Quantum Gravity Bounds on 4d Effective Theories with Minimal Supersymmetry

- 2210.10797 with Luca Martucci and Nicolo Risso
- Earlier works with Seung-Joo Lee, Wolfgang Lerche and with Antonella Grassi

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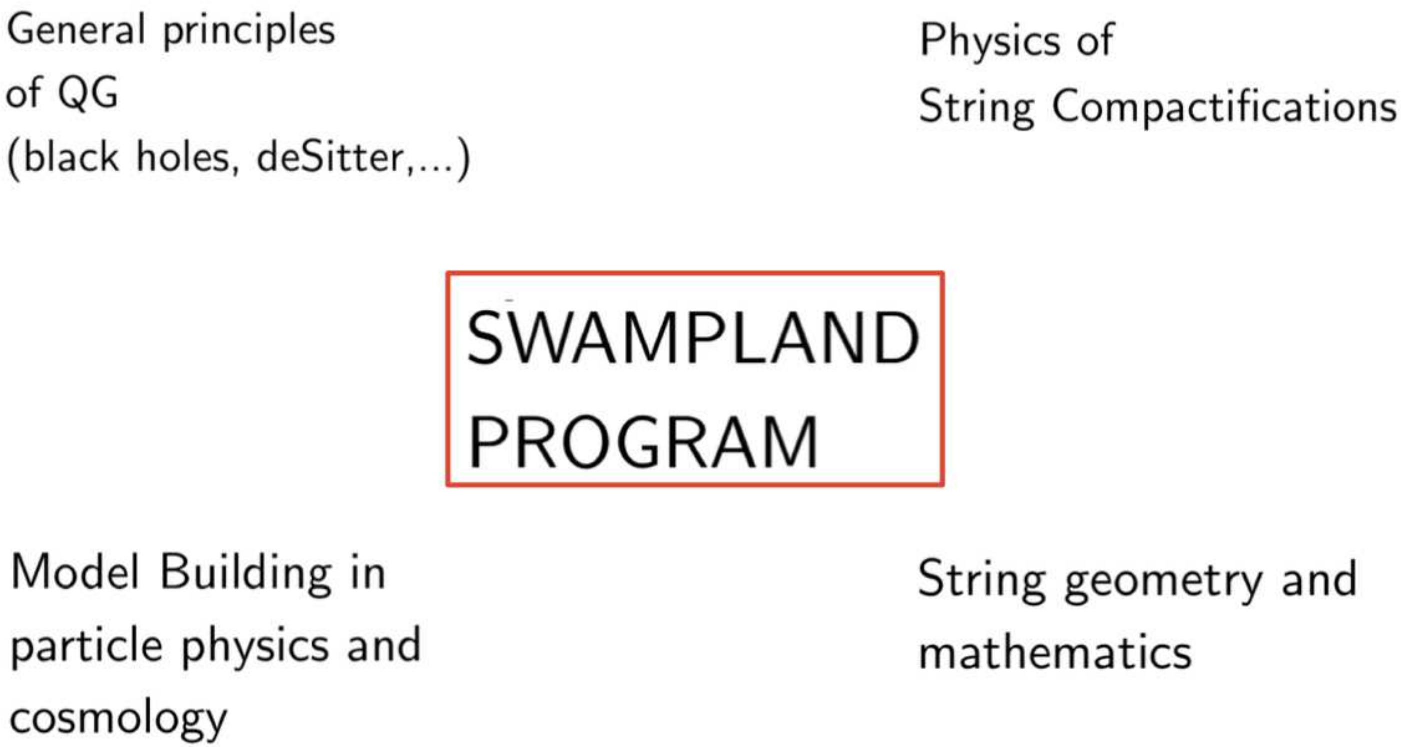
The Swampland Program

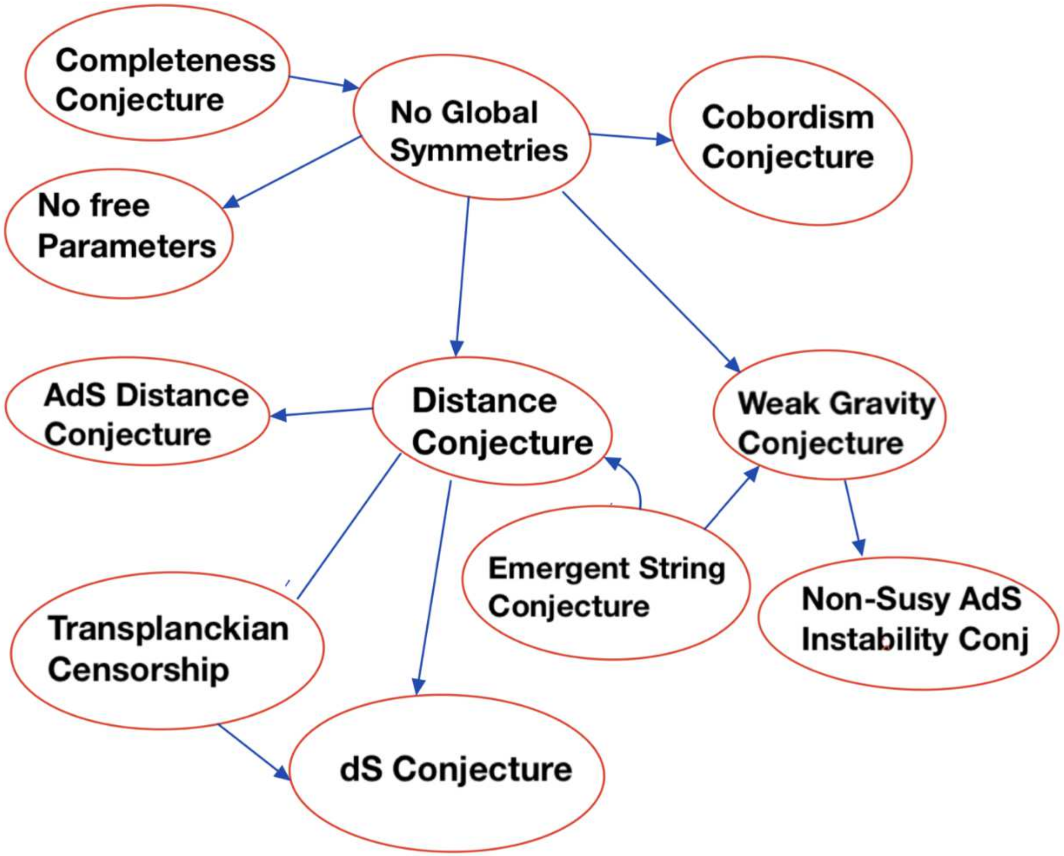
Which EFTs can arise from a consistent theory of QG in $d > 2$?

Swampland [Vafa'05]		Landscape
EFT consistent as QFT but not as QG	$\longleftrightarrow$	EFT fully consistent as QG

This question is at the very center of many developments.



A Web of Conjectures



Review articles:

[Brennon,Carta,Vafa'17] [Palti'19] [Beest,Calderon,Mirfendereski,Valenzuela'21] [Grana,Herraez'21]

[Agmon,Bedroya,Kang,Vafa'22]

Swampland Arguments

Two representative arguments:

1. QG imposes cutoff for EFT parametrically below naive cutoff $M_{\rm Pl}$:

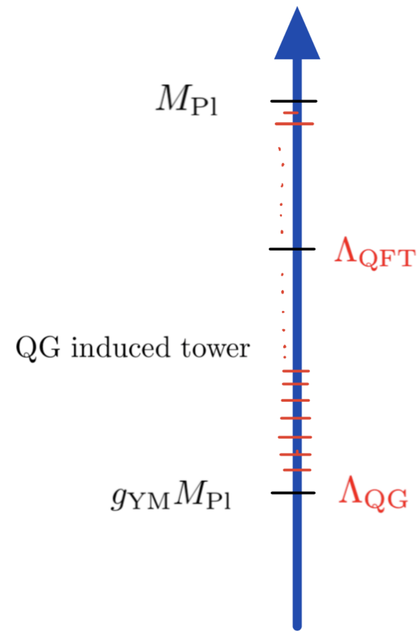
$$\Lambda_{\rm QG} = g_{\rm YM} M_{\rm Pl} \ll M_{\rm Pl} \quad \text{if} \quad g_{\rm YM} \to 0$$

due to light towers of states

$\Longleftrightarrow$ magnetic/asymptotic Weak Gravity Conjecture

$\Longleftrightarrow$ Swampland Distance Conjecture

$\Longleftrightarrow$ Emergent String Conjecture



2. Consistency of additional probe objects enforced by QG constrains the original theory: Can we
   - argue for finiteness of degrees of freedom in QG?
   - bound the rank of the gauge group via gravitational couplings?

Probe argument

- Many EFTs contain **higher p-form gauge fields**
  Example: 6d N=1 supergravity theories: 2-forms B_2

$$S = \frac{M_{\rm Pl}^4}{2} \int_{\mathbb{R}^{1,5}} \sqrt{-g} R + f_{\alpha\beta} dB_2^\alpha \wedge * dB_2^\beta + \dots$$

- Can consider **p-dimensional electrically charged objects**
  Example: strings charged under 2-forms

$$S_{\rm string} = e_\alpha \int_{\mathbb{R}^{1,1}} B_2^\alpha$$

- **QFT:** These may or may not exist as physical states/objects.

  QG: They must exist as dynamical objects if we accept the

  Completeness Conjecture of QG: [Polchinski'03][Banks,Seiberg'10]
  The full charge lattice of a QG in d>2 dimensions is populated by physical states.

Probe argument

Completeness Conjecture of QG: [Polchinski'03] [Banks,Seiberg'10]

The full charge lattice of a QG in d>2 dimensions is populated by physical states.

- Consistency of the p-dim probe $\Longrightarrow$ extra constraints

  Example: Anomaly cancellation on string worldvolume *in addition to* anomaly cancellation in original bulk theory.

$\Longrightarrow$ An EFT not satisfying these is in the Swampland.

Probe argument

Successfully applied in **higher-dimensional and/or higher-SUSY theories:**

- 6d N=1 supergravities

  [Kim,Shiu,Vafa'19] [Lee,TW'19] [Tarazi,Vafa'20] [Angelantonj,Bonnefoy,Condeescu,Dudas'20]

- 5d N=1 supergravities

  [Katz,Kim,Tarazi,Vafa'20] [Cheng,Minasian,Theisen'21]

- 4d N=4 supergravities

  [Kim,Tarazi,Vafa'19]

Example:

6d $N = 1$ theories admit infinite families of theories consistent with anomaly cancellation [Kumar,Taylor'09] [Kumar,Morrison,Taylor'10] [Kumar,Park,Taylor'11], ...

Inconsistency of probe strings in field theory places (some of) them in the Swampland!

This talk

4d $N = 1$ Supergravities: [Martucci,Risso,TW'22]

We will propose new constraints on otherwise consistent $N = 1$ SUGRAs - including non-chiral theories in 4d!

General bounds on rank of gauge algebra in 4d $N = 1$ supergravity theories from EFT strings:

"rank of gauge group $\leq$ gravitational couplings"

Part I:

Motivation in effective field theory, without use of string theory

$\Longrightarrow$ general, modulo certain assumptions

Part II:

Explicit checks in string theory realisations:

New bounds on ranks of gauge groups $\Longrightarrow$ Predictions for geometry

Part I: EFT Bounds from EFT Strings

4d $N = 1$ data

4d $N = 1$ SUGRA with gauge couplings determined by chiral multiplets

bosonic content: $t^i = a^i + \mathrm{i}s^i\,, \qquad a^i \simeq a^i + 1$

Gauge algebra: $G = \prod_A U(1)_A \times \prod_I G_I$

Only consider theories where to leading order in *perturbative* regime:

$$S_{\text{gauge}} = -\frac{1}{4\pi} C_i \int \left(s^i \, F \wedge *F + a^i \, F \wedge F \right) \quad + \quad \text{corrections}$$

- From N=1 SUSY:

$$S_{\text{gauge}} = -\frac{1}{4\pi} \int \left(\operatorname{Im} f \, F \wedge *F + \operatorname{Re} f \, F \wedge F \right) \, .$$

- Assume

$$f(t, \phi) = \langle \mathbf{C}, \mathbf{a} + \mathrm{i}\mathbf{s} \rangle + \Delta f(\phi) + \underbrace{\mathcal{O}(e^{2\pi \mathrm{i} \langle \mathbf{m}, \mathbf{a} + \mathrm{i}\mathbf{s} \rangle})}_{\text{BPS instanton}} \,, \qquad \langle \mathbf{C}, \mathbf{a} + \mathrm{i}\mathbf{s} \rangle \equiv C_i (a^i + \mathrm{i}s^i)$$

4d $N = 1$ data

$$f(t,\phi) = \langle \mathbf{C}, \mathbf{a} + \mathrm{i}\mathbf{s} \rangle + \Delta f(\phi) + \underbrace{\mathcal{O}(e^{2\pi \mathrm{i} \langle \mathbf{m}, \mathbf{a} + \mathrm{i}\mathbf{s} \rangle})}_{\text{BPS instanton}} , \qquad \langle \mathbf{C}, \mathbf{a} + \mathrm{i}\mathbf{s} \rangle = C_i (a^i + \mathrm{i} s^i)$$

- Non-pert. corrections suppressed by

$$|e^{-2\pi m_i s^i}| \qquad m_i : \text{BPS instanton charges}$$

- Perturbative regime:

$$|e^{-2\pi m_i s^i}| \ll 1$$

Expectation:

$$\{\langle \mathbf{C}, \mathbf{s} \rangle\} \geq 0 , \qquad \forall \mathbf{s} \in \text{saxionic cone}$$

4d $N = 1$ data

Higher curvature couplings (to leading order in pert. regime):

$$-\frac{1}{96\pi}\tilde{C}_i \int \left[s^i \,\mathrm{tr}(R \wedge *R) + a^i \,\mathrm{tr}(R \wedge R)\right]$$

No definite expectation if or why

$$\langle \tilde{\mathbf{C}}, \mathbf{s} \rangle > 0$$

but various suggestions in the literature

[Kallosh,Linde,Linde,Susskind'94] [Cheung,Remmen'16] [Etxebarria,Montero,Sousa,Valenzuela'20]

[Aalsma,Shiu'22] [Ong'22]

EFT strings

Dualise axions a^i to 2-forms $B_{2,i}$: $\qquad B_{2,i} \sim *da^i$

$\Longrightarrow$ consider string charged under the 2-form fields:

$$S = \int_{\text{string}} e^i\, B_{2,i} + \ldots \qquad e^i \text{ : string charge} \qquad \frac{T_{\mathbf{e}}}{M_{\text{Pl}}^2} = -\frac{1}{2} e^i \frac{\partial K}{\partial s^i}$$

Completeness Conjecture of Quantum Gravity:

The full charge lattice is populated by such strings.

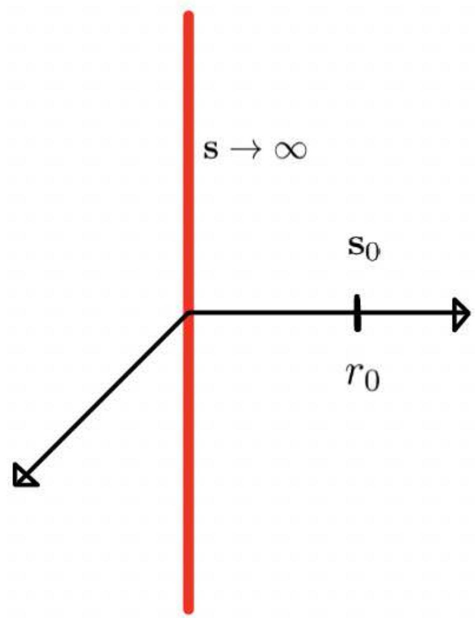
Backreaction of 4d string

(real codimension-two):

[Lanza,Marchesano,Martucci,Valenzuela'20-21]

$$\mathbf{s} = \mathbf{s}_0 + \mathbf{e}\left(\frac{1}{2\pi}\log\frac{r_0}{r}\right)$$

r: radial distance from string

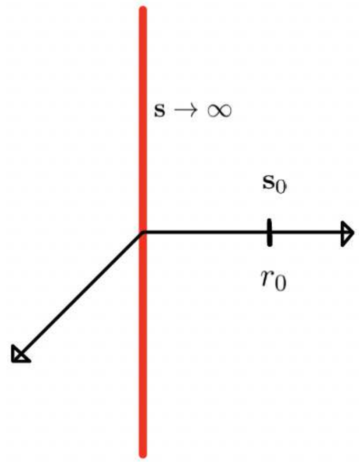


EFT strings

Backreaction of 4d string
(real codimension-two):

$$\mathbf{s} = \mathbf{s}_0 + \mathbf{e}\left(\frac{1}{2\pi}\log\frac{r_0}{r}\right)$$

r: radial distance away from string



Instanton suppression: $|e^{-2\pi\langle \mathbf{m},\mathbf{s}\rangle}| \ll 1$ near string if $\langle \mathbf{m},\mathbf{e}\rangle \geq 0$

$\Longrightarrow$ Field theory becomes weakly coupled in region close to string

Defining property of **EFT string:** [Lanza,Marchesano,Martucci,Valenzuela'20-21]

$$\mathcal{C}_{\rm S}^{\rm EFT} = \{\mathbf{e} \in \mathbb{Z}^{\#\,\text{axions}} \mid \langle \mathbf{m},\mathbf{e}\rangle \geq 0\,,\ \forall \mathbf{m} \in \text{BPS instanton cone}\}$$

Anomaly inflow

- Rewrite the 4d $N = 1$ axionic couplings as

$$S_{\text{bulk}} \supset 2\pi \int a^i I_{4,i} = -2\pi \int h_1^i \wedge I_{3,i}^{(0)}\,, \qquad h_1^i = \mathrm{d}a^i$$

$$I_{4,i} \equiv \mathrm{d}I_{3,i}^{(0)} \equiv -\frac{1}{8\pi^2} C_i\, F \wedge F - \frac{1}{192\pi^2} \tilde{C}_i\, \mathrm{tr}(R \wedge R)$$

- String modifies Bianchi identity:

$$\mathrm{d}h_1^i = e^i \delta_2(W) \qquad W : \text{string worldsheet}$$

- Gauge variance (descent relations) $\delta I_{3,i}^{(0)} = \mathrm{d}I_{2,i}^{(1)}$ cf [Callan,Harvey'85]

$$\delta S_{\text{bulk}} = -2\pi e^i \int_W I_{2,i}^{(1)}$$

= localised anomaly on the string from inflow from bulk

Anomaly inflow

$$\delta S_{\text{bulk}} = -2\pi e^i \int_W I_{2,i}^{(1)}$$

localised anomaly on the string from inflow from bulk must be cancelled by contribution to anomaly on worldsheet:

$$\delta S_W \stackrel{!}{=} +2\pi e^i \int_W I_{2,i}^{(1)}$$

with

$$\begin{aligned} I_{4\,\mathbf{e}}^{\text{ws}} &= e^i I_{4\,i} \\ &= \underbrace{-\frac{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle}{8\pi^2} F_A \wedge F_B}_{\text{abelian gauge}} \quad - \quad \underbrace{\frac{\langle \mathbf{C}^{I}, \mathbf{e}\rangle}{16\pi^2} \mathrm{tr}(F \wedge F)_I}_{\text{non-abelian gauge}} \\ &\quad \underbrace{-\frac{\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle}{192\pi^2} \mathrm{tr}(R_W \wedge R_W)}_{\text{along worldsheet}} \quad + \quad \underbrace{\frac{\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle}{96\pi^2} F_{\mathrm{N}} \wedge F_{\mathrm{N}}}_{\text{rotations normal to WS}} \end{aligned}$$

Strategy

Task:

Derive constraints on the bulk by demanding that this anomaly arises at 1-loop level on string worldsheet!

1. Parametrise massless fields on worldsheet:
   'general' modulo certain assumptions
2. Compute resulting 1-loop anomaly
3. Demand:
   1-loop anomaly matches δS_W

Worldsheet theory

Generically, string preserves 2d N=(0,2) supersymmetry
(follows from κ symmetry) (enhancement to N=(2,2) possible)

Fermion	#	$U(1)_{\rm N}$ charge	$U(1)_A$ charge	G_I repr.	(0,2) multiplet
ρ_+	1	$\frac{1}{2}$	0	$\mathbf{1}$	chiral U
χ_+	$n_{\rm C}$	$-\frac{1}{2}$	$*$	$*$	chiral Φ
ψ_-	$n_{\rm F}$	0	q_A	$\mathbf{r}_I$	Fermi Ψ_-
λ_-	$n_{\rm N}$	$\frac{1}{2}$	0	$\mathbf{1}$	Fermi Λ_-

- Universal multiplet $U = u + \sqrt{2}\theta^+\rho_+ - 2i\theta^+\bar{\theta}^+\partial_{++}u$
  motion in transverse space to string in 4d
- Chiral multiplets Φ: moduli of weakly coupled NLSM target space

$$\Phi = \varphi + \sqrt{2}\theta^+\chi_+ - 2i\theta^+\bar{\theta}^+\partial_{++}\varphi\,,$$

Worldsheet theory

Fermion	#	$U(1)_{\mathrm{N}}$ charge	$U(1)_A$ charge	G_I repr.	(0,2) multiplet
ρ_+	1	$\frac{1}{2}$	0	$\mathbf{1}$	chiral U
χ_+	n_{C}	$-\frac{1}{2}$	$*$	$*$	chiral Φ
ψ_-	n_{F}	0	q_A	$\mathbf{r}_I$	Fermi Ψ_-
λ_-	n_{N}	$\frac{1}{2}$	0	$\mathbf{1}$	Fermi Λ_-

- Chiral multiplets Φ: moduli of weakly coupled NLSM target space
  Generally can be obstructed via superpotential terms:

$$\int \mathrm{d}\theta^+ \Lambda_-^a J_a(\Phi) + \text{c.c.}\,, \qquad \Lambda_-^a \text{ : Fermi multiplets}$$

n_{N} will appear in combination $n_C^{\mathrm{eff}} := n_{\mathrm{C}} - n_{\mathrm{N}}$

Grav. anomaly matching

$$I^{\rm WS}_{4\,\mathbf{e}}\big|_{\textsf{grav}+U(1)_{\rm N}} = \underbrace{-\frac{n_{\rm F}-n_{\rm C}^{\rm eff}-1}{192\pi^2}\,{\rm tr}(R_W\wedge R_W) + \frac{n_{\rm C}^{\rm eff}+1}{32\pi^2}\,F_{\rm N}\wedge F_{\rm N}}_{\textsf{1-loop from WS spectrum}}$$

$$\stackrel{!}{=} \quad -\frac{\langle\tilde{\mathbf{C}},\mathbf{e}\rangle}{192\pi^2}\,{\rm tr}(R_W\wedge R_W) \qquad + \frac{\langle\tilde{\mathbf{C}},\mathbf{e}\rangle}{96\pi^2}\,F_{\rm N}\wedge F_{\rm N}$$

$$\Longrightarrow \quad \langle\tilde{\mathbf{C}},\mathbf{e}\rangle = n_{\rm F}-n_{\rm C}^{\rm eff}-1\,, \qquad \langle\tilde{\mathbf{C}},\mathbf{e}\rangle = 3(n_{\rm C}^{\rm eff}+1)$$

Conclusions:

- Relations

$$\frac{4}{3}\langle\tilde{\mathbf{C}},\mathbf{e}\rangle = n_{\rm F}\geq 0\,, \qquad n_{\rm C}^{\rm eff} = \frac{1}{3}\langle\tilde{\mathbf{C}},\mathbf{e}\rangle - 1 \geq -1$$

- **Quantisation condition and bound:** $\langle\tilde{\mathbf{C}},\mathbf{e}\rangle \equiv \tilde{C}_i e^i$

$$\langle\tilde{\mathbf{C}},\mathbf{e}\rangle \in 3\mathbb{Z}_{\geq 0} \quad \forall\mathbf{e}\in\mathcal{C}_{\rm S}^{\textsf{EFT}}$$

Gauge anomaly matching

$$I_{4\,\mathbf{e}}^{\mathrm{ws}}|_{\mathrm{gauge}} \overset{!}{=} -\frac{\langle \mathbf{C}^{AB}, \mathbf{e}\rangle}{8\pi^2} F_A \wedge F_B - \frac{\langle \mathbf{C}^{I}, \mathbf{e}\rangle}{16\pi^2} \mathrm{tr}(F\wedge F)_I \equiv -\frac{\langle \mathbf{C}^{\mathcal{A}\mathcal{B}}, \mathbf{e}\rangle}{8\pi^2} F_{\mathcal{A}} \wedge F_{\mathcal{B}}$$

1) Anomaly from charged Fermi multiplets

- Anomaly contribution:

$$I_{4\,\mathbf{e}}^{\mathrm{ws}} \supset -\frac{1}{8\pi^2}\, k_{\mathrm{F}}^{\mathcal{A}\mathcal{B}}(\mathbf{e})\, F_{\mathcal{A}} \wedge F_{\mathcal{B}}\,, \qquad k_{\mathrm{F}}^{\mathcal{A}\mathcal{B}} = \sum_{\mathbf{q}\in\mathrm{Fermi}} q^{\mathcal{A}} q^{\mathcal{B}}\,, \qquad \mathrm{rk}(k_{\mathrm{F}}^{\mathcal{A}\mathcal{B}}) \leq n_{\mathrm{F}} = \frac{4}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle$$

2) Anomaly from unobstructed chiral multiplets

Assumption: $n_{\mathrm{unobstr.}} = n_{\mathrm{C}}^{\mathrm{eff}} := n_{\mathrm{C}} - n_{\mathrm{N}}$

- Origin: gauging of shift symmetries of unobstructed scalars

  [Blaszczyk,Groot Nibbelink,Ruehle'11][Quigley,Sethi'11][Adams,Dyer,Lee'12]

- Anomaly contribution

$$I_{4\,\mathbf{e}}^{\mathrm{ws}} \supset -\frac{1}{8\pi^2}\, k_{\mathrm{C}}^{\mathcal{A}\mathcal{B}}(\mathbf{e})\, F_{\mathcal{A}} \wedge F_{\mathcal{B}}\,, \qquad \mathrm{rk}(k_{\mathrm{C}}^{\mathcal{A}\mathcal{B}}) \leq 2n_{\mathrm{C}}^{\mathrm{eff}} = \frac{2}{3}\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2$$

Gauge anomaly matching

$$I_{4\,\mathbf{e}}^{\mathrm{ws}}|_{\mathrm{gauge}} = \quad -\frac{k_{\mathrm{F}}^{\mathcal{AB}} + k_{\mathrm{C}}^{\mathcal{AB}}}{8\pi^2} F_{\mathcal{A}} \wedge F_{\mathcal{B}} \stackrel{!}{=} \quad -\frac{\langle \mathbf{C}^{\mathcal{AB}}, \mathbf{e}\rangle}{8\pi^2} F_{\mathcal{A}} \wedge F_{\mathcal{B}}$$

Goal:

Constrain the **rank of gauge sector coupling to string of charge e:**

$$r(\mathbf{e}) := \mathrm{rank}\{\langle \mathbf{C}^{\mathcal{AB}}, \mathbf{e}\rangle\}$$

$$r(\mathbf{e}) \leq r(\mathbf{e})_{\mathrm{max}} = \mathrm{rk}(k_{\mathrm{F}}^{\mathcal{AB}}) + \mathrm{rk}(k_{\mathrm{C}}^{\mathcal{AB}}) \equiv 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 \qquad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathsf{EFT}}$$

Gauge anomaly matching

Contribution from unobstructed chiral multiplets $n_{\rm C}^{\rm eff} = n_{\rm C} - n_{\rm N}$

- Shift symmetry gauged under $A_{\mathcal{A}} \to A_{\mathcal{A}} + {\rm d}\lambda_{\mathcal{A}}$

$$\tau_r \to \tau_r + \frac{1}{2\pi} N_r{}^{\mathcal{A}} \lambda_{\mathcal{A}} \qquad r = 1, \ldots, n_{\rm A} \leq n_{\rm C}^{\rm eff}$$

- Green-Schwarz like terms on worldsheet (fixed by N=(0,2) SUSY)

[Blaszczyk,Groot Nibbelink,Ruehle'11][Quigley,Sethi'11][Adams,Dyer,Lee'12]

$$S_W \supset -M^{\mathcal{A}r} \int_W {\rm Re}\, \tau_r\, F_{\mathcal{A}} - \frac{1}{8\pi} Q^{AB} \int_W A_{\mathcal{A}} \wedge A_{\mathcal{B}}\,,$$

with

$$Q^{\mathcal{A}\mathcal{B}} = -Q^{\mathcal{B}\mathcal{A}} \equiv (MN)^{\mathcal{A}\mathcal{B}} - (MN)^{\mathcal{B}\mathcal{A}}\,,$$

- Anomalous contribution

$$I \supset -\frac{1}{8\pi^2} k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e})\, F_{\mathcal{A}} \wedge F_{\mathcal{B}} \qquad k_{\rm C}^{\mathcal{A}\mathcal{B}}(\mathbf{e}) \equiv (MN)^{\mathcal{A}\mathcal{B}} + (MN)^{\mathcal{B}\mathcal{A}}\,.$$

$$\Longrightarrow r_{\rm C}(\mathbf{e}) \leq 2 n_{\rm C}^{\rm eff} = \frac{2}{3} \langle \tilde{\mathbf{C}}, \mathbf{e} \rangle - 2$$

Rank bounds

$$r(\mathbf{e}) \leq r(\mathbf{e})_{\max} = \mathrm{rk}(k_{\mathrm{F}}^{\mathcal{AB}}) + \mathrm{rk}(k_{\mathrm{C}}^{\mathcal{AB}}) \equiv 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 \qquad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathsf{EFT}}$$

in a strictly 4d theory with axionic couplings $S \supset -\frac{1}{96\pi}\tilde{C}_i \int_{\mathbb{R}^{1,3}} a^i \, \mathrm{tr} R \wedge R$

- $r(\mathbf{e})$ = rank of bulk gauge sector that couples to string of charge $\mathbf{e}$
  i.e. becomes weakly coupled by backreaction near string
- Applies only to theories whose gauge sector enjoys axionic coupling
- Technical assumption: $n_{\mathrm{C}} - n_{\mathrm{N}} \geq 0$ is number of unobstructed moduli

Caveat:

In [Martucci,Risso,TW'22] we discuss another potential contribution on RHS:

- only present if theory has a 5d origin
- ignored in this talk for simplicity

Caveat: 5d contributions

Bianchi identity: $\mathrm{d}h_1^i|_W = \delta_2(W)|_W = \chi(N_W) = \frac{1}{2\pi}F_{\mathrm{N}} \quad \Longrightarrow F_{\mathrm{N}} = \mathrm{d}A_{\mathrm{N}}$

Can consider additional coupling on string:

$$S_{\mathrm{N}} = -\frac{1}{24}\hat{C}_i(\mathbf{e})\int_W h_1^i \wedge A_{\mathrm{N}} \Longrightarrow \delta S_{\mathrm{N}} = -\frac{1}{48\pi}\hat{C}_i(\mathbf{e})e^i \int_W \lambda_{\mathrm{N}}\, F_{\mathrm{N}}$$

Changes $U(1)_N$ anomaly : $I_{4\,\mathbf{e}}^{\mathrm{ws}}|_{U(1)_N} = \frac{\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle}{96\pi^2} F_{\mathrm{N}} \wedge F_{\mathrm{N}}$

$$\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle \in 3\mathbb{Z}_{\geq 0}\,, \qquad r(\mathbf{e}) \leq 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle + \langle \hat{\mathbf{C}}(\mathbf{e}), \mathbf{e}\rangle - 2 \qquad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathrm{EFT}}$$

$\hat{\mathbf{C}}(\mathbf{e}) \neq 0$ naturally for 5d origin of 4d theory (from 5d CS terms) - confirmed in stringy examples, but generally only plausible

Rank bounds - Example

$$r(\mathbf{e}) \leq r(\mathbf{e})_{\max} = \mathrm{rk}(k_{\mathrm{F}}^{\mathcal{AB}}) + \mathrm{rk}(k_{\mathrm{C}}^{\mathcal{AB}}) \equiv 2\langle \tilde{\mathbf{C}}, \mathbf{e}\rangle - 2 \qquad \forall \mathbf{e} \in \mathcal{C}_{\mathrm{S}}^{\mathsf{EFT}}$$

Example:

4d N=1 SUGRA with single axionic field controlling gauge group G

$$S = -\frac{1}{8\pi}\int (Cs + \ldots)\,\mathrm{tr}(F \wedge *F) - \frac{1}{192\pi}\int (\tilde{C}s + \ldots)\, E_{\mathrm{GB}} * 1$$

Constraints:

$$\tilde{C} \overset{!}{=} 3k \geq 0 \quad \text{with} \quad k \in \mathbb{Z}_{\geq 0}\,,$$

$$\mathrm{rk}(\mathfrak{g}) \overset{!}{\leq} 2\tilde{C} - 2 = 6k - 2\,, \qquad k \in \mathbb{Z}_{\geq 0}$$

$\Longrightarrow$ **A strictly 4d theory with gauge algebra must have $\tilde{C} > 0$!** [a]

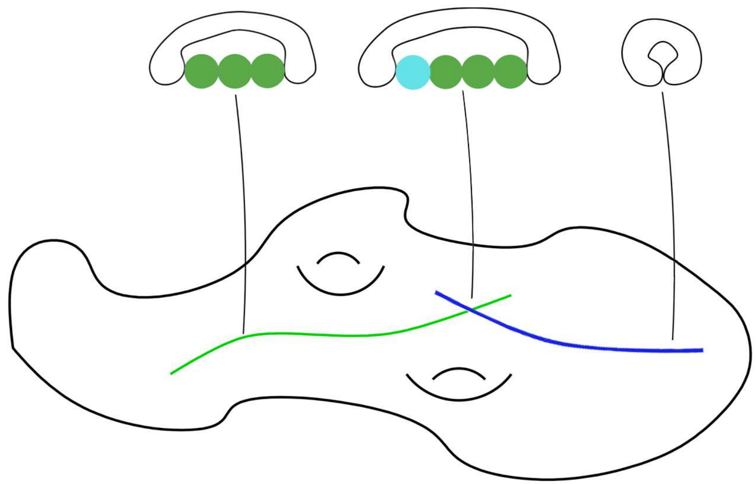
[a] With 5d limit, not all higher derivative grav couplings zero [Martucci,Risso,TW'22]

Part II: Test in String Theory

F-theory in 4d

F-theory in 4d N=1 $\Longleftrightarrow$ Type IIB on $\mathbb{R}^{1,3} \times B_3$ with 7-branes

- B_3 = compact Kähler 3-fold
  $\Longrightarrow$ dynamical gravity
- 7-branes on complex surface $\mathbf{S} \subset B_3$
  $\Longrightarrow$ gauge symmetry



Couplings: (IIB Einstein frame)

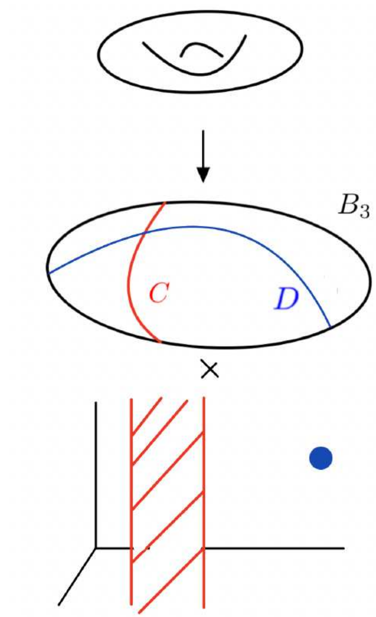
$$\frac{\mathbf{M_{Pl}^2}}{\mathbf{M_{IIB}^2}} = \mathbf{4}\pi\mathcal{V}_{\mathbf{B_3}} \qquad \frac{\mathbf{1}}{\mathbf{g_{YM}^2}} = \frac{\mathbf{1}}{\mathbf{2}\pi}\mathcal{V}_{\mathbf{S}}$$

$\Longrightarrow$ N=1 Kähler moduli space in F-theory

EFT strings from $\mathrm{Mov}_1(B_3)$

N=1 Kähler moduli space in F-theory: [Lanza,Marchesano,Martucci,Valenzuela'20-21]

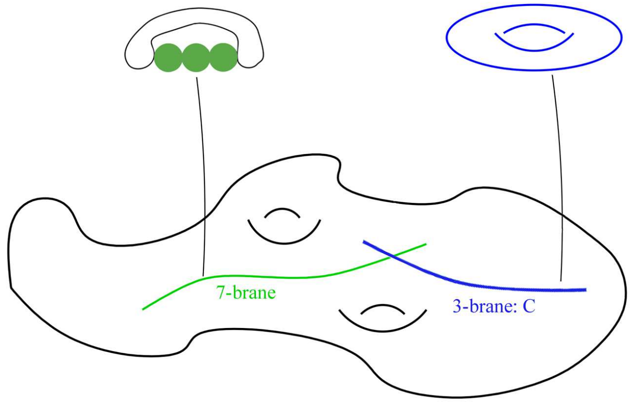
- **Instantons:**
  Euclidean D3 on effective divisors
  $D \in \mathrm{Eff}^1(B_3)$
- **EFT Strings:**
  D3 on curves $\Sigma_{\mathbf{e}}$ in dual cone of **movable curves** $\mathbf{Mov}_1(B_3)$



- **Movable curves** can probe entire base
  (live in a family that covers dense open subset of B_3)
- EFT strings sensitive to gravity

Characterisation of movable curves on B_3 and associated EFT string limits in [Cota,Mininno,TW,Wiesner'22]

EFT strings from $\mathrm{Mov}_1(B_3)$



F-theory on elliptic CY$_4$ with base B_3

D3-brane on $\mathbb{R}^{1,1} \times \Sigma_{\mathbf{e}}$

$\Sigma_{\mathbf{e}}$ a curve in base $\Sigma_{\mathbf{e}} \in \mathrm{Mov}_1(B_3)$

2 important properties of movable curves $\Sigma_{\mathbf{e}}$:

1. Can assume movable C is not contained in discriminant locus

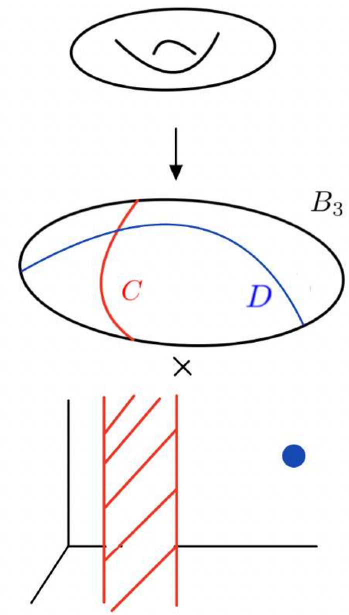
$$\Delta = 12\bar{K}_{B_3} = \text{totality of 7-branes}$$

- $\Sigma_{\mathbf{e}}$ is transverse to 7-branes on B_3
- $\Sigma_{\mathbf{e}}$ intersects 7-branes in isolated points on B_3
  $\Longrightarrow$ charged fermionic modes from 3-7 strings

2. Anti-canonical class $\bar{K}_{B_3} \in \mathrm{Eff}^1(B_3) \longrightarrow \bar{K}_{B_3} \cdot \Sigma_{\mathbf{e}} \geq 0$

Worldsheet Theory

Describe EFT worldsheet theory in F-Theory [Lawrie,Schafer-Nameki,TW'16]
via **topological duality twist** [Martucci'14]



Reduce $N = 4$ SYM on single D3-brane with worldvolume

$$\mathbb{R}^{1,1} \times \Sigma_{\mathbf{e}}$$

$\Longrightarrow$ 2d N=(0,2) theory on worldsheet

In particular:

- $n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)} = \bar{K}_{B_3} \cdot \Sigma_{\mathbf{e}}$
  $=$ number of unobstructed complex geometric deformations of curve $\Sigma_{\mathbf{e}}$ inside B_3
- $n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)} = \bar{K}_{B_3} \cdot \Sigma_{\mathbf{e}} - 1$
  conjectured to agree with number of unobstructed twisted Wilson line moduli
- $n_{\mathrm{F}} = 8\bar{K}_{B_3} \cdot \Sigma_{\mathbf{e}}$
  intersection of 3 and 7-brane

Sharpened Bound

$$r(\mathbf{e}) \leq n_{\mathrm{F}}(\mathbf{e}) + 2n_{\mathrm{C}}^{\mathrm{eff}}$$

$$n_{\mathrm{C}}^{\mathrm{eff}} = (n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)}) + (n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)})$$

- $n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)} = \Sigma_{\mathbf{e}} \cdot \bar{K}$
- $n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)} = \Sigma_{\mathbf{e}} \cdot \bar{K} - 1$

Stronger bound for minimally SUSY F-theory over smooth base B_3

[Martucci,Risso,TW'22]

- $\Phi^{(1)}$: geometric moduli of curve $\Sigma_{\mathbf{e}}$ in B_3
  Under above assumptions, $\Phi^{(1)}$ cannot enjoy gauged shift symmetries
- $\Phi^{(2)}$: Of same origin as charged Fermis in dual M-theory picture
  $\Longrightarrow$ candidates for gauged shift symmetries

$$\boxed{r(\mathbf{e}) \leq n_{\mathrm{F}}(\mathbf{e}) + 2n_{\mathrm{C}}^{\mathrm{eff},(2)} = 10\,\Sigma_{\mathbf{e}} \cdot \bar{K} - 2 = \frac{5}{6}\Sigma_{\mathbf{e}} \cdot \Delta - 2}$$

$\Delta = 12\bar{K}$: discriminant = totality of all 7-branes

EFT vs Kodaira bounds

Compare:

- Known **Geometric Kodaira bound:**

$$\mathrm{rk}(G_{\mathrm{non-ab}}) \leq C \cdot \Delta \qquad \forall C \text{ inside } \mathrm{Mov}_1(B_3)$$

- For EFT curve $C = \Sigma_{\mathbf{e}}$:

$$\mathrm{rk}(\mathbf{e}) \leq \frac{5}{6}\Sigma_{\mathbf{e}} \cdot \Delta - 2$$

What use are the EFT string bounds?

1. Kodaira bound: not sensitive to abelian subgroup
   EFT string bound: includes non-abelian and abelian rank
2. Stronger EFT string bound

$$\boxed{\mathrm{rk}(\mathbf{e}) \leq \frac{5}{6}\Sigma_{\mathbf{e}} \cdot \Delta - 2} \qquad \Longrightarrow \text{Physics predictions for geometry!}$$

not obvious from geometry

Universal bounds in 6d

Bounds constrain rank of gauge algebra to which give EFT string couples

Absolute bounds on (7-brane) group require minimal $\Sigma_{\mathbf{e}}$ in interior of Mov_1:

$$\Sigma_{\mathbf{e}} \cdot D_{\mathrm{eff}} \overset{!}{\geq} 1 \quad \forall D_{\mathrm{eff}} \text{ effective}$$

Simplification for abelian (non-Cartan) U(1)s: [Lee,TW'19]
Suffices to find curve $\Sigma_{\mathbf{e}}$ such that $\Sigma_{\mathbf{e}} \cdot \bar{K}_B \geq 1$

Can be achieved for F-theory on elliptic 3-folds (6d):
Bases of elliptic 3-folds very constrained

B_2: $\mathbb{P}^2$ or (blowup of) Hirzebruch: $B_2 = \mathrm{Bl}^k(\mathbb{F}_n)$ (or Enriques)

Explicit analysis of spectrum $\Longrightarrow$ bound detected by string from curve $\Sigma_{\mathbf{e}}$:

$$r(\mathbf{e})^{\mathrm{strict}}_{\max} = 10\,\Sigma_{\mathbf{e}} \cdot \overline{K}_{B_2} - 2$$

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For (non-Cartan) U(1) groups in 6d this gives a universal bound [Lee,TW'19]:

- $\mathbb{P}^2 : n_{U(1)} \leq 28$
- $\mathrm{Bl}^k(\mathbb{F}_n) : n_{U(1)} \leq 18$

$\Longrightarrow$ bound on rank of Mordell-Weil group of rational sections on ell. CY_3

Current Record: Schoen manifold of Namikawa type [Grassi,TW'21]

$$n_{U(1)} \leq 10$$

Generic Schoen: $n_{U(1)} = 8$ [Schoen'88]
Special Schoen: $n_{U(1)} = 9$ [Morrison,Park,Taylor'18] (12 I_2 fibers in codim-two)
Namikawa type: $n_{U(1)} = 10$ [Namikawa'02] [Grassi,TW'21]

(6 Type IV fibers in codim-two: terminal, non-$\mathbb{Q}$-factorial)

Conclusions

General bounds on rank of gauge algebra in strictly 4d $N=1$ supergravity theories from EFT strings

$$r(\mathbf{e}) \leq 2\langle \tilde{C}, \mathbf{e}\rangle - 2$$

in a theory with axionic couplings $S \supset -\frac{1}{96\pi}\tilde{C}_i \int_{\mathbb{R}^{1,3}} a^i \, \mathrm{tr} R \wedge R$

Applied to F-theory on CY$_4$

Novel sharpened bound: $$r(\mathbf{e}) \leq \frac{5}{6}\Delta \cdot \Sigma_{\mathbf{e}} - 2$$

✓ Stronger than geometric Kodaira bound

✓ Applies to abelian and non-abelian gauge group (from 7-branes)

$\Longrightarrow$ Predictions for arithmetic geometry!

✓ Matches expectations from dual heterotic strings, but more general

Conclusions

Many open questions:

- Prove assumptions on role of uncharged Fermi multiplets - at least for generators of movable cone
- Goal: Translate this into universal bound for rank of gauge group in all 4d N=1 theories comparable to bound to bound on abelian rank in 6d
- What about matter? 6d: cf. [Tazari,Vafa'21]

Appendix

Example: $\mathbb{P}^3$

$H^{1,1}(B_3) = \langle H \rangle \qquad \bar{K} = 4H \qquad \Delta = 12\bar{K} = 48H$

$$\Sigma_e = H \cdot H : \quad r_{\text{tot}} \leq \begin{cases} r(\mathbf{e})_{\text{max}} & = 12\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 = 46\,, \qquad n_F, n_C^{(1)}, n_C^{(2)}, \\ r(\mathbf{e})^{\text{strict}}_{\text{max}} & = 10\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X - 2 = 38\,, \qquad n_F, n_C^{(2)}, \\ r(\mathbf{e})^{\text{F}}_{\text{max}} & = 8\,\Sigma_{\mathbf{e}} \cdot \overline{K}_X \qquad = 32\,, \qquad n_F \end{cases}$$

Maximal rank of $SU(N)$ group in Weierstrass model [Morrison, Taylor '11]

$$SU(N_{\text{max}}) = SU(32) \qquad \text{geometrically}$$

Incidentally, allowed even by bound $r(\mathbf{e})^{\text{F}}_{\text{max}}$, but more generally, at best $r(\mathbf{e})^{\text{strict}}_{\text{max}}$ can be correct:

Examples:

$G = E_6 \times E_7^4 \qquad \text{rank}(G) = 34$

$G = E_6^2 \times E_7^3 \qquad \text{rank}(G) = 33$

Caveats:

Non-minimal fibers → blowups

Flux quantisation → non-trivial flux

Massless Spectrum

Multiplets	(0,2) Type	Origin	Interpretation	Zero-mode Cohomology
U	Chiral	(ϕ_i, Ψ)	Universal	$h^0(C) = 1$
$\Phi^{(1)}$	Chiral	(ϕ_i, Ψ)	Deformations	$n_{\rm C}^{(1)} = h^0(C, N_{C/B_3})$
$\Phi^{(2)}$	Chiral	(A, Ψ)	Twisted Wilson lines	$n_{\rm C}^{(2)} = h^0(C, K_C \otimes \bar{K}_{B_3})$ $= g - 1 + \bar{K}_{B_3} \cdot C$
$\Psi^{(1)}$	Fermi	Ψ	Obstructions	$n_{\rm N}^{(1)} = h^1(C, N_{C/B_3})$ $= h^0(C, N_{C/B_3}) - \bar{K}_{B_3} \cdot C$
$\Psi^{(2)}$	Fermi	Ψ	Obstructions (?)	$n_{\rm N}^{(2)} = h^1(C) = g$
Λ	Fermi	3-7 strings	Charged	$n_{\rm F} = 8\,\bar{K}_{B_3} \cdot C$

- $n_{\rm C}^{(1)} - n_{\rm N}^{(1)} = h^0(C, N_{C/B_3}) - h^1(C, N_{C/B_3}) = \bar{K}_{B_3} \cdot C$
  topological index that agrees with number of unobstructed complex geometric deformations of curve C inside B_3

- $n_{\rm C}^{(2)} - n_{\rm N}^{(2)} = \bar{K}_{B_3} \cdot C - 1$ topological index - conjectured to agree with number of unobstructed twisted Wilson line moduli

General Bound

General bound on rank of gauge group detected by EFT string of charge $\mathbf{e}$

$$r(\mathbf{e}) \leq n_{\mathrm{F}}(\mathbf{e}) + 2n_{\mathrm{C}}^{\mathrm{eff}} = 2\langle \tilde{C}(\mathbf{e}), \mathbf{e}\rangle - 2$$

- $n_{\mathrm{F}}(e)$ number of Fermi multiplets charged under 7-brane gauge group
- $n_{\mathrm{C}}^{\mathrm{eff}} = n_{\mathrm{C}} - n_{\mathrm{N}}$ number of unobstructed chiral multiplets which can experience gauged shift symmetry
- $\tilde{C}$: gravitational higher derivative coupling

Specialisation: [Martucci,Risso,TW'22]

Rank of 7-brane group detected by string from D3 brane on curve $\Sigma_{\mathbf{e}}$:

- $n_{\mathrm{C}}^{\mathrm{eff}} = (n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)}) + (n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)})$
  - $n_{\mathrm{C}}^{(1)} - n_{\mathrm{N}}^{(1)} = \Sigma_{\mathbf{e}} \cdot \bar{K}$
  - $n_{\mathrm{C}}^{(2)} - n_{\mathrm{N}}^{(2)} = \Sigma_{\mathbf{e}} \cdot \bar{K} - 1$
- $n_{\mathrm{F}}(e) = 8\Sigma_{\mathbf{e}} \cdot \bar{K}$

$$\boxed{r(\mathbf{e}) \leq 12\Sigma_{\mathbf{e}} \cdot \bar{K} - 2 = \Sigma_{\mathbf{e}} \cdot \Delta - 2}$$

Consistently:

$\tilde{C} = 6\bar{K}$ from effective action

[Grimm,Taylor'12]

EFT vs Kodaira bounds

$$\{\Delta = 0\} = n_I \mathcal{D}^I + \mathcal{D}' \simeq 12\overline{K} \qquad \text{with} \quad n_I \equiv \mathrm{ord}(\Delta)|_{\mathcal{D}^I}$$

Non-abelian gauge group G_I on divisor $\mathcal{D}^I$ constrained by Kodaira bound cf. [Morrison,Taylor '11]:

$$\mathrm{rk}(G_I) < n_I \equiv \mathrm{ord}(\Delta)|_{\mathcal{D}^I} \,.$$

	$\mathrm{ord}_{\mathcal{D}}(f)$	$\mathrm{ord}_{\mathcal{D}}(g)$	$\mathrm{ord}_{\mathcal{D}}(\Delta)$	singularity
I_0	≥ 0	≥ 0	0	none
$\mathrm{I}_n,\, n \geq 1$	0	0	n	A_{n-1}
II	1	1	≥ 2	none
III	1	≥ 2	3	A_1
IV	≥ 2	2	4	A_2
I_0^*	≥ 2	≥ 3	6	D_4
$\mathrm{I}_n^*,\, n \geq 1$	2	3	$6+n$	D_{4+n}
IV^*	≥ 3	4	8	E_6
III^*	3	≥ 5	9	E_7
II^*	≥ 4	5	10	E_8

For every curve C in interior of movable cone $(C \cdot D_{\mathrm{eff}} \geq 1\, \forall D_{\mathrm{eff}})$

$$\mathrm{rk}(G_{\mathrm{non-ab}}) \leq \sum_I \mathrm{rk}(G_I)(C \cdot D_I) \leq \sum_I n_I (C \cdot D_I) + C \cdot D' = C \cdot \Delta$$

Compare: For EFT curve $C = \Sigma_{\mathbf{e}}$

$$\mathrm{rk}(\mathbf{e}) \leq \Sigma_{\mathbf{e}} \cdot \Delta - 2$$

✓ Conservative EFT bound slightly stronger than geometric upper bound