Conservation, Non-conservation Laws and Gravitational Energy-momentum in General Relativity

From a Geometric perspective

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Abstract

- The spacetime M in general relativity (GR) is curved. Parallel transportations of vectors in M depend on path. This leads to the fact that vectors distributed on different points in M can not be added up to get a sum vector unambiguously. Geometry does not allow talking about matter energy momentum distributed on (passing through) a finite or infinite spacelike (timelike) hypersurface, does not allow talking about the net increase of matter energy momentum in a finite or infinite 4-demensional spacetime region. Even for a simple physical system consisting of only two uncharged mass points, geometry does not allow talking about its energy momentum. Therefore, it is pressing to explore the meaning of conservation of energy momentum in GR from a geometric perspective.
- We also show, in a curved spacetime M, when limited to an infinitesimal spacetime region, vectors distributed at different points can still be added up to get a sum vector unambiguously, if neglecting higher order infinitesimals. For an (r,s)-tensor Q, denoting by J its flux density (r+1,s)-tensor field, the conservation law of Q in curved spacetime M is "the covariant divergence of J vanishes everywhere". It reads, "the net increase of tensor Q in any infinitesimal 4-dimensional neighborhood is zero".
- In particular, "the covariant divergence of T (flux density of matter energy momentum) vanishes everywhere" is the conservation law of matter energy momentum in GR. This is against to mainstream scholar's viewpoint.

§ 1. Motivation

How was the gravitational energy-momentum introduced into general relativity?

- The law of conservation of energy-momentum is the cornerstone of modern physics. Far beyond the scope of physics, it is the bedrock law of nature. So, when Einstein tried to establish his new theory of gravity, the general relativity (GR), his top priority was to ensure the energy-momentum conservation. However, from the fundamental equations of motion of GR, Einstein field equations

$$R^{\alpha\beta}(x) - \frac{1}{2}R(x)g^{\alpha\beta}(x) = \frac{8\pi G}{c^4}T^{\alpha\beta}(x), \forall \alpha, \beta = 0,1,2,3 \quad (1)$$

by using contracted Bianchi identity, he obtained

$$\nabla_\alpha T^{\alpha\beta}(x)|_p = 0, \quad \forall \beta = 0,1,2,3; \quad p \in M \quad (2)$$

that is, the covariant divergence of matter energy-momentum flux density tensor field vanishes everywhere in spacetime *M*. Multiplying it by $\sqrt{-|g(x)|}$, he obtained

$$\frac{\partial}{\partial x^\lambda}\left[\sqrt{-|g(x)|}T^{\lambda\mu}(x)\right] + \sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0 \quad (3)$$

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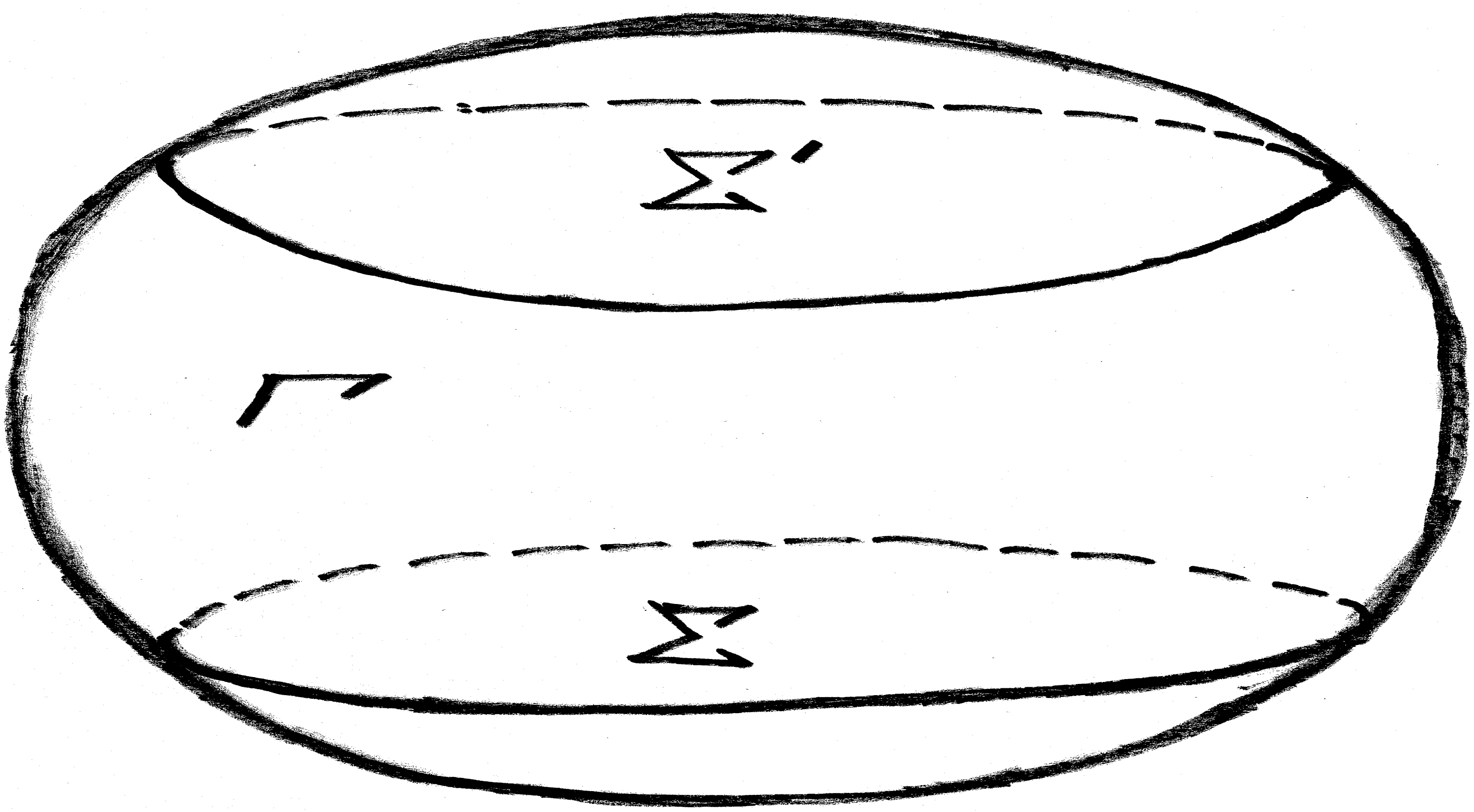
and integrating eqn. (3) over 4-dimensional spacetime domain $\Omega(\subset M)$, he got by using Gaussian theorem

$$\int_{\Omega} d^4x\sqrt{-|g(x)|} \quad T \quad (x) =$$

$$\int_{\partial\Omega} ds_{\lambda}(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x) + \int_{\Omega} d^4x\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x)$$

$$= 0\,, \qquad \forall\, \mu = 0,1,2,3 \tag{4}$$

When the boundary of Ω is composed of a past spacelike hyper-surface Σ, a future spacelike hyper-surface Σ' and a timelike hyper-surface Γ which links the boundaries of Σ and Σ', eqn. (4) can be written as



$$\partial\Omega = \Sigma \cup \Sigma' \cup \Gamma$$

$$\Omega \subset M$$

$$\left\{\int_{\Sigma'} - \int_{\Sigma} - \int_{\Gamma}\right\} ds_{\lambda}(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x) =$$

$$-\int_{\Omega} d^4x\, \sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x), \quad \forall\mu = 0,1,2,3 \qquad (5)$$

Einstein read this as the μ-component of matter energy-momentum distributed on Σ' minus the μ-component of matter energy-momentum distributed on Σ and the μ-component of matter energy-momentum flowing in through Γ equals the integral at left hand side, which is not zero in general. Therefore, he decided that the vanishing of the covariant divergence of matter energy-momentum flux density is a law of non-conservation of matter energy-momentum. In order to save the law of energy-momentum conservation, Einstein rewrite the second term of eqn.(3) as [1]

$$\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = \frac{\partial}{\partial x^{\lambda}}\left[\sqrt{-|g(x)|}t^{\lambda\mu}(x)\right] \qquad \textbf{(6)}$$

where $t^{\lambda\mu}(x)$ is a function of metric field and it's first order derivatives. (Note that, eqn. (6) does not uniquely determine $t^{\lambda\mu}(x)$) Substituting eqn. (6) into eqn. (5), he got

$$\left\{\int_{\Sigma'} - \int_{\Sigma} - \int_{\Gamma}\right\} ds_\lambda(x)\sqrt{-|g(x)|}\,[T^{\lambda\mu}(x) + t^{\lambda\mu}(x)] = 0$$

$$\forall \mu = 0, 1, 2, 3 \tag{7}$$

Now he read the above equations as " the μ-component of matter energy-momentum plus the μ-component of gravitational energy-momentum distributed on Σ' minus the μ-component of matter energy-momentum plus the gravitational energy-momentum distributed on Σ and the μ-component of matter energy-momentum plus the gravitational energy-momentum flowing in through Γ " is zero, or, the μ-component of "the net increase of matter energy-momentum in Ω plus the net increase of gravitational energy-momentum in Ω" is zero. And $t^{\lambda\mu}(x)$was read as the flux

density of gravitational energy-momentum, $\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x)$
was read as the μ-component of the amount of matter energy-momentum which changes into gravitational energy-momentum in per unit 4-dimensional spacetime volume.

Einstein was happy to have saved the law of energy-momentum by introducing gravitational energy-momentum.

Here we point out, the spacetime M in GR is curved, there is no flat coordinate system in M, and in a curved coordinate system, the sum (or integration) of components with the same index of vectors distributed at different points is not the corresponding component of their sum vector in general. And even more, we will show later, the sum vector of vectors distributed at different points in curved spacetime does not exist. The expression $T^{\lambda\mu}(x)\sqrt{-|g(x)|}ds_{\lambda}(x)|_{\Delta\Sigma}$ is indeed the μ −component of matter energy-momentum on infinitesimal hyper-surface element ΔΣ. However, when Einstein read $\int_{\Sigma} ds_{\lambda}(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x)$ as the μ −component of matter energy-momentum on Σ, he was making an elementary mistake. And it causes century long confusion in GR, due to his supremacy in physics.

H. Bauer immediately pointed out that Einstein's gravitational energy-momentum's current density $t^{\alpha\beta}(x)$ is not a tensor and is not localizable [2]. He showed, when $T^{\lambda\mu}(x)|_p = 0, \forall p \in \underline{M}$, $0 \leq \lambda, \mu \leq 3$, Minkowski metric is a solution to Einstein's field equation. There are inertial coordinate systems in Minkowski space. In an inertial coordinate system $\{x^0, x^1, x^2, x^3\}$, $t^{00}(x) \equiv 0$. Switching to spherical coordinate system by a pure spatial coordinate transformation, than

$$t^{00}\left(x^0, r, \theta, \varphi\right) = -\frac{1}{\kappa r^2} \neq 0, \quad \text{and}$$

$$\iiint_{x^0=co} t^{00}\left(x^0, r, \theta, \varphi\right) r^2 \, sin\theta dr d\theta d\varphi = -\infty.$$

Einstein's $t^{\alpha\beta}(x)$ is not symmetric. In the 1930's, Landau and Lifshitz proposed a symmetric $t^{\alpha\beta}(x)$ satisfying the following equations [3]

$$\frac{\partial}{\partial x^{\alpha}}\{-|g(x)|[T^{\alpha\beta}(x)+t^{\alpha\beta}(x)]\}=0, \forall\beta=0,1,2,3 \qquad (8)$$

which is equivalent to

$$\int_{\Omega} d^4x \frac{\partial}{\partial x^{\alpha}}\{-|g(x)|[T^{\alpha\beta}(x)+t^{\alpha\beta}(x)]\}=0 \qquad \forall\beta=0,1,2,3\,.$$

Or $$\int_{\partial\Omega} ds_{\alpha}(x)\{-|g(x)|[T^{\alpha\beta}(x)+t^{\alpha\beta}(x)]\}=0 \qquad \forall\beta=0,1,2,3\,. \qquad (9)$$

However, $ds_{\alpha}(x)\ [-|g(x)|]$ does not have the correct transformation property under general coordinate transformations for a hyper-surface element. It is a wrong expression, and the correct one is $ds_{\alpha}(x)\left[\sqrt{-|g(x)|}\right]$.

After that, several current densities $t^{\alpha\beta}(x)$ of gravitational energy-momentum were proposed [4,5]. But they are all “pseudo-tensors”

$$t^{\alpha\beta}(x) \neq \frac{\partial x^{\alpha}}{\partial y^{\mu}} \frac{\partial x^{\beta}}{\partial y^{\nu}} t^{\mu\nu}(y) \tag{10}$$

and not localizable. Efforts searching for covariant gravitational energy-momentum have never ceased, but have all failed. Some scholars proposed that non-localizability of gravitational energy-momentum is required by the equivalence principle. This viewpoint is generally accepted by relativists. But here I will show it is wrong.

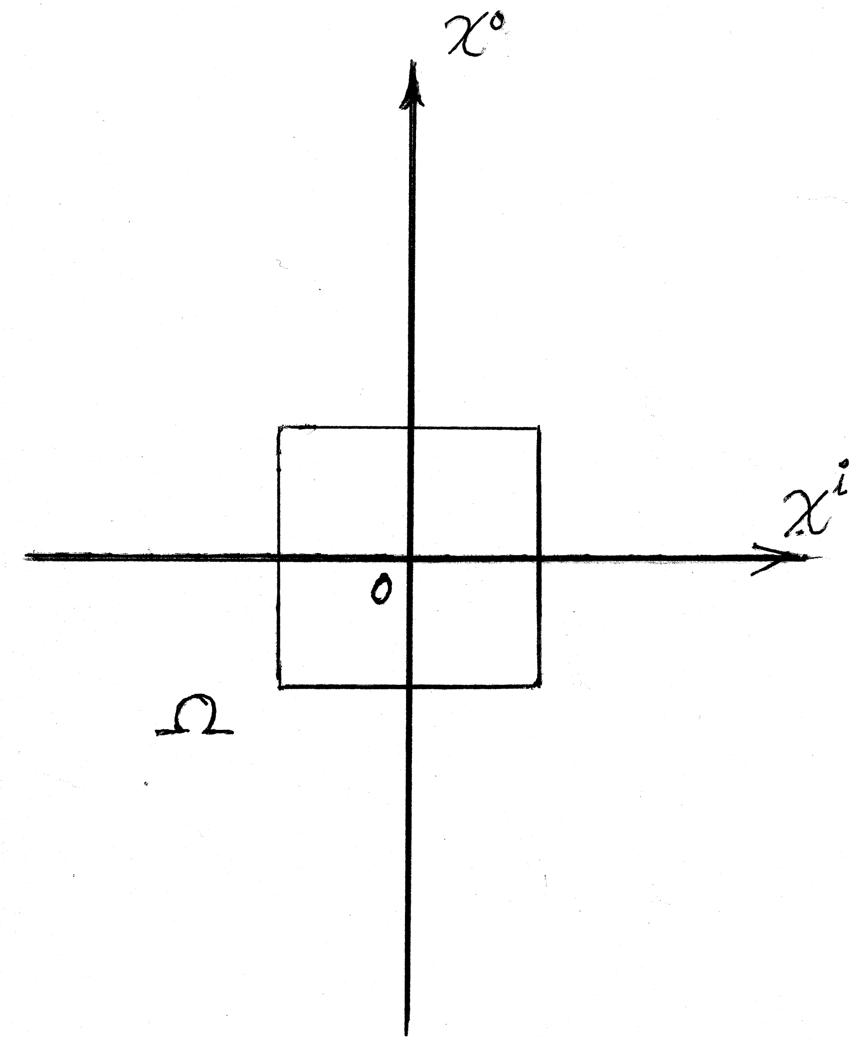
In section 20.4 of their influential book "gravitation" [6], C. Misner, K.S. Thorne, and J.A. Wheeler wrote, *"…One can always find in any given locality a frame of reference in which all local 'gravitational fields' (all Christoffel symbols; all* $\Gamma^{\alpha}_{\mu\nu}$ *) disappear. No* Γ' *s means no 'gravitational field' and no local gravitational field means no 'local gravitational energy-momentum.'…Nobody can deny or wants to deny that gravitational forces make a contribution to the mass-energy of a gravitationally interacting system. …At issue is not the existence of gravitational energy, but the localizability of gravitational energy. It is not localizable. The equivalence principle forbids.*"

From the argument quoted above, it's clear that these authors believe "disappearing of gravitational energy-momentum" and "disappearing of $\Gamma^{\alpha}_{\mu\nu}$" are the same thing. Let me show by using a counter example, their viewpoint breaks the conservation law of energy-momentum.

When $T^{\alpha\beta}(x)|_p = 0, \forall p \in M, 0 \leq \alpha, \beta \leq 3,$ Minkowski metric $g_{\alpha\beta}(x) = \eta_{\alpha\beta}$ is a solution to Einstein's field equation. The corresponding spacetime is Minkowski space which is flat. There exist inertial coordinate systems. In an inertial coordinate system $\{x^0, x^1, x^2, x^3\}$, matter energy-momentum and gravitational energy-momentum vanish everywhere in the whole spacetime. If switching to a new coordinate system $\{y^0, y^1, y^2, y^3\}$, such that outside 4-dimensional domain

$$\Omega =: \left\{p \in M \,\middle|\, \left||x^{\alpha}(p)\right| < 1, \forall 0 \leq \alpha \leq 3\right\}$$

both coordinate systems coincide; and inside Ω, the latter is curved.



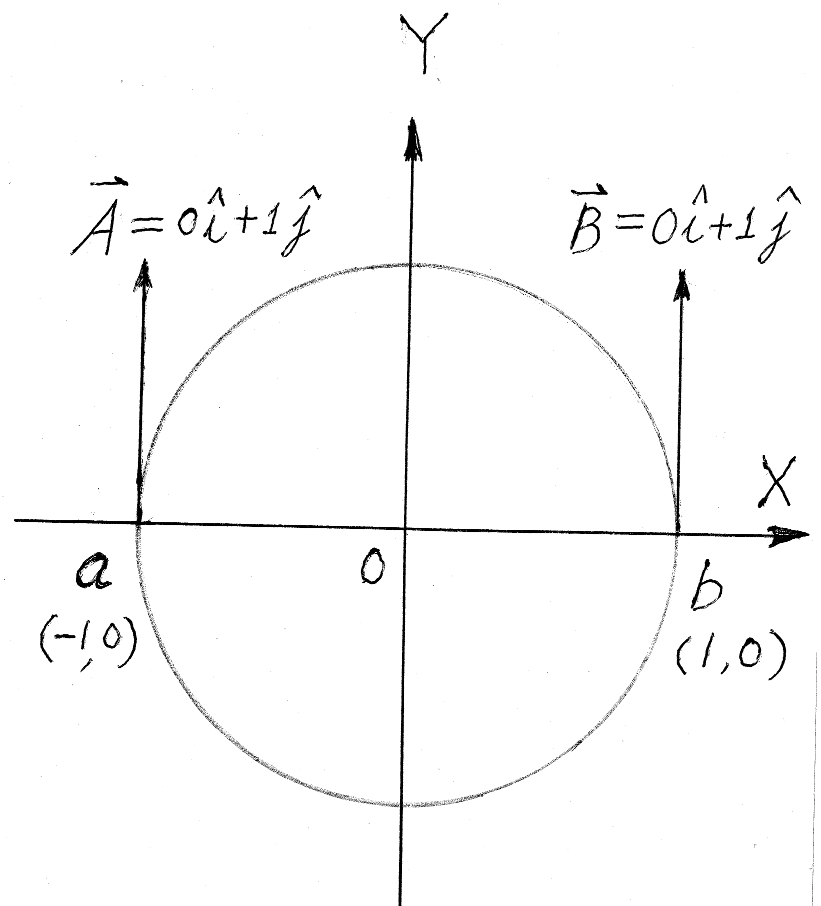
x^0
x^i
0
Ω

In the coordinate system $\{y^0, y^1, y^2, y^3\}$, matter energy-momentum still vanishes in whole spacetime M, and according to the authors of [6], the gravitational energy-momentum would appear in Ω, disappear outside Ω. Where does the gravitational energy-momentum in Ω come from? And where is it gone? Evidently, this breaks the law of energy-momentum conservation.

Some sound facts from geometry

In this section, we will present some sound facts from geometry which are crucial for understanding physics in curved spacetime and are often neglected.

(1) Let us have a look at the following simple example.



In polar coordinate system

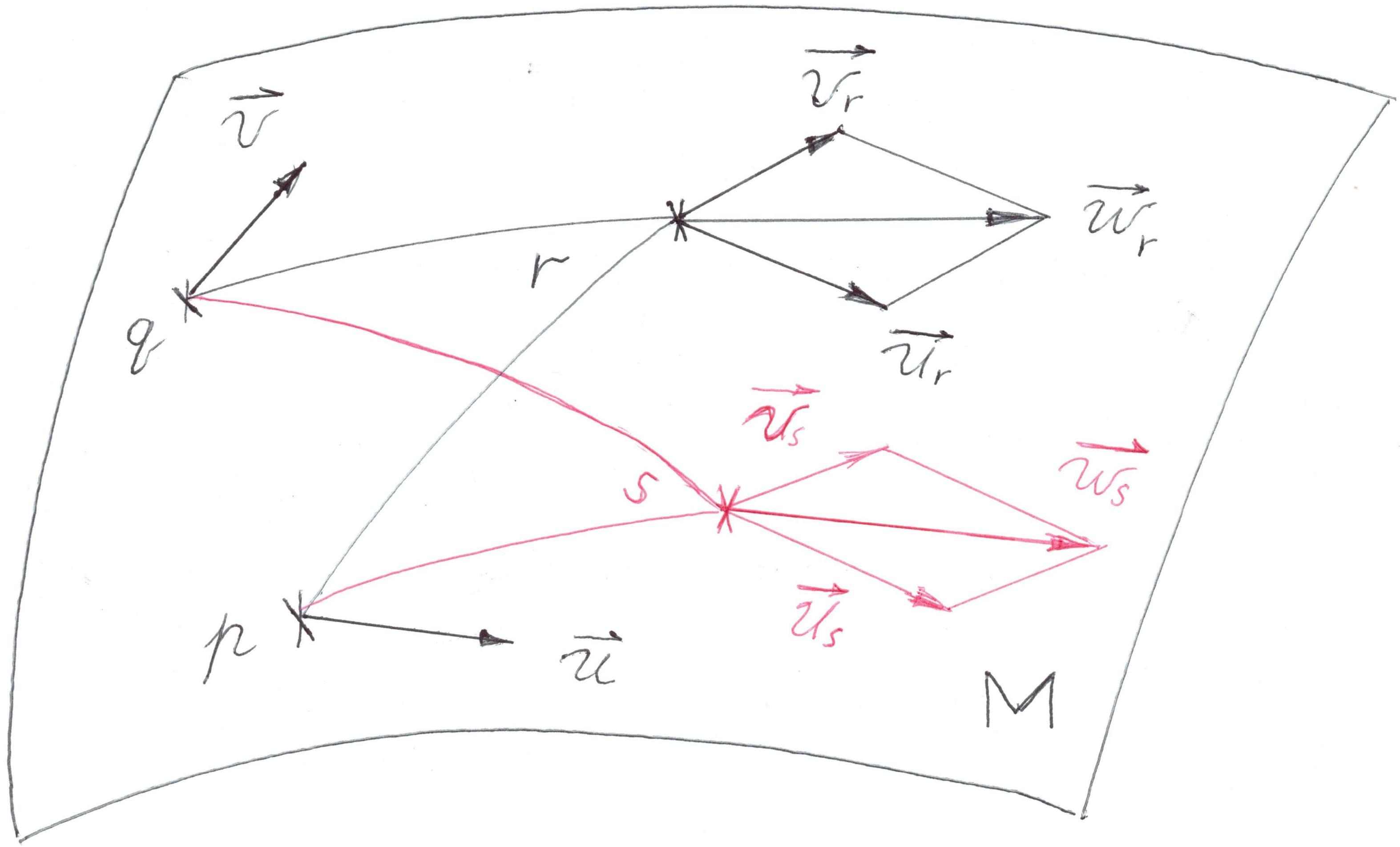
$\vec{A} = 0\,\hat{r}_a - 1\,\hat{\theta}_a, \quad \vec{B} = 0\,\hat{r}_b + 1\,\hat{\theta}_b$

Suppose $a = (-1,0)$, $b = (1,0)$ are two points on Euclidean plane $\mathbb{R}^2$, and there are two vectors $\vec{A}$ at a, and $\overrightarrow{B}$ at b. In the corresponding Descartes coordinate system, $\vec{A} = 0\hat{\imath} + 1\hat{\jmath}$, $\overrightarrow{B} = 0\hat{\imath} + 1\hat{\jmath}$, and $\vec{A} + \overrightarrow{B} = 0\hat{\imath} + 2\hat{\jmath}$.

Switching to polar coordinates, we have $\vec{A} = 0\hat{r}_a - 1\hat{\theta}_a$, $\overrightarrow{B} = 0\hat{r}_b + 1\,\hat{\theta}_b$
however the sum vector $\vec{A} + \overrightarrow{B} \neq 0\hat{r}_p + 0\hat{\theta}_p, \forall p \in \mathbb{R}^2 \backslash \{o\}$.

This shows, even in a flat space, when adopting a curved coordinate system, the sum (or integration) of components with the same index of vectors at different points is not the corresponding component of the sum vector in general. And that is why I said, "when Einstein read $\int_\Sigma d\,s_\lambda(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x)$ as the μ −component of matter energy-momentum on Σ, he was making an elementary mistake".

(2) In the literature of GR, "the mass of a black hole" might be one of the most frequently appearing terms. But a black hole is a huge celestial body in an extremely curved spacetime region. Mass is not a scalar, but the 0-component of energy-momentum 4-vector. Geometry does not allow adding up vectors distributed at different points of a curved spacetime to get a sum vector. Even the energy-momentum of the simple physical system composed of two uncharged mass points does not make sense, let alone "the energy-momentum of a black hole". (See [16,17], or the following)



$\vec{v}$
$\vec{v}_r$
$\vec{w}_r$
$\vec{u}_r$
q
r
$\vec{v}_s$
$\vec{w}_s$
s
$\vec{u}_s$
p
$\vec{u}$
M

Denote by M the spacetime manifold in GR. "M is curved" means "parallel translation of a vector depends on path".
Suppose $p, q \in M$ ($p \neq q$), ($hence\, T_p \neq T_q$), & $U \in T_p, V \in T_q$. No one can add up vectors belonging to different vector spaces. In order to add U, V up, one has to parallelly transport them to same point, say $r \in M$. The transported vectors U_r and V_r now belong to the same tangent space T_r , so one can add them up to get a sum vector $U_r + V_r =: W_r \in T_r$.

But the spacetime M is curved, parallel translation of a vector depends on path. To avoid the ambiguity, one might suggest parallelly transporting U, V to r along geodesics.

But, even so, when one chooses a different point $s \in M$ ($s \neq r$), and parallelly transport U and V along geodesics to s, one gets a sum vector there, $U_s + V_s =: W_s \in T_s$. However, when parallelly transporting W_r along the geodesic to s, the resulting vector is not W_s in general, because geodesics from p (q) to r, plus geodesics from r to s is not the geodesics from p (q) to s in general. We see, one cannot define the sum of vectors at different points in a curved spacetime.

(3) One might ask, "In a flat spacetime, vectors at different points literally belong to different tangent spaces too, why can you add them up?"

In fact, when spacetime M is flat, let $\Phi =: \cup_{p \in M} T_p$, and define a binary relation ~ in Φ, such that, for any $p, q \in M$, and $U \in T_p$, $V \in T_q$, we say $U \sim V$, if and only if parallelly transporting U from p to q will result in V. Because M is flat, parallel translation of vectors does not depend on path, therefore ~ is not ambiguous. It is easy to see that ~ is an equivalence relation and each equivalence class contains one and only one representative in every tangent space of M. Denote the equivalence class containing U by $\underline{U}$ and the quotient set by $\underline{\Phi}$. Because parallel transportation keeps linear relation unchanged, we can define addition and multiplication with real numbers in $\underline{\Phi}$ such that, for any $\underline{U}, \underline{V} \in \underline{\Phi}$, and $\alpha \in \mathbb{R}$, choose a point $p \in M$, and denote by U' the only vector in $T_p \cap \underline{U}$, denote by V' the only vector in $T_p \cap \underline{V}$, then, let $\underline{U} + \underline{V} = \underline{U' + V'}$, and $\alpha\underline{U} = \underline{\alpha U'}$. It's easy to check, these definitions are independent of the choices of p. With these induced operations, $\underline{\Phi}$ is a real vector space. Because parallel translation in generalized Riemannian spaces also keeps scalar product, we can define scalar product in $\underline{\Phi}$ such that, for any $\underline{U}, \underline{V} \in \underline{\Phi}$, choose a point $p \in M$, and denote by U' the only vector in $T_p \cap \underline{U}$, denote by V' the only vector in $T_p \cap \underline{V}$, then, let $\underline{U} \cdot \underline{V} = U' \cdot V'$. This definition is independent of the choice of p. And now $\underline{\Phi}$ is a real scalar product space. For any $p \in M$, the map $U(\in T_p) \mapsto \underline{U} \, (\in \underline{\Phi})$ is an isometric isomorphism of T_p onto $\underline{\Phi}$. This $\underline{\Phi}$ is the so-called scalar product space of free vectors. In the case of a flat spacetime M, when people talk about the sum vector $U + V$ of two tangent vectors $U(\in T_{\square})$ and $V(\in T_q)$, they actually mean $\underline{U} + \underline{V}$ in the scalar product space of free vectors $\underline{\Phi}$. It is clear, when spacetime M is curved, we can not talk about free vectors.

(4) In a curved spacetime M, even the change of energy-momentum 4-vector of an uncharged mass point is not self-evident. Unfortunately, nobody has given it a definition in the literature of GR, as long as I know. (See [16,17] or the following)

Suppose the particle's world line is γ: $\Delta \rightarrow M$, where $\Delta =: [\tau_i, \tau_f]$, $\tau_i(\tau_f)$ is the proper time when the particle is created (annihilated), or $\Delta =: [\tau_i , +\infty)$, $\Delta = : (-\infty, \tau_f]$, $\Delta =: (-\infty, +\infty)$, if its lifespan is infinite. The particle's energy-momentum
4-vectors at proper times $\tau \in \Delta$, $p(\tau)$ is the tangent vector to the world
line at point $\gamma(\tau)$ times its rest mass, $p(\tau) \in T_{\gamma(\tau)}$. When $\tau_1 , \tau_2 \in \Delta$ $(\tau_1 \neq \tau_2)$, $p(\tau_1)$, $p(\tau_2)$ belong to different tangent spaces. We cannot subtract one from the other. For τ_0 , $\tau \in \Delta$, denote by p_{τ_0} (τ) the vector obtained by parallelly transporting $p(\tau 0) \in T_{\gamma(\tau_0)}$ along the world line from $\gamma(\tau_0)$ to $\gamma(\tau)$, p_{τ_0} $(\tau) \in T_{\gamma(\tau)}$.

The change of a particle's energy-momentum 4-vector from proper time τ_1 to τ_2 $(\tau_1 , \tau_2 \in \Delta)$ is a vector field defined only on its world line, such that

$$\delta_{\tau_1 ,\tau_2} p\colon \Delta \rightarrow \cup \tau \in \Delta \; T_{\gamma(\tau)} \qquad (11)$$

$$\delta_{\tau_1 \tau_2}\, p(\tau) =: p_{\tau_2}(\tau) - p_{\tau_1}(\tau) \in T_{\gamma(\tau)}, \; \forall \tau \in \Delta \qquad (12)$$

It is easy to check,

$$\delta_{\tau_1\tau_2}\, p + \delta_{\tau_1\tau_2}\, p = \delta_{\tau_1\tau_3}\, p, \forall\, \tau_1, \tau_2, \tau_3 \in \Delta \qquad (13)$$

That is, the change of a particle's energy-momentum 4-vector from τ_1 to τ_2 plus the change from τ_2 to τ_3, equals the change from τ_1 to τ_3. This is exactly what we expect, but it is not trivial. For, should we define the change of a particle's energy-momentum 4-vector from proper time τ_1 to τ_2 as

$$\delta_{\tau_1\tau_2}\, p(\tau) =: \tilde{p}_{\tau_2}\,(\tau) - \tilde{p}_{\tau_1}\,(\tau) \in T_{\gamma(\tau)}\,, \forall\tau \in \Delta \qquad (14)$$

where $\tilde{p}_{\tau_0}\,(\tau)$ is the vector obtained by parallelly transporting $p(\tau_0)$ along the geodesic from $\gamma(\tau_0)$ to $\gamma(\tau)$, the above self-consistency (13) would fail. Therefore, if we wish to talk about the change of a particle's energy-momentum 4-vector in curved spacetime, (11)+(12) is the only reasonable definition. It is worth noting, this definition does not depend on coordinates.

(6) In (2) of this section, we showed that one can not define the sum of vectors at different points in a curved spacetime M. However, when restricted to a sufficiently small region $\Delta\Omega \sqsubset M$, we can define the sum of vectors at different points in $\Delta\Omega$ when neglecting higher order infinitesimals.

In fact, spacetime M is a generalized Riemannian manifold. Suppose $p \in M$ and $\{X, x\}$ is a compatible coordinate chart of M, such that $p \in X$, and $g_{\alpha\beta}(x)|_p = \eta_{\alpha\beta}$, $\partial_\gamma g_{\alpha\beta}(\mathrm{x})|_p = 0$, where $\eta_{\alpha\beta}$'s are components of Minkowskian metric tensor. Then we call $\{X, x\}$ a local inertial coordinate chart of p. If $p \in \Delta\Omega \sqsubset \mathrm{X}$, and $\Delta\Omega$ is a sufficient small neighborhood of p, then all Christoffel symbols in $\Delta\Omega$ $\Gamma^\alpha_{\beta\gamma}(x)'s$ are infinitesimals. The equations of parallel translation of vector in $\Delta\Omega$ is

$$dv^\alpha(x) = -\Gamma^\alpha_{\beta\gamma}(x)v^\beta(x)dx^\gamma, \forall\alpha = 0,1,2,3, \quad or$$
$$d\,v^\alpha(x) \approx 0, \forall\alpha = 0,1,2,3,$$

when neglecting higher order infinitesimal.

The last equation tells us that parallel transporting a vector in $\Delta\Omega$ means keeping its components in local inertial coordinate system $\{x^0, x^1, x^2, x^3\}$ unchanged, when neglecting higher order infinitesimals. Hence parallel transportation of vector in $\Delta\Omega$ does not depend on path when neglecting higher order infinitesimal. Remembering that parallel translation of vector is a coordinate free concept, we conclude that parallel translation of vector in $\Delta\Omega$ does not depend on path when neglecting higher order infinitesimals, no matter whatever coordinate system you are using. Therefore, when restricted to sufficiently small region $\Delta\Omega \sqsubset M$, we can define the sum of vectors at different points in $\Delta\Omega$ when neglecting higher order infinitesimal

Therefore, when restricted to sufficiently small region $\Delta\Omega \sqsubset M$, we can define the sum of vectors at different points in $\Delta\Omega$, when neglecting higher order infinitesimals. The above result for vectors is also good for (r, s)-tensors.

The concept of flux density tensor field J of an (r, s)-tensor Q relates to addition of (r, s)-tensors at different points. Does it make sense in a curved spacetime? The answer is "yes", because it only relates to addition of (r, s)-tensors at different points in an infinitesimal neighborhood. In particular, the flux density tensor field T of matter energy-momentum P in curved spacetime is meaningful and measurable. $T^{\alpha\beta}(y)\sqrt{-|g(y)|}ds_{\alpha}(y)|_{\Delta\Sigma}$ is the β-component of matter energy-momentum on small spacelike hyper-surface $\Delta\Sigma$, $T^{\alpha\beta}(y)\sqrt{-|g(y)|}ds_{\alpha}(y)|_{\Delta\Gamma}$ is the β-component of matter energy-momentum passing through small timelike hyper-surface $\Delta\Gamma$.

We are now in a position to explore the meaning of conservation and non-conservation in curved spacetime.

§3. Meaning of conservation, non-conservation in curved spacetime

- What is the meaning of $\nabla_\alpha T^{\alpha\beta}(x) = 0$?
- Suppose $\{X, x\}$ is a local inertial coordinate chart of p, $p \in \Delta\Omega \sqsubset$ X, and $\Delta\Omega$ is a sufficiently small neighborhood of p. From $\nabla_\alpha T^{\alpha\beta}(x) = 0$, we have
- $\int_{\Delta\Omega} d^4x\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x) = \int_{\partial\Delta} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) +$
- $\int_{\Delta\Omega} d^4x\sqrt{-|g(x)|}\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0$, $\forall\ \mu = 0, 1, 2, 3$
- Neglecting higher order infinitesimal, we have

  $\int_{\partial\Delta\Omega} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) = 0$, that is
- $\left\{\int \quad - \int \quad - \int \quad \right\} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) = 0$, $\forall\ \mu = 0, 1, 2, 3$

What is the meaning of $\nabla_\alpha T^{\alpha\beta}(x) = 0$?

Suppose $\{X, x\}$ is a local inertial coordinate chart of p, $p \in \Delta\Omega \sqsubset X$, and $\Delta\Omega$ is a sufficiently small neighborhood of p. From $\nabla_\alpha T^{\alpha\beta}(x) = 0$, we have

$$\int_{\Delta\Omega} d^4x\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x) = \int_{\partial\Delta\Omega} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x) + \int_{\Delta\Omega} d^4x\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0,\ \forall\ \mu = 0, 1, 2, 3$$

Neglecting higher order infinitesimal, we have $\int_{\partial\Delta\Omega} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)=0$, that is

$$\left\{\int_{\Delta\Sigma'} - \int_{\Delta\Sigma} - \int_{\Delta\Gamma}\right\} ds_\lambda(x)\sqrt{-|g(x)|}T^{\lambda\mu}(x)=0,\ \forall\ \mu = 0, 1, 2, 3$$

We see, $\nabla_\alpha T^{\alpha\beta}(x) = 0$ means, "for a sufficiently small neighborhood, in a local inertial coordinate system, when neglecting higher order infinitesimals, the μ-component of matter energy-momentum on $\Delta\Sigma'$, minus the μ-component of matter energy-momentum on $\Delta\Sigma'$ and the μ-component of matter energy-momentum passing through $\Delta\Gamma$ equals zero, or, the μ-component of the net increase of matter energy-momentum in a sufficiently small neighborhood is zero".

Or equivalently,

“for a sufficiently small neighborhood, when neglecting higher order infinitesimals, the matter energy-momentum on $\Delta\Sigma'$, minus the matter energy-momentum on $\Delta\Sigma'$ and the matter energy-momentum passing through $\Delta\Gamma$ equals zero, or, the net increase of matter energy-momentum in a sufficiently small neighborhood is zero”.

This is a coordinate free conclusion. It is also good for any (r,s)-tensor Q.

The conservation law for an (r,s)-tensor *Q is*

$$\nabla_\sigma J^{\sigma\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}(\mathrm{x})=0,\ \forall 0\leq \alpha_i, \beta_j \leq 3 \qquad (19)$$

It reads "the increase of (r, s)-tensor Q in any infinitesimal 4-dimensional spacetime neighborhood is zero." When and only when $r + s = 0$, this is equivalent to continuum integral equation

$$\int_{\partial\Omega} d\, s_\sigma(x)\sqrt{-|g(x)|}J^\sigma(x) = 0 \qquad (20)$$

§6. Conclusion

$$We\ have\ proven\ that$$

$$\nabla_\alpha T^{\alpha\beta}(x)|_p = 0, \quad \forall\beta = 0,1,2,3, \&\, p \in M$$

is the conservation law of matter energy-momentum in GR, and it should be read as the increase of matter energy-momentum in any infinitesimal 4-neighborhood is zero. Introducing gravitational energy-momentum does not save but breaks the law of energy-momentum. Matter and matter exchange energy-momentum with each other, they don't exchange energy-momentum with things which are not matter, including gravitational field. Spacetime metric field does not exchange energy-momentum with matter particles and matter fields. We say it does not carry energy-momentum. In physics, force or interaction always means exchange of energy-momentum. The so-called gravitational field (actually the metric field of spacetime) is not a force field, and gravity is not a natural force. Spacetime metric field is not a special matter field, it is only the geometric aspect of the 4-dimensional moving matter continuum.