Accelerator & Technology Sector
Beams Department
Accelerator Beam Physics Group

Particle Accelerators and Beam Dynamics

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Summer Student Lectures 2023

Disclaimer

Based on:

- Y. Papaphilippou : "Introduction to Accelerators"
- Summer student lectures:
  - B. Holzer, V. Kain, and M. Schaumann
- CERN accelerator school (CAS):
  - F. Tecker: *"Longitudinal beam dynamics"*
- Joint Universities Accelerator School (JUAS):
  - F. Antoniou, H. Bartosik and Y. Papaphilippou*: "Linear imperfections"* and *"nonlinear dynamics"*
- Books:
  - K. Wille: *"The Physics of Particle Accelerators"*
  - S.Y. Lee: *"Accelerator Physics"*
  - A. Wolski: *"Beam Dynamics in High Energy Particle Accelerators"*
  - H. Wiedemann: "Particle Accelerator Physics"

Images: cds.cern.ch

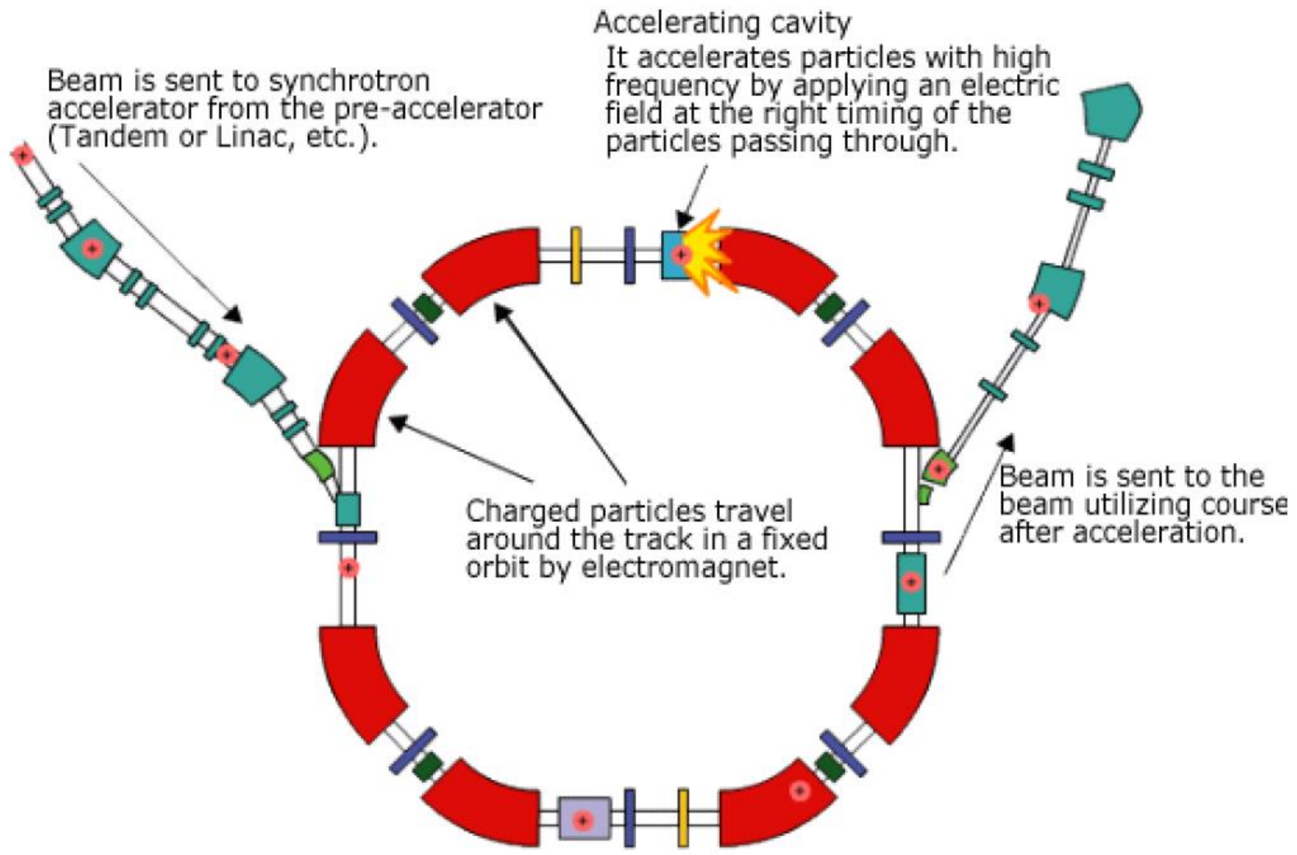
Overview

I. Introduction to Accelerators

II. Accelerator beam dynamics
- Transverse beam dynamics
  - Optics functions
  - Tune and resonances
- Longitudinal beam dynamics
  - Acceleration
  - Synchrotron motion

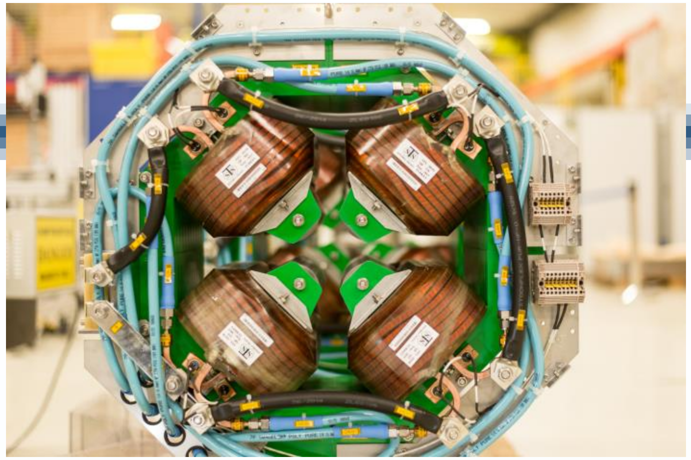
III. CERN accelerator complex

Reminder – Synchrotron



The most common accelerator

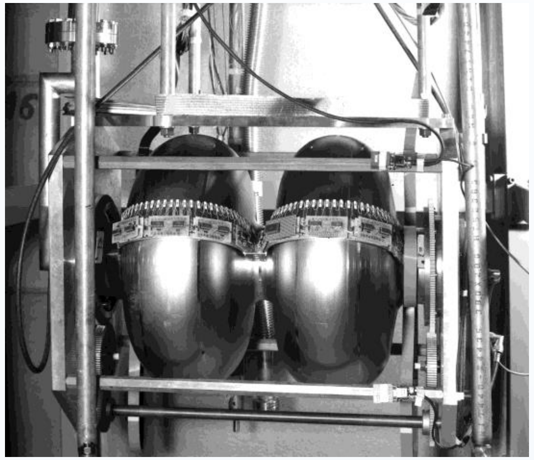
- Fixed beam trajectory | magnetic field changes synchronous to the energy
- Magnets around the beam path to control the motion | **bending** (dipoles) & **focusing** (quadrupoles)
- Electric fields used to **accelerate** (RF cavity) the beam



How do particles move under the influence of these elements?

→ Transverse & Longitudinal Beam Dynamics

Charges in electromagnetic fields

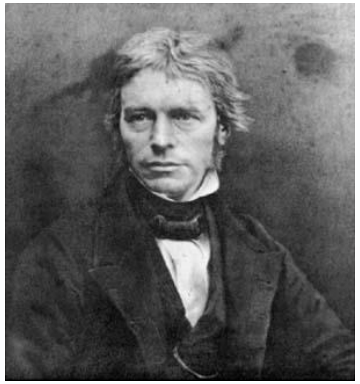


Maxwell's equations for electromagnetism

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Gauss law for electricity

electric field diverges from electric charges



$$\nabla \cdot \mathbf{B} = 0$$

Gauss law for magnetism

no isolated magnetic poles

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$$

Faraday's law of induction

changing magnetic fields produce electric fields

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}$$

Ampere-Maxwell law

changing electric fields and currents produce circulating magnetic fields

Lorentz force

- Force acting on **charged particles** moving under the influence of **electromagnetic fields**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- Kinetic energy (T) change is caused by the **electric field** – ***acceleration***

$$\frac{dT}{dt} = \mathbf{v} \cdot \mathbf{F} = q\mathbf{v} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = q\mathbf{v} \cdot \mathbf{E}$$

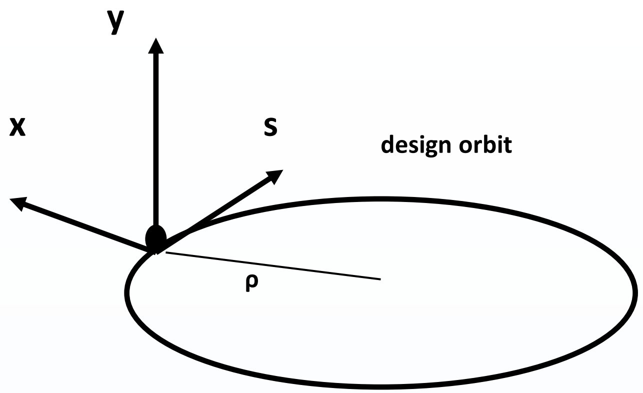
- **Horizontal component** of the Lorentz force (particle moving on the longitudinal plane

$$\mathbf{F_x} = q(E_x - v_z B_y)$$

- For high energy (relativistic limit): $\upsilon_z \approx c$ & $\upsilon_z B_y >> E_x$ (1 T corresponding to 300 MV/m)

  → **Magnetic fields** much more efficient for ***steering***

Transverse motion – Field expansion



- In a synchrotron we want to study particles on the design orbit
- Magnetic fields are present all along s
- The magnetic field at the vicinity of the particle can be expanded as:

$$\frac{e}{p}B_y(x) = \frac{e}{p}B_{y0} + \frac{e}{p}\frac{dB_y}{dx}x + \frac{1}{2!}\frac{e}{p}\frac{d^2B_y}{dx^2}x^2 + \frac{1}{3!}\frac{e}{p}\frac{d^3B_y}{dx^3}x^3 + \cdots = \frac{1}{r} + kx + \frac{1}{2!}mx^2 + \frac{1}{3!}ox^3 + \cdots$$

Linear terms

Dipole | ***Quadrupole***

Sufficient terms for a synchrotron

Higher order terms

Sextupole | ***Octupole*** | …

Transverse motion – Dipoles

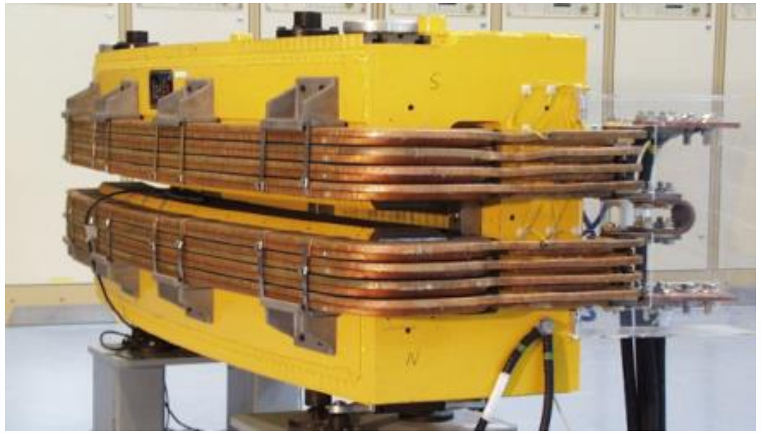
In a circular accelerator of energy ***E***, with ***N*** dipoles, each of length ***L***

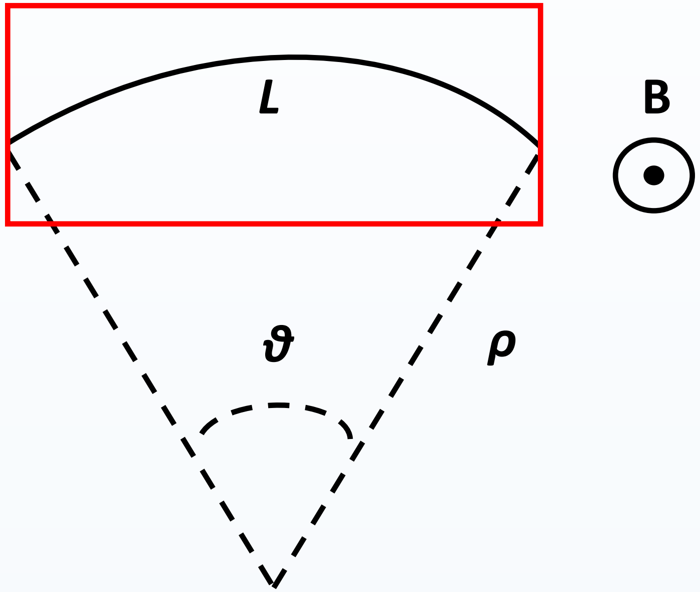
- Bending angle: $$\theta = \frac{2\pi}{N}$$
- Bending radius: $$\rho = \frac{L}{\theta}$$
- Dipole field: $$B = 2\rho p/(qNl)$$

→ Choosing a dipole magnetic field: the length is determined (and vice versa)

→ For higher fields, smaller and fewer dipoles can be used

→ Ring circumference (cost) depends on field selection





Example LHC:

7000 GeV Proton storage ring
dipole magnets N = 1232
l = 15 m
q = +1 e

$$\int B\,dl \approx N\,l\,B = 2\pi\ p/e$$

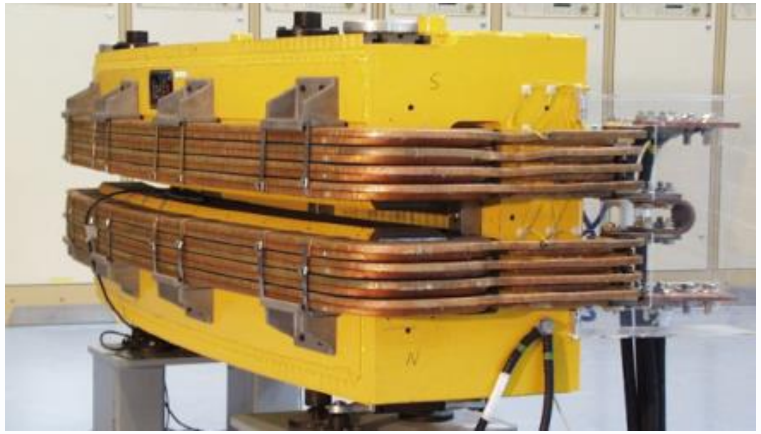
$$B \approx \frac{2\pi\ \ 7000\ \ 10^{9}\,eV}{1232\ \ 15\,m\ \ 3\ 10^{8}\,\frac{m}{s}\ \ e} = 8.3\ Tesla$$

Transverse motion – Dispersion

Reminder:

- From the RF cavities | **bunches formation**:
  - *The particles forming a bunch have a spread of momenta around the reference particle*

  → *Off-momentum particles ($\Delta p/p$,* with respect to the reference*)*

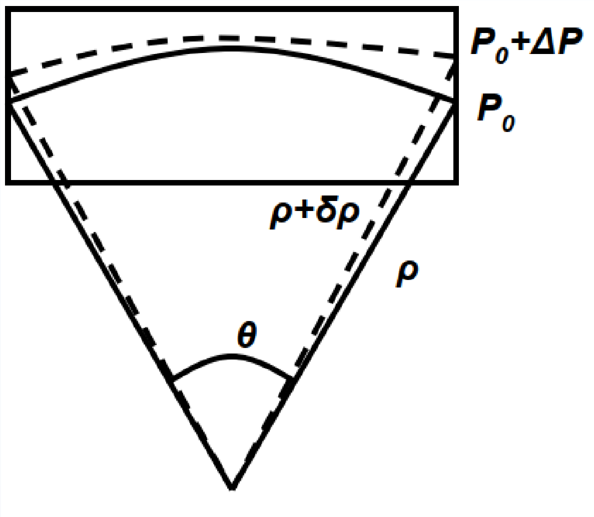


- From the **beam rigidity** (& dipole field):
  - *The synchrotron has a constant radius if the field follows the momentum*

  → *Off-momentum particles:* $B(\rho + \Delta\rho) = \dfrac{P_0 + \Delta P}{q} \Rightarrow \dfrac{\Delta\rho}{\rho} = \dfrac{\Delta P}{P_0}$

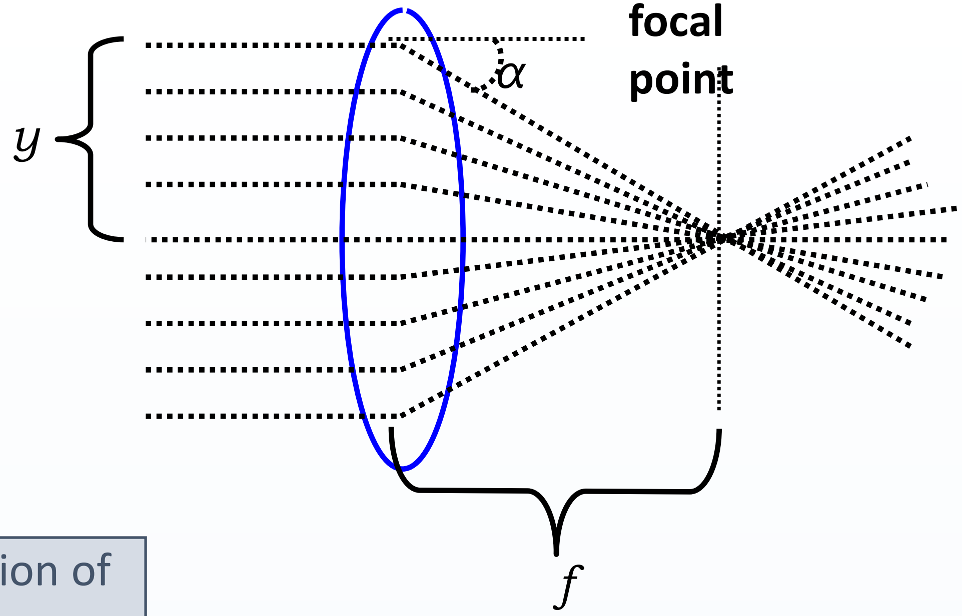
- ***The off-momentum particles follow a different orbit than the reference!***

→ The different orbit when *$\Delta p/p = 1$* is called: ***Dispersion***



Transverse motion – Quadrupoles

- Quadrupoles can have a focusing effect similar to lenses, where: $\alpha = -\frac{y}{f}$



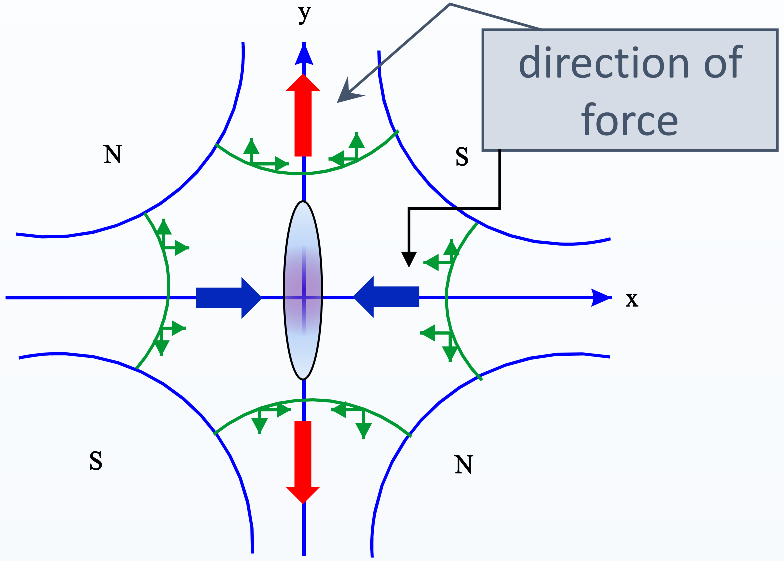
Quadrupole with field: $(B_x, B_y) = G \cdot (y, x)$

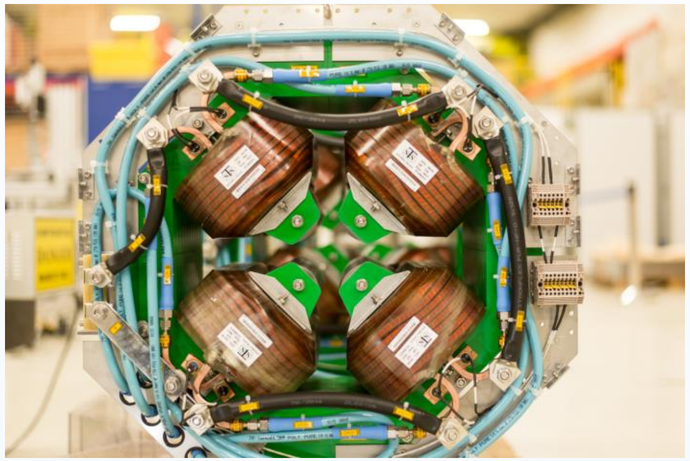
and force: $(F_x, F_y) = k \cdot (-x, y)$, $k = \frac{G}{B\rho}$

- Acts as a lens with focal length: $f = \frac{1}{k \cdot l_Q}$

Reminder:

- **Quadrupoles with a focusing effect in one plane have a defocusing in the other**





Transverse motion – FODO

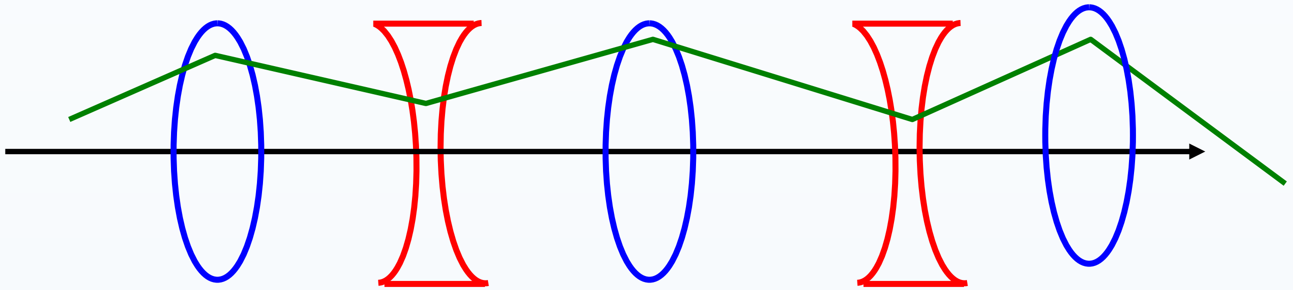
Alternating gradient focusing:

- ***Alternating focusing and defocusing lenses can have an overall focusing effect***
- Combination of lenses with focal lengths, f_1 and f_2 in a distance d gives a focal length:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 \ f_2}$$

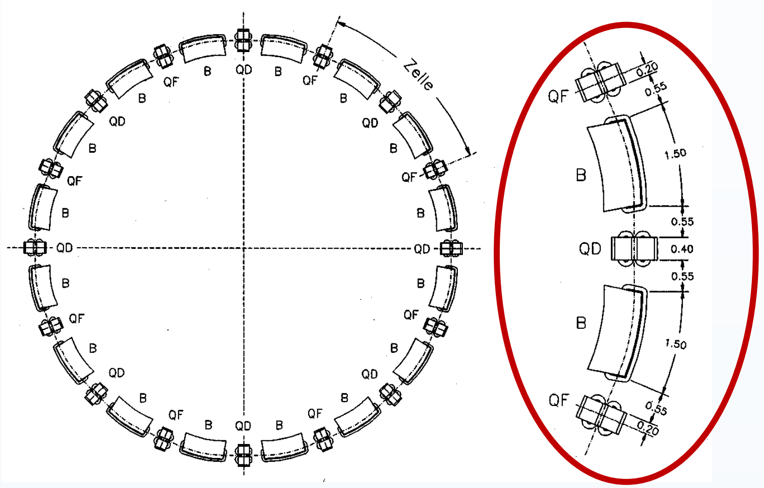
- if $f_1 = -f_2$, we get an overall focusing effect:

$$\frac{1}{f} = \left|\frac{d}{f_1 \ f_2}\right|$$

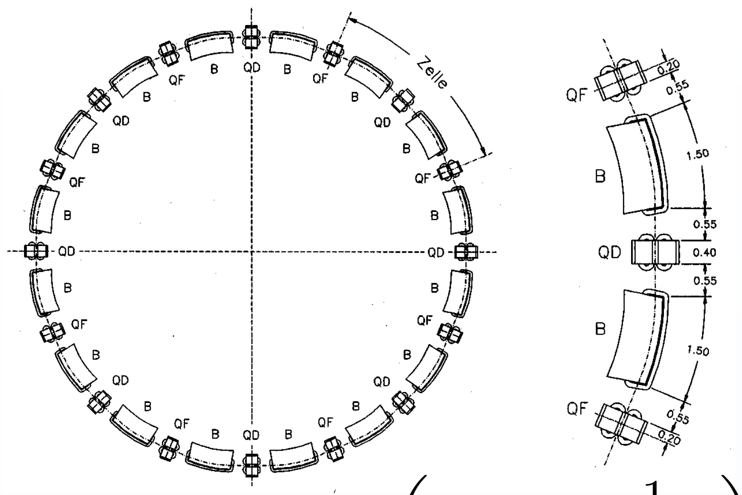


FODO structure

- *"**Cell**" of alternating focusing and defocusing elements (along with drifts, dipoles etc)*
- Structure repeats itself giving a strong periodicity in the ring



Transverse motion – FODO



The equations of motion moving along the FODO structure:

$$x'' + \left(k(s) + \frac{1}{\rho(s)^2}\right)x = \frac{1}{\rho(s)}\frac{\Delta p}{p}$$

$$y'' - k(s)\,y = 0$$

Dispersive contribution

Quadrupole contribution (strong focusing)

Dipole contribution (weak focusing)

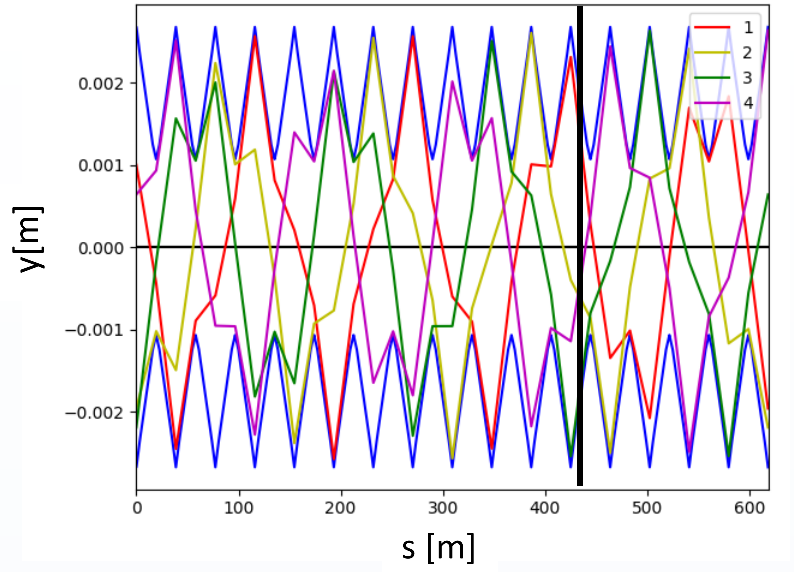
Setting $K_x(s) = \left(k(s) + \frac{1}{\rho(s)^2}\right)$ and $K_y(s) = -k(s)$ we obtain *Hill's equations, with $K_{x,y}(s)$* depending on the position *s and $K_{x,y}(s+L) = K_{x,y}(s)$* periodic functions, where *L* is the periodicity

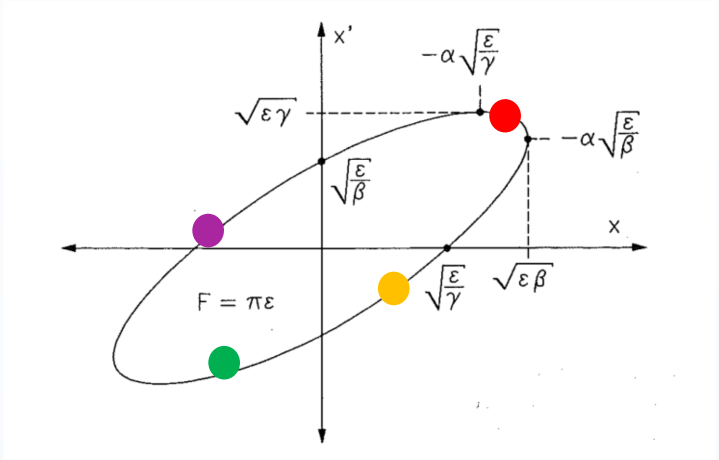
- Solutions describe a quasi harmonic oscillation, where **amplitude**, **phase** (and **dispersion**) depend on the position s in the ring

$$y(s) = \sqrt{\varepsilon_y \beta_y(s)}\cos\left(\varphi_y(s)\right)$$

$$x(s) = \sqrt{\varepsilon_x \beta_x(s)}\cos\left(\varphi_x(s)\right) + D(s)\frac{\Delta p}{p}$$

Transverse motion – betatron oscillations





- Particles perform **oscillations (betatron)** around the **design orbit**
- The motion is bound from the **envelope** ($\sqrt{\boldsymbol{\varepsilon_y \beta_y(s)}}$),
  - $\boldsymbol{\beta_y(s)}$: **beta function** characteristic of the ring
  - $\boldsymbol{\varepsilon_y}$: **emittance** is a **constant** of the motion (Liouville's theorem: the area is preserved)
    - It defines an **ellipse in the phase space** (Courant-Snyder invariant)

      $\varepsilon_y = \gamma_y(s)y^2(s) + 2\alpha_y(s)y'(s)y(s) + 2\beta_y(s){y'}^2(s)$,

      α,β,γ: optics functions
    - It cannot be changed by the optics functions
  - The envelope gives the **beam size** of a particle ensemble
- # of oscillations per turn, **tune:** $Q = \frac{1}{2\pi}\int \frac{ds}{\beta(s)}$

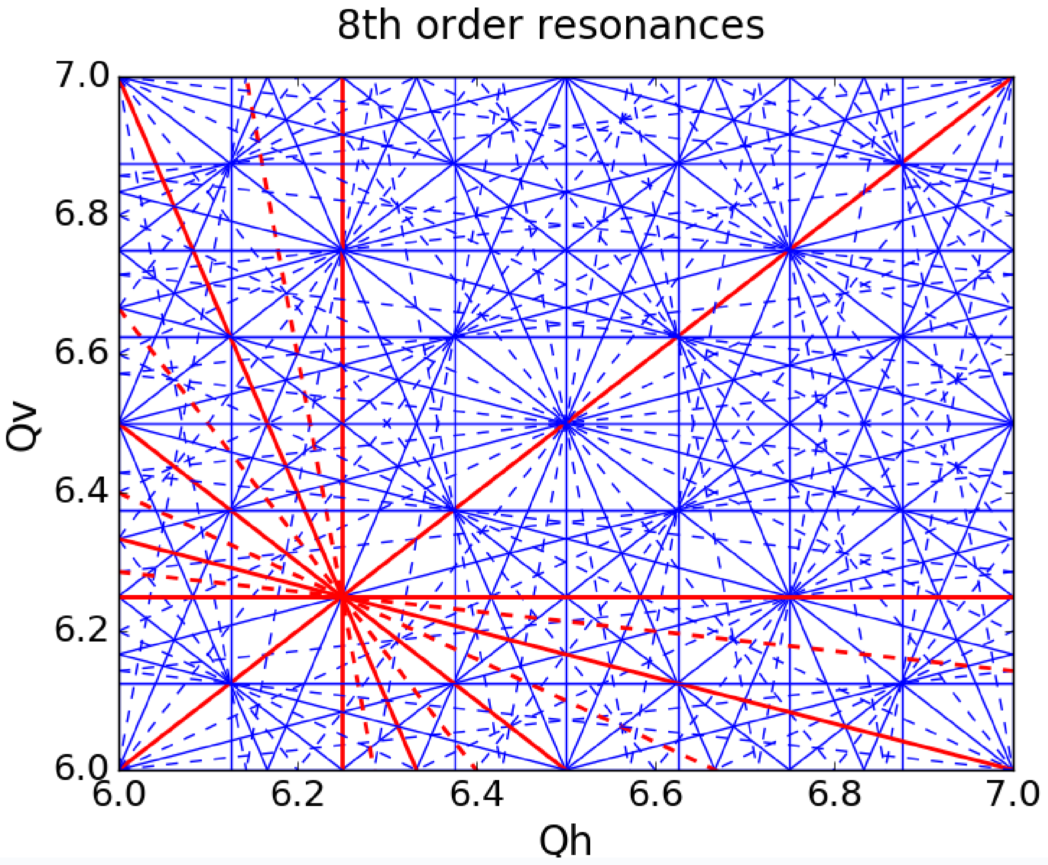
Resonances

The tunes in the respective planes: $(\boldsymbol{Q_x}, \boldsymbol{Q_y})$

Define resonance conditions described by: $\boldsymbol{mQ_x + nQ_y = l}$,

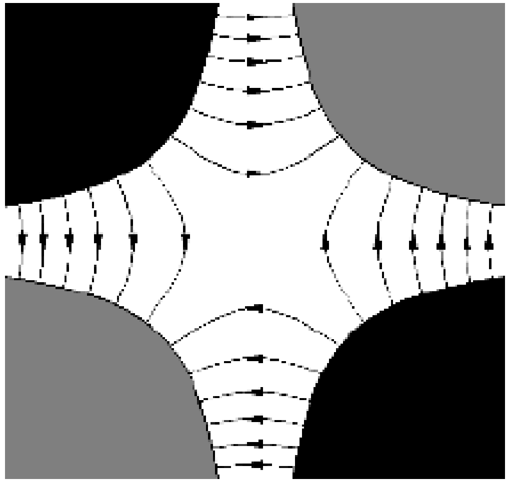
- where m,n,l integers
- $m|+|n|$ the resonance order
- If the above condition is satisfied:
  - ***Particle losses***
  - ***Emittance increase***

Resonances	Machine Periodicity	
Magnetic Field Component		
	Systematic	Non Systematic
Skew	red dashed line	blue dashed line
Normal	red solid line	blue solid line



Magnetic Field Component

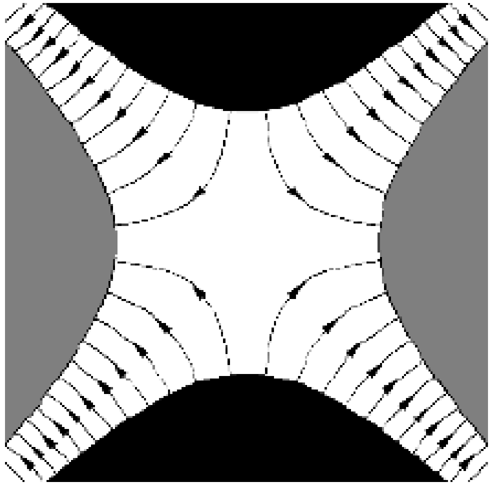
Normal components become skew when rotated by half the rotation symmetry

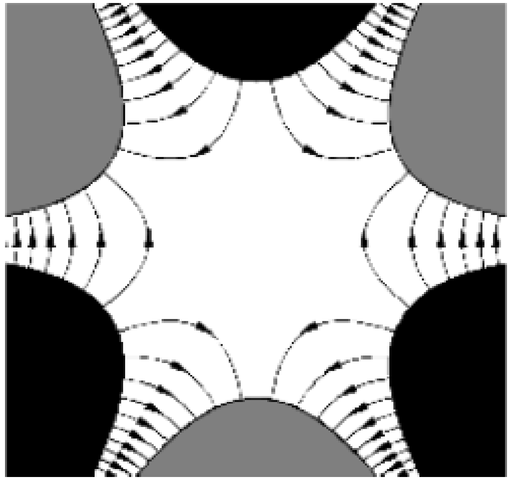


Normal Quadrupole

45° Rotation

Skew Quadrupole

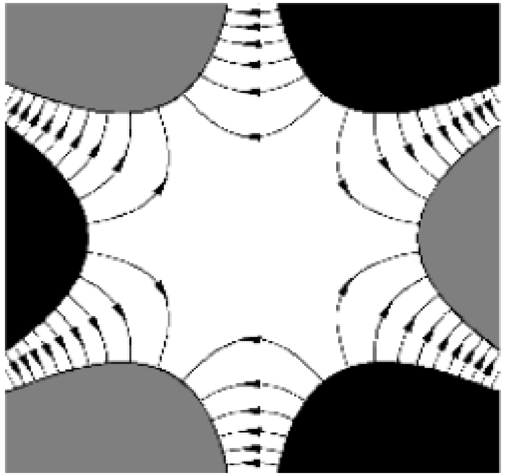




Normal Sextupole

30° Rotation

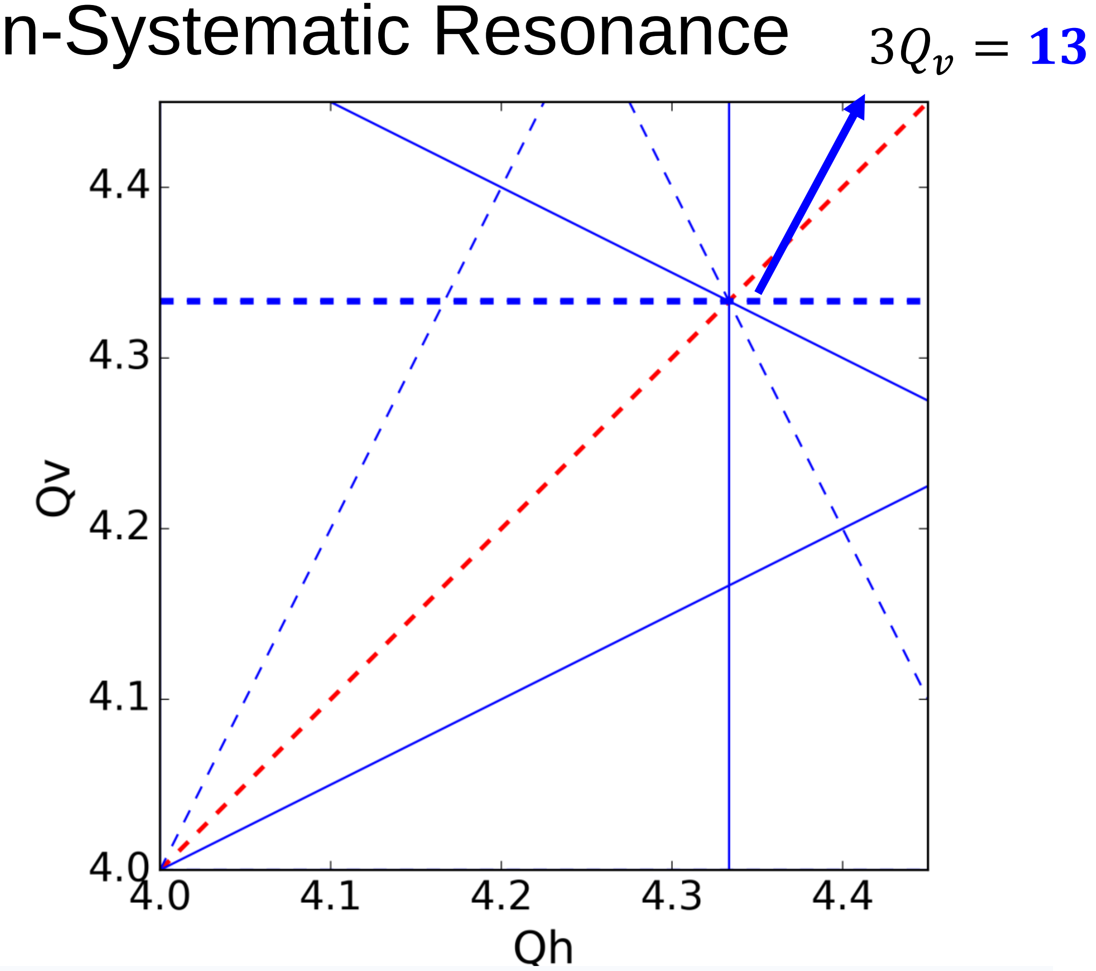
Skew Sextupole

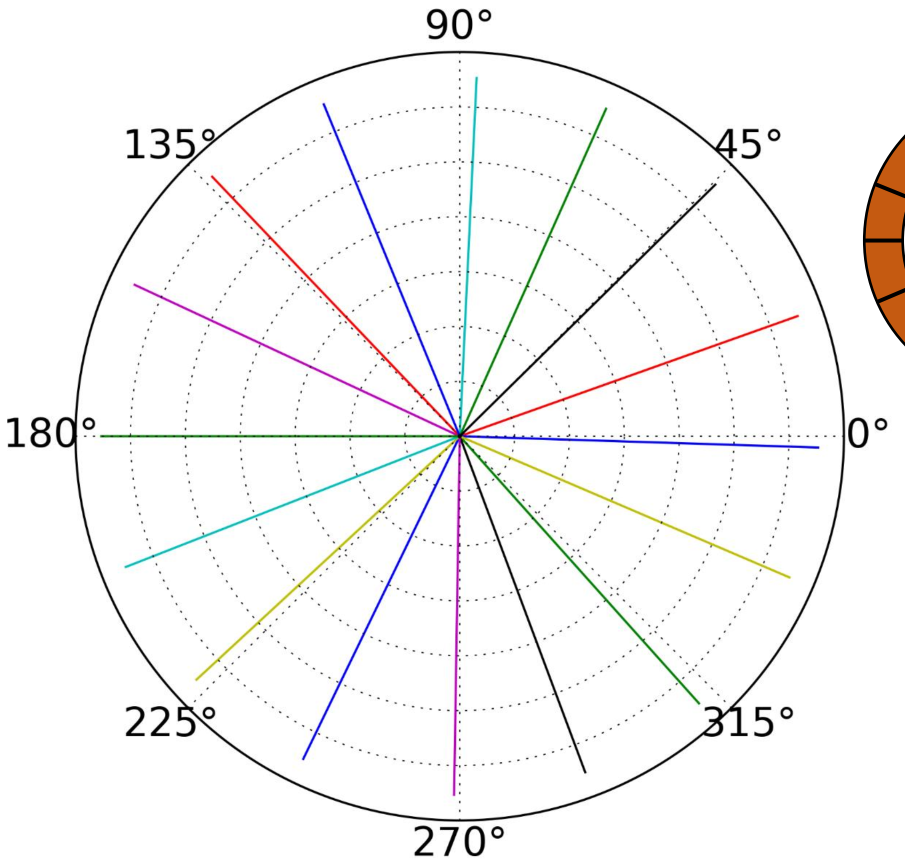


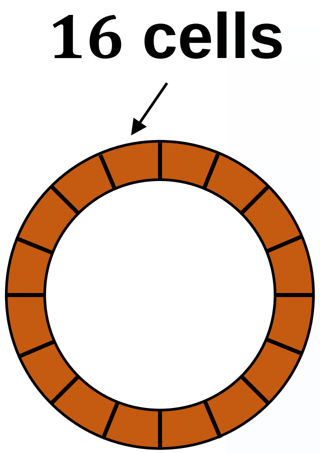
Plots from:
T. Satogata *et al.*, "*Magnets and Magnet Technology*", USPAS2013

Machine Periodicity

Non-Systematic Resonance



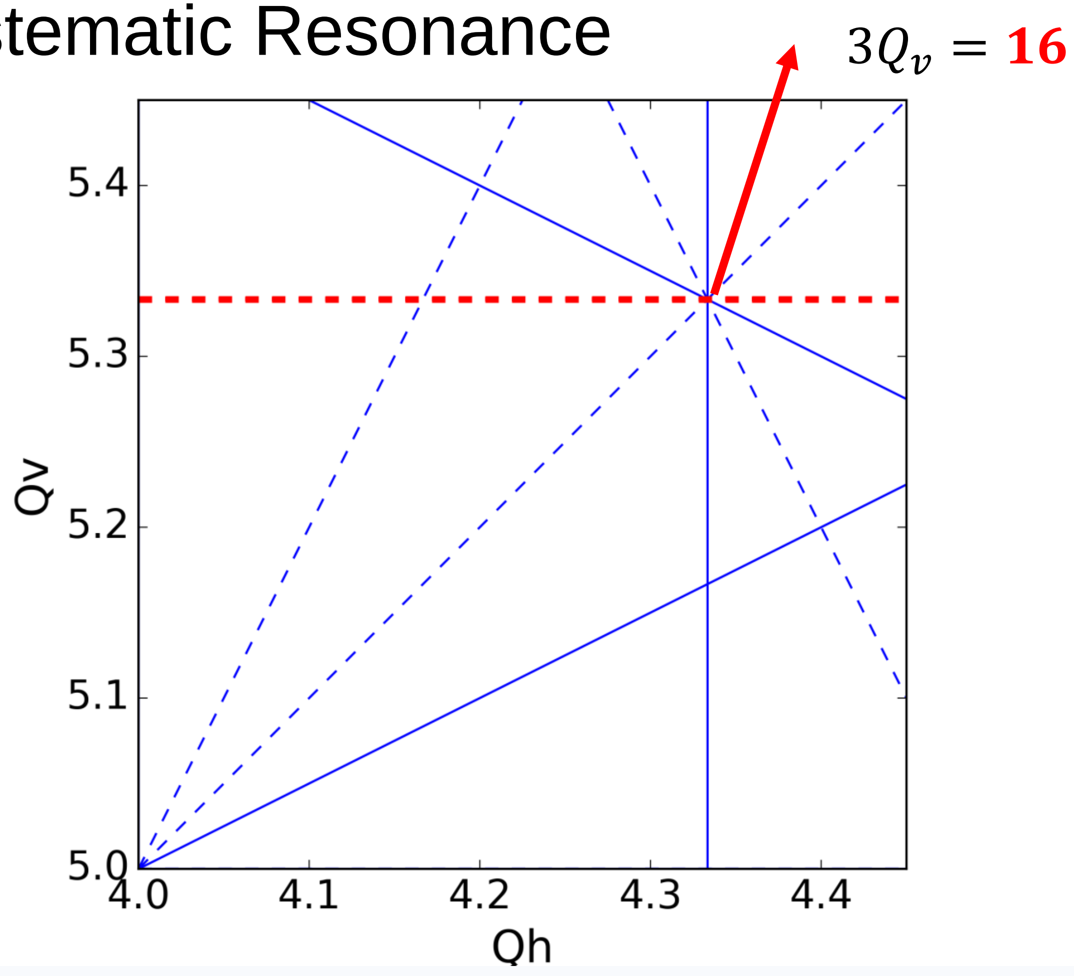


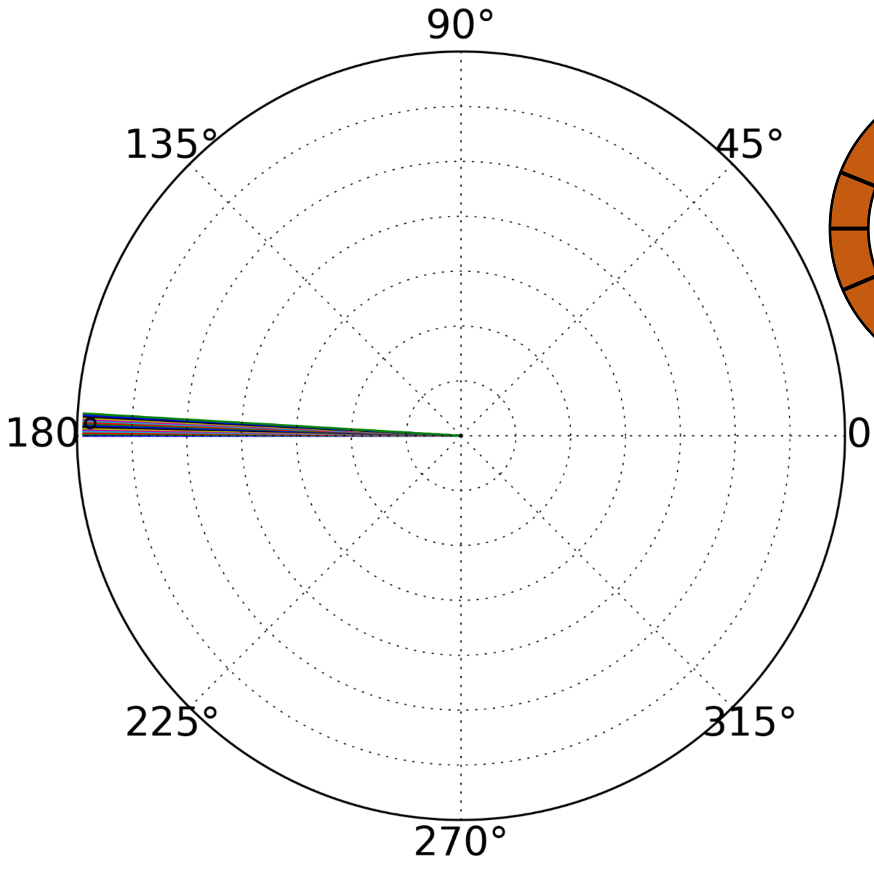


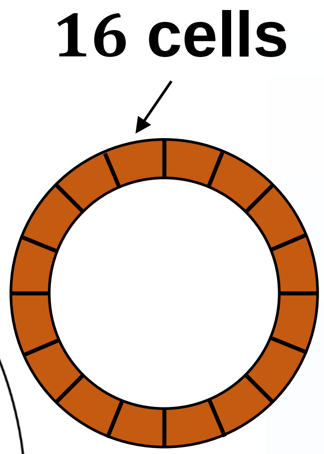
The errors **cancel out due to the periodicity**

Machine Periodicity

Systematic Resonance





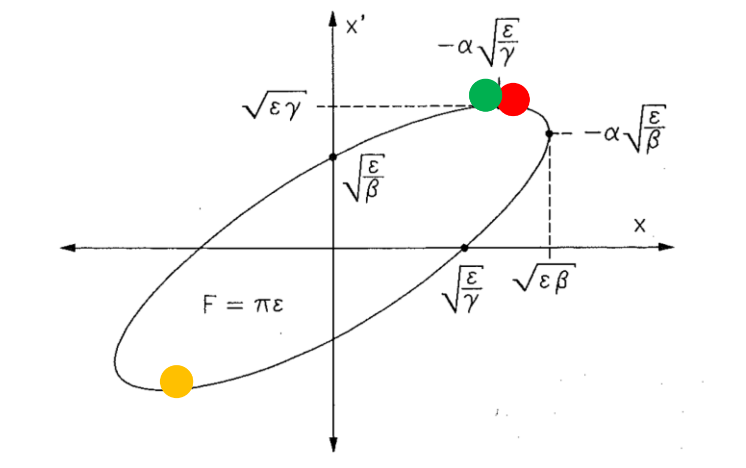


The errors **sum up** – strong resonance

Transverse motion – betatron resonances

Under normal conditions the emittance is preserved turn after turn:

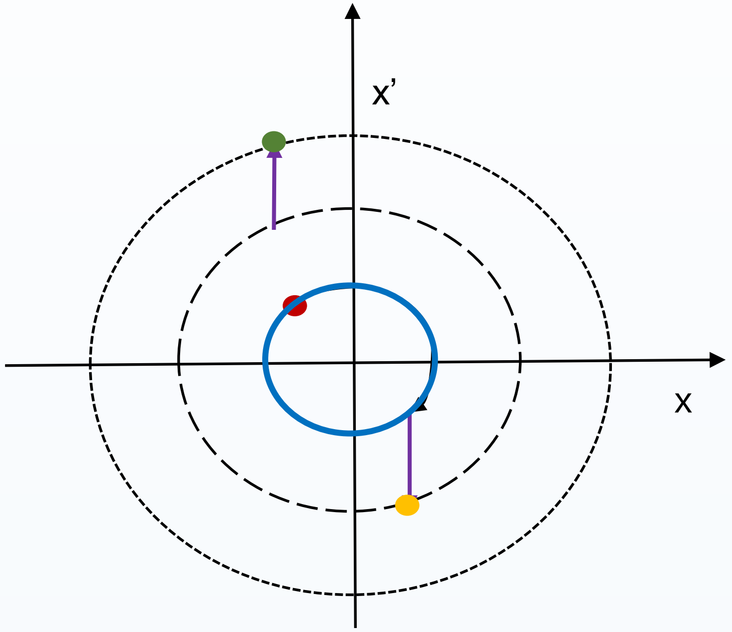
- Observing the phase space turn-by-turn we get the emittance ellipse



In the presence of a strong **resonance:**

- Emittance of a particle on the **1st turn**
- Emittance increases on the **2nd turn**
- Emittance increases further on the **3rd turn**

Emittance (and amplitude) will keep increasing until the particle is lost



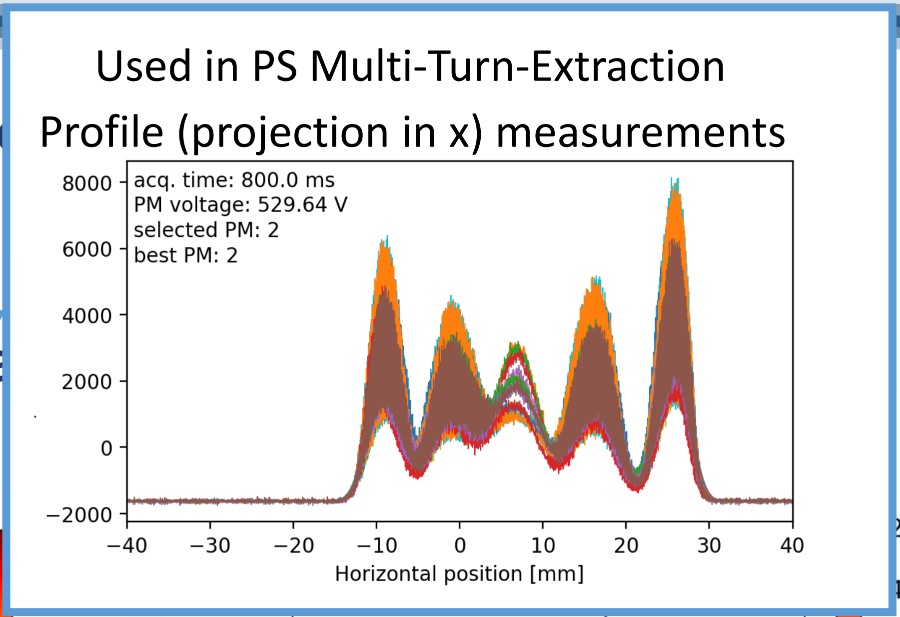
Mitigation measures:

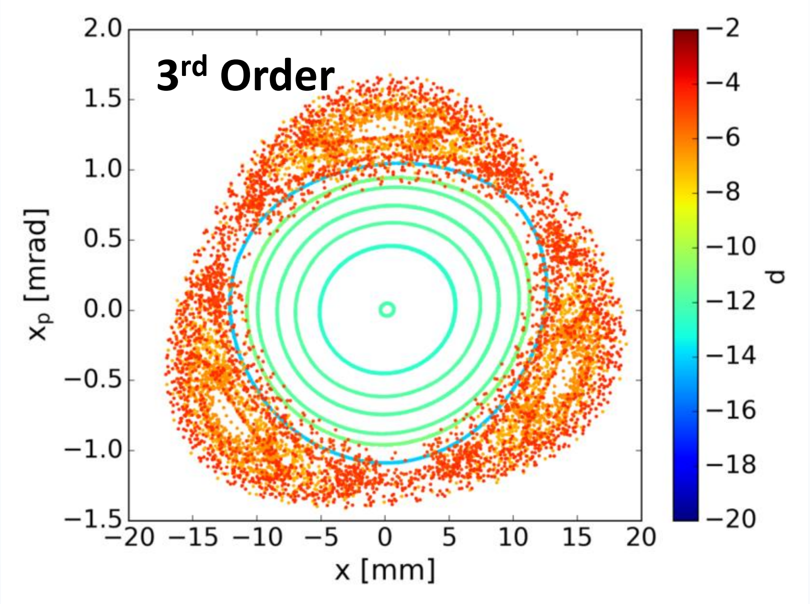
1. **Careful tune choice – avoid resonance condition**
2. **Higher order elements – corrections to cancel the effect of the resonance**

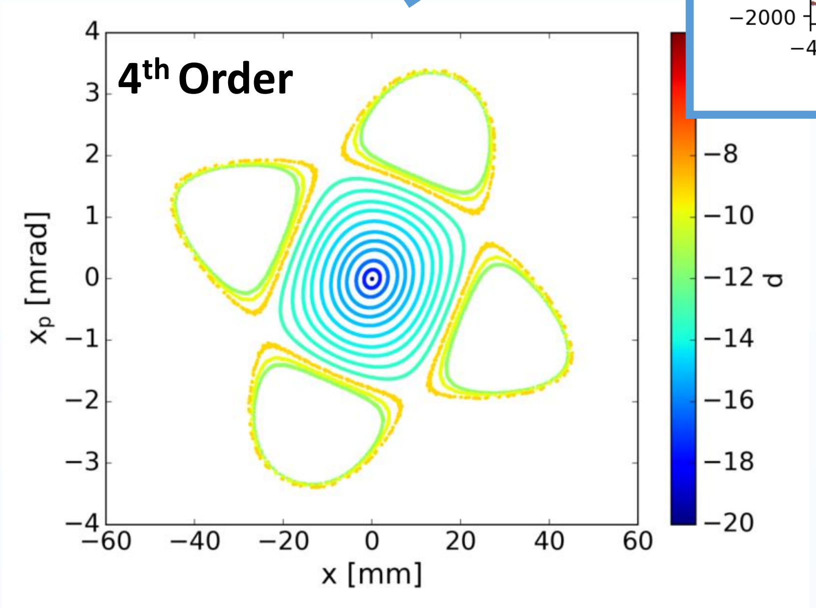
Transverse motion – phase space

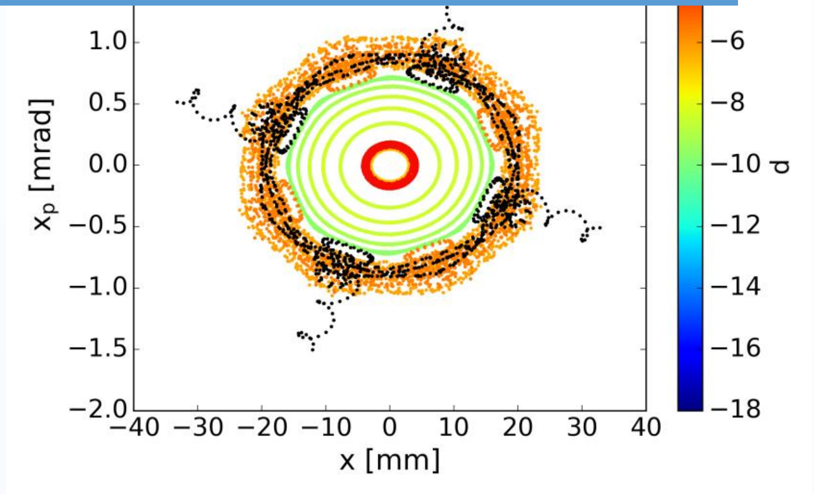
The phase space gets distorted in the vicinity of driven reso

- *Changing our tune to approach a resonance:*
  - Starting with well defined ellipses
  - Approaching the resonances – ***forming islands depe***









Longitudinal motion - Acceleration

Reminder:

- Acceleration in a synchrotron is achieved in the **RF cavities**, using a voltage V
- During operations, we have a **synchronous RF phase** for which the **energy gain** fits the **increase of the magnetic field** at each turn. → *condition for constant radius*

- Energy gain per turn:

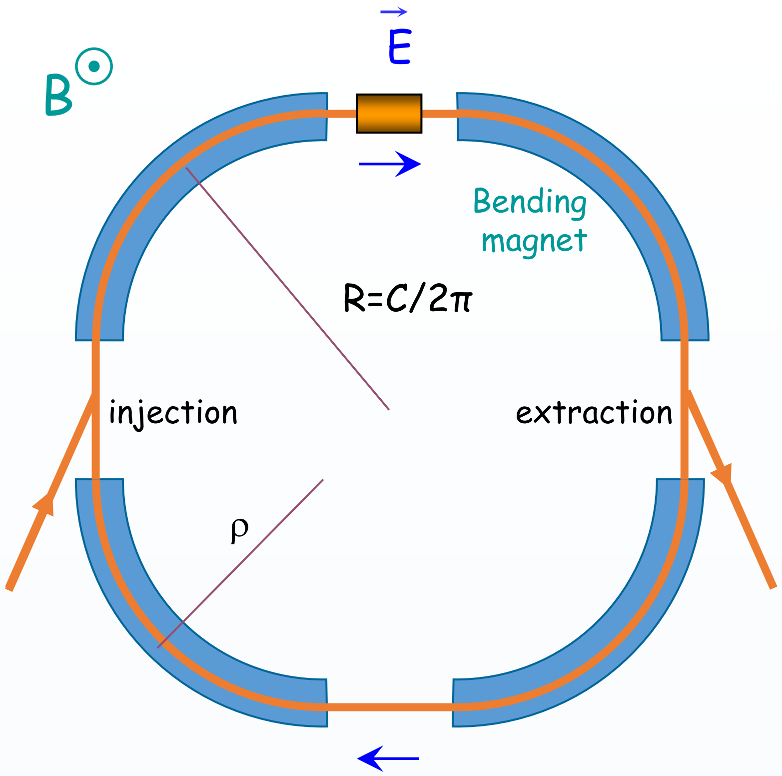
$$qV \sin\phi = qV \sin\omega_{RF} t$$

- synchronous phase:

$$\phi = \phi_s = const$$

- RF synchronism - frequency must be on the revolution frequency (1 turn around the ring):

$$\omega_{RF} = h\omega_{rev}$$



h (integer): **harmonic number**

- number of RF cycles per revolution
- ***Defines the maximum number of bunches in the synchrotron*** *(available RF buckets)*

Longitudinal motion – f_{RF} and ϕ_s change

During acceleration “***ramping***” energy & the magnetic field are changing:

- The revolution frequency changes: $\omega(B, R_s)$
- From the synchronism condition RF frequency needs to follow (using $p(t) = eB(t)r,\ E^2 = (m_0c^2)^2 + p^2c^2$):

Can be omitted at the relativistic limit where B >> $m_0c^2/(ecr)$

$$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{r}{R_s}B(t) = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0c^2/ecr)^2 + B(t)^2}\right\}^{1/2}$$

- Similarly, the phase, $\boldsymbol{\varphi_s}$ needs to follow
- From Bρ:

$$(\Delta p)_{turn} = e\rho\dot{B}T_r = \frac{2\pi e\rho R\dot{B}}{v} \Rightarrow (\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R\dot{B} = e\hat{V}\sin\phi_s$$

$$\phi_s = \arcsin\left(2\pi\rho R\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

Longitudinal motion – Dispersion effects

Reminder:

Off-momentum particles follow a different orbit than the design: ***Dispersion***

- The ***orbit length*** is different – the ***momentum compaction factor*** shows the variation of the orbit length with respect to the variation of the momentum:

$$\alpha_c = \frac{dR/R}{dp/p}$$

- From different momentum, different velocity & different path: different time (& revolution frequency) to arrive to the RF cavity – ***slip factor***, variation of the revolution frequency with respect to the variation of the momentum:

$$h = \frac{\mathrm{d}f_r/f_r}{\mathrm{d}p/p}$$

Longitudinal motion – Dispersion effects

- The revolution frequency change depends both on the **orbit** and **velocity** change:

$$\frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R} = \frac{d\beta}{\beta} - \alpha_c \frac{dp}{p}$$

- For $\frac{d\beta}{\beta}$: $p = mv = \beta\gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-1/2}}{\left(1-\beta^2\right)^{-1/2}} = \underbrace{\left(1-\beta^2\right)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$

- Finally, we get the relation between momentum compaction and slip factor: $\eta = \frac{1}{\gamma^2} - \alpha_c$

- The energy in which $\eta = 0$, is called ***transition energy***: $\boldsymbol{\gamma_t = \frac{1}{\sqrt{\alpha_c}}}$

➢ **Below γ_t (η>0)** the arrival at the cavity depends on the **velocity**

➢ **At γ_t (η=0)** the velocity change and the path length change **compensate each other**

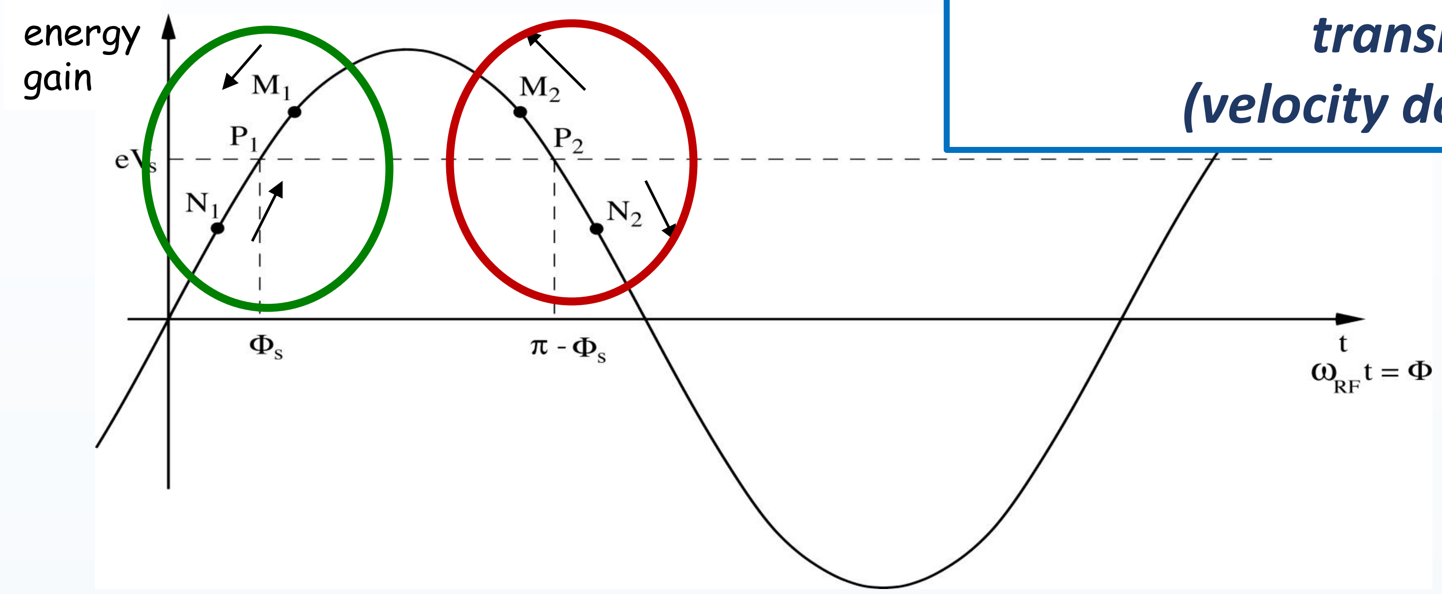
➢ **Above γ_t (η<0)** the arrival at the cavity depends ***only*** on the **path length**

Longitudinal motion – Phase stability

Reminder:

- Phase focusing: bunches are formed as particles arriving at the cavity before or after the synchronous particle are “brought closer” to it
- This stands for $\varphi_s < \pi/2$ as:
  - M_1 & N_1 will move towards P_1 => **stable**
  - M_2 & N_2 will go away from P_2 => **unstable**

This behaviour is true below transition (velocity dominates)

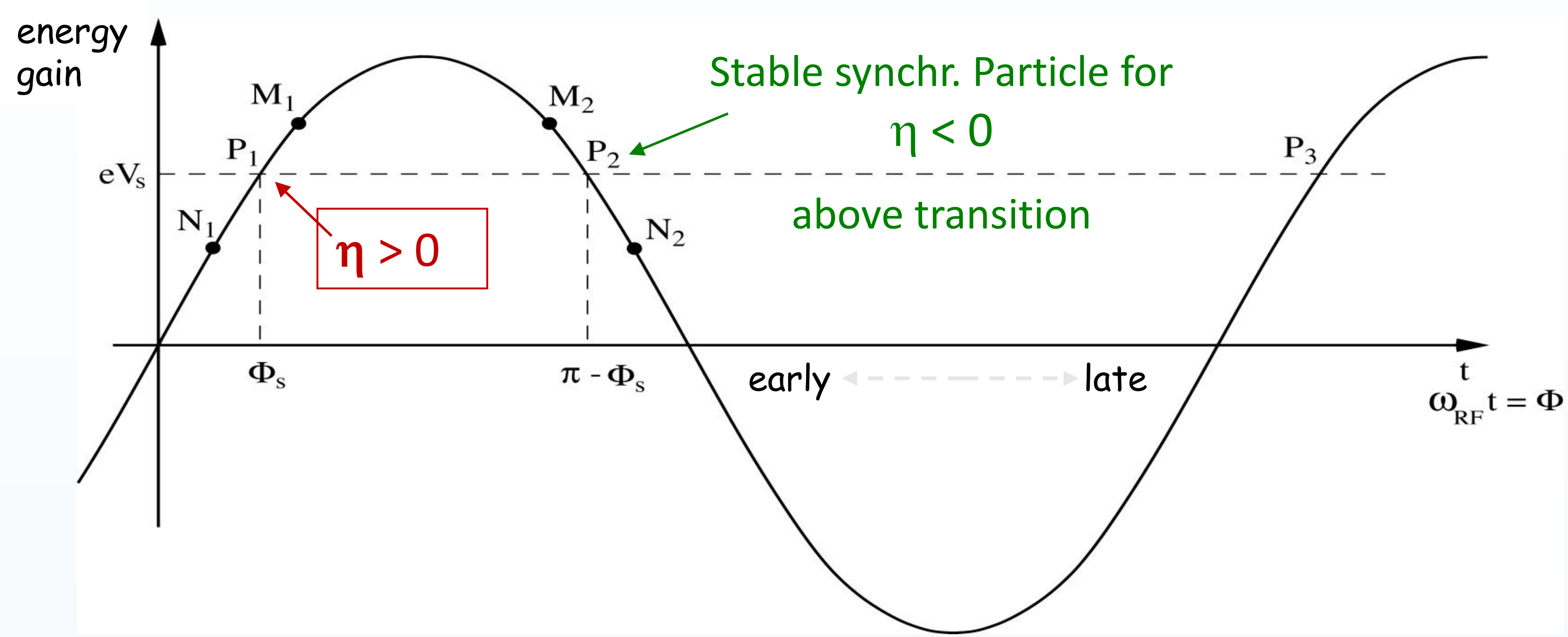


Longitudinal motion – Phase stability

Since:

- **Below γ_t (η>0)** the arrival at the cavity depends on the **velocity**
- **Above γ_t (η<0)** the arrival at the cavity depends ***only*** on the **path length**

➢ **The behaviour for the phase stability is reversed around transition crossing**



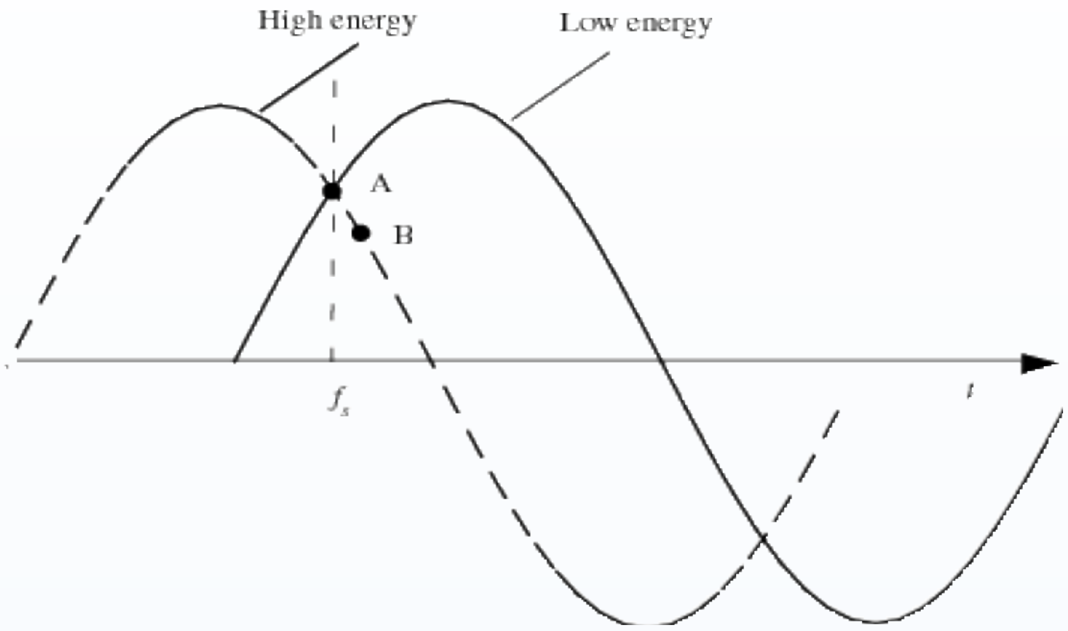
Longitudinal motion – Transition crossing

- Change of stable phase implies:

- ***Crossing transition during acceleration makes the previous stable synchronous phase unstable.***

➢ The RF system needs to make a rapid change of the RF phase, a "phase jump".

- ***Such a manipulation is needed at the CERN PS***

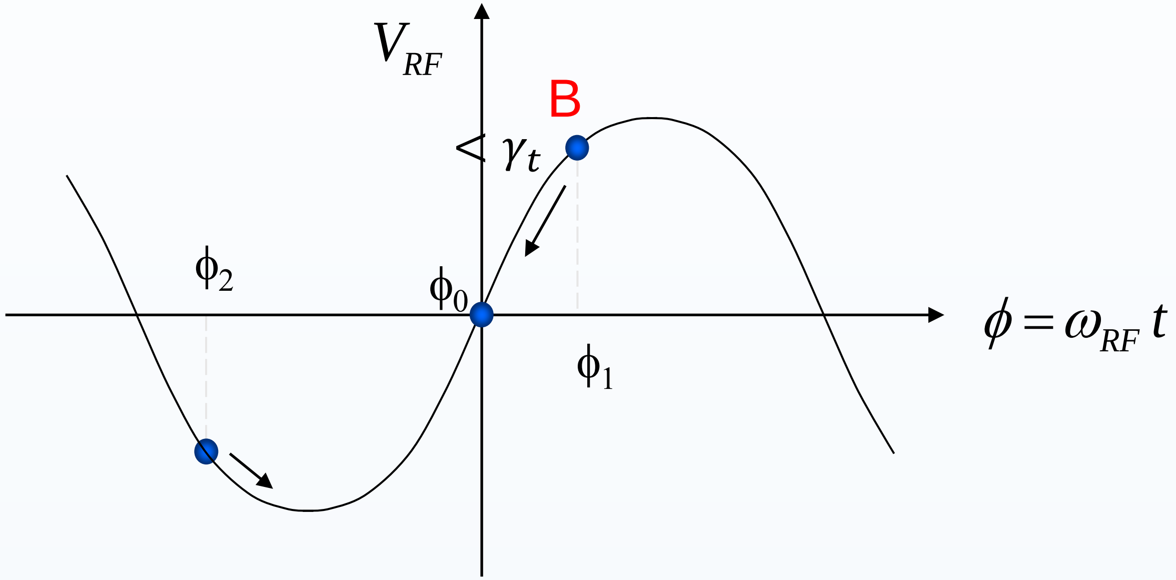


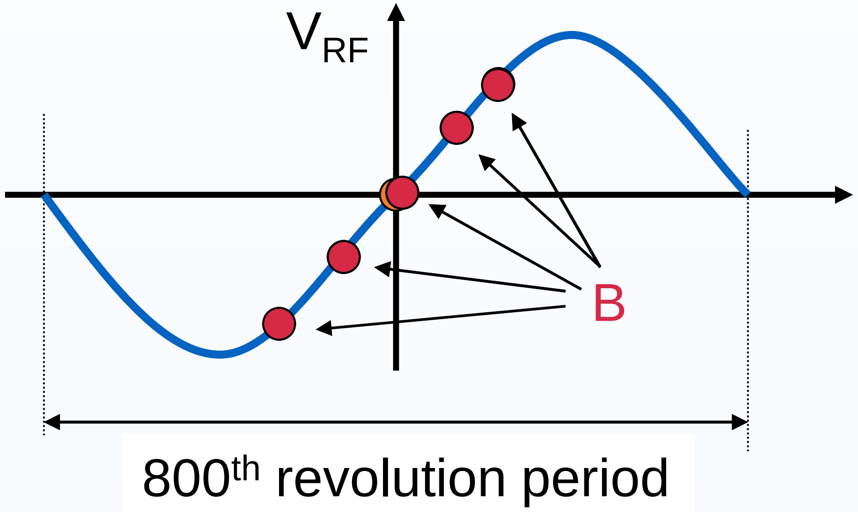
Longitudinal motion – Synchrotron oscillations

Operating below transition & at constant energy (and B)

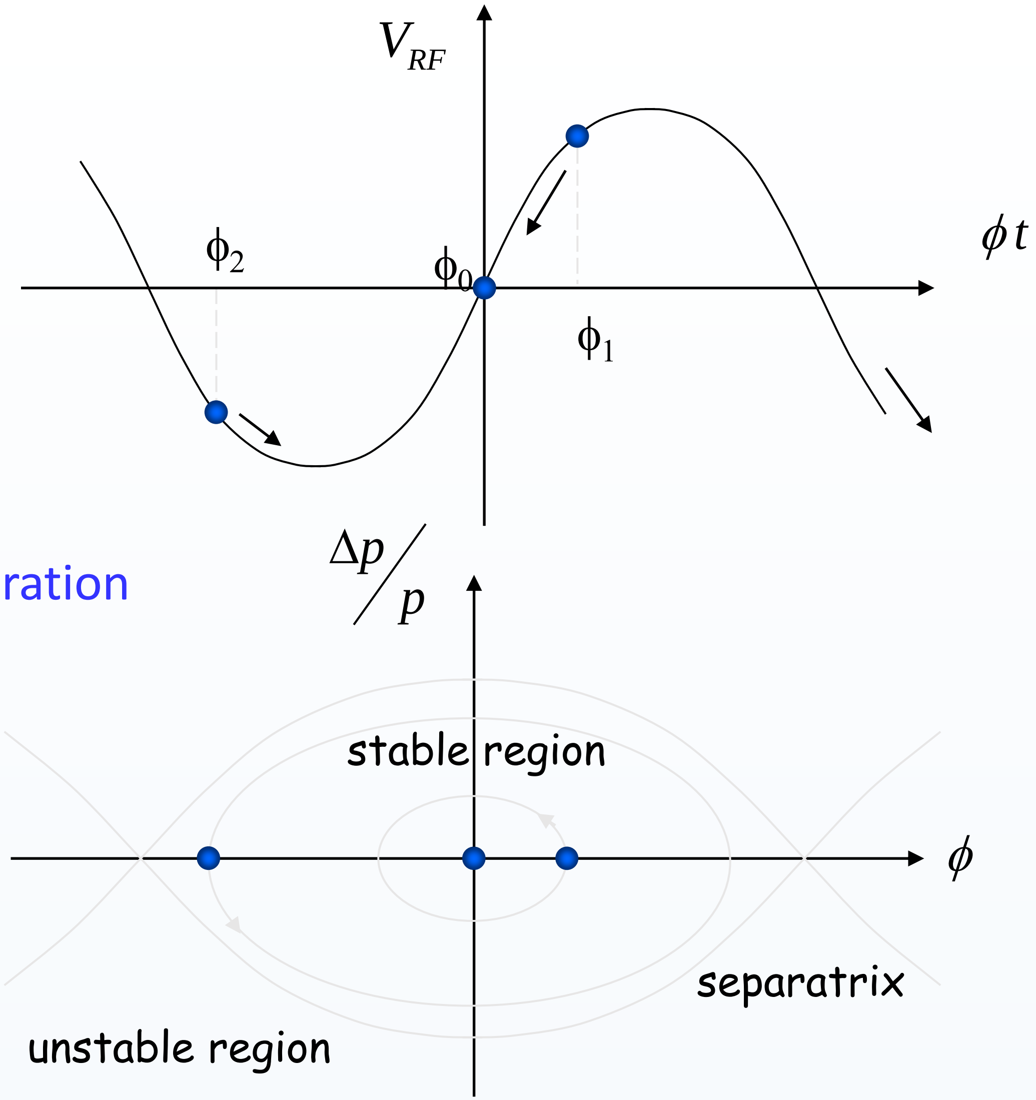
- Synchronous phase ϕ_0=0
- Particle with a $\phi>\phi_0$: particle gets accelerated and moves towards ϕ_0
- Particle with a $\phi<\phi_0$: particle gets decelerated and moves towards ϕ_0

➢ ***Particles will start performing oscillations around the synchronous particle***

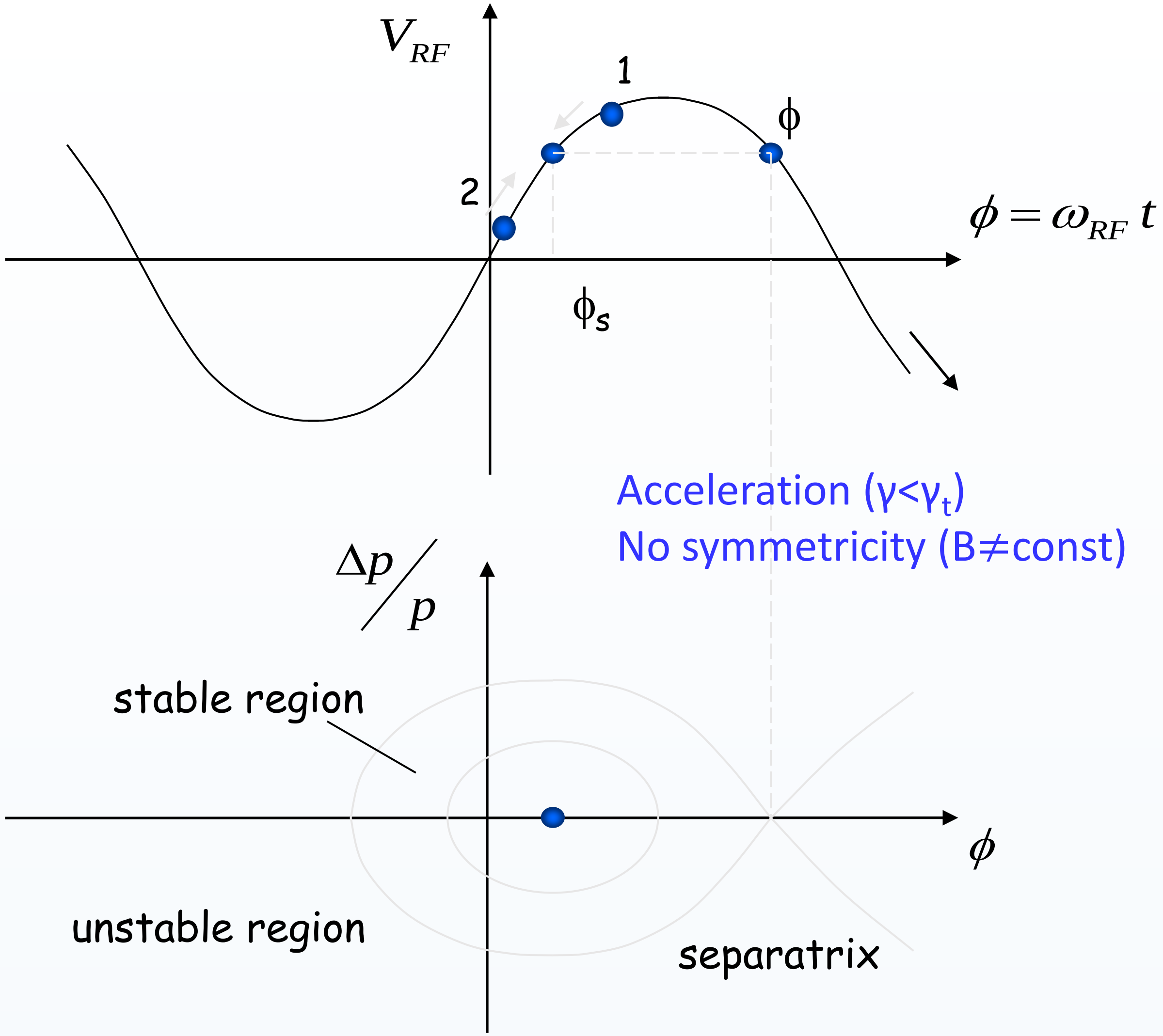




Longitudinal motion – Phase space



No Acceleration



Acceleration ($\gamma<\gamma_t$)
No symmetricity ($B \neq const$)

Takeaways

Transverse motion

- The beam moves in ***FODO structure***
- Particles perform oscillations around the design orbit called ***betatron***
- Turn-by-turn the ellipse formed in the phase space is called ***emittance***
- The number of betatron oscillations in 1 turn is called **tune**
- Emittance remains **constant** for "normal" conditions
- In the presence of ***resonances*** in the tune space, the emittance increases
- The ***beam size*** is defined as $\sqrt{\varepsilon_y \beta_y(s)}$

Longitudinal motion

- ***Synchronism:*** RF frequency needs to be locked to revolution frequency
- During acceleration the ***phase and frequency*** need to adjust to the ***energy & B increase***
- Path length changes with momentum – ***momentum compaction factor***
- Frequency changes with momentum – ***slip factor***
- Phase stability depends on ***transition energy***
- ***Phase jump*** to cross transition
- Particles perform oscillations around ϕ_s called ***synchrotron***