PINNflation

solving the dynamics of Inflation using Physics Informed Neural Nets

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Overview

- Physics informed neural networks
- Harmonic oscillator
  - Adding Data
  - Uncertainty estimation
- Inflation

PINNs

- Idea: machine learning can be enhanced with physics information
- Neural Networks as universal approximators
- Needs only a few labeled data points

$$\dot{\mathbf{u}}(t) = F(\mathbf{u}, t)$$
$$\mathbf{u}^i = \mathbf{g}^i$$

$$\mathscr{L}_{ini} = \left|\mathbf{u}_\theta^i - \mathbf{g}^i\right|^2 = \left|\mathbf{u}_\theta^i - \mathbf{u}^i\right|^2$$
$$\mathscr{L}_{\mathrm{ODE}} = \left|\dot{\mathbf{u}}_\theta(t) - F(\mathbf{u}_\theta, t)\right|^2$$
$$\mathscr{L} = (1-\alpha)\mathscr{L}_{ini} + \alpha\mathscr{L}_{\mathrm{ODE}}$$

[Raise et al., 2018,
Cuomo et al., 2022]

Harmonic oscillator

$$\ddot{x} + Ax = 0, \quad x(0) = (1,0)^T, \quad \dot{x}(0) = (0,1)^T$$

$$\dot{Y} = -BY = -\begin{pmatrix} 0 & -\mathbb{I} \\ A & 0 \end{pmatrix} Y \qquad\qquad Y(t) = \exp(-Bt)\, Y(0)$$

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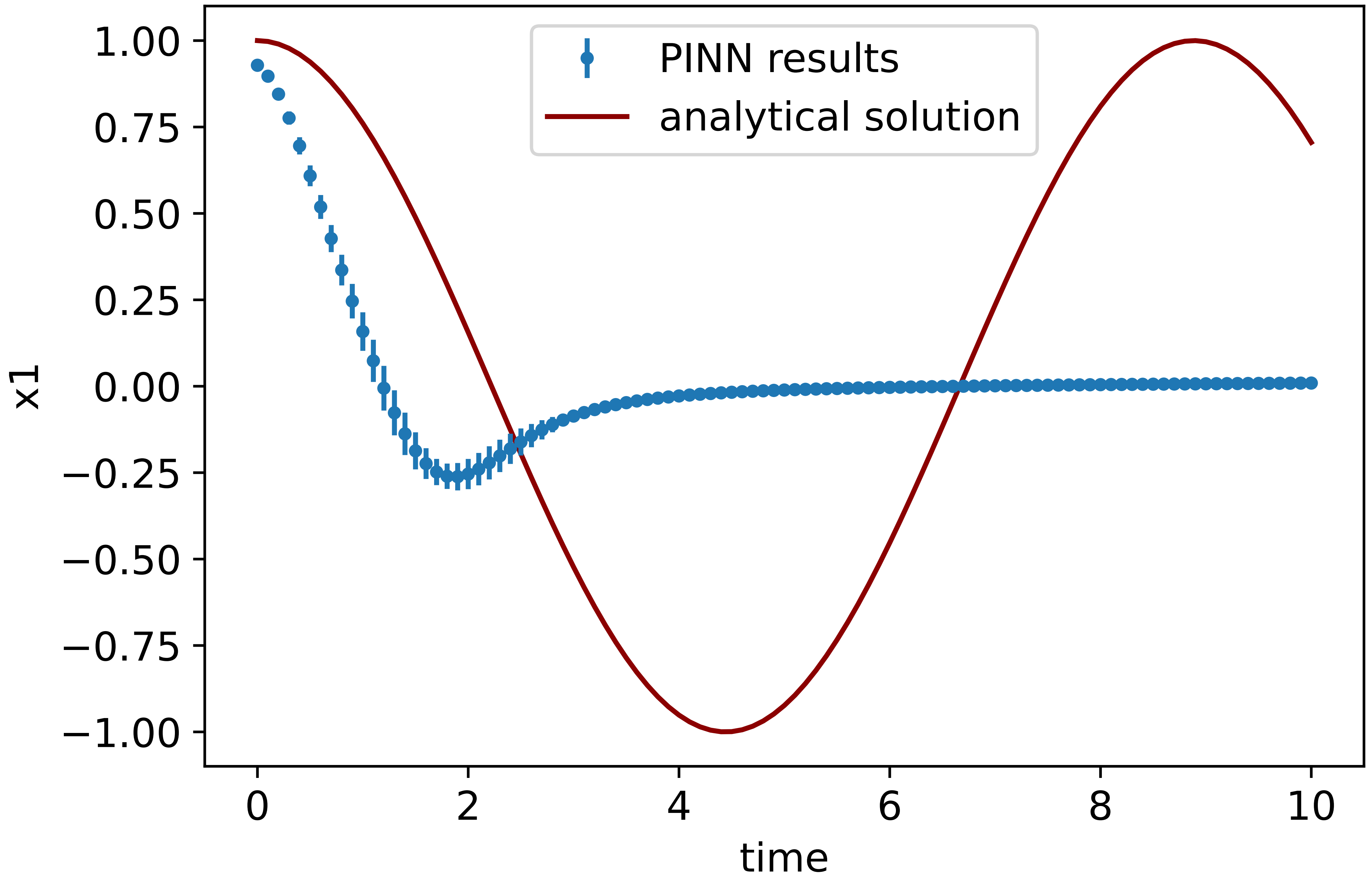

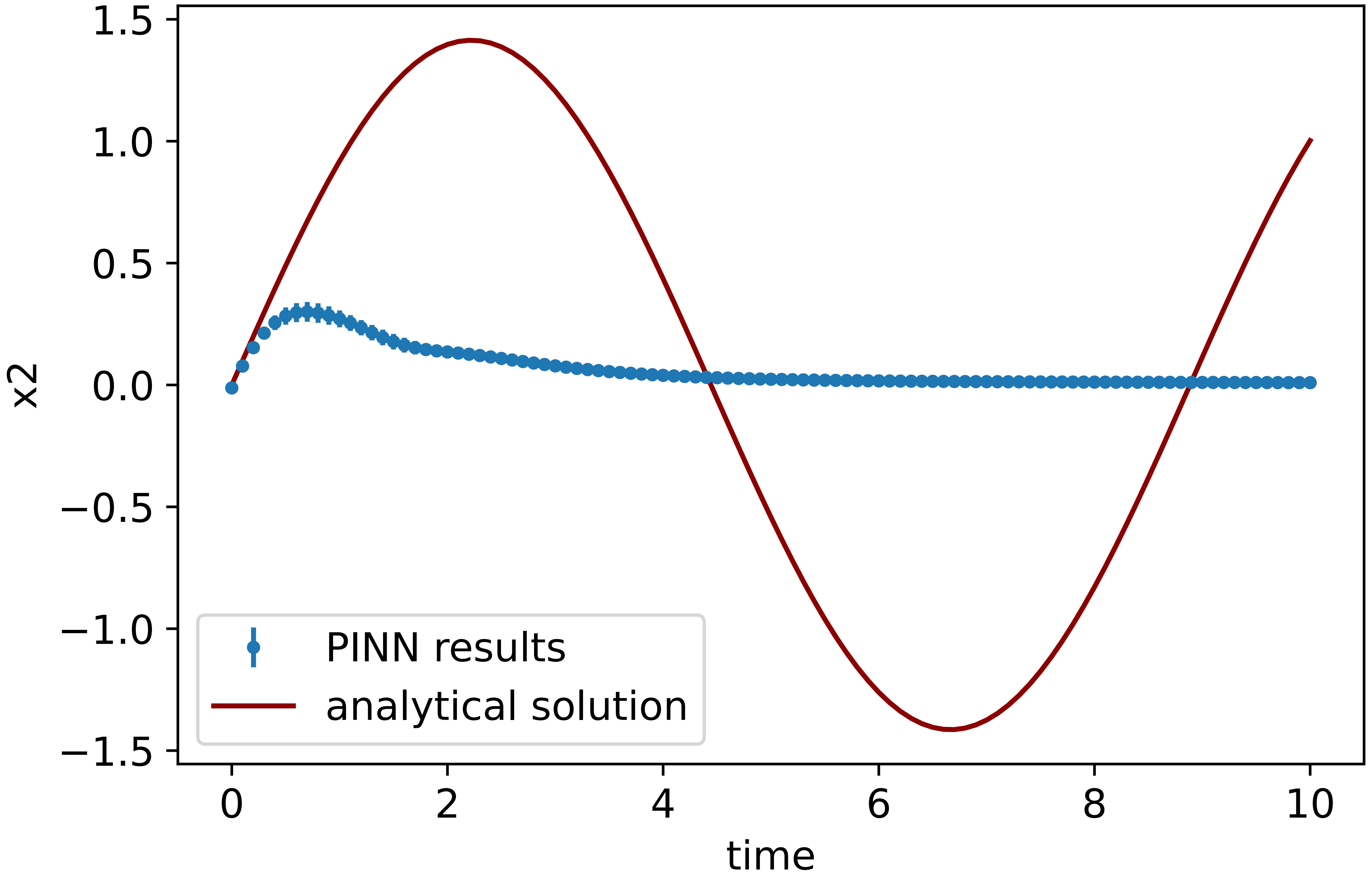

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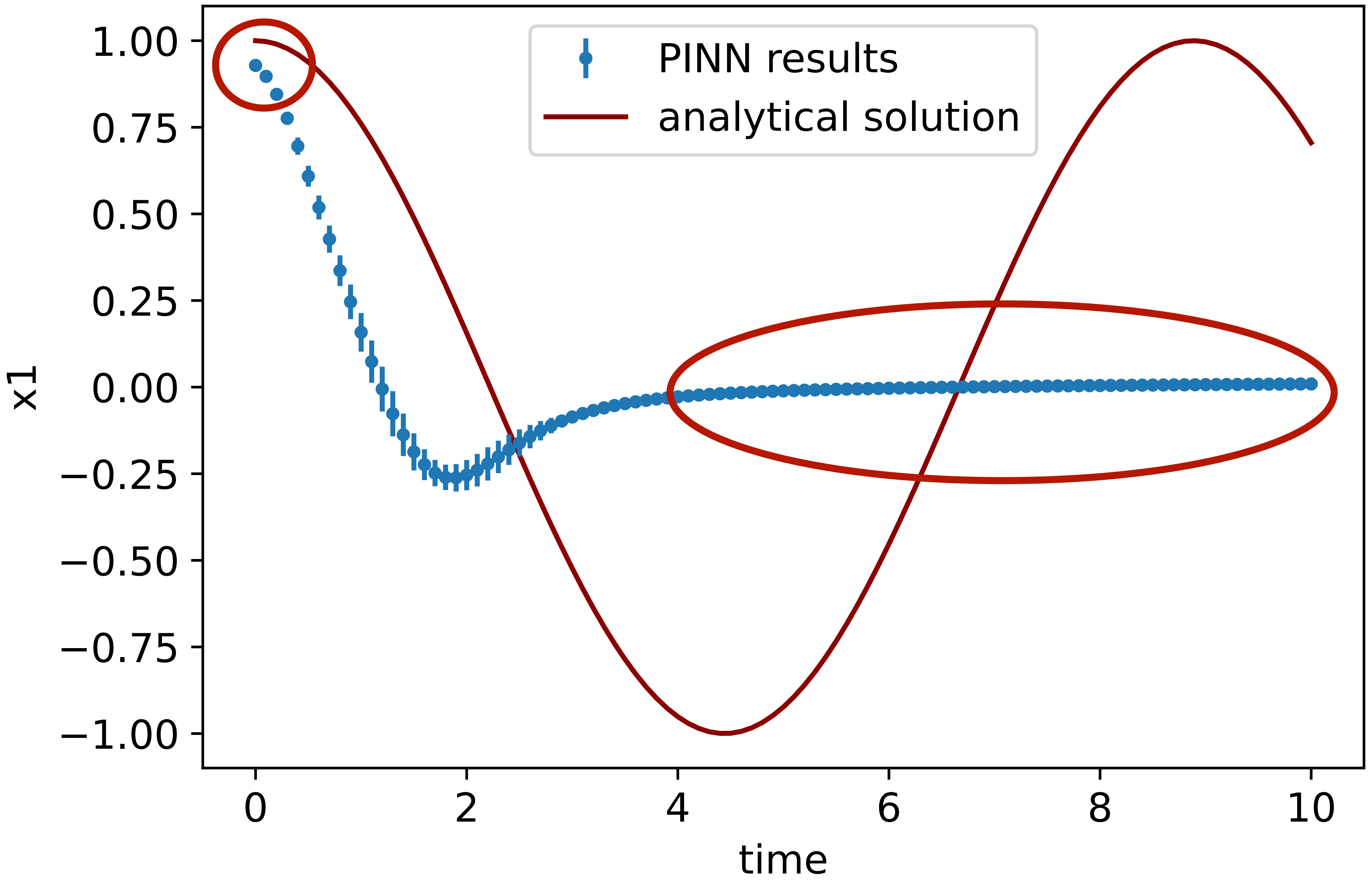
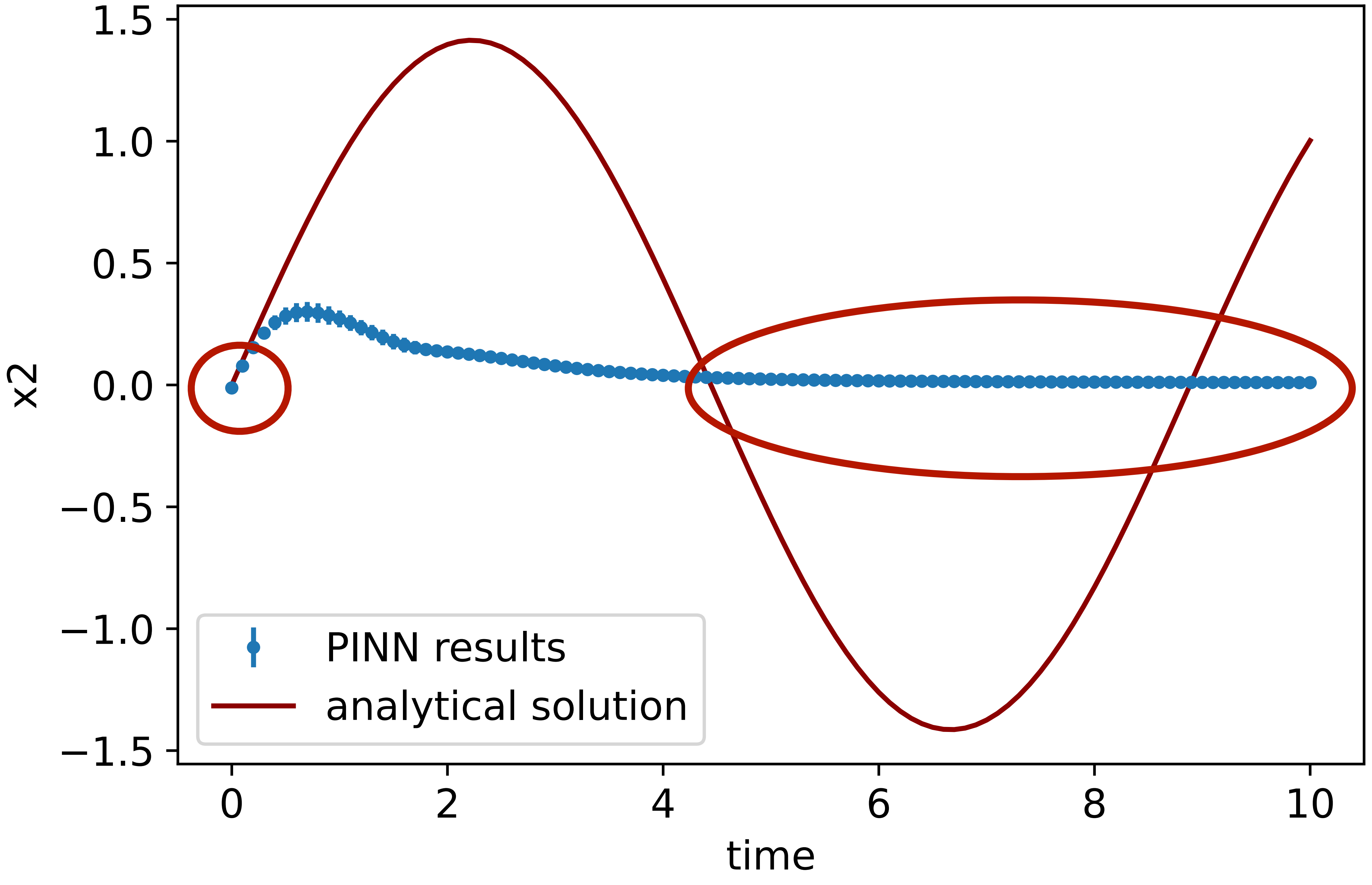
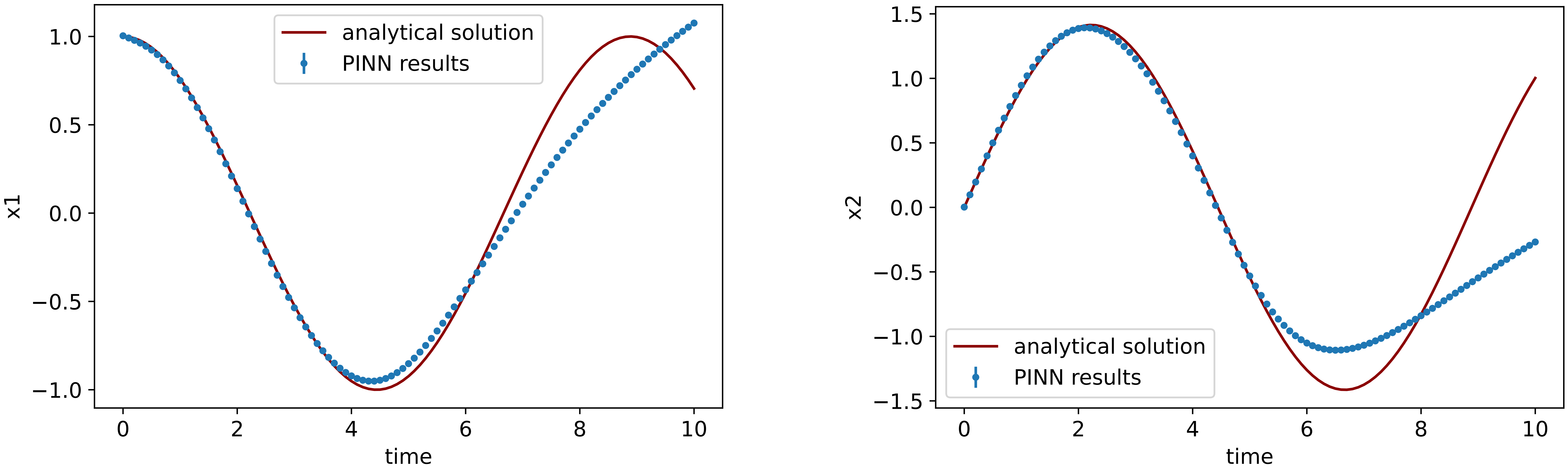
Correct solution! (approximately)





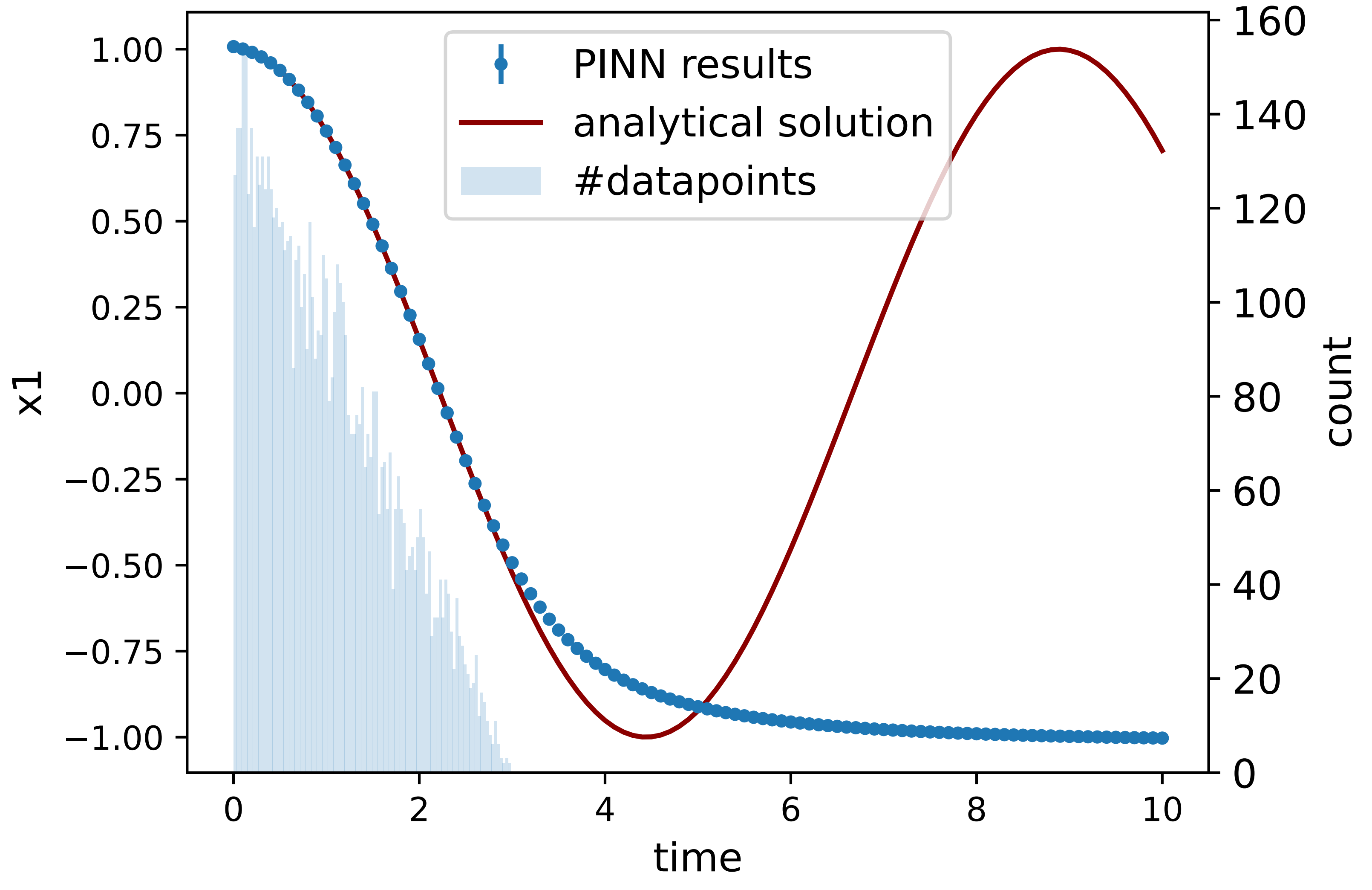
Reduce complexity

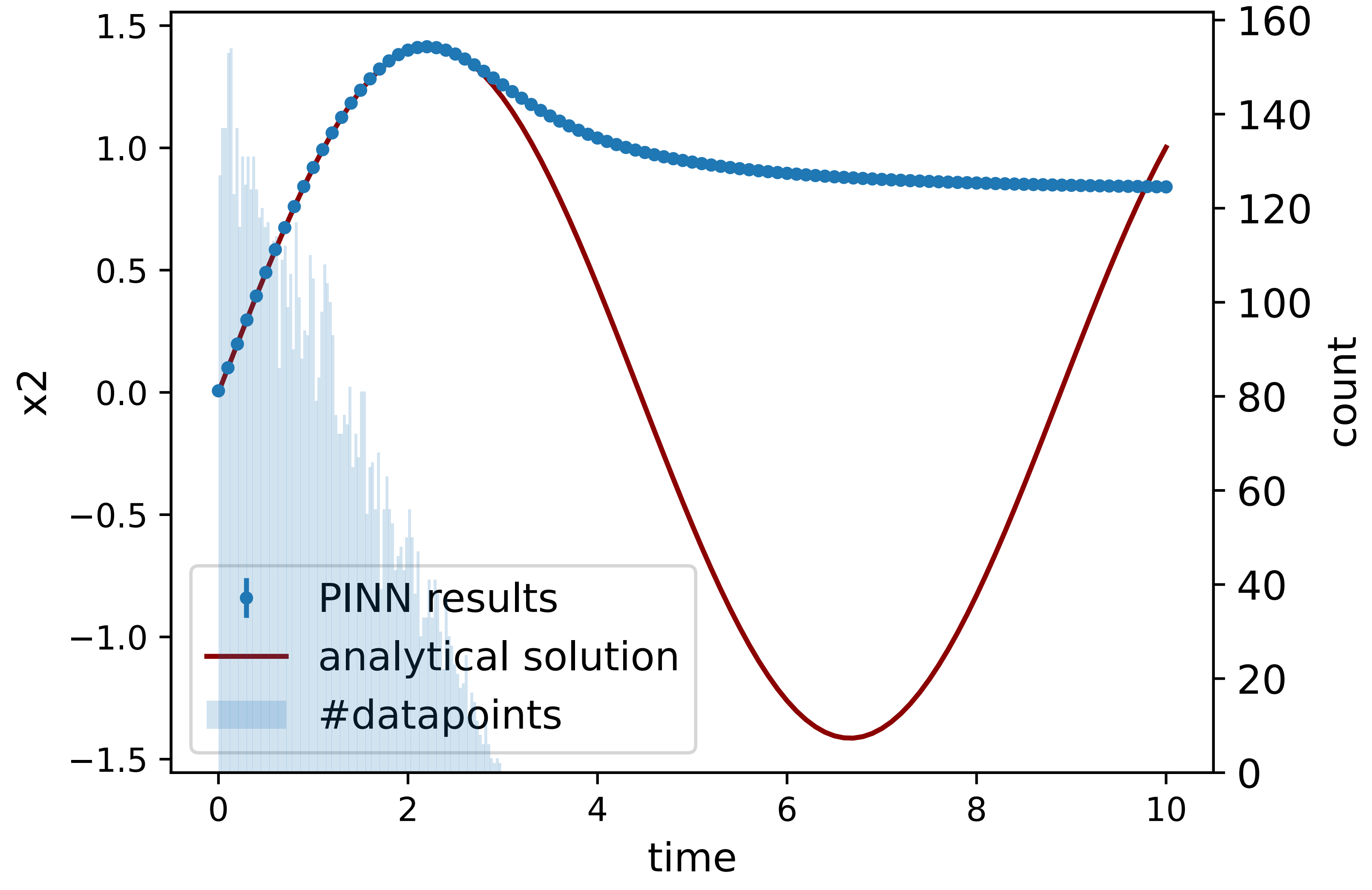
$$\ddot{x} + \frac{1}{2}x = 0, \quad x(0) = (1{,}0)^T, \quad \dot{x}(0) = (0{,}1)^T$$



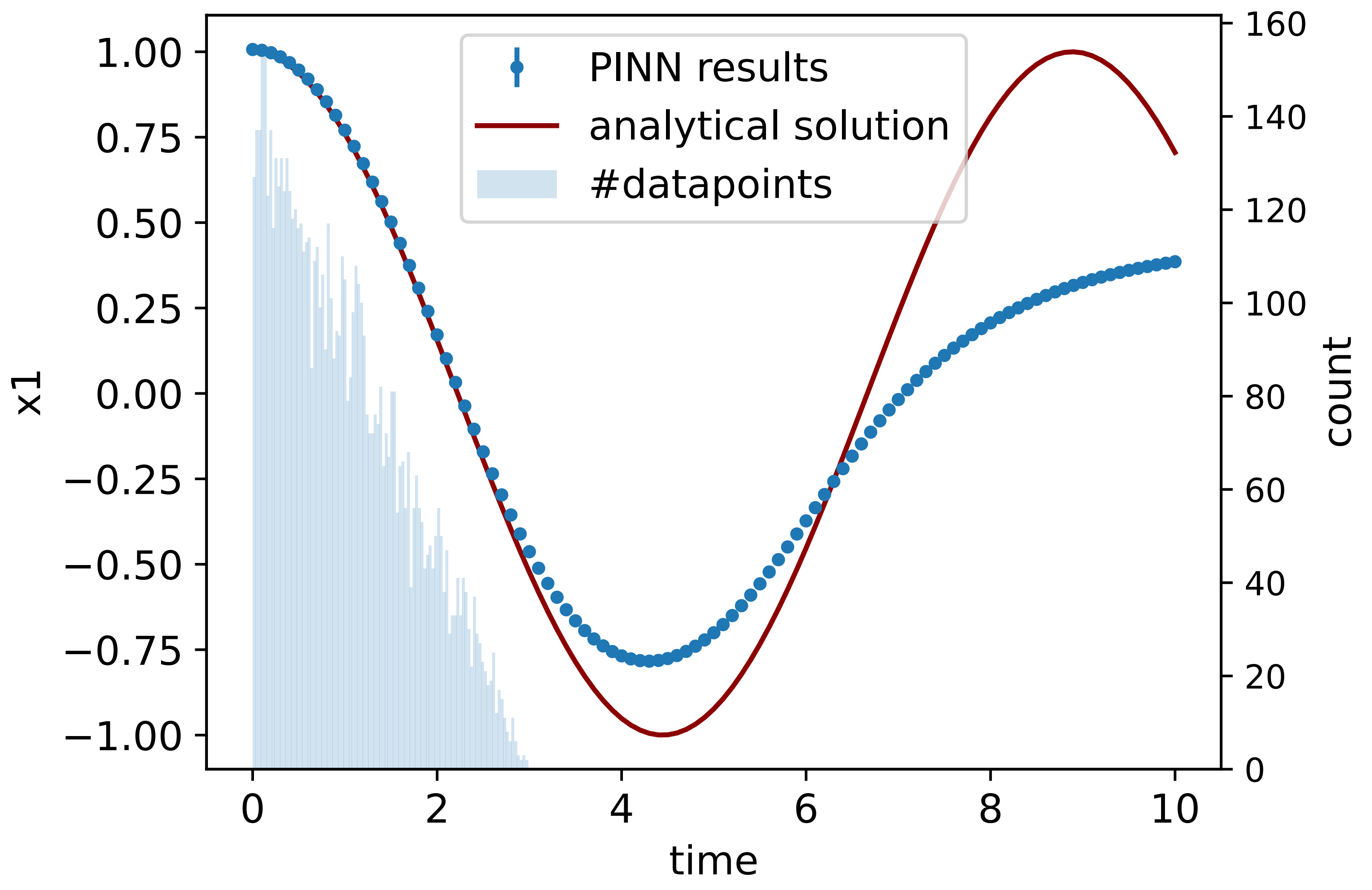
Adding labeled Data

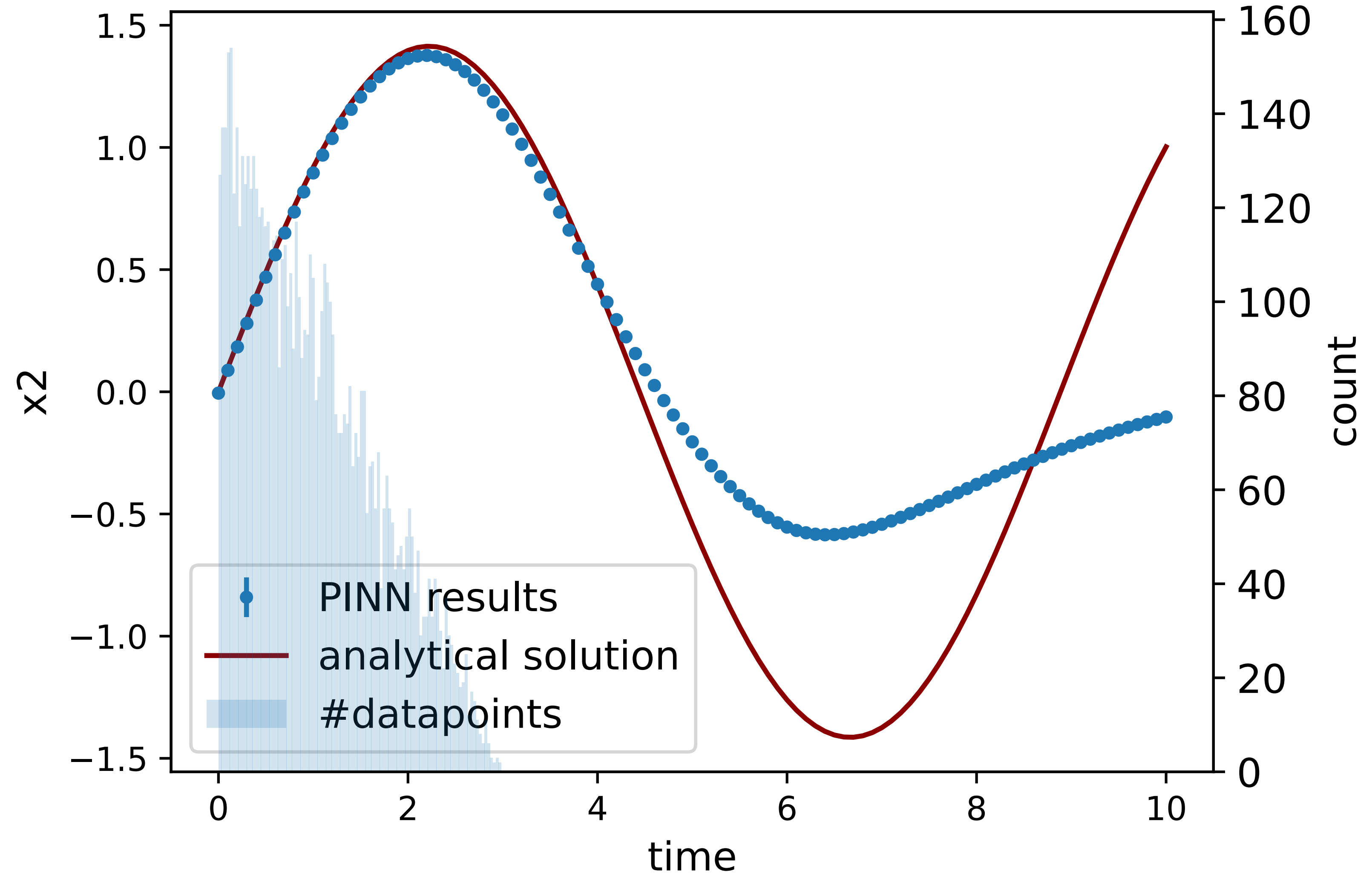
Data





+ ODE

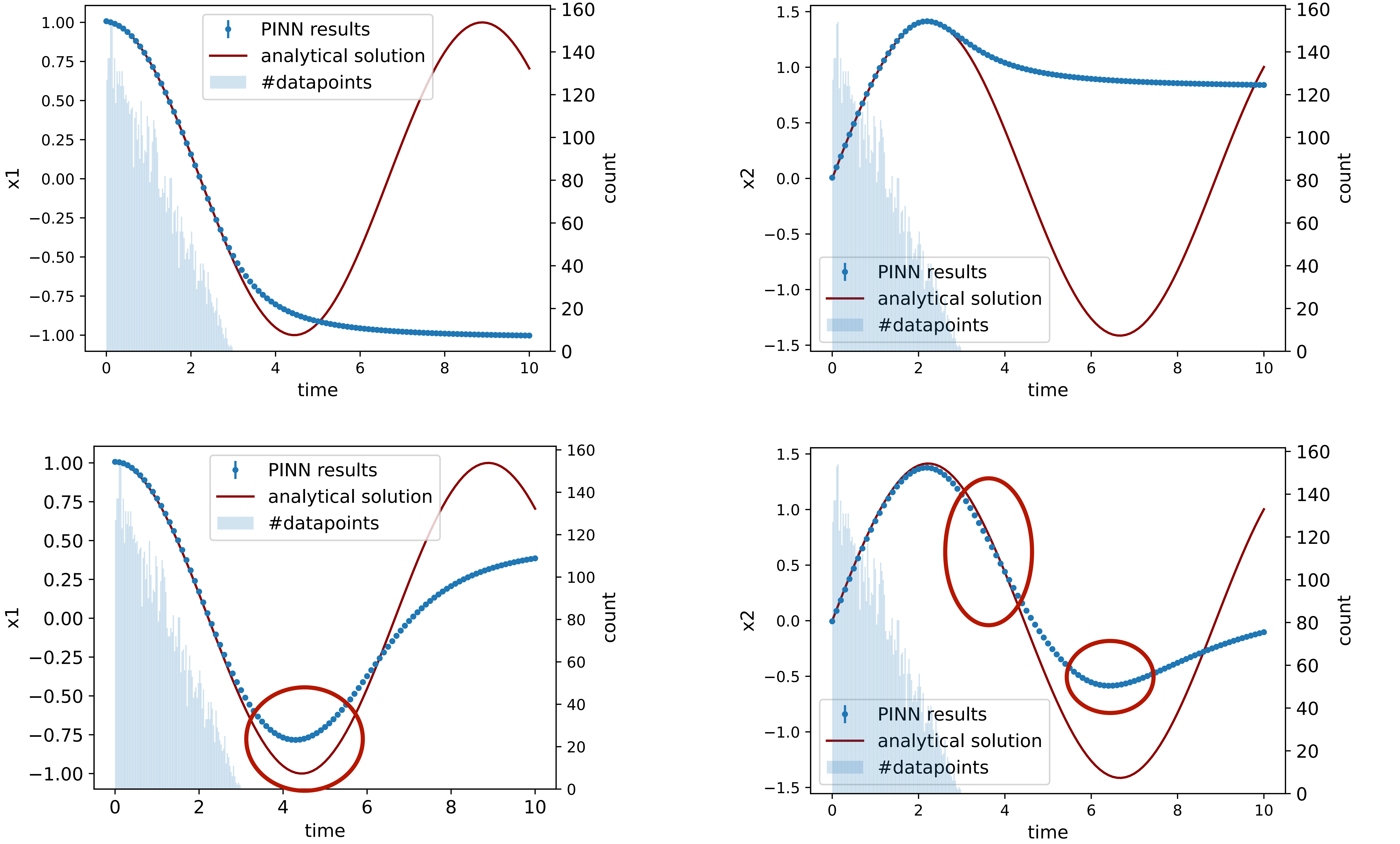




Adding labeled Data

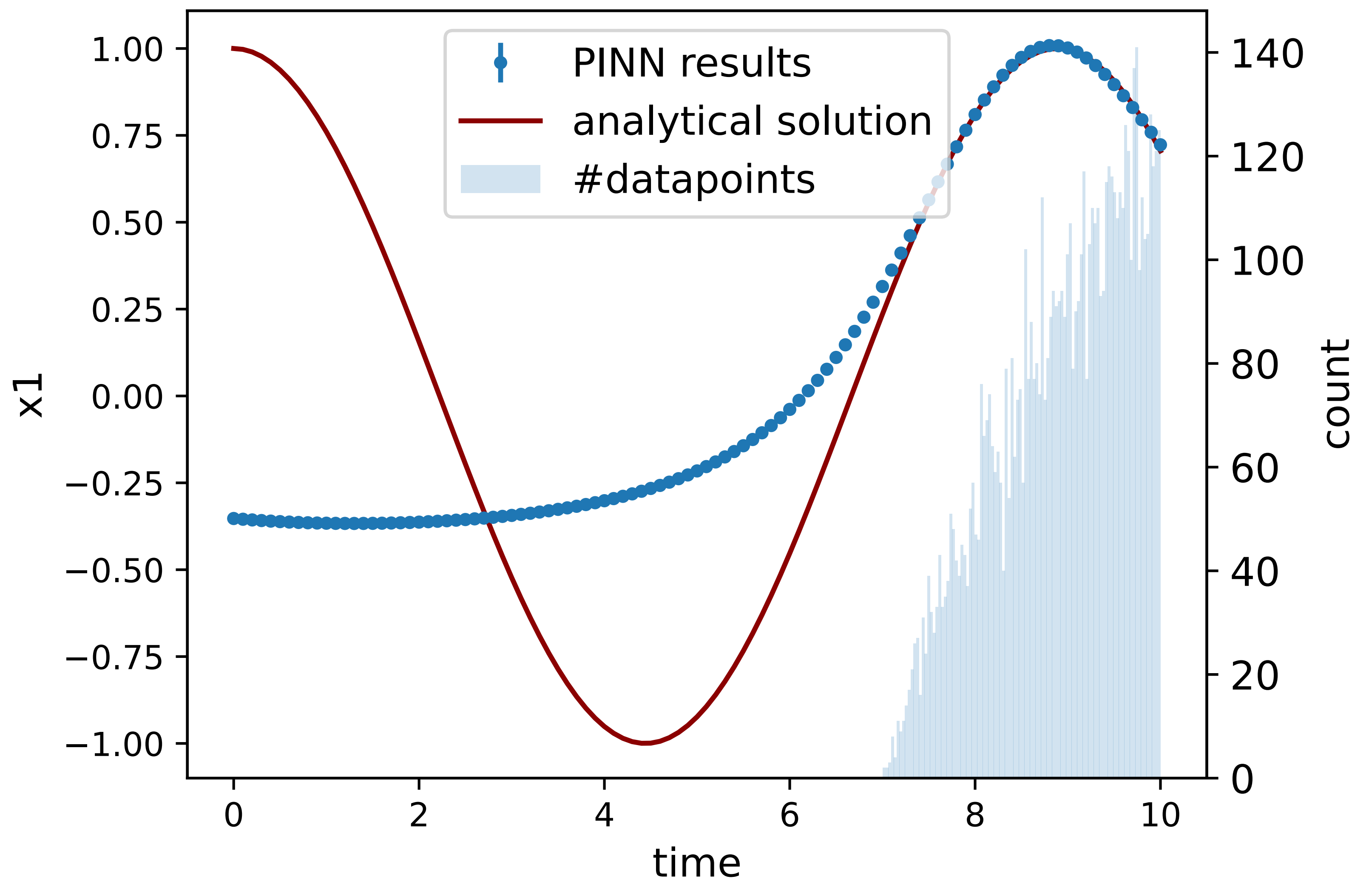
Data

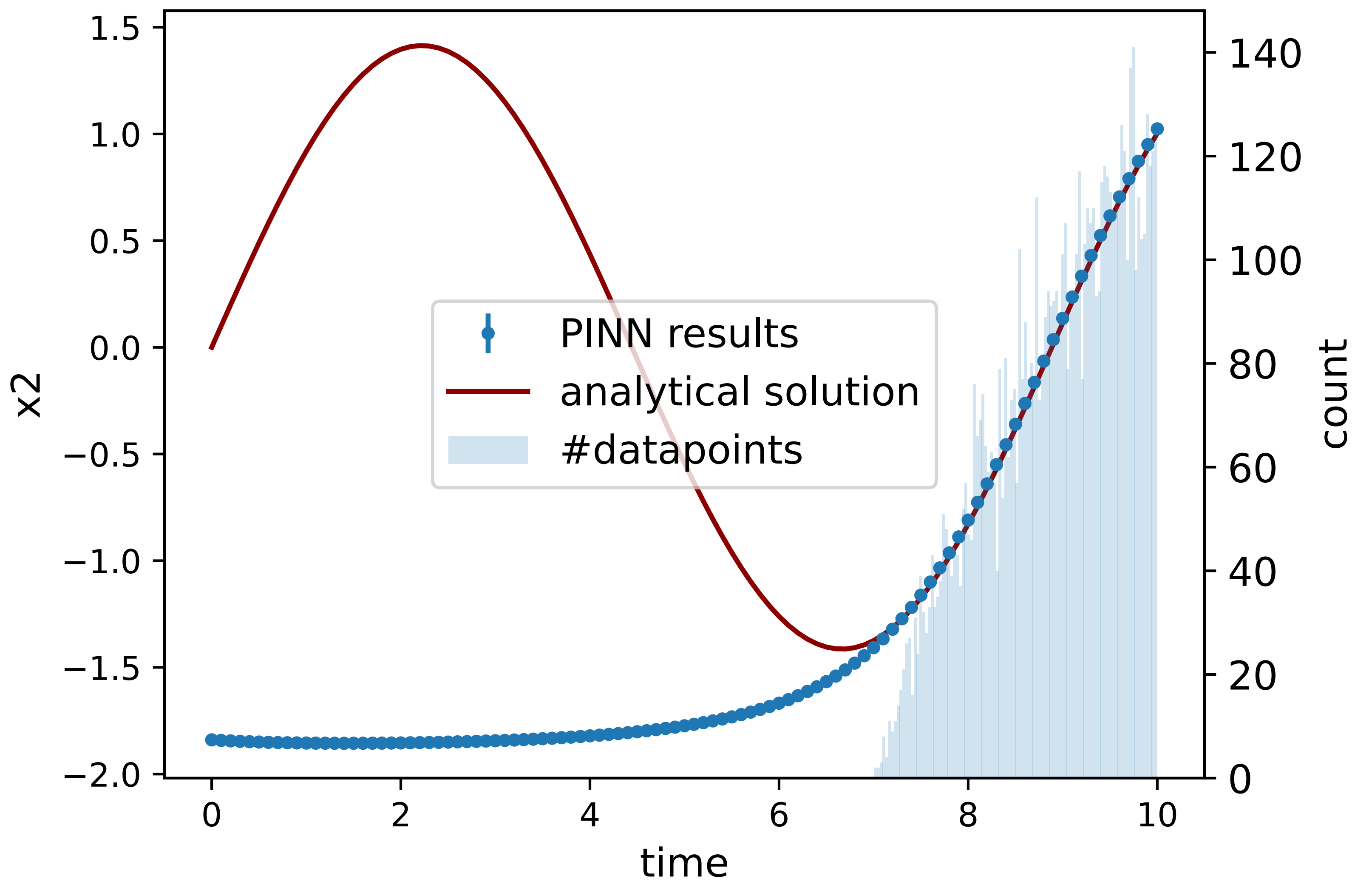
+ ODE



Adding labeled Data

Data





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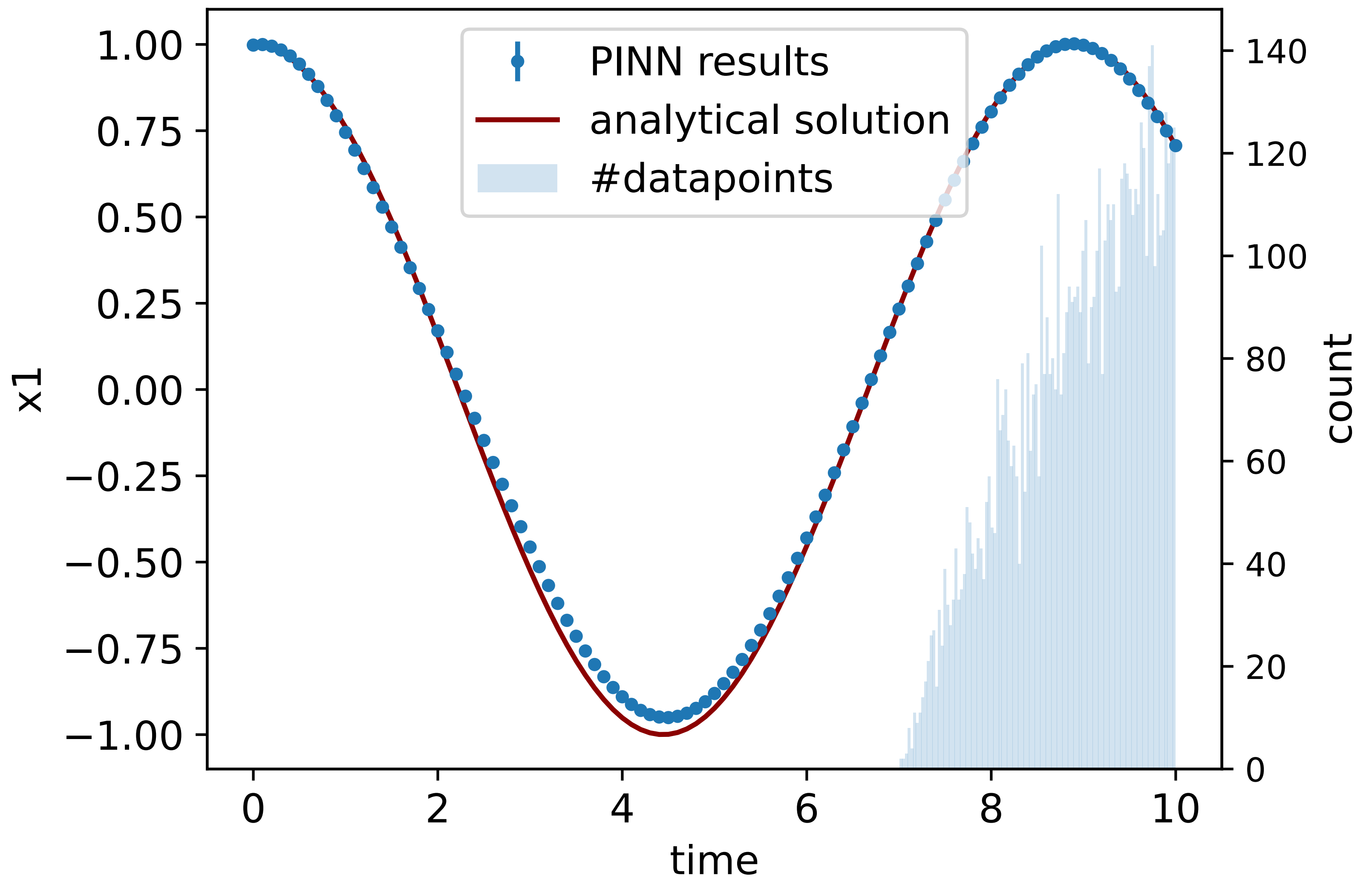

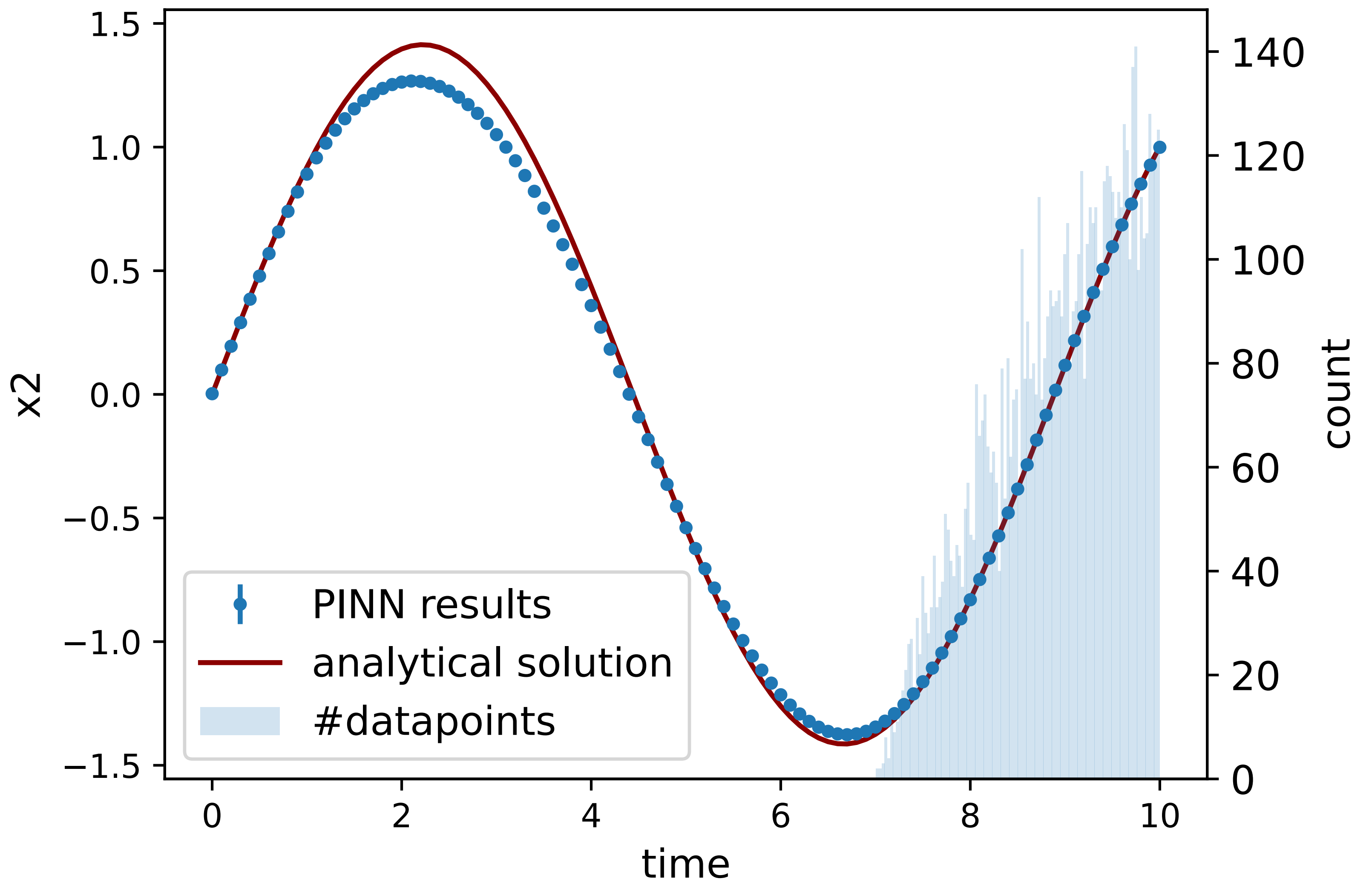

Uncertainty

- Uncertainty of the model given the data
- Construct as information of uncorrelated normal distribution

$$\mathscr{L} = \sum_{Data} -\log p(f(x_i)\quad f_\theta(x_i))$$

$$\mathscr{L}_{heteroscedastic} = \sum_{Data} -\log \left[\prod_{dim,k} \frac{1}{\sqrt{2\pi\sigma_\theta^k(x_i)}} \exp\left(-\frac{(f^k(x_i) - f_\theta^k(x_i))^2}{2\left(\sigma_\theta^k(x_i))^2\right)} \right) \right]$$

Uncertainty

- Uncertainty of the model given the data

- Construct as information of uncorrelated normal distribution

$$\mathscr{L} = \sum_{Data} - \log p(f(x_i)\ \ f_\theta(x_i))$$

$$\mathscr{L}_{heteroscedastic} = \sum_{Data} \left[\sum_{dim,k} \left(\frac{(f^k(x_i) - f^k_\theta(x_i))^2}{2\left(\sigma^k_\theta(x_i)\right)^2} + \log \sigma^k_\theta(x_i) \right) + \frac{n}{2} \log 2\pi \right]$$

$$\mathscr{L}_{homoscedsatic} = \sum_{dim,k} \left(\frac{\sum_{Data} (f^k(x_i) - f^k_\theta(x_i))^2}{2\left(\sigma^k_\theta\right)^2} + N_{Data} \log \sigma^k_\theta \right) + \frac{nN_{Data}}{2} \log 2\pi$$

Uncertainty

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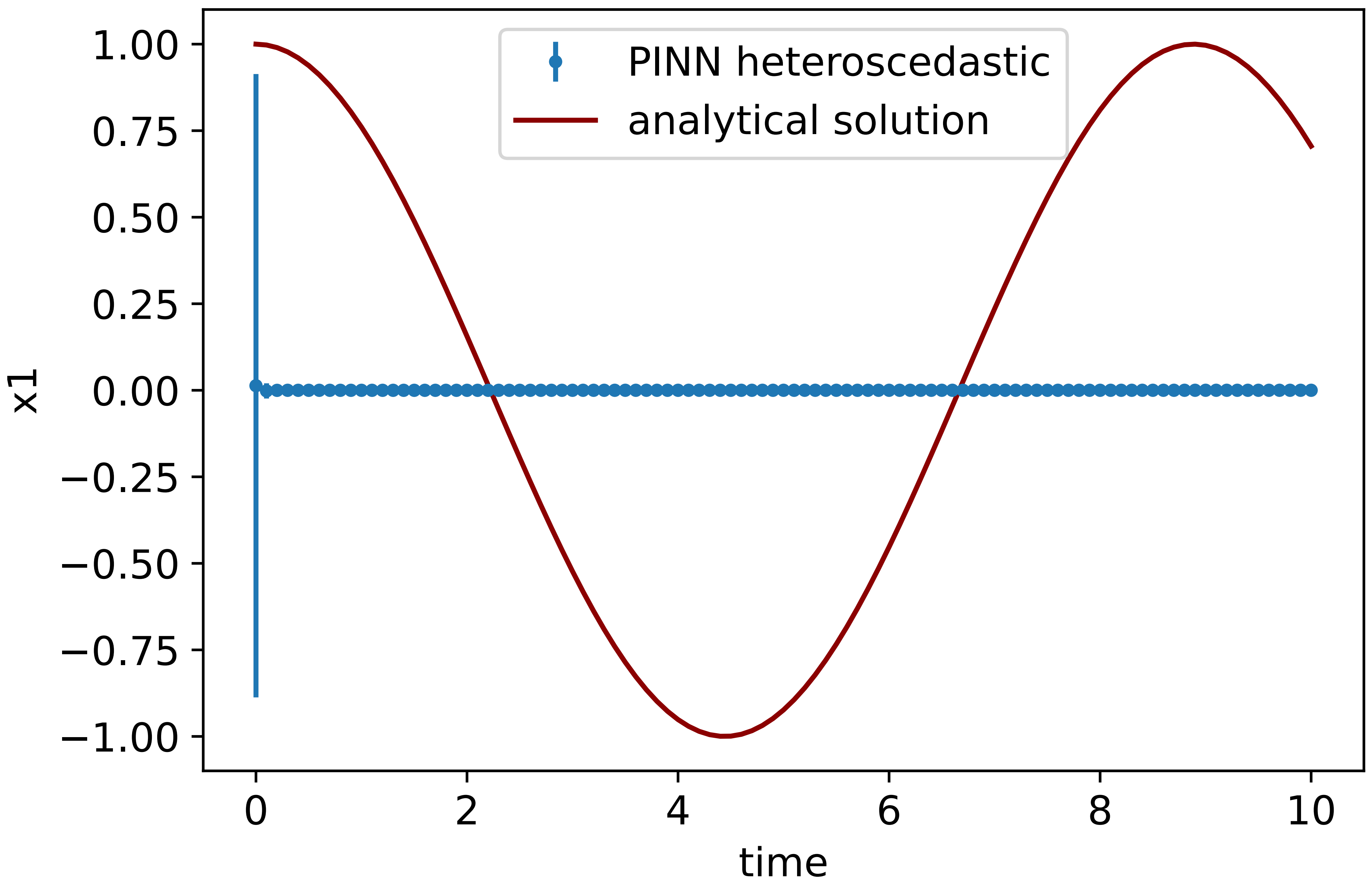
Uncertainty in PINNs

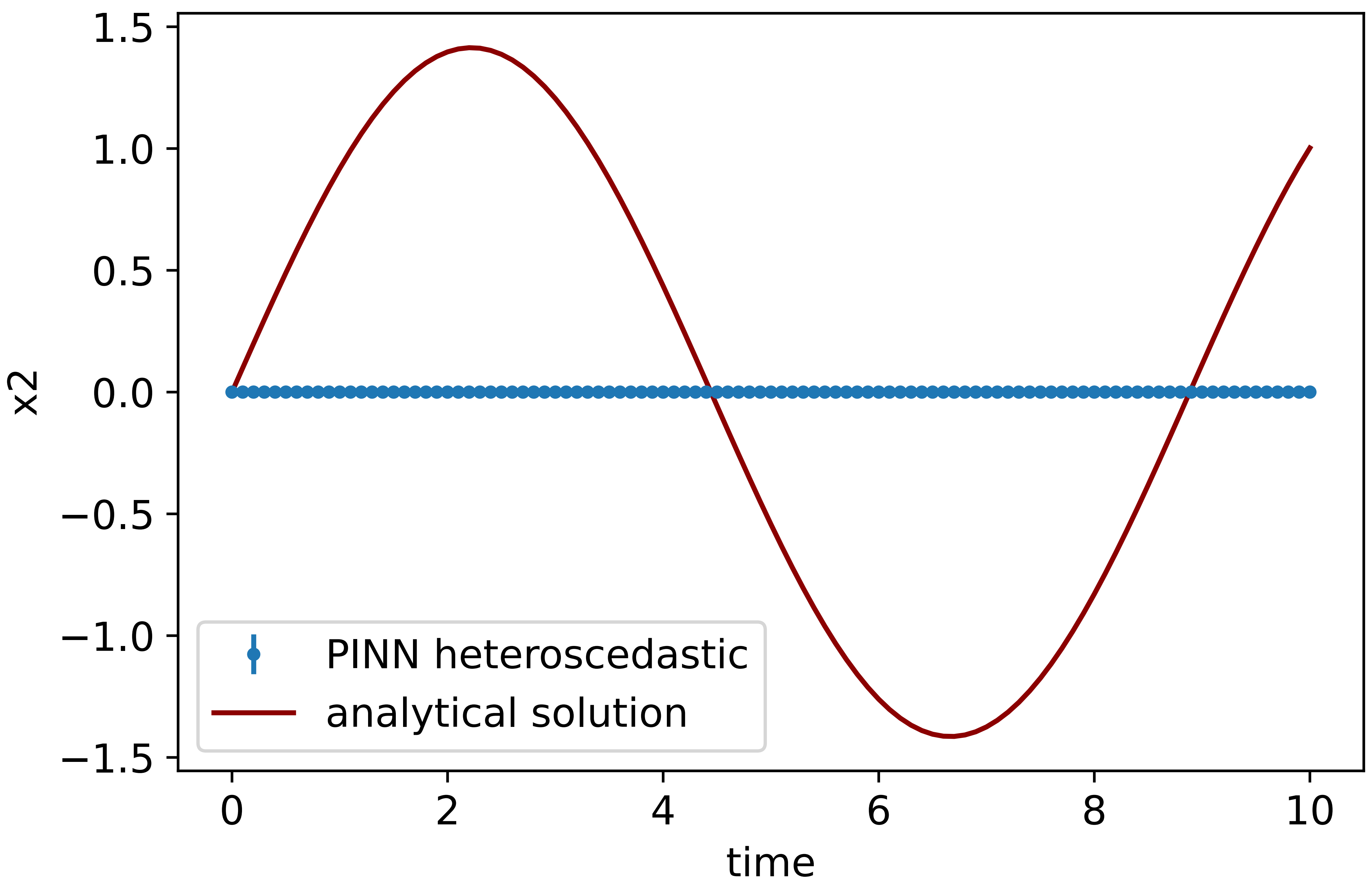
- Uncertainty as a solution to the differential equation
- Implement: double the output parameters

$$\begin{aligned}\dot{\mathbf{u}}(t) &= F(\mathbf{u}, t)\\ \mathbf{u}^i &= \mathbf{g}^i\end{aligned}$$

$$\mathscr{L}_{he} = \sum_{Data}\left[\sum_{dim,k}\left(\frac{\left(\dot{u}_\theta^k(x_i) - F^k(u_\theta(x_i))\right)^2}{2\left(\sigma_\theta^k(x_i)\right)^2} + \log \sigma_\theta^k(x_i)\right) + \frac{n}{2}\log 2\pi\right]$$

Heteroscedastic loss

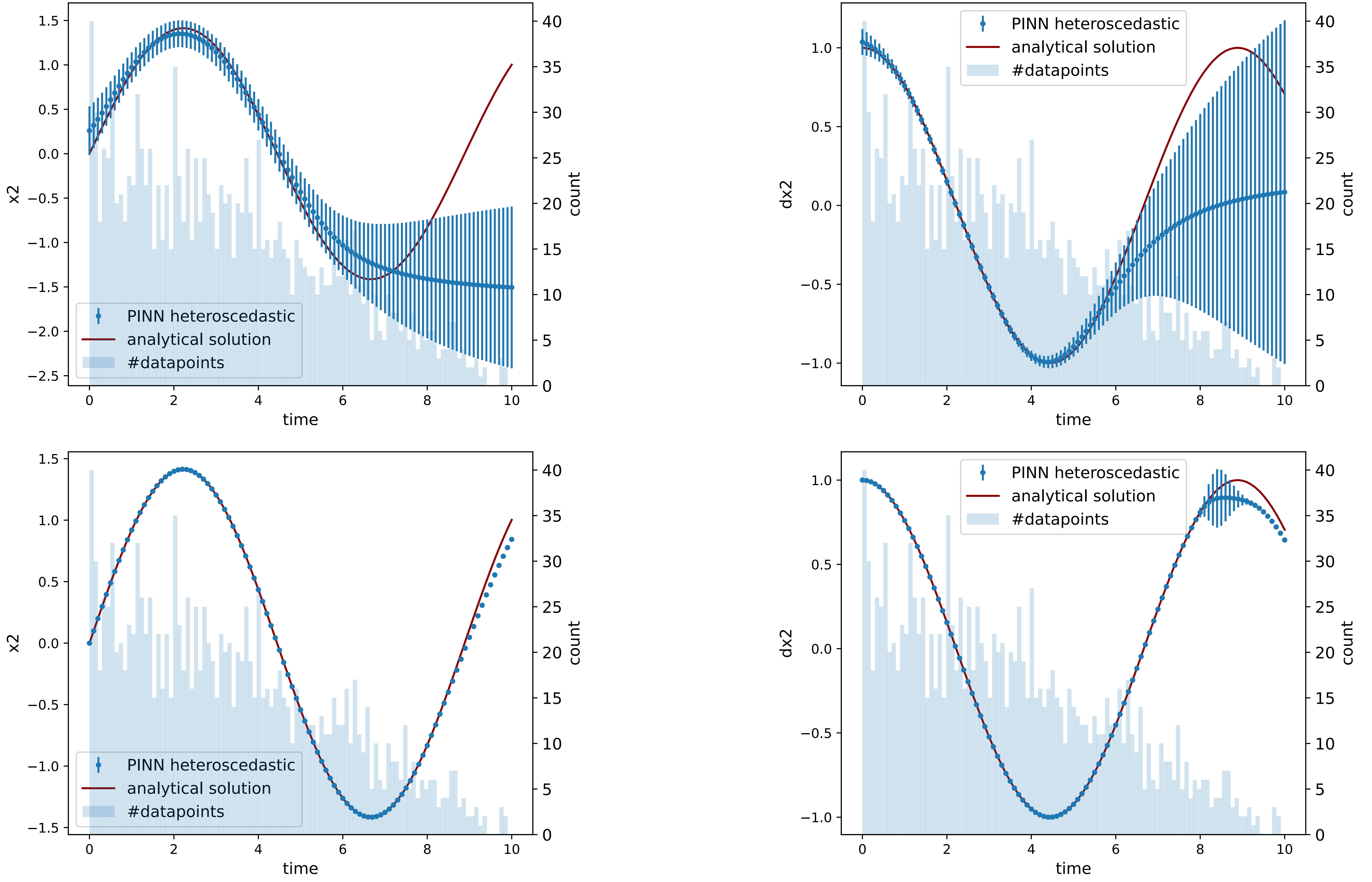




Heteroscedastic loss

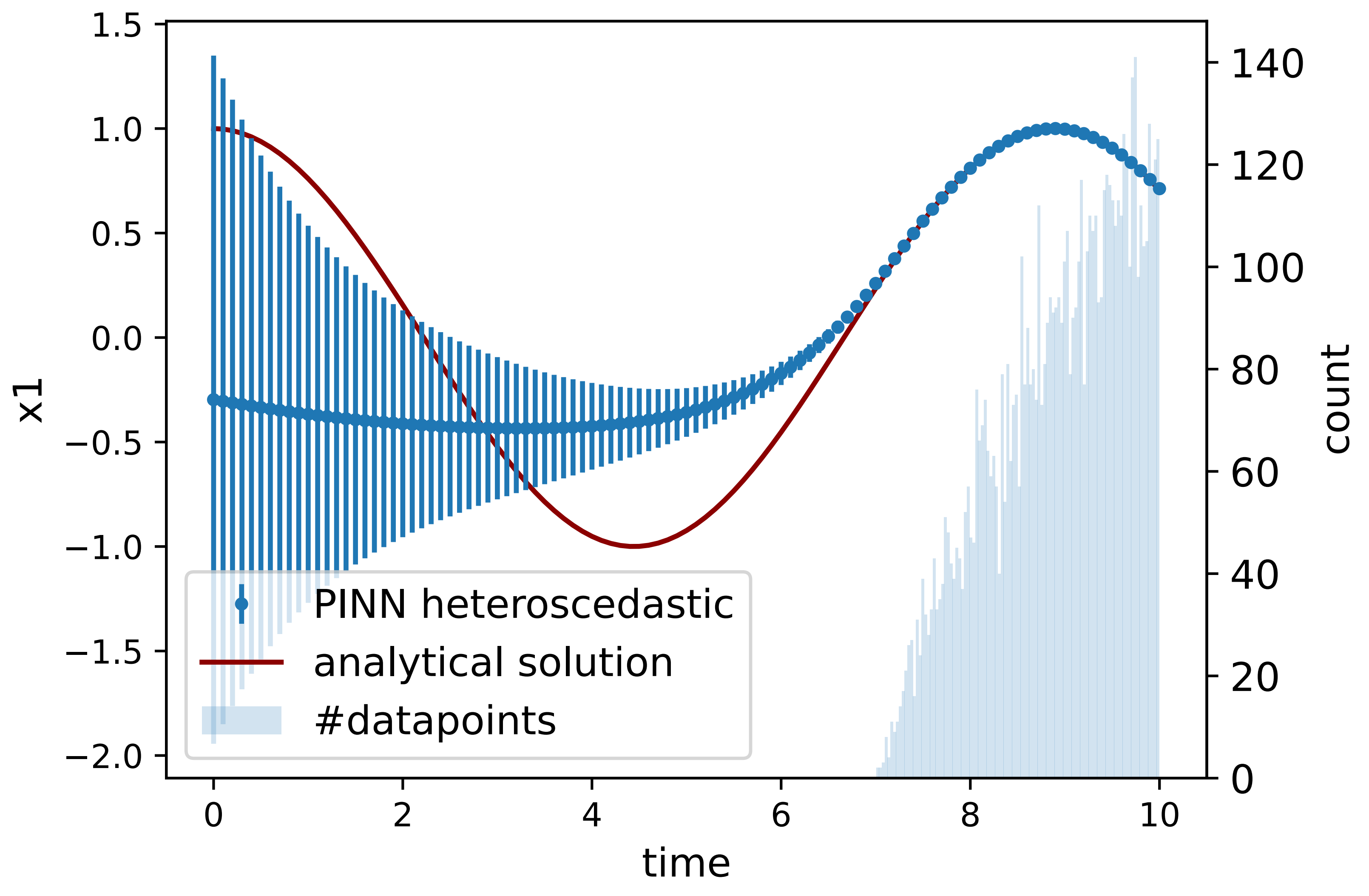
Data

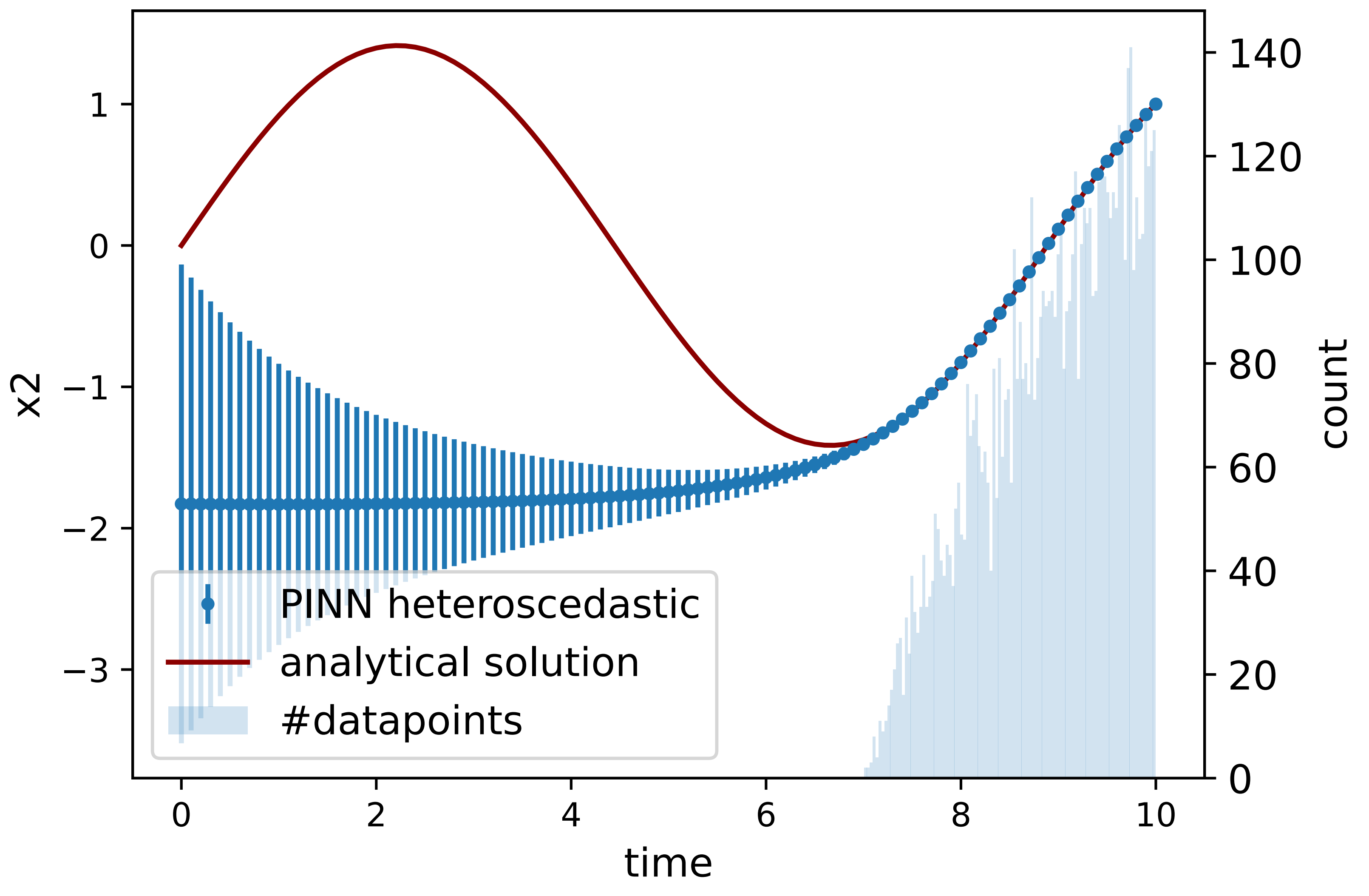
+ ODE



Heteroscedastic loss

Data





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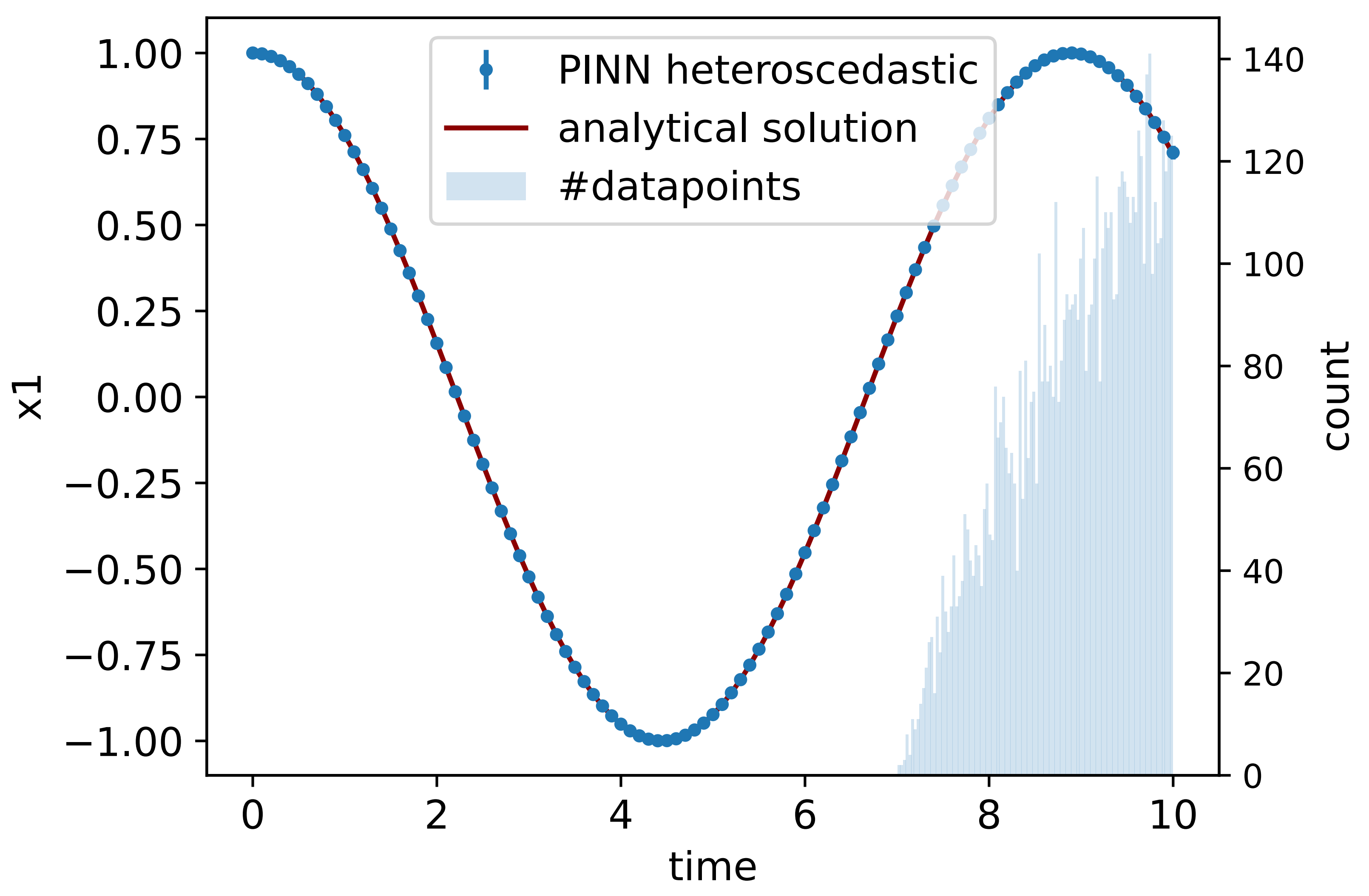
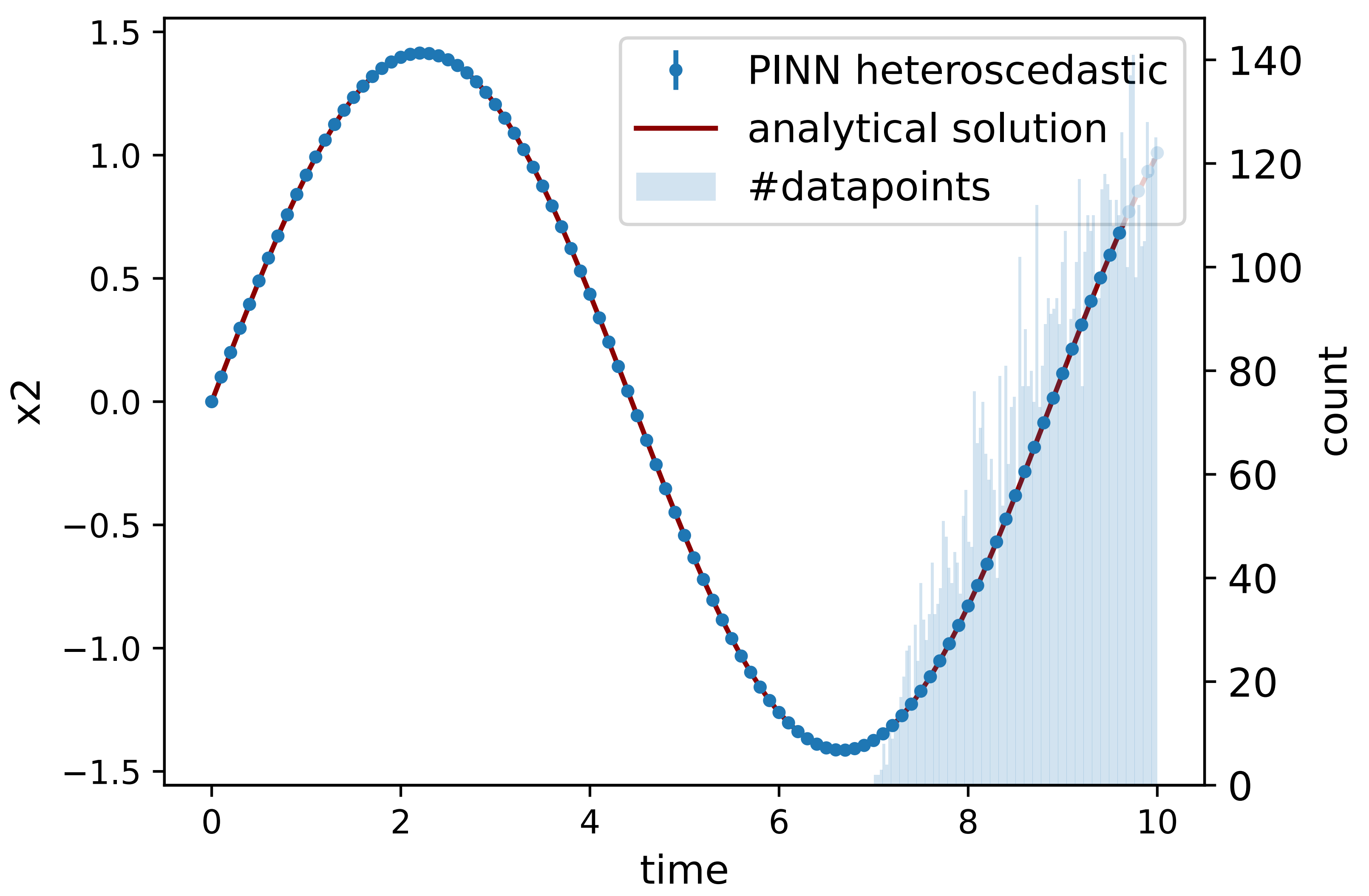


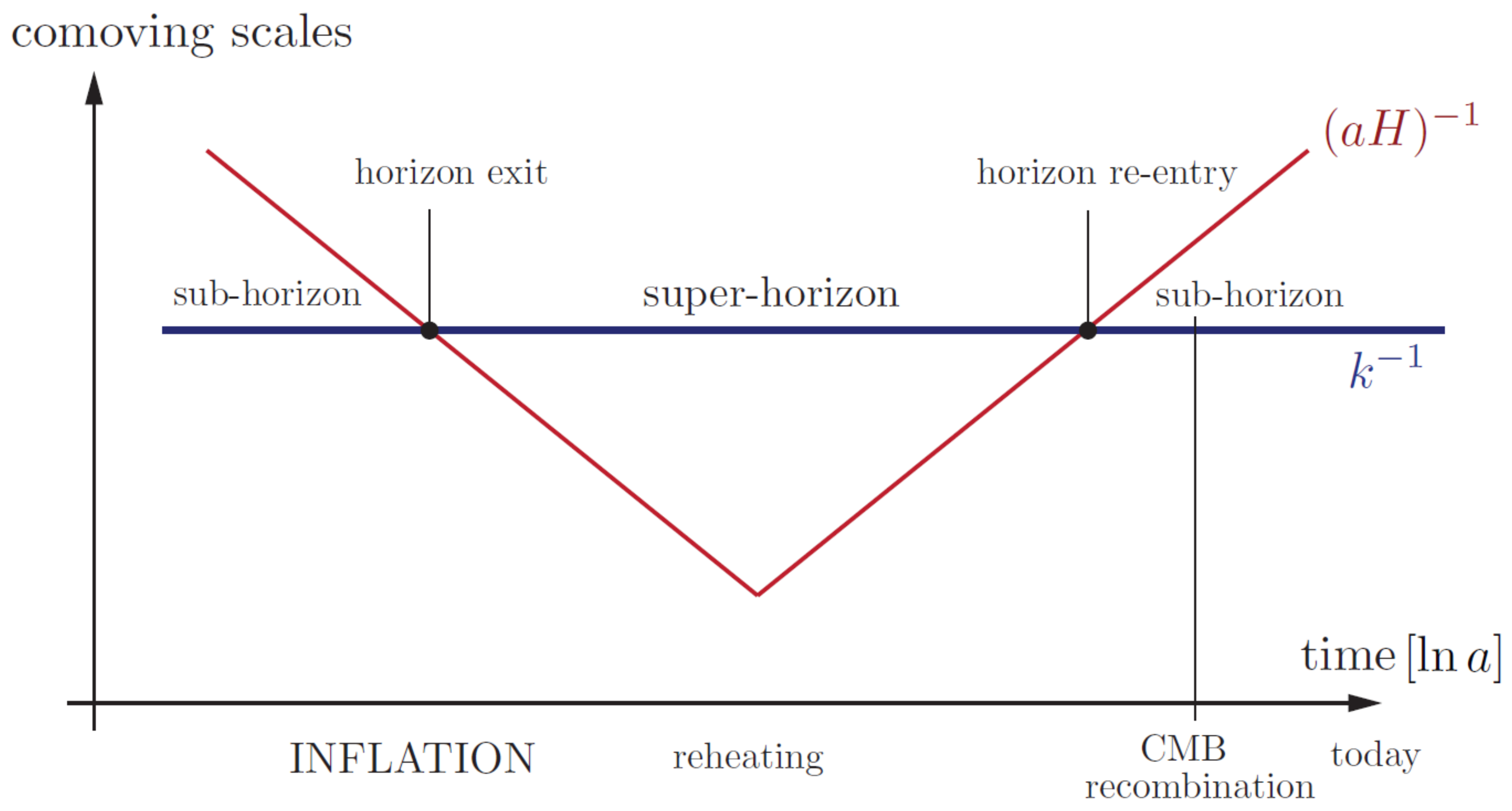


Inflation

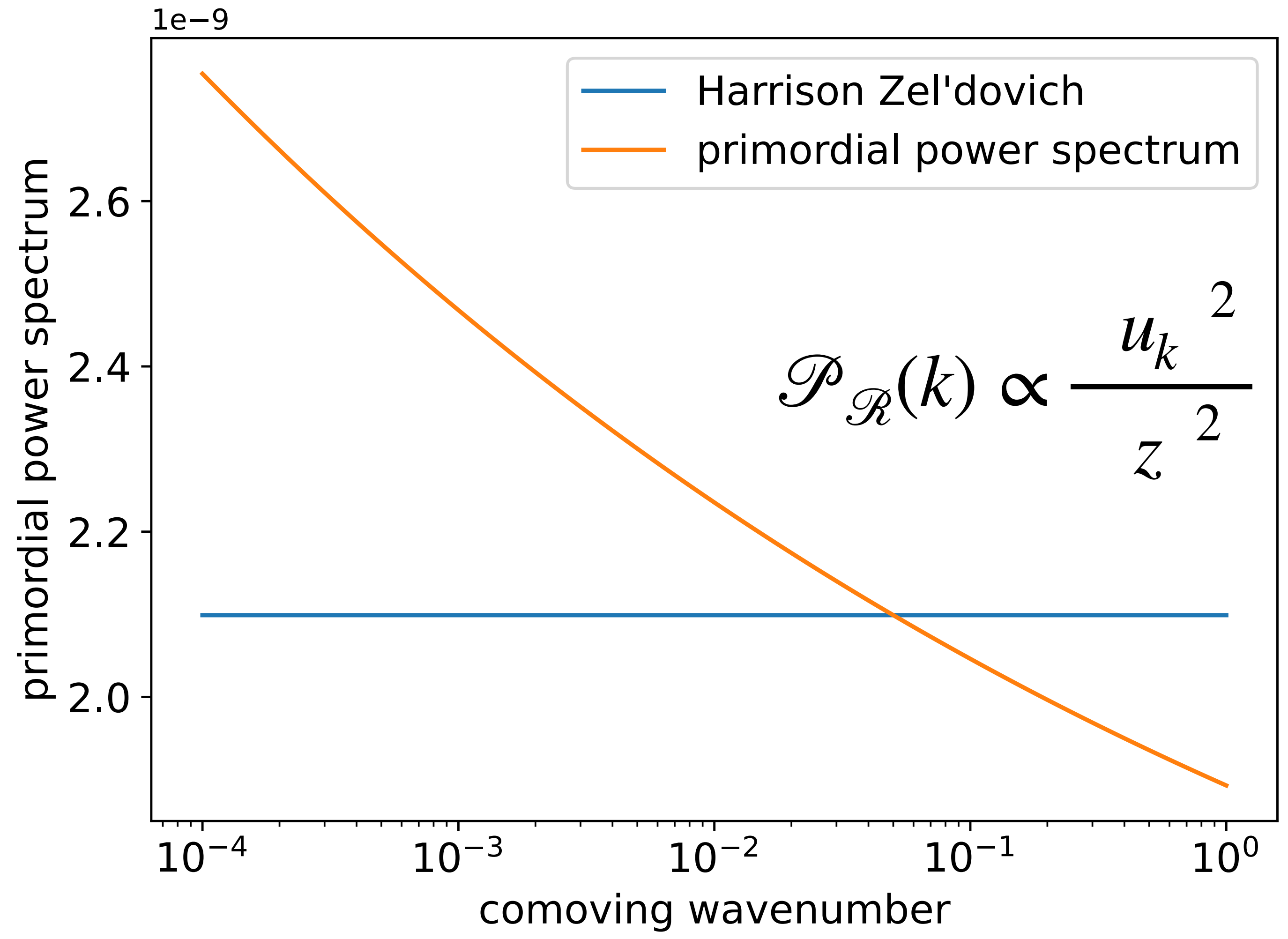
$$S_\varphi = \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \right]$$

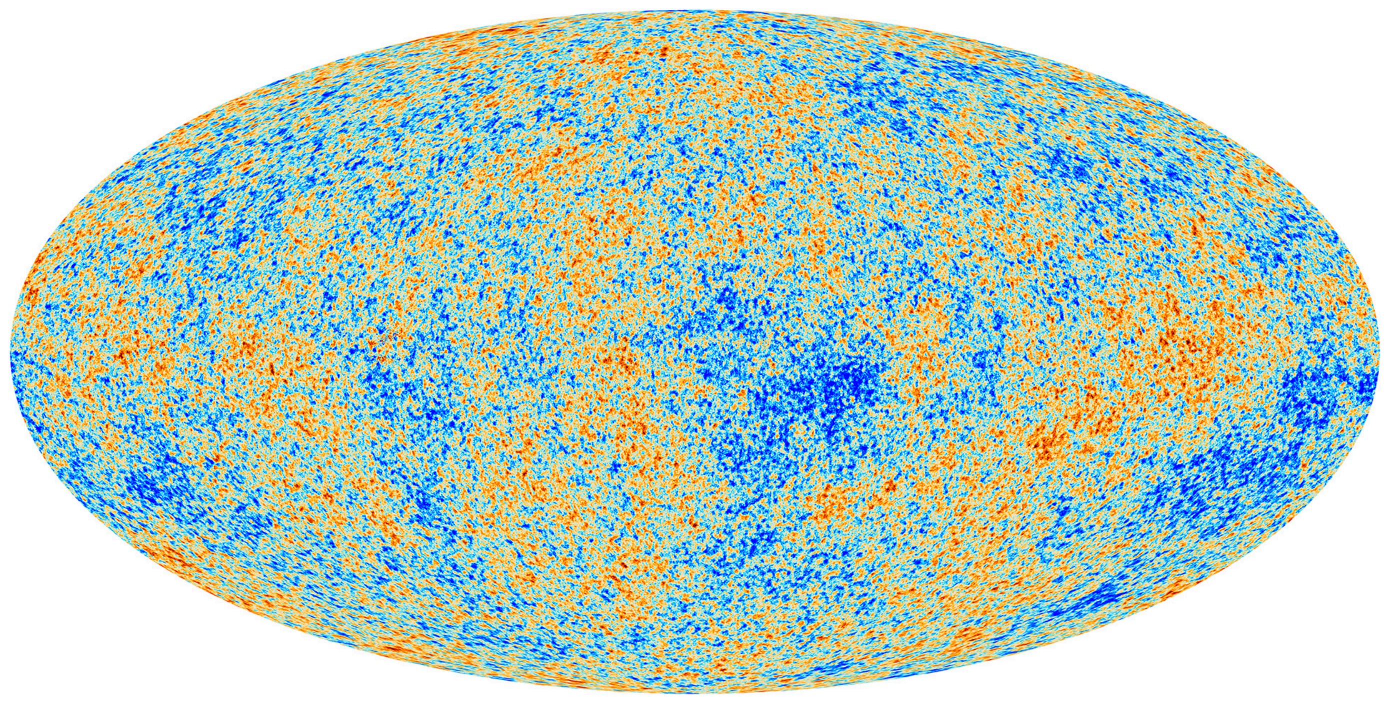
$\delta\varphi$

$$\ddot{u}_k = F(u_k, \dot{u}_k, \varphi)$$



Baumann ICTS 2011





Cosmic Microwave Background ESA

Inflation

$$V(\varphi) = V_0 \exp\left(\sqrt{\frac{2}{p}}\frac{\varphi}{M_{pl}}\right)$$

background

$$\boxed{\begin{aligned} \varphi'' + \left(\frac{H'}{H} - 3\right)\varphi' &= -\frac{1}{H^2}\partial_\varphi V \\ H^2 &= \frac{2V(\varphi)}{\frac{6}{M_{pl}^2} - (\varphi')^2} \end{aligned}}$$

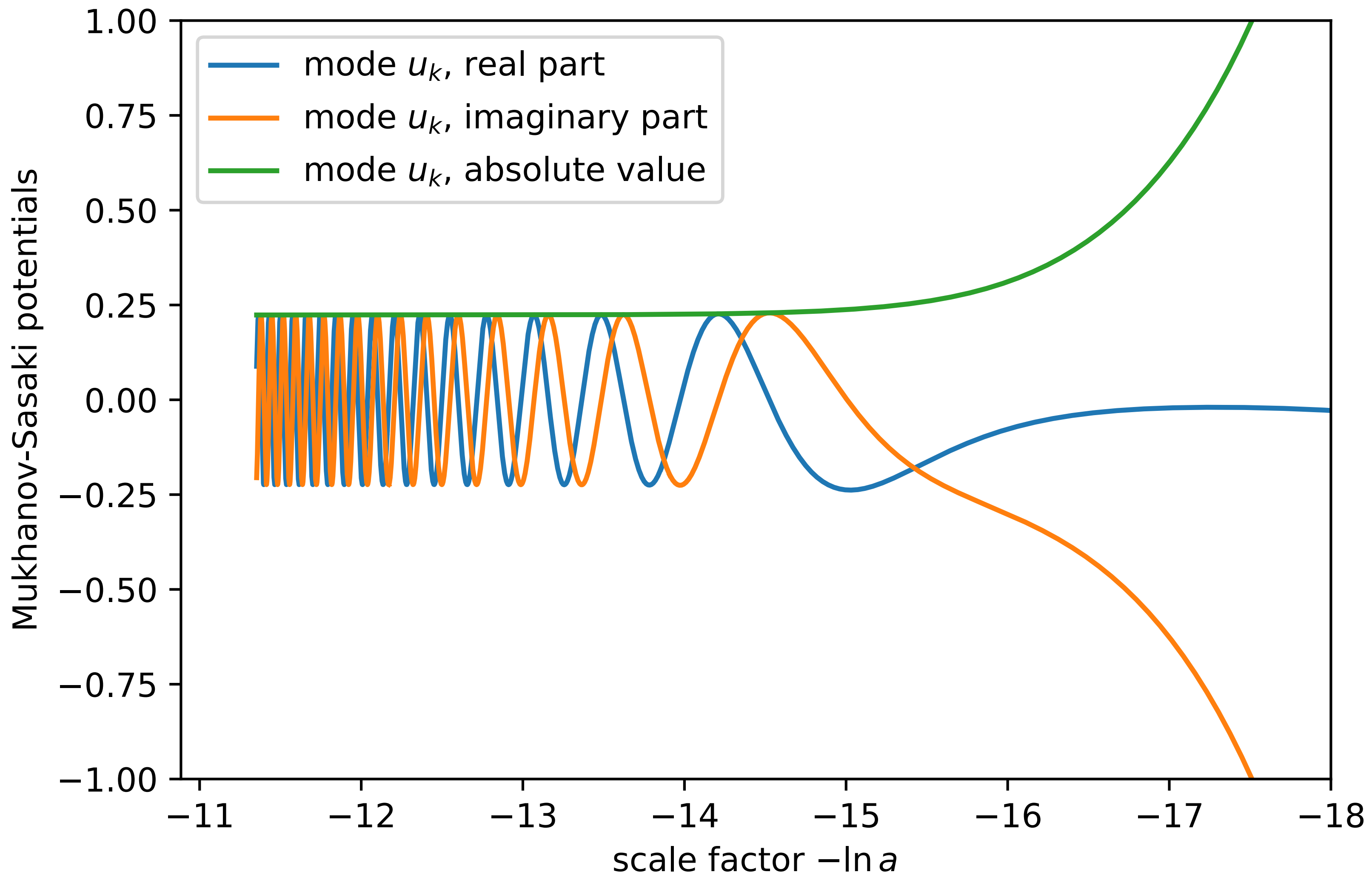
$$u_k'' + \left(\frac{H'}{H} - 1\right)u_k' + \left[\frac{k^2}{a^2H^2} - \left(2 - 2\left(\frac{H'}{H}\right)^2 - 4\frac{H'}{H}\frac{\varphi''}{\varphi'} + 5\frac{H'}{H} - \frac{\partial_\varphi^2 V}{H^2}\right)\right]u_k = 0$$

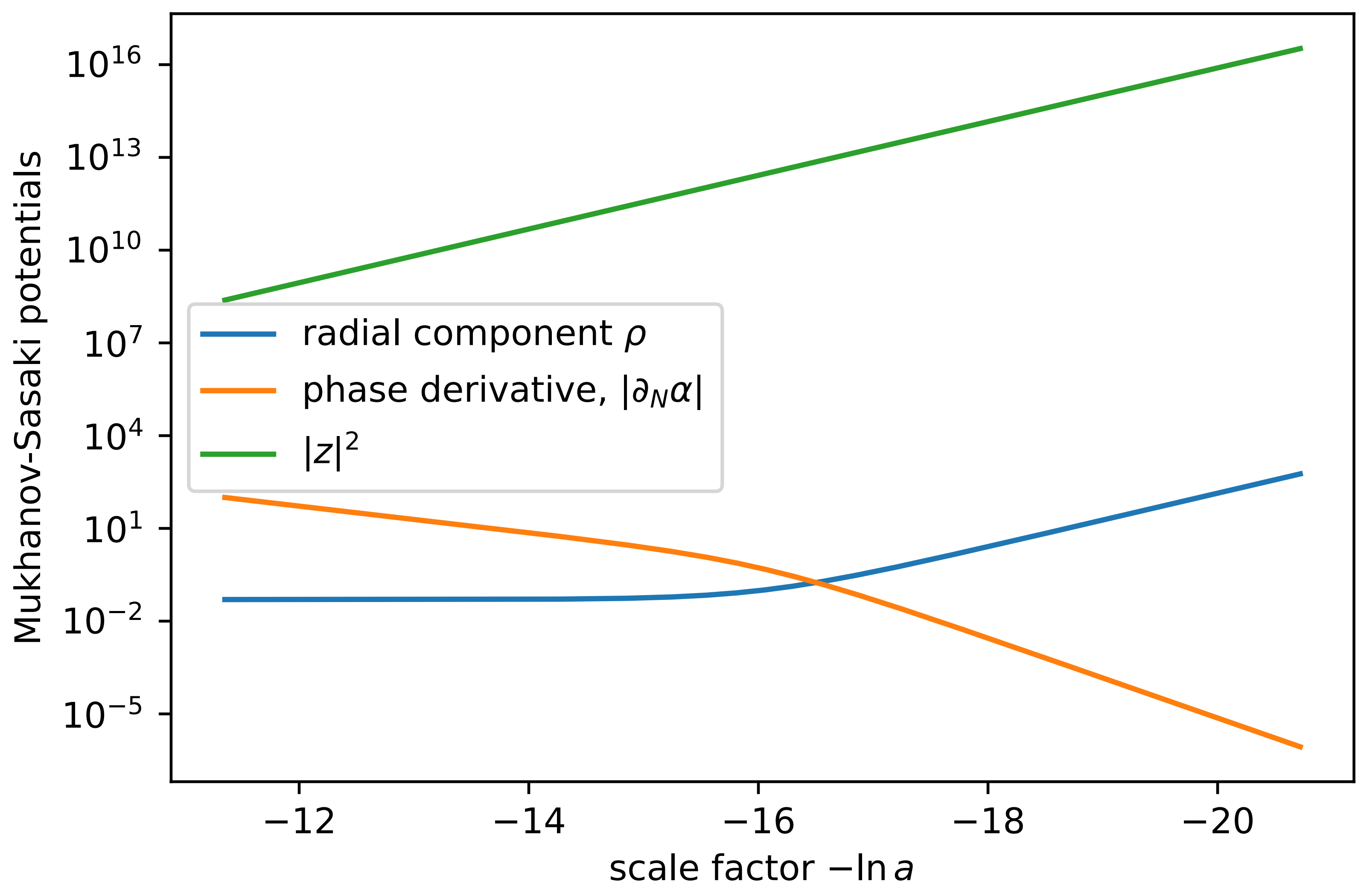
modes

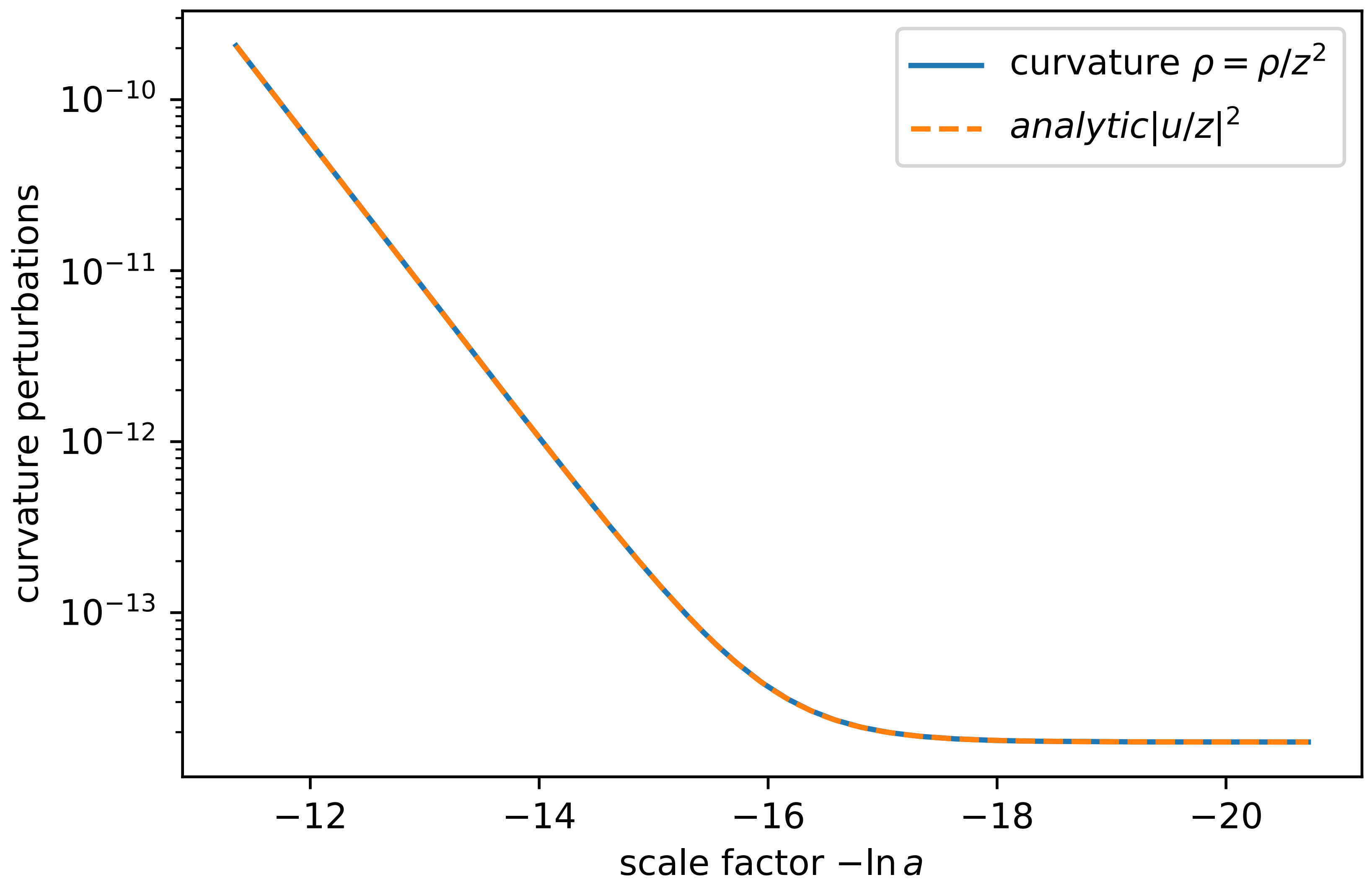
Madelung modes

Madelung transformation

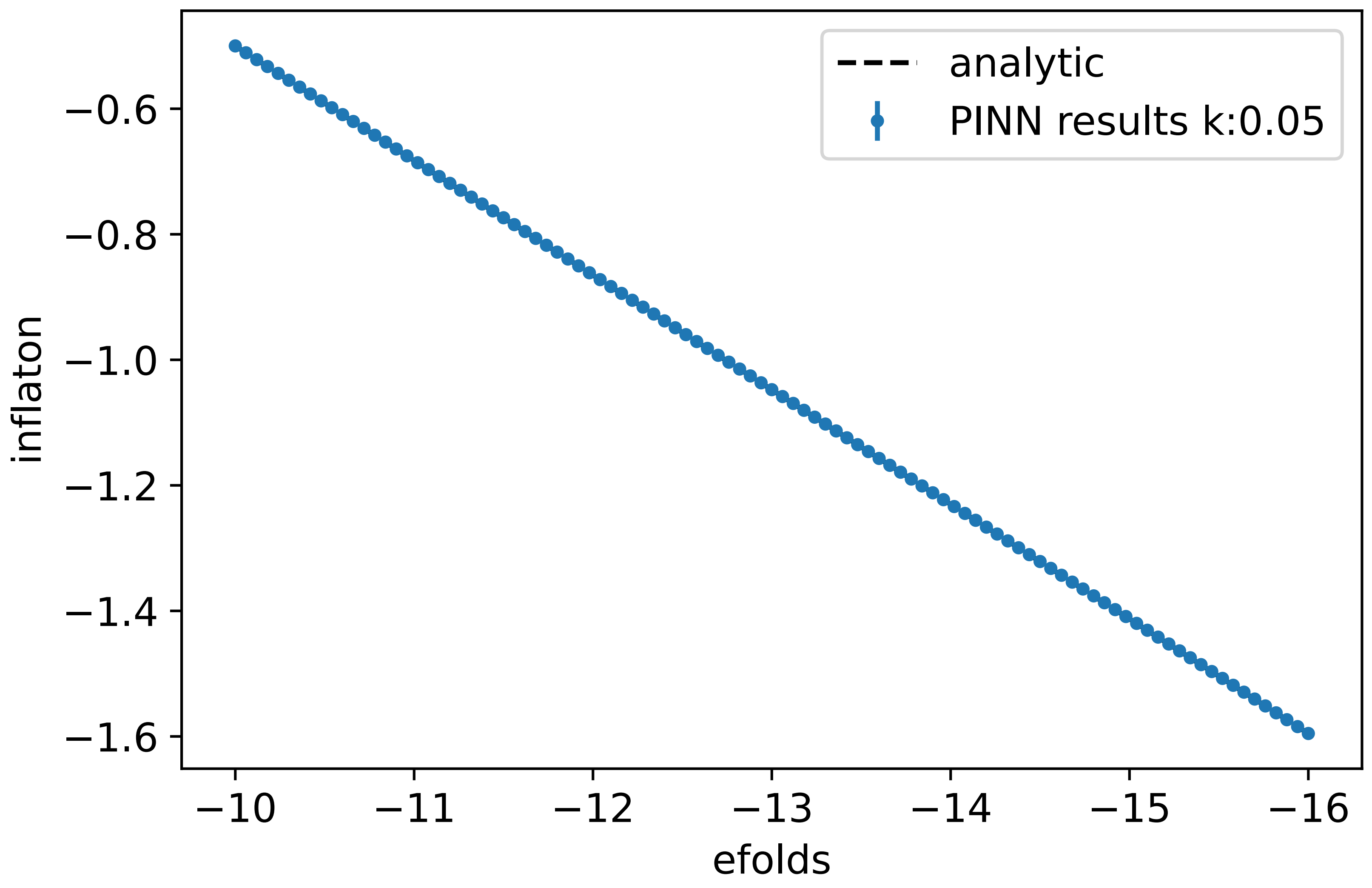
Rescaling

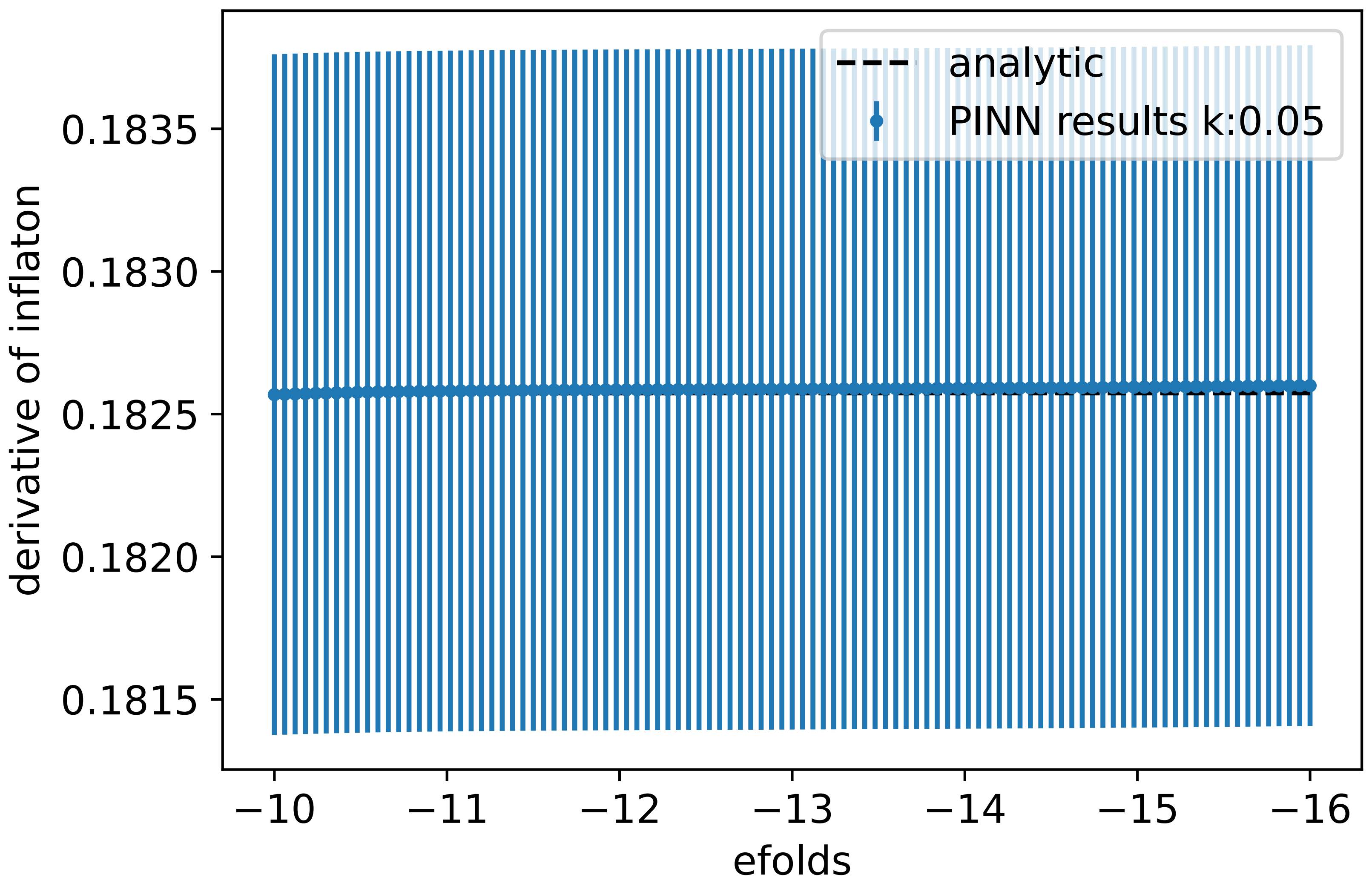


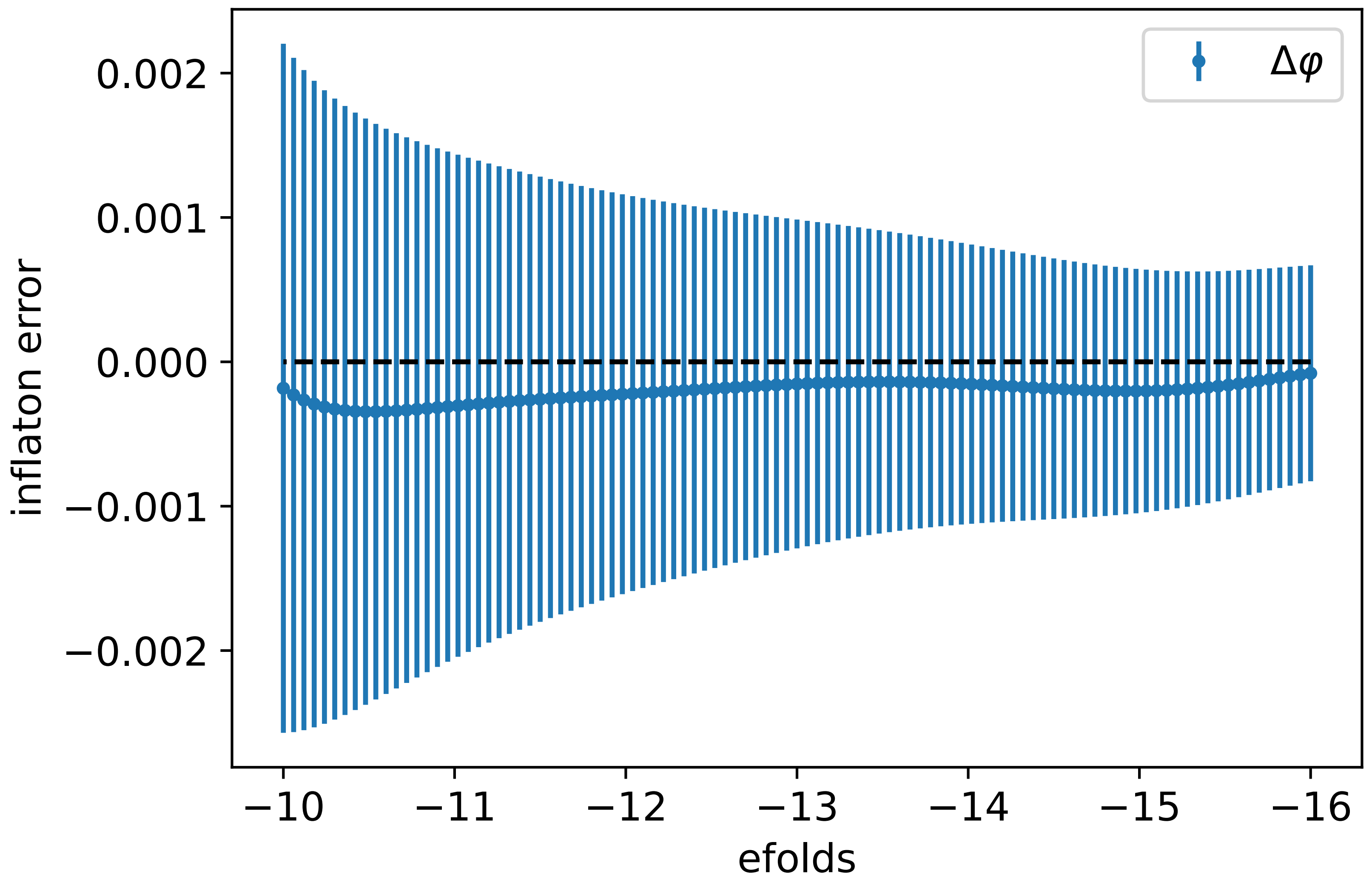


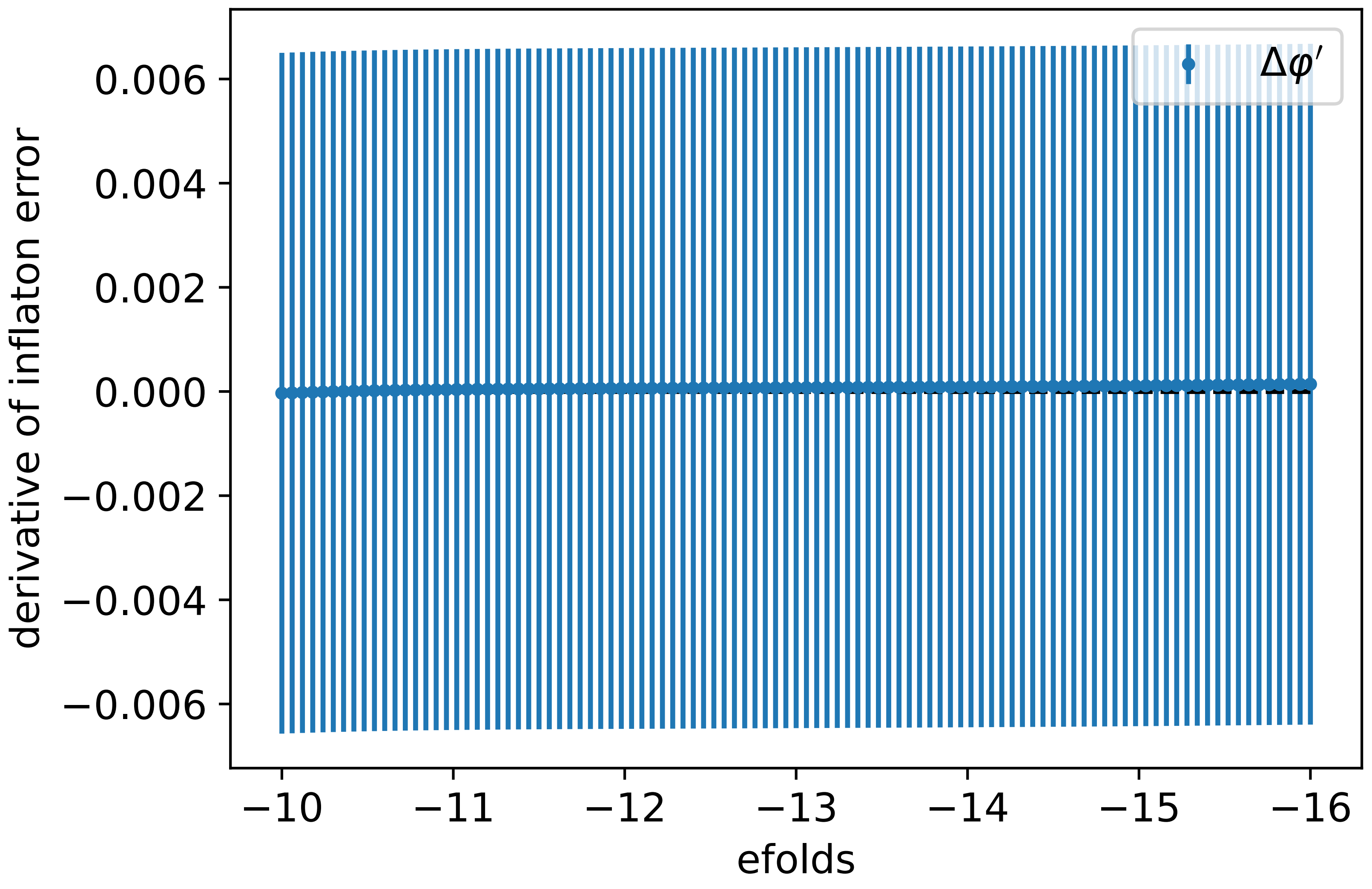


PINNflation — background

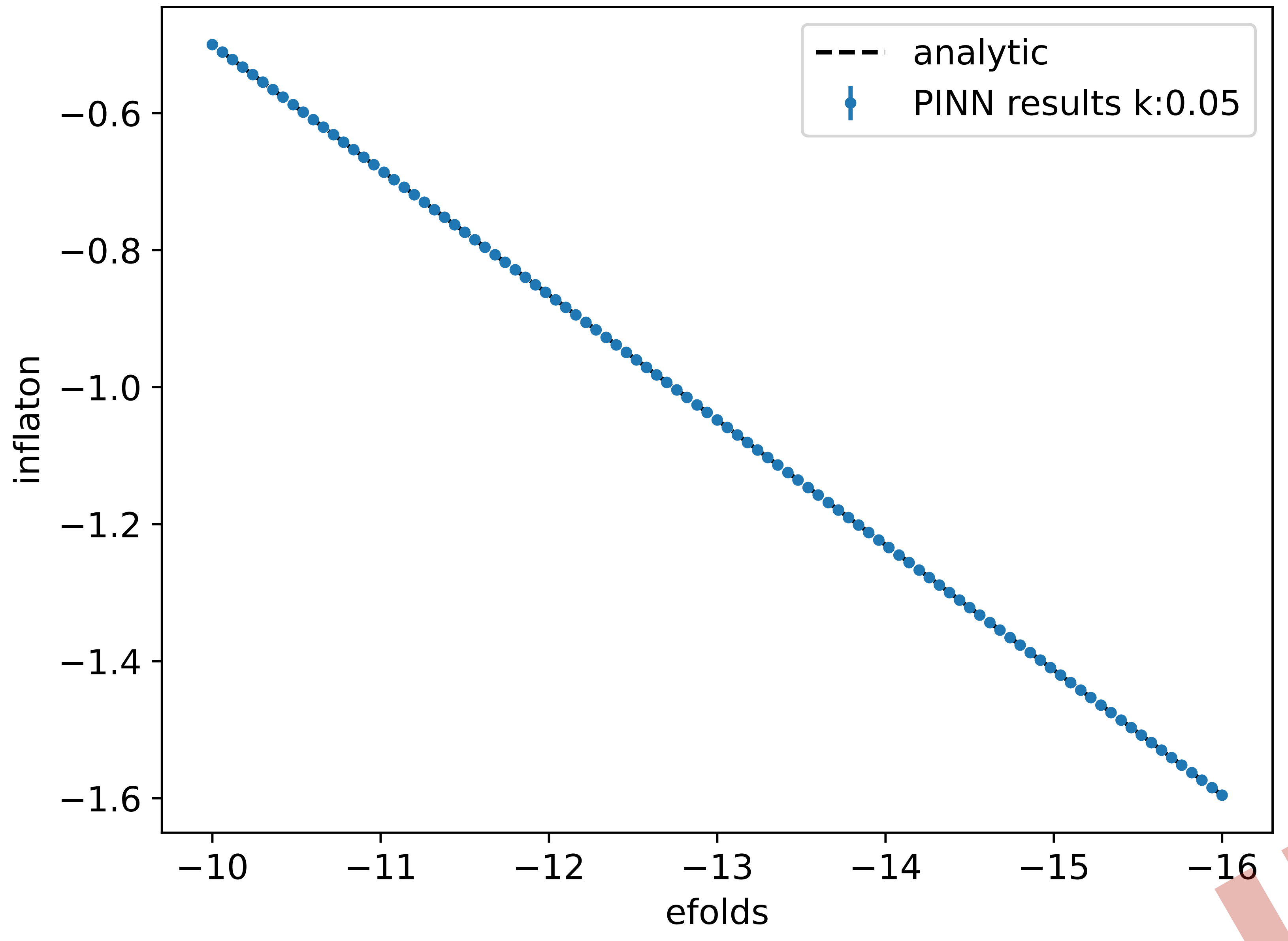


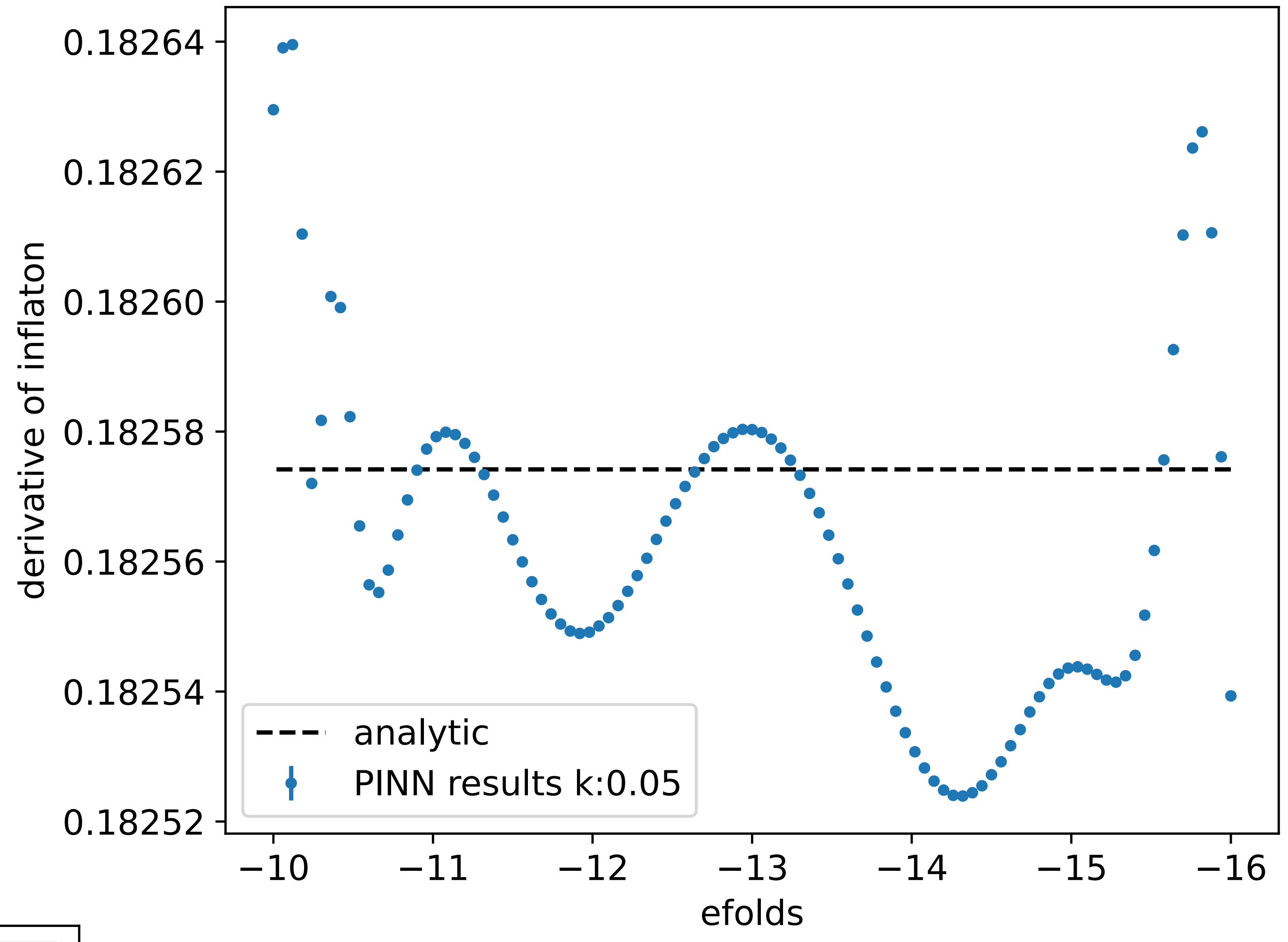


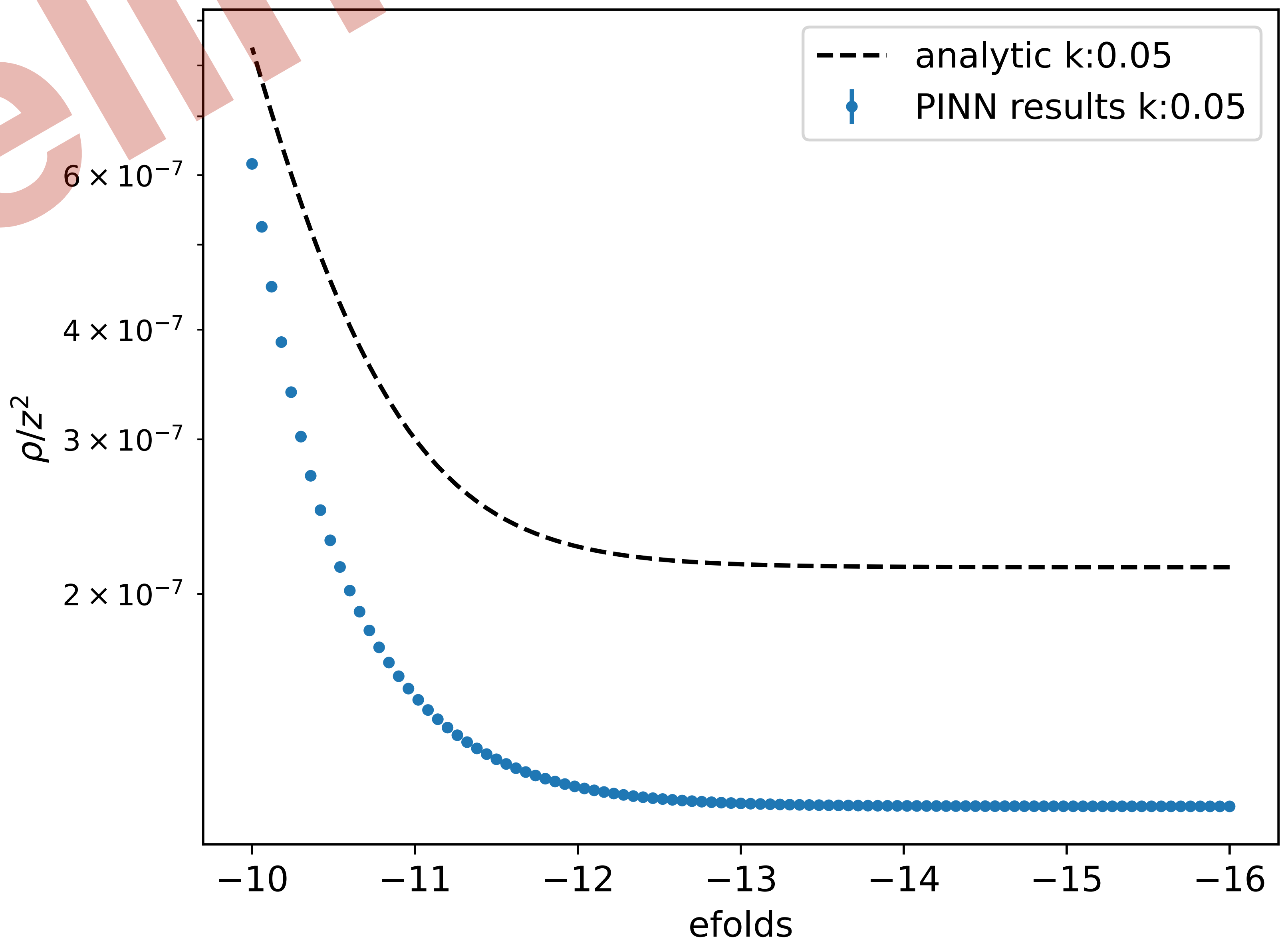




PINNflation — modes







Preliminary!

Conclusion

- PINNs are potentially useful for:
  - Extrapolation
  - Interpolation with a gap
- Watch out for:
  - Loss has local minima at ODE solutions
- Outlook: inverse problems