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Standard Model | **Homework 1** | **Date: 03.07.2023**

Feel free to send me (`christophe.grojean@desy.de`) your solutions and I'll give you feedback.

Exercice 1: Lorentz-invariance of space-time measure

In QFT, the action corresponds to integration of whole space-time of the Lagrangian

$$S = \int c\,dt\,d^3x\,\mathcal{L}$$

By construction, the Lagragian for scalars, fermions and gauge bosons are invariant under Lorentz transformations. Prove that the space-time integration measure, $c\,dt\,d^3x$, is also invariant under Lorentz transformations.

Exercice 2: Generic boost transformation

For a boost transformation along the x direction, the space-time coordinates transform as

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} t' = t \\ x' = -\beta_0 ct + x \\ y' = y \\ z' = z \end{pmatrix}$$

Show that the generic boost transformation for a relative speed $c\vec{\beta}$ between the two systems of coordinates is given by

$$\begin{pmatrix} ct \\ \vec{x} \end{pmatrix} \to \begin{pmatrix} ct' = \gamma\left(ct + \vec{\beta}\cdot\vec{x}\right) \\ \vec{x}' = \vec{x} + \vec{\beta}\left(\gamma ct + \frac{(\gamma-1)\vec{\beta}\cdot\vec{x}}{\beta^2}\right) \end{pmatrix}.$$

You'll simply show that this transformation leave $\Delta^2 \equiv (ct)^2 - \vec{x}^2$ invariant.

Exercice 3: EM action for photons

The photon field strength is constructed from the potential 4-vector $A^\mu = (\phi, \vec{A})$ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

We use a mostly minus metric: $\eta_{\mu\nu} = \text{diag}(1,-1,-1,-1)$.
a) From the classical EM definition of the electric and magnetic fields from the scalar and vector potential

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\wedge\vec{A}$$

show that

$$\vec{E}_i = -F_{i0} \quad \vec{B}_i = -\frac{1}{2}\epsilon_{ijk}F_{jk},$$

where ϵ_{ijk} is the totally antisymmetric 3-tensor normalised to $\epsilon_{123} = 1$. Conclude that

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\vec{E}_x & -\vec{E}_y & -\vec{E}_z \\ \vec{E}_x & 0 & -\vec{B}_z & \vec{B}_y \\ \vec{E}_y & \vec{B}_z & 0 & -\vec{B}_x \\ \vec{E}_z & -\vec{B}_y & \vec{B}_x & 0 \end{pmatrix}.$$

b) Derive the expression of the gauge field Lagrangian density, $-\frac{1}{4\pi}F_{\mu\nu}F^{\mu\nu}$, in terms of the electric and magnetic fields and recognise the usual expression of the energy density stored in the electromagnetic fields.

c) It is helpful to introduce the dual field strength

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally anti-symmetric tensor with 4 indices normalised such that $\epsilon^{0123} = 1$. Show that

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -\vec{B}_x & -\vec{B}_y & -\vec{B}_z \\ \vec{B}_x & 0 & \vec{E}_z & -\vec{E}_y \\ \vec{B}_y & -\vec{E}_z & 0 & \vec{E}_x \\ \vec{B}_z & \vec{E}_y & -\vec{E}_x & 0 \end{pmatrix}.$$

d) There is another Lorentz-invariant Lagrangian density that one can construct from the electromagnetic field:

$$\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}.$$

Compute this Lagrangian density in terms of $\vec{E}$ and $\vec{B}$. How does this Lagrangian density behave under a spatial parity transformation?

e) In the vacuum, the Maxwell equations read

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0$$

and are therefore manifestly invariant under the exchange $F^{\mu\nu} \leftrightarrow \tilde{F}^{\mu\nu}$. How does this transformation act on the electric and magnetic fields? This transformation is actually part of what is called the $SL(2,Z)$ eletric-magnetic duality.

Exercice 4: Rapidity and pseudo-rapidity

We define the rapidity, y, and the pseudo-rapidity, η, of a particle of mass m produced in a collider:

$$y = \frac{1}{2}\log\frac{E+p_L}{E-p_L}, \quad \eta = -\log\tan\theta/2,$$

where E is the energy of the particle, p_L its momentum along the beam axis, and θ the angle of the particle with the beam axis.

a) Show that in the limit $m \to 0$, $\eta \to y$.
b) Show that the rapidity is bounded by: $|y| \leq \log \frac{2E}{m}$.
c) Show that the difference of rapidities of two particles is invariant under Lorentz boosts along the beam axis.

Exercice 5: U(1) field strength

For a U(1) gauge theory, the covariant derivate acting on a scalar field of charge q is defined as $D_\mu \phi = \partial_\mu + ieqA_\mu \phi$. Check the identity:

$$[D_\mu, D_\nu]\phi = ieqF_{\mu\nu}\phi,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Exercice 6: Non-Abelian field strength

In the lecture, we derived the gauge transformation of a non-Abelian gauge field:

$$A_\mu \to A'_\mu = UA_\mu U^\dagger + \frac{i}{g}\left(\partial_\mu U\right)U^\dagger.$$

a) Check explicitly that the tensor field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$$

transforms homogeneously

$$F_{\mu\nu} \to UF_{\mu\nu}U^\dagger.$$

b) Check that, like in the Abelian case, the field strength is associated to the commutator of covariant derivatives

$$[D_\mu, D_\nu]\phi = igF_{\mu\nu}\phi,$$

where the covariant derivative acting on ϕ as: $D_\mu\phi = \partial_\mu\phi + igA_\mu\phi$.