Quantum Coherence and Antidistinguishability

Nathaniel Johnston,
Shirin Moein, Rajesh Pereira, Sarah Plosker, (coherence)
Vincent Russo, and Jamie Sikora (antidistinguishability)

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Distinguishability

A **pure quantum state** is a unit vector $|\phi\rangle \in \mathbb{C}^d$.

If we are given a single copy of an arbitrary pure state (in a lab, not on paper), we cannot figure out exactly which one was given to us: measuring it gives some information but causes the state to collapse.

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Suppose we are given (on paper) a set of potential states:

$$S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\} \subset \mathbb{C}^d.$$

Then we are given (in a lab) one of those n states.

Theorem

*It is possible to determine which $|\phi_j\rangle$ was given to us (i.e., S is **distinguishable**) if and only if the members of S are mutually orthogonal (i.e., $\langle\phi_i|\phi_j\rangle = 0$ whenever $i \neq j$).*

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Antidistinguishability

What if, instead of wanting to determine which state from S was given to us, we just want to determine some state from S that was *not* given to us?

In other words, we want to determine whether or not S is **antidistinguishable**.

- If S is distinguishable then it is antidistinguishable.
- If $n = 2$ then S is distinguishable iff S is antidistinguishable.
- If $n \geq 3$ then there are antidistinguishable sets that are not distinguishable...

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Antidistinguishability Example

For example, consider the set of "trine" states:

$$S = \left\{ |0\rangle, \ \ -\tfrac{1}{2}\big(|0\rangle + \sqrt{3}|1\rangle\big), \ \ -\tfrac{1}{2}\big(|0\rangle - \sqrt{3}|1\rangle\big) \right\} \subset \mathbb{C}^2.$$

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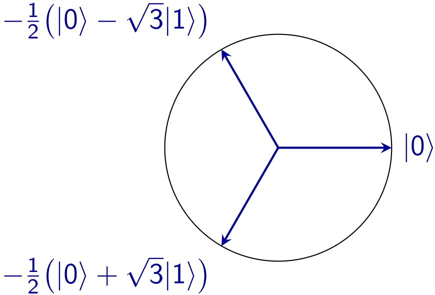

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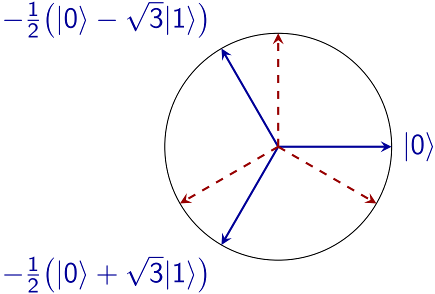

Some Inner Product Bounds

Fact: Whether or not a set S is antidistinguishable depends only on the inner products between the $|\phi_j\rangle$'s.

If the inner products are large then S is not antidistinguishable:

Theorem (Bandyopadhyay–Jain–Oppenheim–Perry '14)

Let $n \geq 2$ be an integer and let $S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$. If

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Conversely, if the inner products are small (e.g., all less than $1/2$) then S is antidistinguishable.

Conjecture (Havlívcek–Barrett '20)

Let $n \geq 2$ be an integer and let $S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$. If

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Refutation of the Conjecture

Via a couple years of computer search, the $n = 4$ case of this conjecture was recently disproved by Russo and Sikora:

- The conjecture says that if $|\langle \phi_i | \phi_j \rangle| \leq 2/3 = 0.6666\ldots$ for all $i \neq j$ then S is antidistinguishable.
- They numerically found a non-antidistinguishable set of states with $|\langle \phi_i | \phi_j \rangle| \leq 0.6451$ for all $i \neq j$.

So what is the "correct" bound for $n = 4$?

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Positive Semidefiniteness

A matrix $X \in M_n(\mathbb{C})$ is **positive semidefinite (PSD)** iff...

- ...it is Hermitian ($X^* = X$) and has non-negative eigenvalues.
- Equivalent: ...there exist vectors $\mathrm{v_1}, \mathrm{v_2}, \ldots, \mathrm{v_n} \in \mathbb{C}^d$ such that

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k-Incoherence

Let $k \geq 1$ be an integer. A matrix $X \in M_n(\mathbb{C})$ is k-incoherent iff...

- ...there exist vectors $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_\mathrm{d} \in \mathbb{C}^n$, each with at most k non-zero entries, such that

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- Equivalent: ...we can write

$$X = \sum_{j} X_j$$

for some PSD matrices X_1, X_2, ... that are 0 outside of a single $k \times k$ principal submatrix.

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Let's look at some particular values of k...

- When $k = 1$: a matrix is 1-incoherent if and only if it is diagonal with non-negative (real) diagonal entries.

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

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- When $k = n$: a matrix is n-incoherent if and only if it is PSD.
- When $n = 3$, $k = 2$: the following matrix is 2-incoherent:

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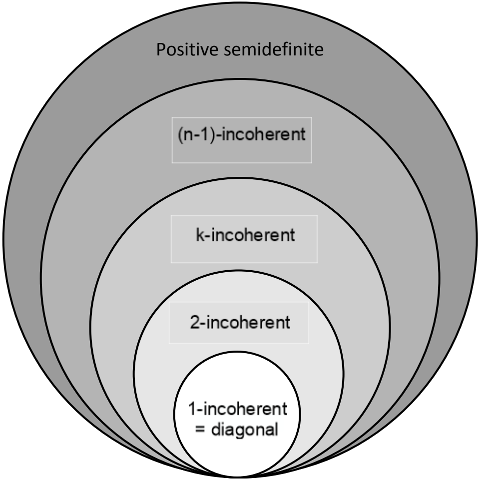

Connection with $(n-1)$-Incoherence

It turns out that antidistinguishability is equivalent to k-incoherence in the $k = n - 1$ case:

Theorem (J.–Russo–Sikora '23)

Let $n \geq 2$ be an integer and let $S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$. Then S is antidistinguishable if and only if the Gram matrix

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Cool! This provides a connection with $(n-1)$-incoherence that is useful for a few reasons...

- $(n-1)$-incoherence can be checked (reasonably...) quickly via semidefinite programming. So antidistinguishability can too.
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Let $n \geq 2$ be an integer, let $S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the Gram matrix G. If

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then G is $(n-1)$-incoherent, so S is antidistinguishable. Furthermore, this inequality is tight.

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Let $n \geq 2$ be an integer, let $S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$, and let G be the Gram matrix of S. If

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$$\|G\|_{\mathrm{F}} \leq \frac{n}{\sqrt{2}}$$

then G is $(n-1)$-incoherent, so S is antidistinguishable. Furthermore, this inequality is tight.

Correction of the Conjecture

We then get a correction to the antidistinguishability conjecture:

Theorem (J.–Russo–Sikora '23)

Let $n \geq 2$ be an integer and let $S = \{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle\}$. If

$$\big|\langle\phi_i|\phi_j\rangle\big| \leq \frac{1}{\sqrt{2}}\sqrt{\frac{n-2}{n-1}} \quad \text{for all} \quad i \neq j$$

then S is antidistinguishable.

When $n = 2, 3$, or 4, the RHS bound is 0, $1/2$, or $1/\sqrt{3}$ respectively, which are tight.

Unknown if it's tight for $n \geq 5$.

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Thank You!

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antidistinguishability: coming soon