From quantum resource theories to discrete dynamical systems

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Theory Canada 15

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This idea is made mathematically rigorous with resource theories.

Outline

Section 1

Resource theories

What is a resource?

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Oil as a resource

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It becomes more valuable if there isn't much of it.

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    - the tensor product of free operations is free.

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Main question

Can ρ be converted into σ with free operations?

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- symmetry and reference frames [Bartlett et al.].

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States of A are processes from the trivial system to A.

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Free states of A are free processes from the trivial system to A.

Section 2

Discrete dynamical systems

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We work in the finite case: a finite set S, with a generator of dynamics $\phi : S \to S$.

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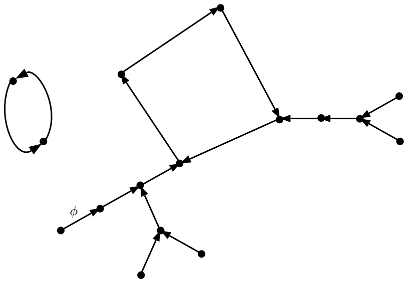
- Stochastic states: probability vectors $\boldsymbol{p}$.
- The evolution stays deterministic: $\boldsymbol{p}$ evolves under ϕ.
- We may allow stochastic influences: stochastic maps $f : \mathfrak{S} \to \mathfrak{S}$, with $\mathfrak{S}$ simplex of probability vectors on S.

Features of a DDS



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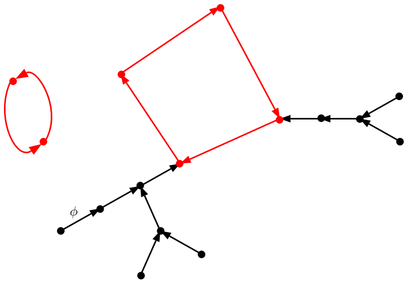

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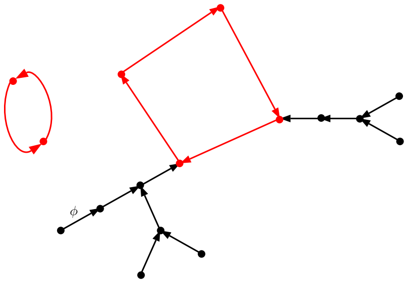

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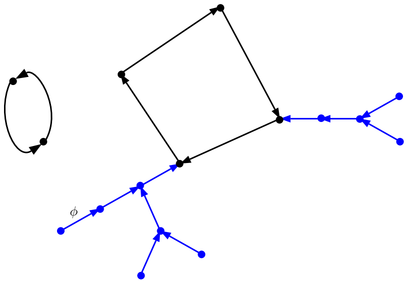

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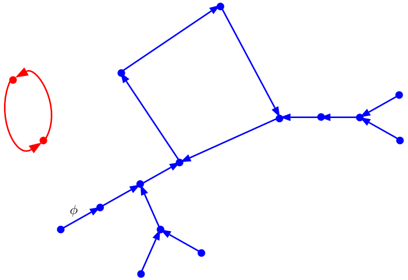
- An attractor is reached after enough time steps.
- There may be transient states too.

Three important notions

S is partitioned into basins of attraction.



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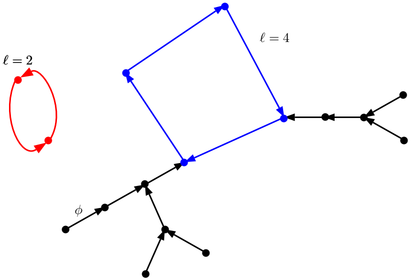

The length ℓ of a state is the length of the cycle in its basin of attraction.

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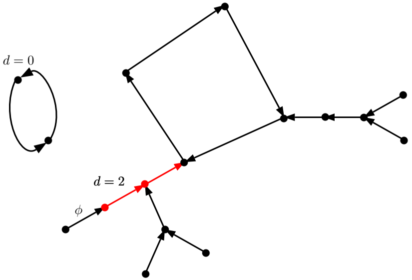

The transient progeny d of a state is the number of steps necessary to reach its attractor.

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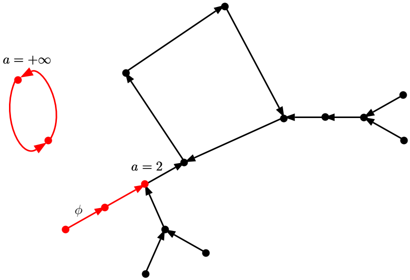

The ancestry $a(s)$ of a state s is the number of steps necessary to reach s from its farthest predecessor.

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Theorem

Covariant influences form a wide symmetric monoidal subcategory of the category of influences.

Conversions in the deterministic case [CMS et al.]

We have a resource theory of covariant influences.

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2. $d' \leq d$, *and*
3. $a\left(\phi^{n}\left(s'\right)\right) \geq a\left(\phi^{n}\left(s\right)\right)$ *for* $n = 0, \ldots, d' - 1$.

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- We can still say something general about the stochastic case...
- The entries of F are $p(s'|s)$: the probability of jumping from s to s' (deterministic states).

The entries of a covariant F [CMS et al.]

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Randomness in covariant influences is a resource: it activates transitions between deterministic states.

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- An example are DDSs.
- We were able to find universal results in the presence of covariant influences.
- What other non-quantum areas could we apply them to?

References

- S D Bartlett, T Rudolph, R W Spekkens, *Rev. Mod. Phys.* **79** (2), 555 (2007).
- E Chitambar, G Gour, *Rev. Mod. Phys.* **91** (2), 025001 (2019).
- B Coecke, T Fritz, R W Spekkens, *Inf. Comput.* **250**, 59–86 (2016).
- J Goold et al., *J. Phys. A* **49** (14), 143001 (2016).
- R Horodecki et al., *Rev. Mod. Phys.* **81** (2), 865 (2009).
- M Lostaglio, *Rep. Prog. Phys.* **82** (11), 114001 (2019).
- M B Plenio, S Virmani, *Quant. Inf. Comput.* **7** (1&2), 1 (2007).
- CMS, G Gour, B C Sanders, *Phys. Rev. E* **107** (1), 014203 (2023).
- A Streltsov, G Adesso, M B Plenio, *Rev. Mod. Phys.* **89** (4), 041003 (2017).