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Standard Model | **Homework 2** | **Date: 04.07.2023**

Exercice 1: Natural units

a) Show that $[\hbar] = M \cdot L^2 \cdot T^{-1}$ and $[c] = L \cdot T^{-1}$.
b) Check the consistency of the classical/quantum correspondence at the dimensional level:

$$E \to i\hbar \frac{\partial}{\partial t} \quad \& \quad p \to i\hbar \frac{\partial}{\partial x}$$

c) Show that

$$1\,\text{s} = 1.52 \cdot 10^{27}\,\hbar/\text{TeV}, \quad 1\,\text{m} = 5.1 \cdot 10^{18}\,\hbar c/\text{TeV}, \quad 1\,\text{kg} = 5.61 \cdot 10^{23}\,\text{TeV}/c^2$$

d) Using the Newton constant, $\hbar$ and c, construct a mass scale, a length scale and a time scale. They are defining the Planck scales. Compute the matter density of the universe today (10^{-29} g/cm^3) in Planck units.
e) The Schwarzschild radius of an object of mass m is the measure of its mass in Planck units. The Compton wavelength is defined as $\hbar/(mc)$. Compute the Schwarzschild radius of the Earth, the Sun, a neutron star, a stellar black-hole, a super-massive BH, a micro-BH (you'll check on Wikipedia the characteristic mass of these objects). What do you conclude? Compute the Schwarzschild radius of a micro-BH assuming that the Planck scale has been reduced to 1 TeV. What do you conclude?
f) Using e, m_e and c, construct a length scale. This is the classical radius of the electron.
Using e, m_e and $\hbar$, construct a length scale. This is the Bohr radius of the electron.
g) The pion Compton wavelength in natural units is $\lambda_\pi = \hbar/(m_\pi c)$. A typical hadronic cross section is of order $\sigma \simeq \lambda_\pi^2 \simeq 1/(140\,\text{MeV})^2$. Express this quantity in units of barns (1 barn = $10^{-28}\,\text{m}^2$).

Exercice 2: Value of e in HEP units

The electromagnetic fine-structure constant was defined by A. Sommerfeld in 1916. It is given by

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c},$$

where $e = 1.6 \times 10^{-19}\,\text{C}$ is the unit electric charge, $\epsilon_0 = 8.8 \times 10^{-12}\,\text{F} \cdot \text{m}^{-1}$ is the vacuum permittivity.
a) Compute the value of α. Check that it is a dimensionless quantity (we remind that $1\,\text{F} = 1\,\text{C}^2.\text{J}^{-1}$)
b) Deduce the value of the electric charge e in the HEP units ($\hbar = c = \epsilon_0 = 1$).

Exercice 3: Order of magnitude estimates

a) Estimate the energy of the cosmic rays given that the lifetime of a muon is about 1 μs.

b) A 1 cm^3 piece of ice melts in about 40 minutes under the sun. Compute the volume of oil to burn 1 cm-thick ice cap surrounding the sun at a distance of 150 million kilometres (the effects of the atmosphere will be neglected). Assuming that all the energy radiated by the Sun would be of chemical origin, what would be the maximal lifetime of the Sun? What do you conclude concerning the origin of the energy radiated by the Sun? We recall that burning 1 liter of oil produces about 30 MJ and that 333 kJ are needed to melt 1kg of ice.

Exercice 4: Average temperatures on the planets of the Solar system

We'll assume that the sun and the solar system planets are perfect black-bodies, and we'll neglect any effects of the planet atmospheres.

a) Using the Stefan–Boltzmann law, compute the luminosity of the sun (we recall that the average surface temperature of the Sun in $\langle T_\odot^{\text{surface}} \rangle = 5778\,\text{K}$).

b) Assuming that the planets radiate away all the energy received from the Sun, estimate the average temperature on the different planets. Compare with the data you can find on Wkipedia?

Exercice 5: Hawking Black Hole radiation

a) S. Hawking understood that the laws of quantum mechanics imply that a BH is actually radiating particles, hence energy. Based on dimensional arguments, find the Hawking temperature of a BH of mass M. There is a priori a 1D infinite family of solution, you'll retain the solution that scales with a single power of $\hbar$. The exact formula derived by Hawking is smaller by a factor $1/(8\pi)$ compared to the naive estimate.

b) Assuming that a BH is a perfect black-body, use the Stefan–Boltzmann law ($P \propto T^4$) to derive the luminosity of a BH of mass M. Numerically, compute the power of a BH of solar mass (we recall that the Stefan–Boltzmann constant is equal to $\pi^2 k_B^4/(60\hbar^3 c^2)$).

c) Using the conservation of energy, derive the differential equation controlling the time evolution of the BH mass. Integrate this equation to obtain the BH life time.

d) What is the lower bound on the mass of a BH to be as old as the Universe?