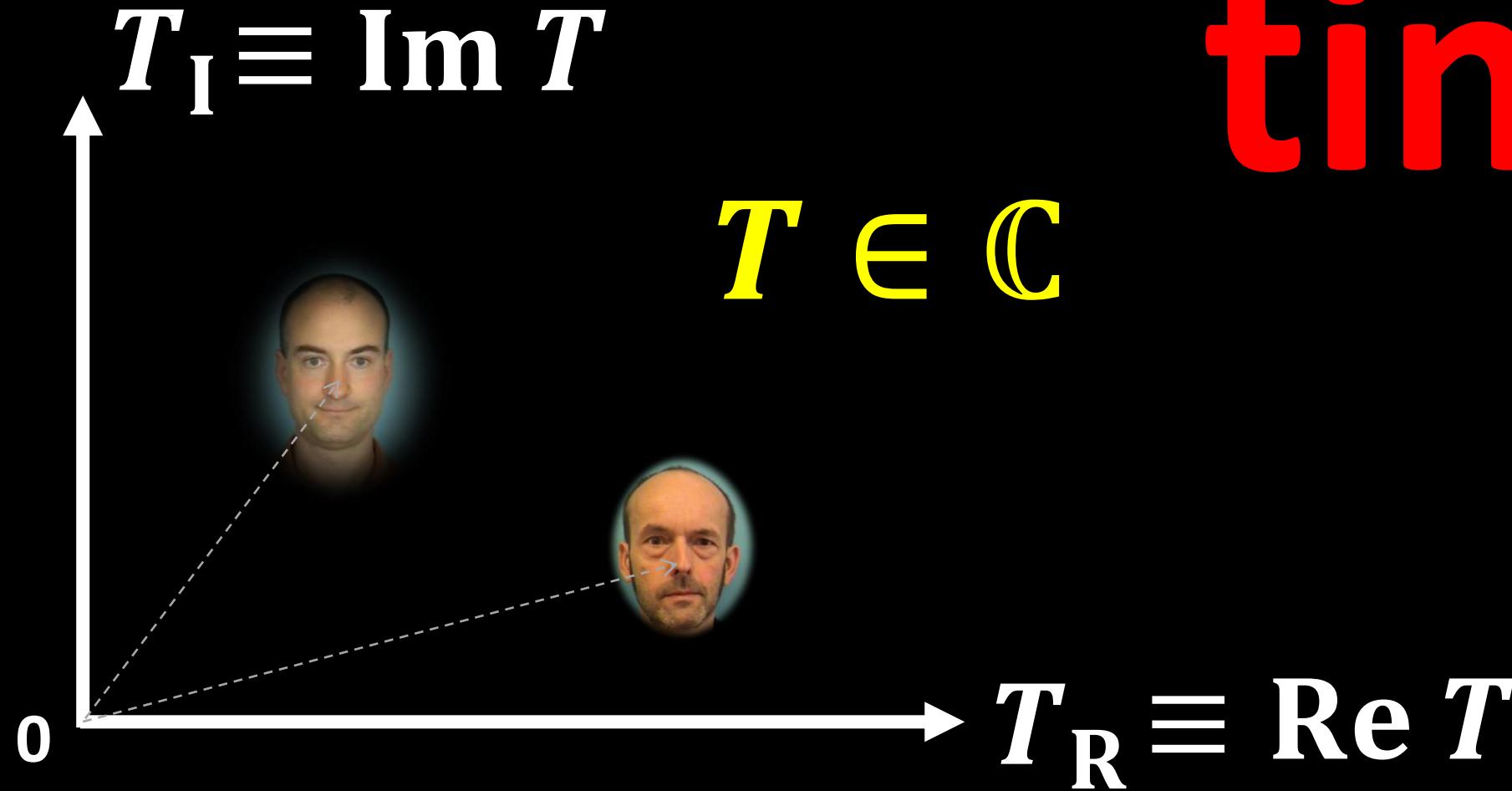
$$T_R \equiv \text{Re } T$$

Hawking's
imaginary-time
cosmology

Complex

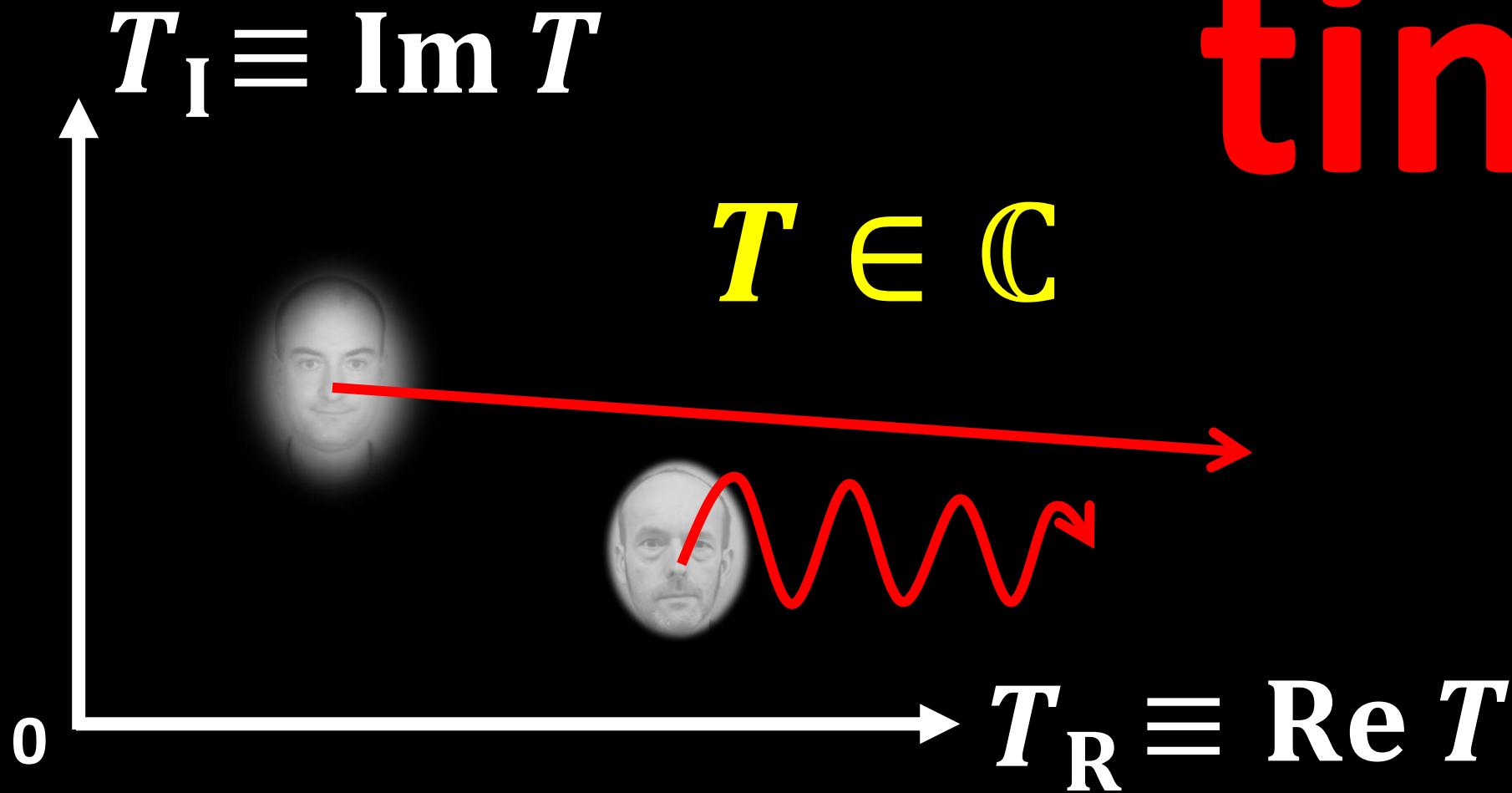
times

$$T \in \mathbb{C}$$



Complex

times

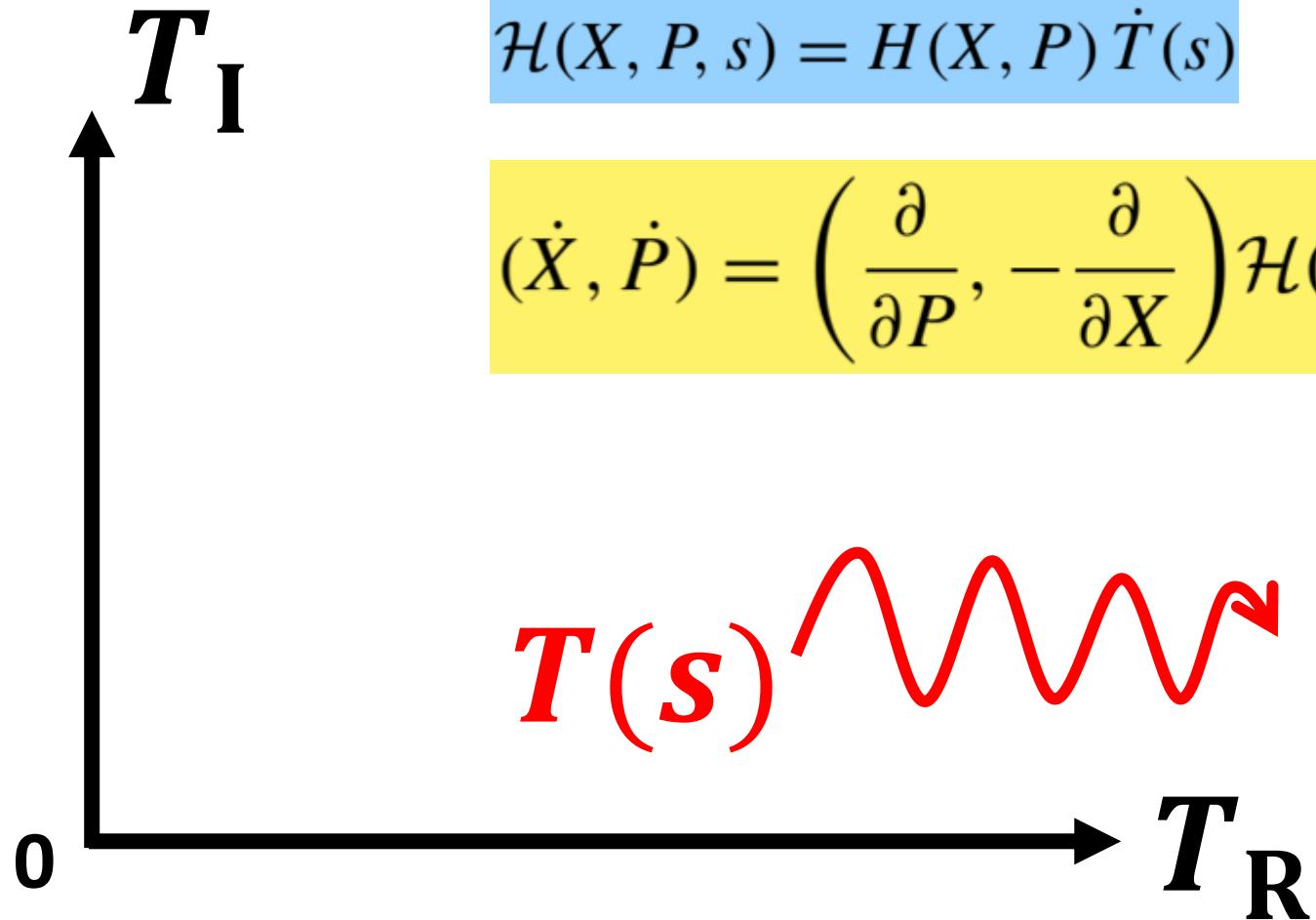


Complex-time classical mechanics

$$H(X, P) = \frac{P^2}{2m} + V(X) = \underbrace{\left(\frac{P_R^2 - P_I^2}{2m} + V_R(X) \right)}_{H_R(X, P)} + i \underbrace{\left(\frac{P_R P_I}{m} + V_I(X) \right)}_{H_I(X, P)}$$

$$\mathcal{H}(X, P, s) = H(X, P) \dot{T}(s)$$

$$(\dot{X}, \dot{P}) = \left(\frac{\partial}{\partial P}, -\frac{\partial}{\partial X} \right) \mathcal{H}(X, P, s)$$



Cauchy-Riemann conditions

$$\text{Re} \frac{dF}{dZ} = \frac{\partial F_R}{\partial Z_R} = \frac{\partial F_I}{\partial Z_I}$$

$$\text{Im} \frac{dF}{dZ} = \frac{\partial F_I}{\partial Z_R} = -\frac{\partial F_R}{\partial Z_I}$$

$$s \in \mathbb{R}$$

$$\dot{A} = \frac{dA}{ds} = \frac{dA}{dT} \dot{T}$$

Complex-time classical mechanics

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T_I

$$\mathcal{H}(X, P, s) = H(X, P) \dot{T}(s)$$

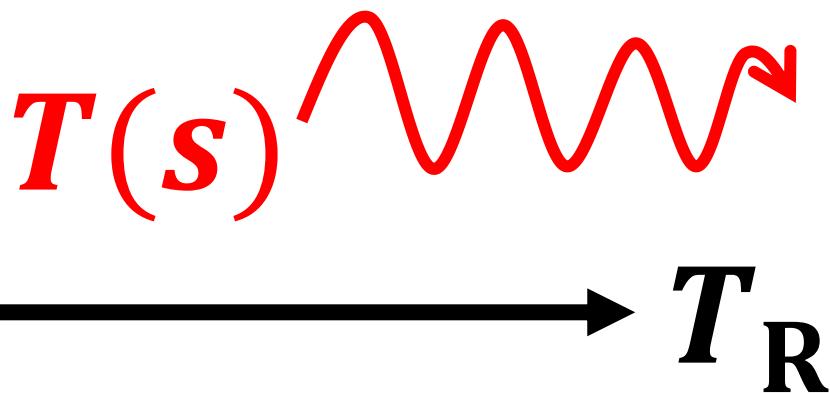
$$(\dot{X}_R, \dot{P}_R) = \left(\frac{\partial}{\partial P_R}, -\frac{\partial}{\partial X_R} \right) \mathcal{H}_R(X, P, s),$$

$$(\dot{X}_I, -\dot{P}_I) = \left(\frac{\partial}{\partial (-P_I)}, -\frac{\partial}{\partial X_I} \right) \mathcal{H}_R(X, P, s)$$

↔

$$(\dot{X}_R, \dot{P}_R) = \left(\frac{\partial}{\partial P_I}, -\frac{\partial}{\partial X_I} \right) \mathcal{H}_I(X, P, s)$$

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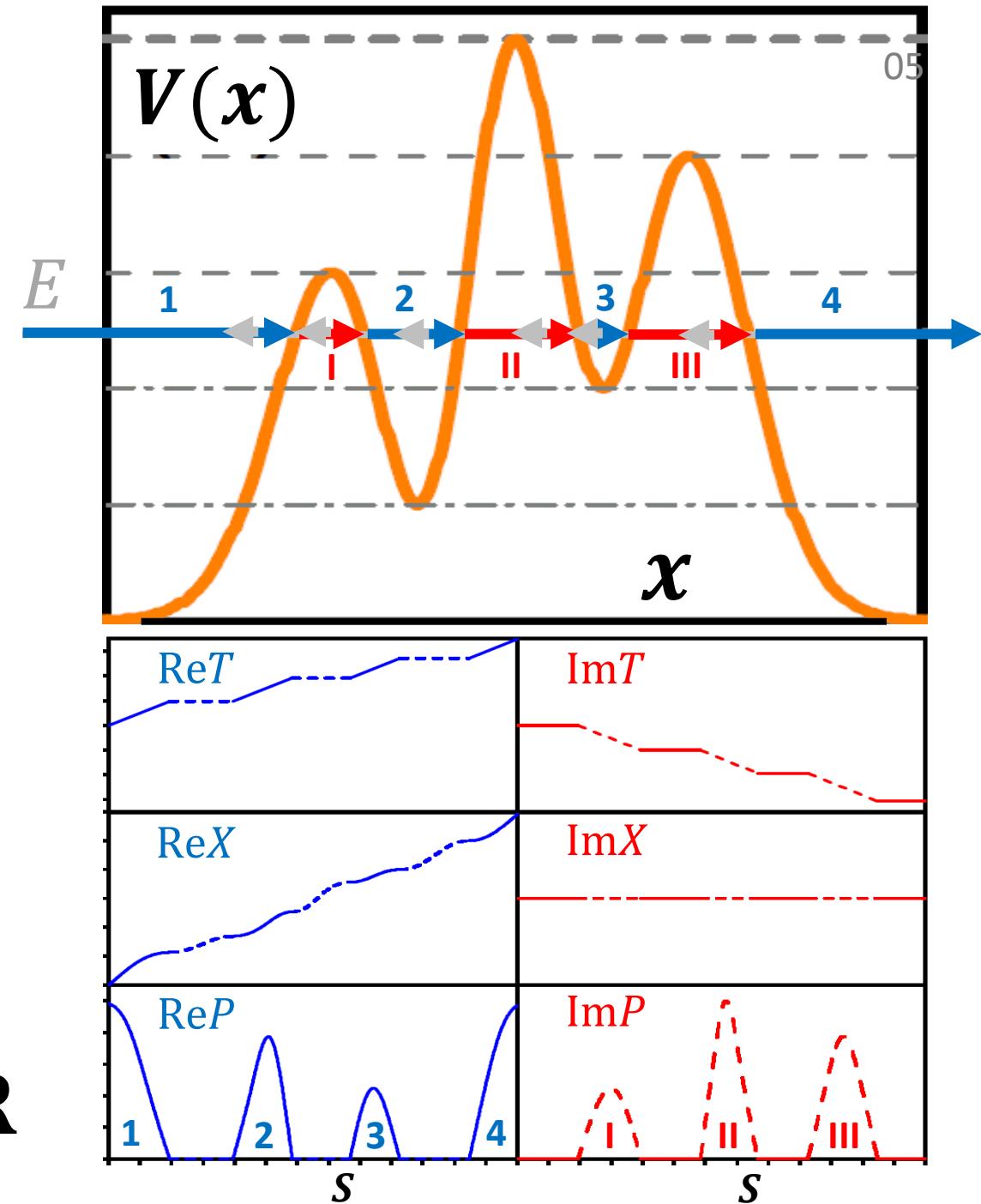
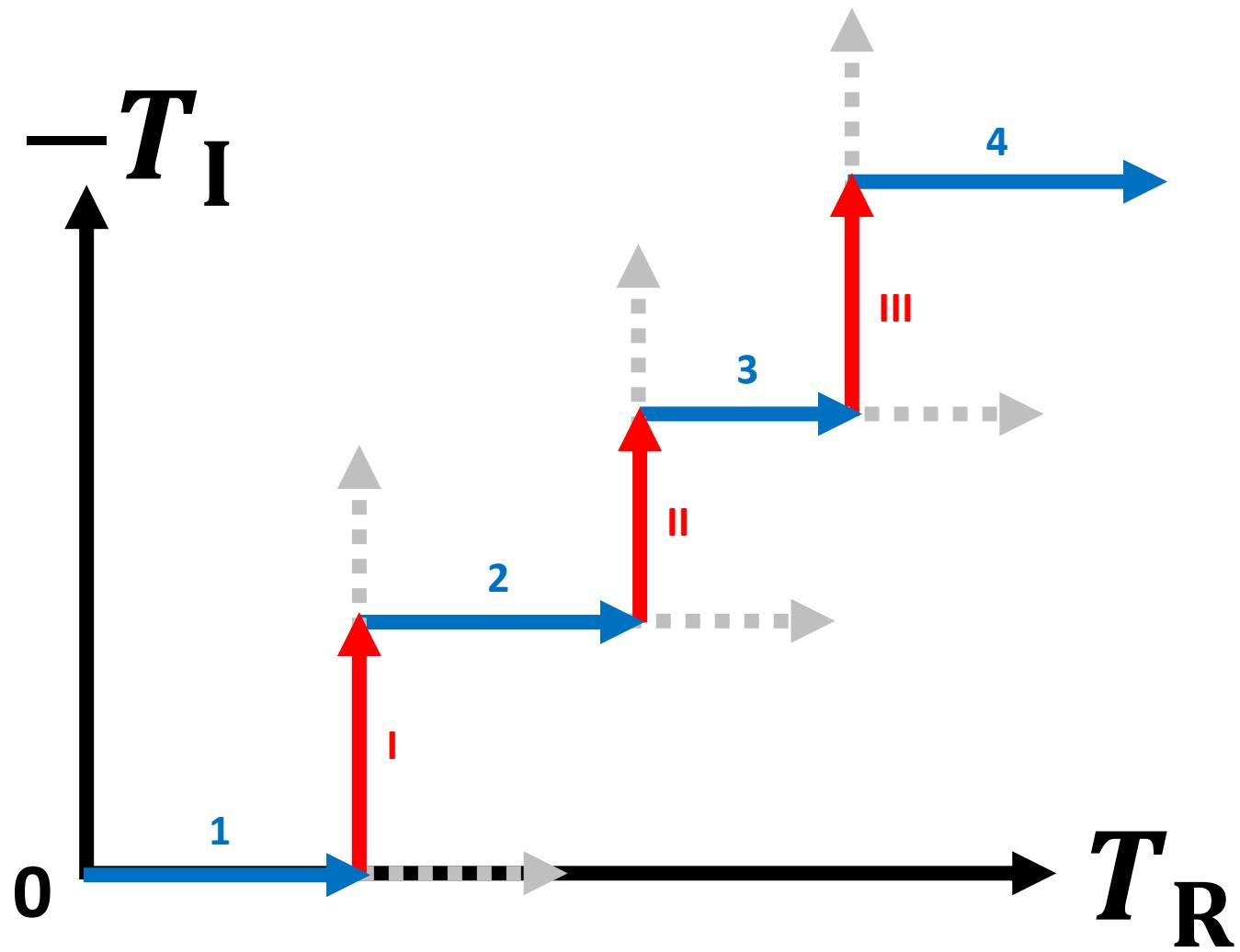


$$s \in \mathbb{R}$$

$$\dot{A} = \frac{dA}{ds} = \frac{dA}{dT} \dot{T}$$

Classical tunneling trajectories

Look for keywords: instanton, Wick rotation ...



Tunneling semiclassics

The genuinely quantum process of tunneling allows for a **semiclassical description** !

transmission amplitude $\mathcal{T}(E) = e^{i\Phi(E)}$

(1) **complex continuum density of states**

$$\Delta\rho(E) = \frac{i}{\pi} \lim_{\epsilon \rightarrow 0} \text{Tr} \left[\frac{1}{E + i\epsilon - \hat{H}} - \frac{1}{E + i\epsilon - \hat{H}_0} \right]$$

(2) **complex tunneling phase**

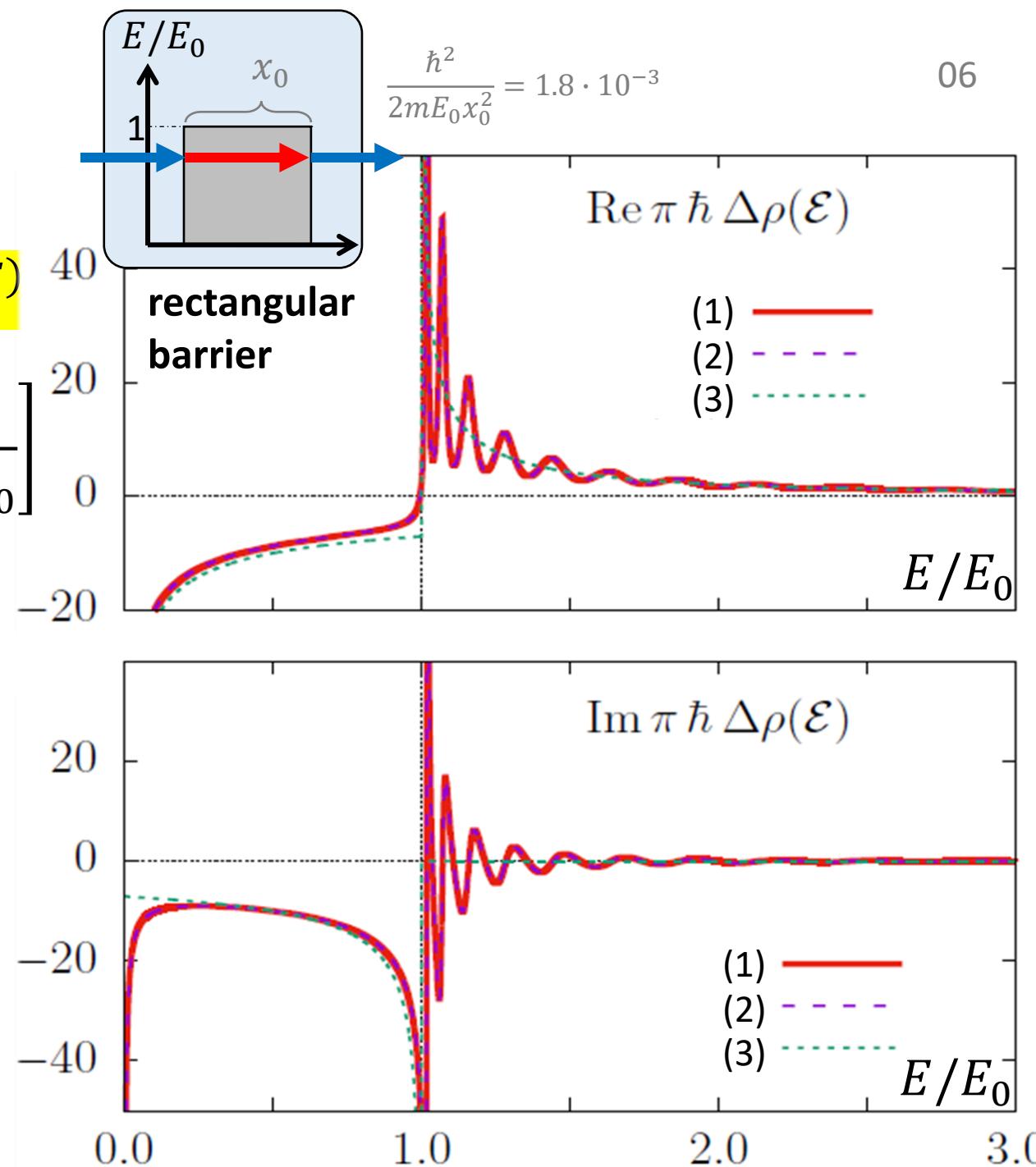
$$= \frac{1}{\pi} \frac{d}{dE} \Phi(E)$$

(3) **complex tunneling time**

$$= \frac{1}{\pi\hbar} \Delta T \approx \frac{1}{\pi\hbar} (T_{\text{cl}} - T_{\text{cl0}}) = \Delta\bar{\rho}(E)$$

smoothed (semiclassical)
continuum density of states

complex extension of Eisenbud-Wigner time shift
(1948,1955): difference of classical times of flight
of the tunneling and free particles



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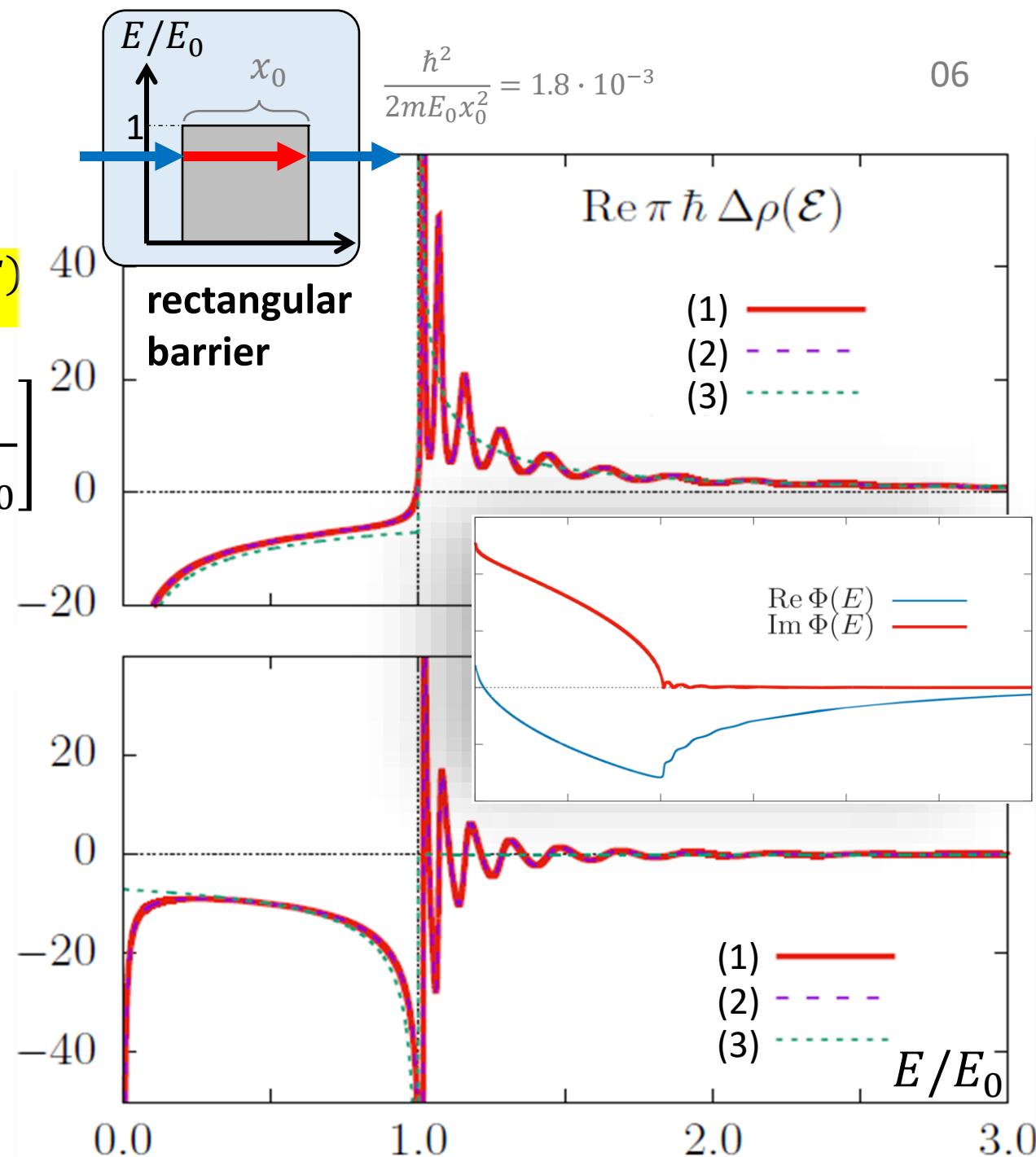
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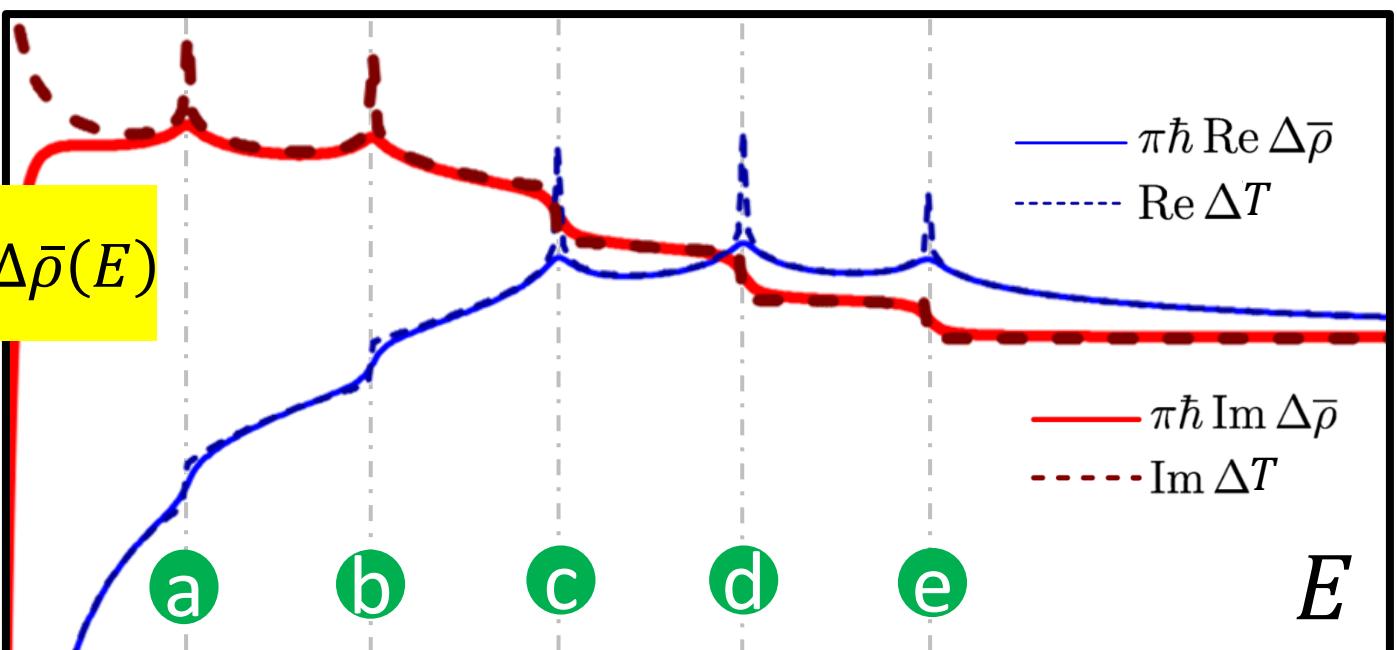
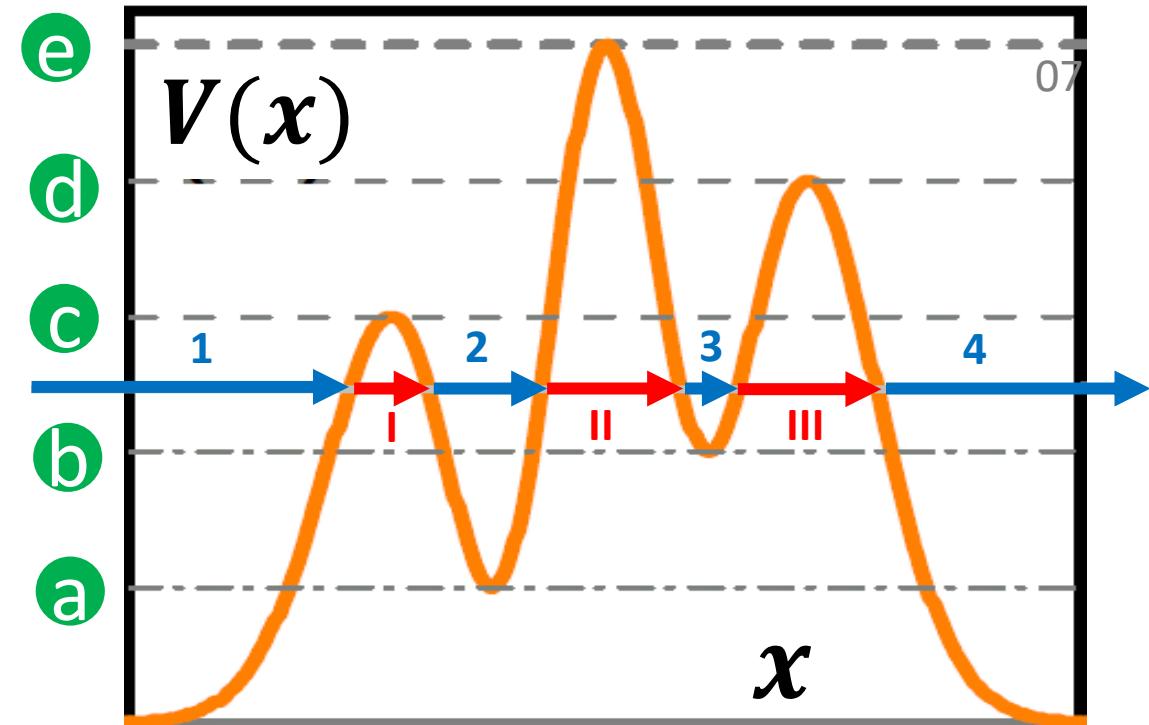
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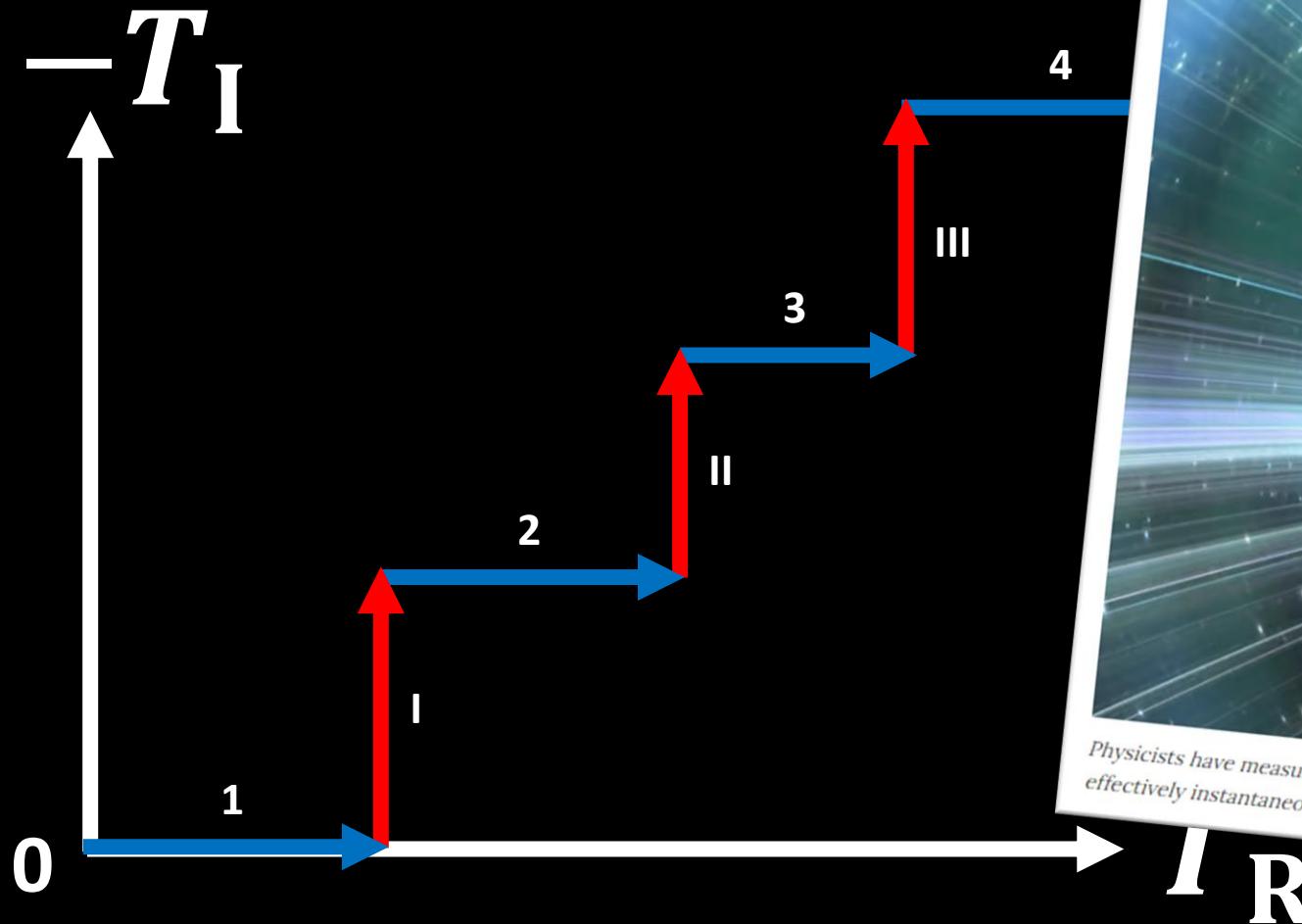
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⇒ **tunneling singularities** due to **stationary points** of the potential



Problem of tunneling time

How long does the particle spend in the forbidden region of the potential? Many experiments indicate that the tunneling time is very small, close to zero...



Physicists measure quantum tunneling time to be near-instantaneous

By Michael Irving
March 18, 2019

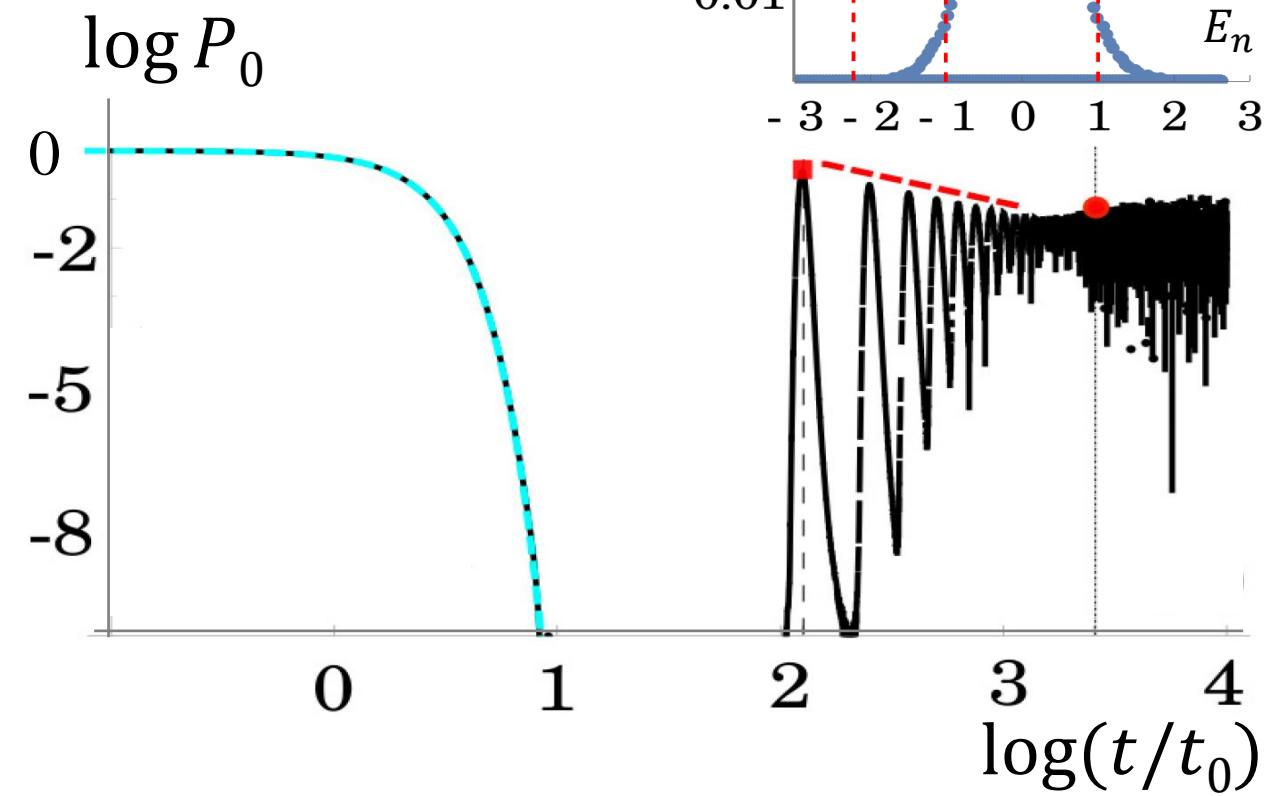
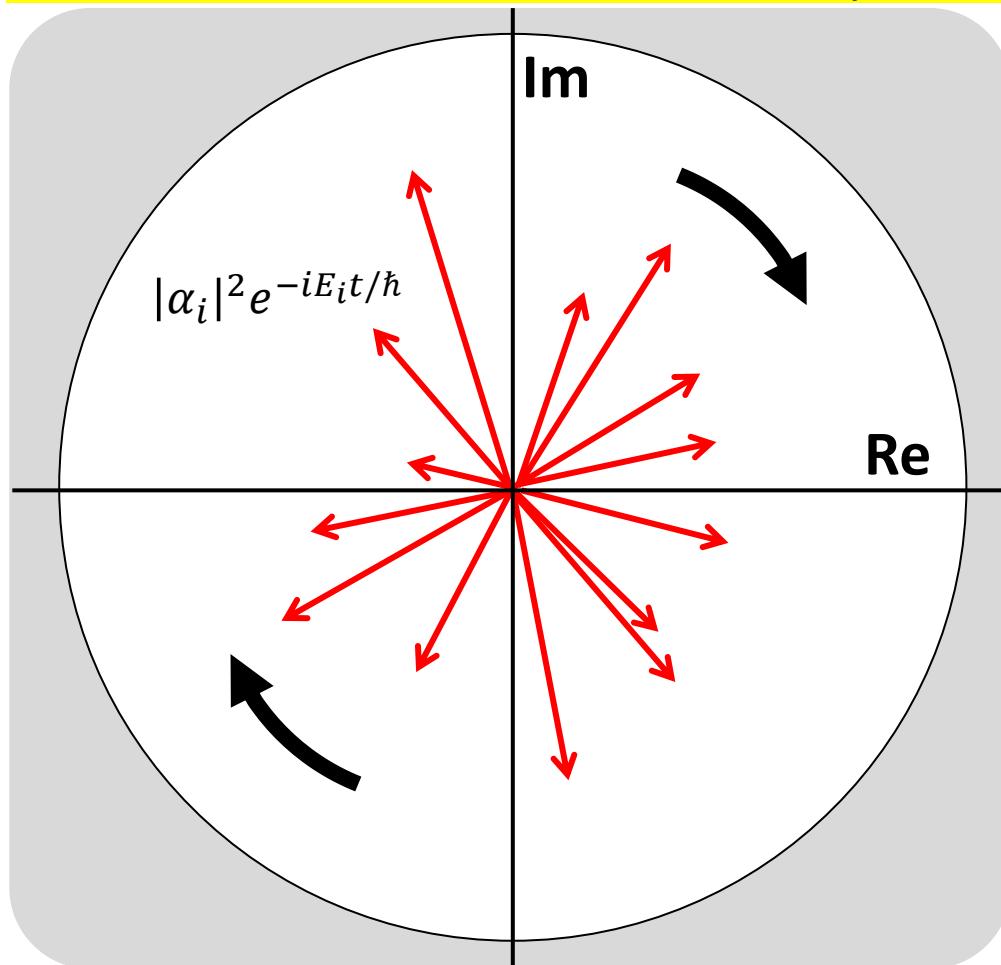


Physicists have measured how long it takes particles to quantum tunnel through barriers, and found it to be effectively instantaneous fredmantel/Depositphotos

Quantum survival probability

Quantum evolution of the survival probability of an initial state $|\psi\rangle = \sum_n \alpha_n |E_n\rangle$

$$P_0(t) = |\langle\psi|e^{-i\hat{H}t/\hbar}|\psi\rangle|^2 = \left| \sum_n |\alpha_n|^2 e^{-iE_n t/\hbar} \right|^2$$



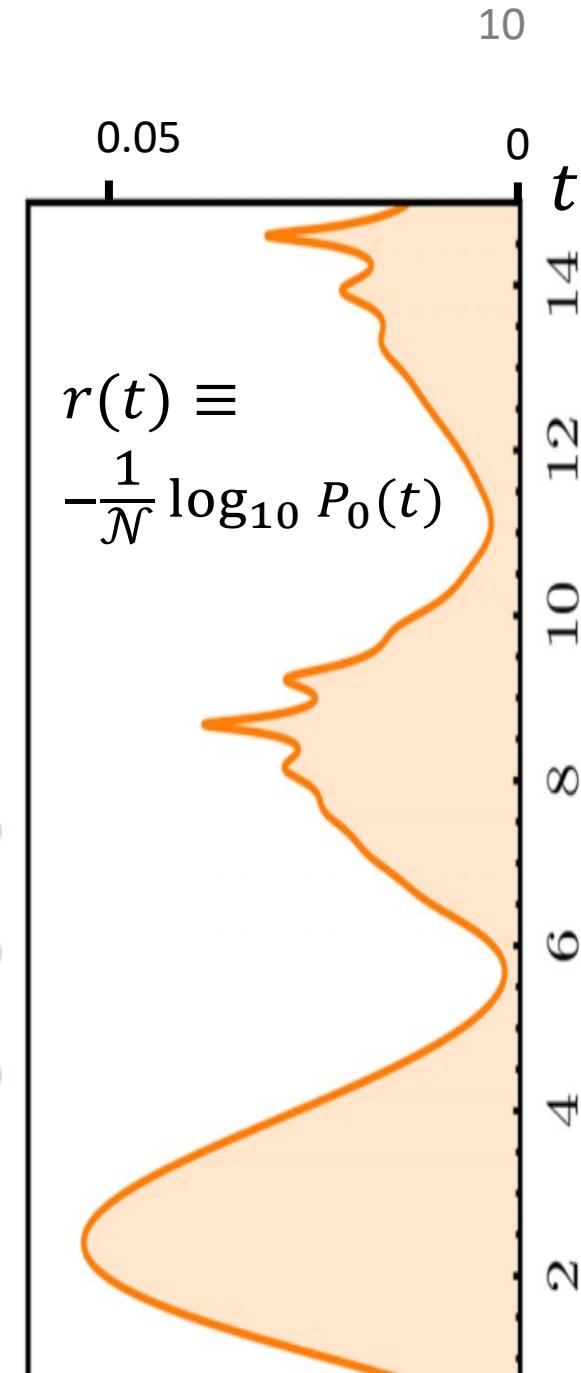
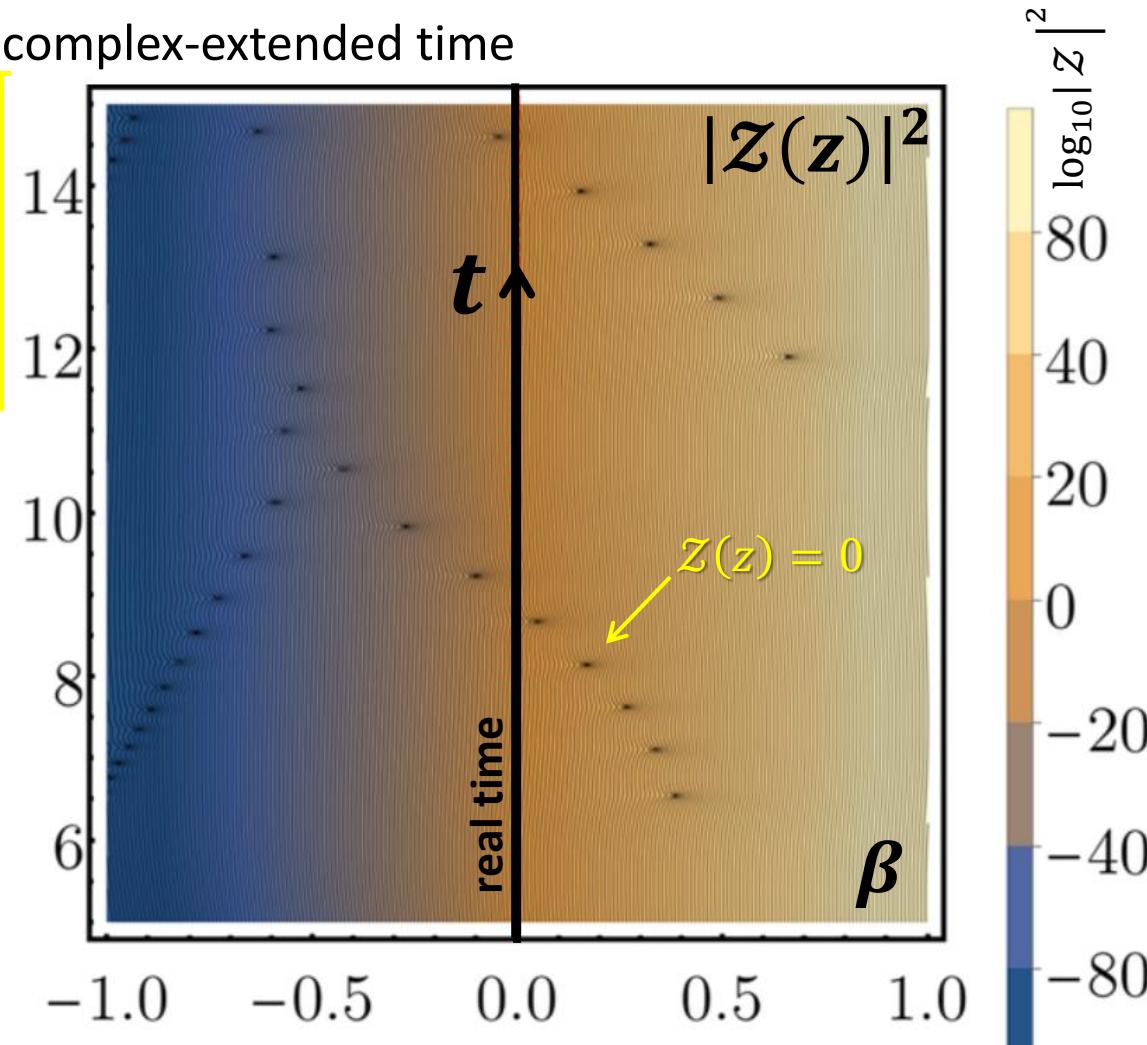
Example: M. Kloc, P. Stránský, P. C., PRA 98, 013836 (2018)

Quantum survival probability in complex time

Quantum survival amplitude in complex-extended time

$$\mathcal{Z}(z) = \sum_n |\alpha_n|^2 e^{-zE_i}$$
$$z = \beta + it \in \mathbb{C}$$

Analog of partition function $Z(\beta) = \sum_n d_n e^{-\beta E_n}$ in complex-extended inverse temperature β . **Zeros** of $Z(\beta)$ near the real β -axis indicate **thermal phase transitions** in the infinite-size limit $\mathcal{N} \rightarrow \infty$. Similarly, zeros of $\mathcal{Z}(z)$ near the imaginary z -axis indicate **dynamical quantum phase transitions** ...





Thank you!

<https://tenor.com/view/ghost-forest-spooky-gif-8926095>