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Feel free to send me (`christophe.grojean@desy.de`) your solutions and I'll give you feedback.

Exercice 1: Solar neutrino flux

The main source of energy of the Sun is the thermonuclear reaction:

$$4\,{}^1_1H \to {}^4_2He + 2\,e^+ + 2\,\nu_e + \gamma$$

where ${}^1_1H$ and ${}^4_2He$ denote the nuclei of the hydrogen and helium atoms respectively.
a) The mass of Helium is 4.002602 atomic unit, the mass the proton is 1.007276466879 atomic unit (or 938.2720813 MeV). Compute the amount of energy carried away by the photon is the reaction above (you'll consider that all the particles are at rest).
b) From the luminosity of the Sun estimated in the lecture ($4 * 10^{26}$ W), compute the amount of matter lost and transformed into energy every second in the Sun. Is it now compatible with the age of the Sun?
c) From the value of the luminosity of the Sun, estimate the number of neutrinos produced by the Sun every second.
d) Compute the flux of neutrinos emitted by the Sun and received on Earth.

Exercice 2: Chirality matrix

In 4d space-time, the four 4×4 Dirac matrices obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_4.$$

It is useful to define the *chirality* matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

a) Show that the chirality matrix anti-commutes with the four Dirac matrices.
b) Compute the square of the chirality matrix.
c) Conclude that $P_L = (1_4 + \gamma^5)/2$ and $P_R = (1_4 - \gamma^5)/2$ are two projector operators.

Exercice 3: Dirac, Weyl and Majorana representations of the Dirac matrices

In the lecture, it was stressed that the representation of the Dirac matrices is not unique. Three standard representations are the following (σ^i, for $i = 1, 2, 3$, are the three 2×2 Pauli matrices):

$$\text{Dirac: } \gamma^0 = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}.$$

$$\text{Weyl: } \gamma^0 = \begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix}, \ \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}.$$

$$\text{Majorana: } \gamma^0 = \begin{pmatrix} & \sigma^2 \\ \sigma^2 & \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} i\sigma^3 & \\ & i\sigma^3 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} & -\sigma^2 \\ \sigma^2 & \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} -i\sigma^1 & \\ & -i\sigma^1 \end{pmatrix}.$$

a) Verify that these representations indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_4$.
b) Compute the chirality matrix, γ^5, in these 3 representations.
c) In each of these representations, find the charge conjugation matrix B such that $B\gamma^{\mu *} = -\gamma^\mu B$.

Exercice 4: Weyl and Majorana fermions

a) A Weyl or chiral spinor is defined to be an eigenstate of the chirality matrix γ^5:

$$\text{Weyl fermion:} \quad \gamma^5 \psi = \pm \psi.$$

Show that this condition is Lorentz-invariant (you'll consider, here and in the rest of the exercise, *small* Lorentz transformations only: $\psi(x) \to \psi'(x) = (1_4 + [\gamma^\mu, \gamma^\nu]\omega_{\mu\nu}/8)\psi(x)$, with $\omega_{\mu\nu} = -\omega_{\nu\mu}$).
b) Is the condition $\psi^* = \psi$, that would naively define a *real* spinor, Lorentz-invariant?
c) For any spinor ψ, we define the charge-conjugated spinor ψ_C as

$$\psi_C = B\psi^*,$$

where B is the charge conjugation matrix that satisfies $B\gamma^{\mu *} = -\gamma^\mu B$. Show that ψ and ψ_C transform in the way under Lorentz transformations. Conclude that the Majorana reality condition:

$$\text{Majorana fermion:} \quad \psi_C = \psi,$$

is Lorentz-invariant.
d) Show that if a spinor satisfies both Weyl and Majorana conditions at the same time, it has to vanish.
e) In the lecture, we have said that the Dirac mass term

$$\mathcal{L}_{\text{Dirac mass}} = m\psi^\dagger \gamma^0 \psi$$

is Lorentz-invariant. Write explicitly this Dirac mass operator in terms of the two chiralities of the fermion $\psi = \psi_L + \psi_R$, where $\psi_{L/R} = (1_4 \pm \gamma^5)\psi/2$.
f) From the result of **c)**, argue that the Majorana mass term

$$\mathcal{L}_{\text{Majoran mass}} = m\psi_C^\dagger \gamma^0 \psi$$

is Lorentz-invariant.
g) Write the Majorana mass operator in terms of the two chiralities of the fermion $\psi = \psi_L + \psi_R$.

Exercice 5: Scalar, pseudo-scalar, vector and pseudo-vector fermion currents
For two generic spinors $\psi_{1,2}$, show that under under Lorentz transformation and space-parity

$$\bar{\psi}_1\psi_2 \text{ transforms as a scalar,}$$
$$\bar{\psi}_1\gamma^5\psi_2 \text{ transforms as a pseudo-scalar,}$$
$$\bar{\psi}_1\gamma^\mu\psi_2 \text{ transforms as a vector,}$$
$$\bar{\psi}_1\gamma^\mu\gamma^5\psi_2 \text{ transforms as a pseudo-vector.}$$

We recall that $\bar{\psi} = \psi^\dagger\gamma^0$.