Conservation, Non-conservation Laws and Gravitational Energy-momentum in General Relativity

From a Geometric Perspective

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Abstract

The spacetime M in general relativity (GR) is curved. Parallel transportations of vectors in M depend on path. This leads to the fact that vectors distributed on different points in M can not be added up to get a sum vector unambiguously. Geometry does not allow talking about matter energy momentum distributed on (passing through) a finite or infinite spacelike (timelike) hypersurface, does not allow talking about the net increase of matter energy momentum in a finite or infinite 4-demensional spacetime region. Even for a simple physical system consisting of only two uncharged mass points, geometry does not allow talking about its energy momentum. Therefore, it is pressing to explore the meaning of conservation of matter energy momentum in GR from a geometric perspective.

We show, in a curved spacetime M, when limited to an infinitesimal spacetime region, vectors distributed at different points can still be added up to get a sum vector unambiguously, if neglecting higher order infinitesimals. For an (r, s)-tensor Q, denoting by J its flux density (r+1, s)-tensor field, the conservation law of Q in curved spacetime M is "the covariant divergence of J vanishes everywhere". It reads, "the net increase of tensor Q in any infinitesimal 4-dimensional neighborhood is zero".

In particular, "the covariant divergence of T (flux density of matter energy momentum) vanishes everywhere" is the conservation law of matter energy momentum P in GR. Introducing gravitational energy momentum does not save but breaks the law of energy momentum conservation in GR.

Noether's theorem for GR is re-visited. It is shown, all the Noether's conserved currents corresponding to 1-parameter local group of diffeomorphisms of M onto itself generated by smooth vector fields on M are vector fields on M, hence they are conservation laws of scalars. It is also shown, all these Noether's conservation laws together are equivalent to the equations of motion of GR.

§1. Motivation

The law of conservation of energy-momentum is the cornerstone of modern physics. Far beyond the scope of physics, it is the bedrock law of nature. So, when Einstein tried to establish his new theory of gravity, the general relativity (GR), his top priority was to ensure the energy-momentum conservation. However, from the fundamental equations of motion of GR, Einstein field equations

$$R^{\alpha\beta}(x) - \frac{1}{2}R(x)g^{\alpha\beta}(x) = \frac{8\pi G}{c^4}T^{\alpha\beta}(x), \forall \alpha, \beta = 0,1,2,3 \qquad (1)$$

by using contracted Bianchi identity, he obtained

$$\nabla_\alpha T^{\alpha\beta}(x)|_p = 0, \ \ \forall \beta = 0,1,2,3; \ \ p \in M \qquad (2)$$

that is, the covariant divergence of matter energy-momentum flux density tensor field vanishes everywhere in spacetime M. Multiplying it by $\sqrt{-|g(x)|}$, he obtained

$$\frac{\partial}{\partial x^\lambda}\left[\sqrt{-|g(x)|}T^{\lambda\mu}(x)\right] + \sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = 0 \qquad (3)$$

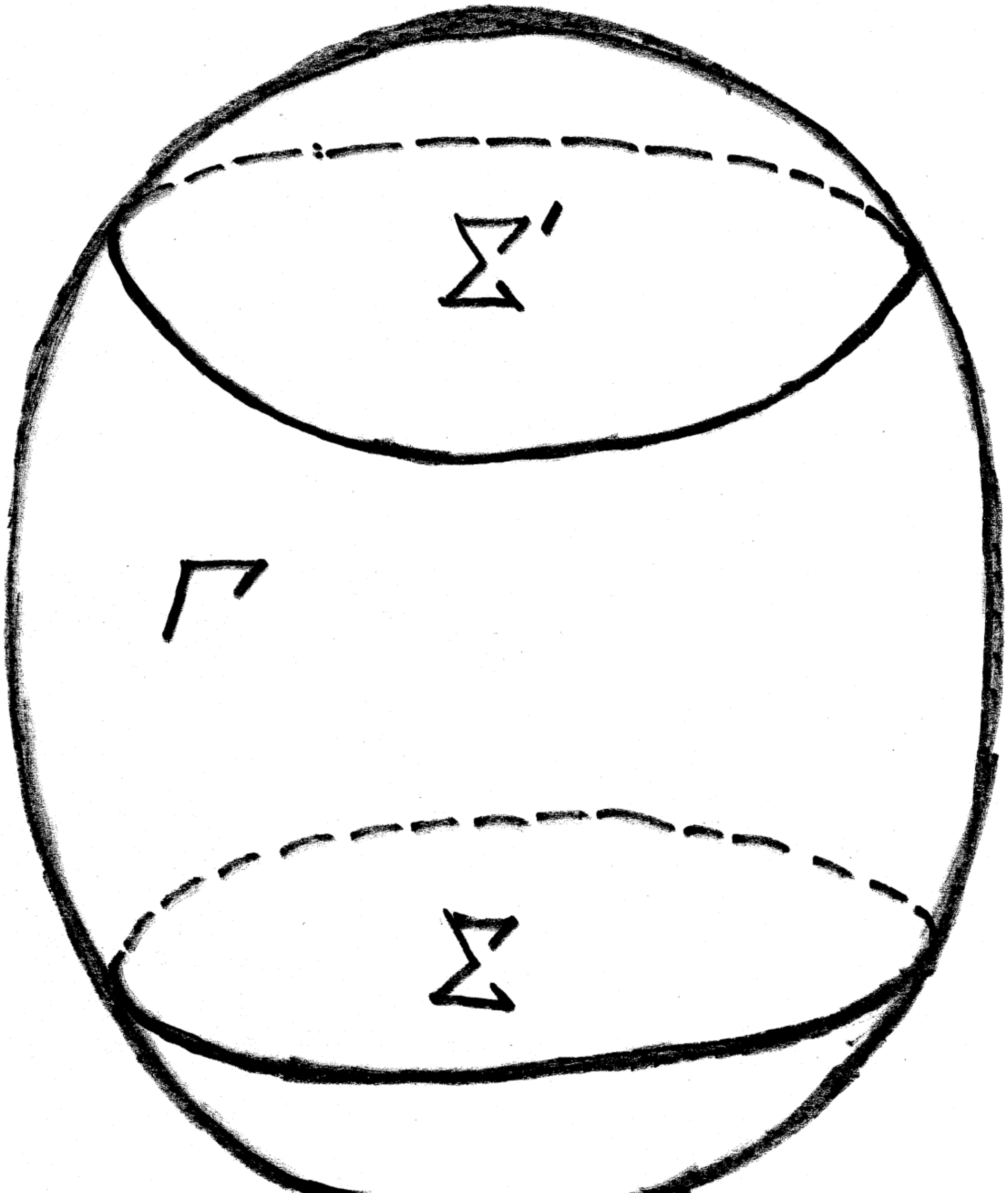
and integrating eqn. (3) over 4-dimensional spacetime domain $\Omega(\subset M)$, he got by using Gaussian theorem

$$\int_\Omega d^4x\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x) =$$

$$\int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x) + \int_\Omega d^4x\sqrt{-|g(x)|}\Gamma^{\mu}_{\lambda\sigma}(x)T^{\lambda\sigma}(x)$$

$$= 0\,, \qquad \forall\, \mu = 0, 1, 2, 3 \qquad (4)$$

When the boundary of Ω is composed of a past spacelike hyper-surface Σ, a future spacelike hyper-surface Σ' and a timelike hyper-surface Γ which links the boundaries of Σ and Σ', eqn. (4) can be written as



$$\partial\Omega = \Sigma \cup \Sigma' \cup \Gamma$$

$$\Omega \subset M$$

$$\left\{\int_{\Sigma'} - \int_{\Sigma} - \int_{\Gamma} \right\} ds_\lambda(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x) =$$

$$-\int_\Omega d^4x\, \sqrt{-|g(x)|}\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x), \quad \forall\mu = 0,1,2,3 \qquad (5)$$

Einstein read this as the μ-component of matter energy-momentum distributed on Σ' minus the μ-component of matter energy-momentum distributed on Σ and the μ-component of matter energy-momentum flowing in through Γ equals the integral at left hand side, which is not zero in general. Therefore, he decided that the vanishing of the covariant divergence of matter energy-momentum flux density is a law of non-conservation of matter energy-momentum. In order to save the law of energy-momentum conservation, Einstein rewrite the second term of eqn.(3) as [1]

$$\sqrt{-|g(x)|}\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x) = \frac{\partial}{\partial x^\lambda}\left[\sqrt{-|g(x)|}t^{\lambda\mu}(x)\right] \qquad (6)$$

where $t^{\lambda\mu}(x)$ is a function of metric field and it's first order derivatives. (Note that, eqn. (6) does not uniquely determine $t^{\lambda\mu}(x)$) Substituting eqn. (6) into eqn. (5), he got

$$\{\int_{\Sigma'} - \int_{\Sigma} - \int_{\Gamma} \} ds_\lambda(x)\sqrt{-|g(x)|}\, [T^{\lambda\mu}(x) + t^{\lambda\mu}(x)] = 0$$

$$\forall\mu = 0,1,2,3 \qquad (7)$$

Now he read the above equations as "the μ-component of matter energy-momentum plus the μ-component of gravitational energy-momentum distributed on Σ' minus the μ-component of matter energy-momentum plus the gravitational energy-momentum distributed on Σ and the μ-component of matter energy-momentum plus the gravitational energy-momentum flowing in through Γ" is zero, or, the net increase of matter energy-momentum in Ω plus the net increase of gravitational energy-momentum in Ω is zero. And $t^{\lambda\mu}(x)$ was read as the flux density of gravitational energy-momentum, $\Gamma^\mu_{\lambda\sigma}(x)T^{\lambda\sigma}(x)$ was read as the μ-

component of the amount of matter energy-momentum which changes into gravitational energy-momentum in per unit 4-dimensional spacetime volume. Einstein was happy to have saved the law of energy-momentum by introducing gravitational energy-momentum.

Here we point out, the spacetime M in GR is curved, there is no flat coordinate system in M, and in a curved coordinate system, the sum (or integration) of components with the same index of vectors distributed at different points is not the corresponding component of their sum vector in general. And we will show later, the sum vector of vectors distributed at different points of curved spacetime does not exist. The expression $T^{\lambda\mu}(x)\sqrt{-|g(x)|}ds_{\lambda}(x)|_{\Delta\Sigma}$ is indeed the μ −component of matter energy-momentum on infinitesimal hyper-surface element ΔΣ. However, when Einstein read $\int_{\Sigma} d\, s_{\lambda}(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x)$ as the μ −component of matter energy-momentum on Σ, he was making an elementary mistake. And it causes century long confusion in GR, due to his supremacy in physics.

H. Bauer immediately pointed out that Einstein's gravitational energy-momentum's current density $t^{\alpha\beta}(x)$ is not a tensor and is not localizable [2]. He showed, when $T^{\lambda\mu}(x)|_{p} = 0, \forall p \in M, 0 \leq \lambda, \mu \leq 3$, Minkowski metric is a solution to Einstein's field equation. There are inertial coordinate systems in Minkowski space. In an inertial coordinate system $\{x^0, x^1, x^2, x^3\}$, $t^{00}(x) \equiv 0$. Switching to spherical coordinate system by a pure spatial coordinate transformation, than $t^{00}(x^0, r, \theta, \varphi) = -\frac{1}{\kappa r^2} \neq 0$, and

$$\iiint_{x^0=const} t^{00}(x^0, r, \theta, \varphi) r^2\, sin\theta dr d\theta d\varphi = -\infty.$$

Einstein's $t^{\alpha\beta}(x)$ is not symmetric. In the 1930's, Landau and Lifshitz proposed a symmetric $t^{\alpha\beta}(x)$ satisfying the following equations [3]

$$\frac{\partial}{\partial x^{\alpha}}\{-|g(x)|[T^{\alpha\beta}(x) + t^{\alpha\beta}(x)]\} = 0, \forall\beta = 0,1,2,3 \qquad (8)$$

which is equivalent to

$$\int_{\Omega} d^4x \frac{\partial}{\partial x^{\alpha}}\{-|g(x)|[T^{\alpha\beta}(x) + t^{\alpha\beta}(x)]\} = 0 \qquad \forall\beta = 0, 1, 2, 3 .$$

Or $$\int_{\partial\Omega} ds_{\alpha}(x)\{-|g(x)|[T^{\alpha\beta}(x) + t^{\alpha\beta}(x)]\} = 0 \qquad \forall\beta = 0, 1, 2, 3 . \qquad (9)$$

However, $ds_\alpha(x)\ [-|g(x)|]$ does not have the correct transformation property under general coordinate transformations for a hyper-surface element. It is a wrong expression, and the correct one is $ds_\alpha(x)\ \left[\sqrt{-|g(x)|}\right]$.

After that, several current densities $t^{\alpha\beta}(x)$ of gravitational energy-momentum were proposed [4,5]. But they are all "pseudo-tensors"

$$t^{\alpha\beta}(x) \neq \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} t^{\mu\nu}(y) \tag{10}$$

and not localizable. Efforts searching for covariant gravitational energy-momentum have never ceased, but have all failed. Some scholars proposed that non-localizability of gravitational energy-momentum is required by the equivalence principle. This viewpoint is generally accepted by relativists. But here I will show it is wrong.

In section 20.4 of their influential book "gravitation" [6], C. Misner, K.S. Thorne, and J.A. Wheeler wrote, *"…One can always find in any given locality a frame of reference in which all local 'gravitational fields' (all Christoffel symbols; all $\Gamma^\alpha_{\mu\nu}$) disappear. No Γ' s means no 'gravitational field' and no local gravitational field means no 'local gravitational energy-momentum.'…Nobody can deny or wants to deny that gravitational forces make a contribution to the mass-energy of a gravitationally interacting system. …At issue is not the existence of gravitational energy, but the localizability of gravitational energy. It is not localizable. The equivalence principle forbids.*"

From the argument quoted above, it's clear that these authors believe "disappearing of gravitational energy-momentum" and "disappearing of $\Gamma^\alpha_{\mu\nu}$" are the same thing. Let me show by using a counter example, their viewpoint breaks the conservation law of energy-momentum.

When $T^{\alpha\beta}(x)|_p = 0, \forall p \in M, 0 \le \alpha, \beta \le 3$, Minkowski metric $g_{\alpha\beta}(x) = \eta_{\alpha\beta}$ is a solution to Einstein's field equation. The corresponding spacetime is Minkowski space which is flat. There exist inertial coordinate systems. In an inertial coordinate system $\{x^0, x^1, x^2, x^3\}$, matter energy-momentum and gravitational energy-momentum vanish everywhere in the whole spacetime. If switching to a new coordinate system $\{y^0, y^1, y^2, y^3\}$, such that outside 4-dimensional domain $\Omega =: \{p \in M \,||x^\alpha(p)| < 1, \forall 0 \le \alpha \le 3\}$, both coordinate systems coincide; and inside Ω, the latter is curved. In the coordinate system $\{y^0, y^1, y^2, y^3\}$, matter energy-momentum still vanishes in whole spacetime M, and according to the authors of [6], the gravitational energy-momentum would appear in Ω, disappear outside Ω. Where does the gravitational energy-momentum in Ω come from? And where is it gone? Evidently, this breaks the law of energy-momentum conservation.

Still believing in the existence of gravitational energy-momentum, some scholars switched to search for the “total gravitational energy-momentum” when spacetime is asymptotically flat at spacelike and light-like infinities [7,8,9]. The proof of mass-energy positivity theorem is considered one of the greatest theoretical achievements in GR [10,11]. For this, Shing -Tung Yau, and Edward Witten were awarded Fields medals. It inspired relativists to search for quasi-local quantities. But this turned out to be surprisingly difficult [12]. As for me, the geometry in [10,11] is great, but not the physics. After all, Bondi mass and ADM mass are not the total gravitational energy.

100 years after Einstein founded his new theory of gravity GR, which predicted the existence of gravitational waves, the first successful direct detection of gravitational wave GW150914 was eventually reported by LIGO [13]. And the Nobel prize in physics 2017 was then awarded to this great contribution. Long before LIGO, ever since 1960's, enormous efforts have been made to detect gravitational waves. Most of them have followed the strategy proposed by R. Feynman at Chapel Hill conference 1957 [14]. For Feynman and his followers, to detect gravitational waves means to detect the energy carried by the passing gravitational waves. This is different from LIGO's strategy, which is to perform a pure geometrical measurement directly on the metric field of spacetime. It is worth noting that all the experiments detecting the energy carried by passing gravitational waves have failed so far, while over the past 7 years more and more successful direct geometrical measurements of gravitational waves have been reported by LIGO and Virgo [15]. Is this huge contrast just an accidental coincident?

While LIGO’s successful measurement of spacetime metric ripple is certainly the greatest experimental breakthrough in the history of GR, there are still fundamental theoretical issues in GR: (i) Whether or not matter energy-momentum conserves? (ii) Is the non-localizable gravitational energy-momentum pseudo-tensor the inevitable consequence of the equivalence principle? (iii) Whether or not gravitational field carries energy-momentum?

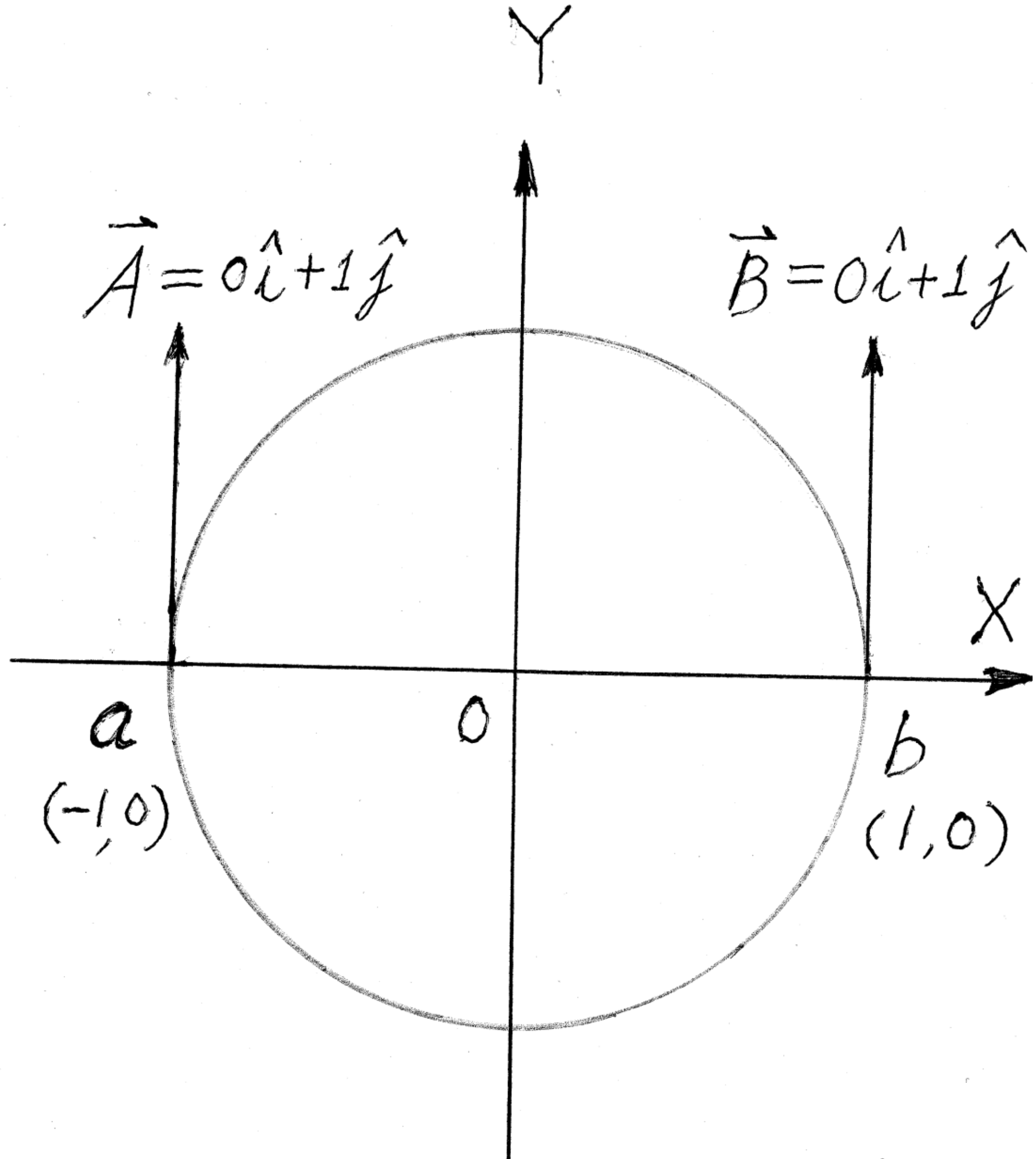
These long existing issues are crucial for cosmology, astrophysics, and whole physics. They must be solved now and can be solved now with the help of modern geometry, which is much more powerful than the classical tensor analysis 100 years ago. The purpose of the present article is to present solutions to these issues from a geometrical perspective.

Our whole argument will be a deduction based on three starting points: variational principle, Einstein-Hilbert action and Riemannian geometry. It does not contain any phenomenological assumption and approximation. It does not contain any revision to fundamental motion equations of Einstein’s GR. The present article can be considered as a revised and enlarged version of my earlier works published and unpublished.

§2. Some sound facts from geometry

In this section, we will present some sound facts from geometry which are crucial for understanding physics in curved spacetime and are often neglected.

(1) Let us have a look at the following simple example.



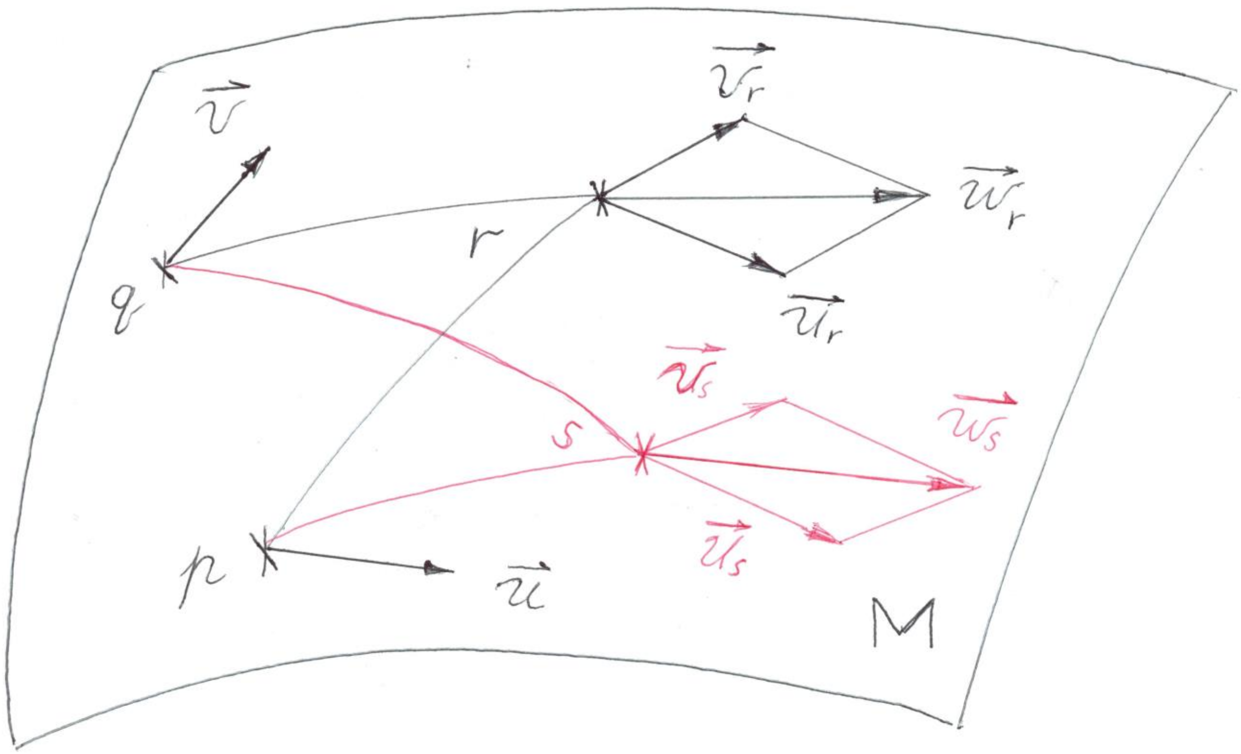
In polar coordinate system

$$\vec{A} = 0\,\hat{r}_a - 1\,\hat{\theta}_a, \qquad \vec{B} = 0\,\hat{r}_b + 1\,\hat{\theta}_b$$

Suppose $a = (-1{,}0), b = (1{,}0)$ are two points on Euclidean plane $\mathbb{R}^2$, and there are two vectors $\vec{A}$ at a, and $\vec{B}$ at b. In the corresponding Descartes coordinate system, $\vec{A} = 0\hat{i} + 1\hat{j}, \vec{B} = 0\hat{i} + 1\hat{j}$, and $\vec{A} + \vec{B} = 0\hat{i} + 2\hat{j}$. Switching to polar coordinates, we have $\vec{A} = 0\hat{r}_a - 1\hat{\theta}_a, \vec{B} = 0\hat{r}_b + 1\hat{\theta}_b$, however the sum vector $\vec{A} + \vec{B} \neq 0\hat{r}_p + 0\hat{\theta}_p, \forall p \in \mathbb{R}^2 \backslash (0{,}0)$.

This shows, even in a flat space, when adopting a curved coordinate system, the sum (or integration) of components with the same index of vectors at different points is not the corresponding component of the sum vector. And that is why I said, "when Einstein read $\int_\Sigma \ d\, s_\lambda(x)\sqrt{-|g(x)|}\, T^{\lambda\mu}(x)$ as the μ −component of matter energy-momentum on Σ, he was making an elementary mistake".

(2) In the literature of GR, "the mass of a black hole" might be one of the most frequently appearing terms. But a black hole is a huge celestial body in an extremely curved spacetime region. Mass is not a scalar, but the 0-component of energy-momentum 4-vector. Geometry does not allow adding up vectors distributed at different points of a curved spacetime to get a sum vector. Even the energy-momentum of the simple physical system composed of two uncharged mass points does not make sense, let alone "the energy-momentum of a black hole". (See [16,17], or the following)



Denote by M the spacetime manifold in GR. "M is curved" means "parallel translation of a vector depends on path". Suppose $p, q \in M$ ($p \neq q$), ($hence\ T_p \neq T_q$) & $U \in T_p, V \in T_q$. No one can add up vectors belonging to different vector spaces. In order to add U, V up, one has to parallelly transport them to same point, say $r \in M$. The transported vectors U_r and V_r now belong to the same tangent space T_r , so one can add them up to get a sum vector $U_r + V_r =: W_r \in T_r$. But the spacetime M is curved, parallel translation of a vector depends on path. To avoid the ambiguity, one might suggest parallelly transporting U, V to r along geodesics. But, even so, when one chooses a different point $s \in M$ ($s \neq r$), and parallelly transport U and V along geodesics to s, one gets a sum vector there, $U_s + V_s =: W_s \in T_s$. However, when parallelly transporting W_r along the geodesic to s, the resulting vector is not W_s in general, because geodesics from p (q) to r, plus geodesics from r to s is not the geodesics from p (q) to s in general. We see, one cannot define the sum of vectors at different points in a curved spacetime.

In general, the sum of (r, s)-tensors distributed at different points of a curved spacetime does not make sense unless r+s=0, because parallel translation of scalars does not depend on path. In a flat spacetime, the parallel translation of (r, s)-tensors does not depend on path, the above ambiguities do not happen. Therefore (r, s)-tensors distributed at different points of a flat spacetime can be added up.

(3) One might ask, "In a flat spacetime, vectors at different points literally belong to different tangent spaces too, why can you add them up?"

In fact, when spacetime M is flat, let $\Phi =: \cup_{p\in M} T_p$, and define a binary relation $\sim$ in Φ, such that, for any $p, q \in M$, and $U \in T_p$, $V \in T_q$, we say $U \sim V$, if and only if parallelly transporting U from p to q will result in V. Because M is flat, parallel translation of vectors does not depend on path, therefore $\sim$ is not ambiguous. It is easy to see that $\sim$ is an equivalence relation and each equivalence class contains one and only one representative in every tangent space of M. Denote the equivalence class containing U by $\underline{U}$ and the quotient set by $\underline{\Phi}$. Because parallel transportation keeps linear relation unchanged, we can define addition and multiplication with real numbers in Φ such that, for any $\underline{U}, \underline{V} \in \underline{\Phi}$, and $\alpha \in \mathbb{R}$, choose a point $p \in M$, and denote by $U'$the only vector in $T_p \cap \underline{U}$, denote by $V'$the only vector in $T_p \cap \underline{V}$, then, let $\underline{U} + \underline{V} = \underline{U' + V'}$, and $\alpha\underline{U} = \underline{\alpha U'}$. It's easy to check, these definitions are independent of the choices of p. With these induced operations, $\underline{\Phi}$ is a real vector space. Because parallel translation in generalized Riemannian spaces also keeps scalar product, we can define scalar product in $\underline{\Phi}$ such that, for any $\underline{U}, \underline{V} \in \underline{\Phi}$, choose a point $p \in M$, and denote by $U'$the only vector in $T_p \cap \underline{U}$, denote by $V'$the only vector in $T_p \cap \underline{V}$, then, let $\underline{U} \cdot \underline{V} = U' \cdot V'$. This definition is independent of the choice of p. And now $\underline{\Phi}$ is a real scalar product space. For any $p \in M$, the map $U(\in T_p) \mapsto \underline{U}(\in \underline{\Phi})$ is an isometric isomorphism of T_p onto $\underline{\Phi}$. This $\underline{\Phi}$ is the so-called scalar product space of free vectors. In the case of a flat spacetime M, when people talk about the sum vector $U + V$ of two tangent vectors $U(\in T_p)$ and $U(\in T_q)$, they actually mean $\underline{U} + \underline{V}$ in the scalar product space of free vectors $\underline{\Phi}$. It is clear, when spacetime M is curved, we can not talk about free vectors.

(4) In a curved spacetime M, even the change of energy-momentum 4-vector of an uncharged mass point is not self-evident. Unfortunately, nobody has given it a definition in the literature of GR, as long as I know. (See [16,17] or the following)

Suppose the particle's world line is $\gamma: \Delta \to M$, where $\Delta =: [\tau_i, \tau_f]$, τ_i (τ_f) is the proper time when the particle is created (annihilated), or $\Delta =: [\tau_i, +\infty)$, $\Delta =: (-\infty, \tau_f]$, $\Delta =: (-\infty, +\infty)$, if its lifespan is infinite. The particle's energy-momentum 4-vectors at proper times $\tau \in \Delta$, $p(\tau)$ is the tangent vector to the world line at point $\gamma(\tau)$ times its rest mass, $p(\tau) \in T_{\gamma(\tau)}$. When $\tau_1, \tau_2 \in \Delta$ ($\tau_1 \neq \tau_2$) , $p(\tau_1), p(\tau_2)$ belong to different tangent spaces. We cannot subtract one from the other. For τ_0 , $\tau \in \Delta$, denote by $p_{\tau_0}(\tau)$ the vector obtained by parallelly transporting $p(\tau_0) \in T_{\gamma(\tau_0)}$ along the world line from $\gamma(\tau_0)$ to $\gamma(\tau)$, $p_{\tau_0}(\tau) \in T_{\gamma(\tau)}$

The change of a particle's energy-momentum 4-vector from proper time τ_1 to τ_2 ($\tau_1,\ \tau_2 \in \Delta$) is a vector field defined only on its world line

$$\delta_{\tau_1\tau_2}p\colon \Delta \to \cup_{\tau\in\Delta}\, T_{\gamma(\tau)} \qquad (11)$$

such that

$$\delta_{\tau_1\tau_2}p(\tau) =: p_{\tau_2}(\tau) - p_{\tau_1}(\tau) \in T_{\gamma(\tau)}, \forall\tau \in \Delta \qquad (12)$$

It is easy to check,

$$\delta_{\tau_1\tau_2}p + \delta_{\tau_2\tau_3}p = \delta_{\tau_1\tau_3}p,\ \forall\, \tau_1, \tau_2, \tau_3 \in \Delta \qquad (13)$$

That is, the change of a particle's energy-momentum 4-vector from τ_1 to τ_2 plus the change from τ_2 to τ_3, equals the change from τ_1 to τ_3. This is exactly what we expect, but it is not trivial. For, should we define the change of a particle's energy-momentum 4-vector from proper time τ_1 to τ_2 as

$$\delta_{\tau_1\tau_2}p(\tau) =: \tilde{p}_{\tau_2}(\tau) - \tilde{p}_{\tau_1}(\tau) \in T_{\gamma(\tau)}, \forall\tau \in \Delta \qquad (14)$$

where $\tilde{p}_{\tau_0}(\tau)$ is the vector obtained by parallelly transporting $p(\tau_0)$ along the geodesic from $\gamma(\tau_0)$ to $\gamma(\tau)$, the above self-consistency (13) would fail. Therefore, if we wish to talk about the change of a particle's energy-momentum 4-vector in curved spacetime, (11)+(12) is the only reasonable definition. It is worth noting, this definition does not depend on coordinates.

(5) Both mathematics and physics pursuit objective truth. And that's why a good concept in linear algebra does not depend on the choice of basis of the linear space, all the good concepts in geometry, such as tangent vectors, cotangent vectors on a smooth manifold, parallel translation of vectors, etc. do not depend on coordinates, and any good concept in physics does not depend on reference coordinate system. The difference between math and physics is, "a branch of math, such as Euclidean geometry, can be any consistent logic system, and its concepts need not refer to real things, and need not be measurable experimentally; while a branch of theoretical physics, such as GR, is a consistent logic system too, but all its concepts must refer to real things in nature, and must be experimentally measurable".

The following expressions are frequently seen in the literature of GR

$$\int_\Omega d^4x\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x), \qquad \int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x).$$

But they are highly dependent on coordinates. In modern geometry, the integrand of an integral on l-dimensional manifold is an l-dimensional form field to ensure that the integral is independent of coordinates. $\because\ d^4x\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x)$ is not a 4-form field, $\therefore \int_\Omega d^4x\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x)$ and $\int_\Omega d^4y\sqrt{-|g(y)|}\,\nabla_\lambda T^{\lambda\mu}(y)$ do not refer to the same geometric, physical object. $\because ds_\lambda(x)\sqrt{-|g(x)|}\,\nabla_\lambda T^{\lambda\mu}(x)$ is not a 3-form field, $\therefore \int_{\partial\Omega} ds_\lambda(x)\sqrt{-|g(x)|}\,T^{\lambda\mu}(x)$ and $\int_{\partial\Omega} ds_\lambda(y)\sqrt{-|g(y)|}\,T^{\lambda\mu}(y)$ do not refer to the same geometric, physical object.

Neglecting this would cause mistaking different geometric, physical objects as one and the same.

(6) In (2) of this section, we show that one can not define the sum of vectors at different points in a curved spacetime M. However, when restricted to sufficiently small region $\Delta\Omega \sqsubset M$, we can define the sum of vectors at different points in $\Delta\Omega$ when neglecting higher order infinitesimal.

In fact, spacetime M *is* a generalized Riemannian manifold. Suppose $p \in M$ and $\{X, x\}$ is a compatible coordinate chart of M, such that $p \in X$, and $g_{\alpha\beta}(x)|_p = \eta_{\alpha\beta}$, $\frac{\partial}{\partial x^\gamma} g_{\alpha\beta}(x)|_p = 0$, where $\eta_{\alpha\beta}$'s are components of Minkowskian metric tensor. Then we call $\{X, x\}$ a local inertial coordinate chart of p. If $p \in \Delta\Omega \sqsubset$ X, and $\Delta\Omega$ is a sufficient small neighborhood of p, then all Christoffel symbols in $\Delta\Omega$ $\Gamma^\alpha_{\mu\nu}(x)'$s are infinitesimal. The equations of parallel translation of vector in $\Delta\Omega$ is

$$dv^\alpha(x) = -\Gamma^\alpha_{\mu\nu}(x)v^\mu(x)dx^\nu, \forall\alpha = 0,1,2,3$$

$$or \qquad dv^\alpha(x) \approx 0, \forall\alpha = 0,1,2,3, \qquad (16)$$

neglecting higher order infinitesimal.

The last equation tells us that parallel transporting a vector in $\Delta\Omega$ means keeping its components in local inertial coordinate system $\{x^0, x^1, x^2, x^3\}$ unchanged, when neglecting higher order infinitesimal. Hence parallel transportation of vector in $\Delta\Omega$ does not depend on path when neglecting higher order infinitesimal. Remembering that parallel translation of vector is a coordinate free concept, we conclude that parallel translation of vector in $\Delta\Omega$ does not depend on path when neglecting higher order infinitesimal, no matter whatever coordinate system you are using.

Therefore, when restricted to sufficiently small region $\Delta\Omega \sqsubset M$, we can define the sum of vectors at different points in $\Delta\Omega$ when neglecting higher order infinitesimal.

The above result for vectors is good for (r, s)-tensors.

The concept of flux density tensor field J of an (r, s)-tensor Q relates to addition of (r, s)-tensors at different points. Does it make sense in a curved spacetime? The answer is "yes", because it only relates to addition of (r, s)-tensors at different points in an infinitesimal neighborhood. In particular, the flux density tensor field T of matter energy-momentum P in curved spacetime is meaningful and measurable. $T^{\alpha\beta}(y)\sqrt{-|g(y)|}ds_\alpha(y)|_{\Delta\Sigma}$ is the β-component of matter energy-momentum on small spacelike hyper-surface ΔΣ, $T^{\alpha\beta}(y)\sqrt{-|g(y)|}ds_\alpha(y)|_{\Delta\Gamma}$ is the β-component of matter energy-momentum passing through small timelike hyper-surface ΔΓ.

We are now in a position to explore the meaning of conservation and non-conservation in curved spacetime.

§3. Meaning of conservation, non-conservation in curved spacetime

What is the meaning of $\nabla_\alpha T^{\alpha\beta}(x)=0$?

Suppose $\{X, x\}$ is a local inertial coordinate chart of p, $p \in \Delta\Omega \sqsubset X$, and ΔΩ is a sufficiently small neighborhood of p. From $\nabla_\alpha T^{\alpha\beta}(x)=0$, we have

$$\int_{\Delta\Omega} d^4x\sqrt{-|g(x)|}\nabla_\alpha T^{\alpha\beta}(x)$$

$$= \int_{\partial\Delta\Omega} ds_\alpha(x)\sqrt{-|g(x)|}T^{\alpha\beta}(x) + \int_{\Delta\Omega} d^4x\sqrt{-|g(x)|}\Gamma^\beta_{\alpha\sigma}(x)T^{\alpha\sigma}(x)=0\ ,\ \forall\ \beta = 0, 1, 2, 3 \quad (17)$$

Neglecting higher order infinitesimal, we have

$$\int_{\partial\Delta\Omega} ds_\alpha(x)\sqrt{-|g(x)|}T^{\alpha\beta}(x)=0,$$

that is

$$\left\{\int\ -\int\ -\int\ \right\} ds_\alpha(x)\sqrt{-|g(x)|}T^{\alpha\beta}(x)=0,\ \forall\ \beta = 0, 1, 2, 3 \quad (18)$$

We see, $\nabla_\alpha T^{\alpha\beta}(x) = 0$ means, "in a local inertial coordinate system, for a sufficiently small neighborhood ΔΩ, when neglecting higher order infinitesimals, the β-component of matter energy-momentum on ΔΣ′, minus the β-component of matter energy-momentum on ΔΣ′ and the β-component of matter energy-momentum passing through ΔΓ equals zero, or, the net increase of the β-component of matter energy-momentum in a sufficiently small neighborhood ΔΩ is zero", that is "for a sufficiently small neighborhood ΔΩ, when neglecting higher order infinitesimal, the matter energy-momentum on ΔΣ′, minus the matter energy-momentum on $\Delta\Sigma'$ and the matter energy-momentum passing through $\Delta\Gamma$ equals zero, or, the net increase of matter energy-momentum in a sufficiently small neighborhood $\Delta\Omega$ is zero" From the latter, we see this is a coordinate free proposition.

Therefore $\nabla_\alpha T^{\alpha\beta}(x)= 0$ is the conservation law for matter energy-momentum in curved spacetime. It tells us, matter does not exchange energy-momentum with any thing which is not matter, including gravitational field.

The above argument for matter energy-momentum 4-vector is also good for

(r, s)-tensor Q. The conservation law for Q is the covariant divergence of its flux density (r+1,s)-tensor field vanishes everywhere in spacetime:

$$\nabla_\sigma J^{\sigma\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s}(x)=0, \ \forall 0\leq \alpha_i, \beta_j \leq 3 \qquad (19)$$

It reads "the increase of (r, s)-tensor Q in any infinitesimal 4-dimensional spacetime neighborhood is zero." When and only when $r + s = 0$, this is equivalent to continuum integral equation

$$\int_{\partial\Omega} d\, s_\sigma(x)\sqrt{-|g(x)|}J^\sigma(x) = 0 \qquad (20)$$

§4. What is space-time?

In Einstein's classical theory of general relativity, spacetime M is a 4-dimensional Lorentzian generalized Riemannian manifold. Geometry tells us, a topological manifold becomes a differential manifold when it is given a differential structure; and it further becomes a generalized Riemannian manifold, when it is further given a metric field g. If g is positively definite or negatively definite, we call it a Riemannian manifold. If the signature of g is (p; 1) (p > 0) or (1; q) (q > 0), we call it a Lorentzian generalized Riemannian manifold, or simply a Lorentzian manifold.

(1) Physical realization of the differential structure of space-time

In physics, a chart (U; u) of spacetime manifold M (U is a non-empty open subset of M) is constructed as follows. Placing infinite point-like (small) clocks such that their world lines cover U and do not intersect with each other within U (that is, for any $q \in U$, there is one and only one point-like clock's world line passing through q). Each point-like clock is given three specific ordered real numbers $u^1, u^2, u^3;$, and the clock's numerical reading monotonically changes in its own way. If some event q happens within U, at a point-like clock labeled u^1, u^2, u^3, when the point-like clock's reading is u^0/c, then (u^0, u^1, u^2, u^3) will be the coordinates of this event (spacetime point) q (∈ U).

When we have a collection of charts of M, $\{U(a), u(a) | a \in A\} =: \mathcal{A}'$, A is some set of indices) such that [$\cup_{a\in A} U(a)$ = M, and all its members are c^{∞}-compatible with each other, then the c^{∞}-compatible atlas of M, $\mathcal{A}'$, decides a unique smooth differential structure of M, $\mathcal{A}$, the collection of all charts of M that are c^{∞}-compatible to all members of $\mathcal{A}'$.

(2) Physical measurement of spacetime metric field

Every concept in physics should be measurable. The Lorentzian signatured metric field g of spacetime manifold M is no exception. For any spacetime point $p \in M$, and its neighbourhood coordinate chart $(X;\ x)$, we can measure all the components of g at p, $\{g_{\alpha\beta}(x) | 0 \le \alpha, \beta \le 3\}$, by using several standard clocks, light pulse signal emitters and receivers.

Place a standard clock at the point-like clock labeled x^1, x^2, x^3, before the latter's numerical reading passes through $\frac{x^0}{c}$, write down the standard clocks' numerical readings at p (x^0, x^1, x^2, x^3), τ, and write down the two clocks' numerical readings $\tau + \Delta\tau, \frac{x^0+\Delta x^0}{c}$ again shortly.

Because a standard clock tells its proper time, we have (neglecting higher order infinitesimals)

$$-c^2(\Delta\tau)^2 = g_{\alpha\beta}(x)\Delta x^{\alpha}\Delta x^{\beta} = g_{00}(x)\Delta x^0\Delta x^0 \tag{21}$$

hence

$$g_{00}(x) = -c^2(\Delta\tau)^2/(\Delta x^0)^2 \tag{22}$$

In order to determine the other components, emit a light pulse signal at p. When this signal arrives at point-like clock labeled $x^1 + \Delta x^1, x^2, x^3$ $(x^1 - \Delta x^1, x^2, x^3)$, write down the clock's numerical reading $x^0/_c + \xi$ ($x^0/_c + \xi$'). Because the world lines of light pulse signals are light-like, we have (neglecting higher order infinitesimals)

$$0 = g_{00}(x)c^2\xi^2 + 2g_{01}(x)c\xi\Delta x^1 + g_{11}(x)(\Delta x^1)^2 \tag{23}$$

$$0 = g_{00}(x)c^2\xi'^2 - 2g_{01}(x)c\xi'\Delta x^1 + g_{11}(x)(\Delta x^1)^2 \tag{24}$$

Subtracting eqn. (24) from eqn. (23) gives

$$g_{01}(x) = \frac{c(\xi'-\xi)}{2\Delta x^1} g_{00}(x) = -\frac{c(\xi'-\xi)}{2\Delta x^1} c^2(\Delta\tau)^2/(\Delta x^0)^2 \tag{25}$$

$$g_{11}(x) = -\frac{c^2\xi'\xi}{(\Delta x^1)^2} g_{00}(x) = \frac{c^2\xi'\xi}{(\Delta x^1)^2}/(\Delta x^0)^2 \tag{26}$$

Similarly, if when the above-mentioned light signal arrives at point-like clock labeled $x^1, x^2 + \Delta x^2, x^3$ $(x^1, x^2 - \Delta x^2, x^3)$ $\{x^1, x^2, x^3 + \Delta x^3$ $(x^1, x^2, x^3 - \Delta x^3)\ \}$ The numerical reading is $x^0/_c + \eta, (\ x^0/_c + \eta')\left\{\ x^0/_c + \zeta\ , (x^0/_c + \zeta')\right\}$, then we have

$$g_{02}(x) = \frac{c(\eta'-\eta)}{2\Delta x^2} g_{00}(x) = -\frac{c(\eta'-\eta)}{2\Delta x^2} c^2(\Delta\tau)^2/(\Delta x^0)^2 \tag{27}$$

$$g_{22}(x) = -\frac{c^2\eta'\eta}{(\Delta x^2)^2} g_{00}(x) = \frac{c^2\eta'\eta}{(\Delta x^2)^2}/(\Delta x^0)^2 \tag{28}$$

$$g_{03}(x) = \frac{c(\zeta'-\zeta)}{2\Delta x^3} g_{00}(x) = -\frac{c(\zeta'-\zeta)}{2\Delta x^3} c^2(\Delta\tau)^2/(\Delta x^0)^2 \tag{29}$$

$$g_{33}(x) = -\frac{c^2\zeta'\zeta}{(\Delta x^3)^2} g_{00}(x) = \frac{c^2\zeta'\zeta}{(\Delta x^3)^2}/(\Delta x^0)^2 \tag{30}$$

Let us determine the rest. If when the above-mentioned signal arrives at point-like clock labeled $x^1 + \Delta x^1, x^2 + \Delta x^2, x^3$ $(x^1, x^2 + \Delta x^2, x^3 + \Delta x^3; x^1 + \Delta x^1, x^2, x^3 + \Delta x^3)$, the clock's numerical reading is $x^0/_c + \lambda$ $(x^0/_c + \mu;\ x^0/_c + \nu)$, then we have

$$\begin{gathered} g_{00}(x)c^2\lambda^2 + 2g_{01}(x)c\lambda\Delta x^1 + 2g_{02}(x)c\lambda\Delta x^2 + g_{11}(x)\Delta x^1\Delta x^1 \\ +2g_{12}(x)\Delta x^1\Delta x^2 + g_{22}(x)\Delta x^2\Delta x^2 = 0 \end{gathered} \tag{31}$$

hence

$$\begin{aligned} g_{12}(x) &= -\frac{c^2}{2\Delta x^1\Delta x^2}[\lambda^2 + \lambda(\xi' - \xi) + \lambda(\eta' - \eta) - \xi'\xi - \eta'\eta]g_{00}(x) \\ &= \frac{c^2}{2\Delta x^1\Delta x^2}[\lambda^2 + \lambda(\xi' - \xi) + \lambda(\eta' - \eta) - \xi'\xi - \eta'\eta]\frac{c^2\tau^2}{(\Delta x^0)^2} \end{aligned} \tag{32}$$

Similarly, we have

$$\begin{aligned} g_{23}(x) &= -\frac{c^2}{2\Delta x^2\Delta x^3}[\mu^2 + \mu(\eta' - \eta) + \mu(\zeta' - \zeta) - \eta'\eta - \zeta'\zeta]g_{00}(x) \\ &= \frac{c^2}{2\Delta x^2\Delta x^3}[\mu^2 + \mu(\eta' - \eta) + \mu(\zeta' - \zeta) - \eta'\eta - \zeta'\zeta]\frac{c^2\tau^2}{(\Delta x^0)^2} \end{aligned} \tag{33}$$

$$\begin{aligned} g_{13}(x) &= -\frac{c^2}{2\Delta x^1\Delta x^3}[\nu^2 + \nu(\xi' - \xi) + \nu(\eta' - \eta) - \xi'\xi - \eta'\eta]g_{00}(x) \\ &= \frac{c^2}{2\Delta x^1\Delta x^2}[\nu^2 + \nu(\xi' - \xi) + \nu(\eta' - \eta) - \xi'\xi - \eta'\eta]\frac{c^2\tau^2}{(\Delta x^0)^2} \end{aligned} \tag{34}$$

Thus, for any spacetime point $p \in M$, we can determine the spacetime metric at p, $g|_p$, by using the above-mentioned finite measurement results. These measurement results are the least measurement results needed for determining

$g|_p$. Note that eqns. (21) through (34) are inhomogeneous linear equations for the 10 unknowns $\{g_{\alpha\beta}(x)|0 \le \alpha \le \beta \le 3.$ Because eqn. (21) ⟺ eqn. (22), eqn. (23) and eqn. (24) ⟺ eqn. (25) and eqn. (26), eqn. (31) ⟺ eqn. (32), there are 10 independent equations.

The procedure for determining g discussed above can be taken as the working definition of space-time

Whether more similar measurement results would lead to contradiction or not? If they would, the fundamental concept of spacetime in GR would be wrong. I do not believe this could happen, because Einstein's spacetime concept is logically so beautiful, and there have been no experimental observation definitely against GR so far. Nevertheless, people can do more measurements to examine GR's spacetime concept. We see, doing more than necessary above-mentioned measurements provides a way to prove whether g (determined in the way described above) is a $(0,2)$-tensor field or not.

In geometry, the metric field contains all geometric information of a generalized Riemannian manifold. Particularly, in GR, the metric field g contains all geometric information of the spacetime. The dynamic variable that describes gravitation in GR is spacetime metric field. Clearly, this metric field g (the so-called gravitational field in GR) is not a new kind of matter field itself, it is the geometric aspect of the moving matter 4-demensional continuum, it is a pure geometric notion.

§6. Conclusion

In a curved spacetime M, the sum (r, s)-tensor Q on a finite or infinite hyper-surface does not make sense unless $r = s = 0$. The conservation law for (r, s)-tensor Q is $\nabla \cdot J = 0$, where J is the current density $(r + 1, s)$-tensor field of Q. $\nabla \cdot J = 0$ reads the increase of Q in any infinitesimal 4-spacetime neighborhood is zero. It is equivalent to the continuum integral equation

$$\int_{\partial\Omega} d\, s_\sigma(x)\sqrt{-|g(x)|}J^\sigma(x) = 0$$

when and only when $r = s = 0$.

In particular, "The covariant divergence of matter energy-momentum vanishes in whole spacetime"

$$\nabla_\alpha T^{\alpha\beta}(x)|_p = 0, \ \ \forall\beta = 0,1,2,3, \& \ p \in M$$

is the conservation law of matter energy-momentum in GR, and it should be read as the net increase of matter energy-momentum in any infinitesimal 4-neighborhood is zero. Introducing gravitational energy-momentum does not save but breaks the law of energy-momentum. Because the conservation law of matter energy-momentum holds in GR, it means matter only exchange energy-momentum with each other, matter does not exchange energy-momentum with things which is not matter including gravitational field. The name we call metric field of spacetime "gravitational field" sounds as if it's a force field, not non-material. But spacetime metric field does not exchange energy-momentum with matter particles and matter fields. We say it does not carry energy-momentum. In physics, force or interaction always means exchange of energy-momentum. The so-called gravitational field (actually the metric field of spacetime) is not a force field, and gravity is not a natural force. Spacetime metric field is not a special material field, it is only the geometric aspect of the 4-demensional moving matter continuum.

Appendix

In (4) of section 2, we defined the concept "change of a matter point's energy-momentum" [eqns. (11)+(12)], it is a coordinate free concept. In section 3, we proved, the conservation law of matter energy-momentum is $\nabla_\alpha T^{\alpha\beta}(x) = 0$. And a freely falling mass point's equation of motion is $\frac{D}{d\tau} p^\rho(\tau) = 0$. Here in this appendix, we will show all the above-mentioned results are consistent with each other.

The last result tells us, a freely falling mass point's world line $\gamma: \mathbb{R} \to M$ is a time-like geodesics, and only a freely falling mass point's world line is a time-like geodesic.

The current density of matter energy-momentum contributed by a mass point is

$$T_m^{\alpha\beta}(x) = \int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau} \frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}} m \frac{dX^\beta(\tau)}{d\tau} \qquad \text{(A1)}$$

where $X^\alpha(\tau) =: X^\alpha[\gamma(\tau)]$, and τ is the proper time.

Eqns. (11)+(12) tells us the mass point's energy-momentum does not change when and only when its world line is a time-like geodesics.

Proposition: $\nabla_\alpha T_m^{\alpha\beta}(x) = 0$, if and only if $x^\alpha = X^\alpha(\tau) =: X^\alpha[\gamma(\tau)]$ is a time-like geodesic.

Proof: $\nabla_\alpha T_m^{\alpha\beta}(x) = \partial_\alpha T_m^{\alpha\beta}(x) + \Gamma^\alpha_{\alpha\rho}(x) T_m^{\rho\beta}(x) + \Gamma^\beta_{\alpha\sigma}(x) T_m^{\alpha\sigma}(x)$

$$= \int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau} \frac{\partial}{\partial x^\alpha}\left[\frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}}\right] m \frac{dX^\beta(\tau)}{d\tau} + \Gamma^\alpha_{\alpha\rho}(x) T_m^{\rho\beta}(x)$$

$$+ \int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau} \frac{\delta^4(x - X(\tau))}{\sqrt{-|g(x)|}} m \Gamma^\beta_{\alpha\sigma}(x) \frac{dX^\sigma(\tau)}{d\tau}$$

$$=:<1>+<2>+<3> \tag{A2}$$

$$<1>=\int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau}\frac{\partial}{\partial x^\alpha}\left[\frac{1}{\sqrt{-|g(x)|}}\right]\delta^4(x-X(\tau))m\frac{dX^\beta(\tau)}{d\tau}$$

$$+\int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau}\frac{1}{\sqrt{-|g(x)|}}\frac{\partial}{\partial x^\alpha}\delta^4(x-X(\tau))m\frac{dX^\beta(\tau)}{d\tau}$$

$$=\int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau}\frac{-1}{\sqrt{-|g(x)|}}\Gamma^\rho_{\rho\alpha}(x)\delta^4(x-X(\tau))m\frac{dX^\beta(\tau)}{d\tau}$$

$$+\int_{-\infty}^{\infty} d\tau \frac{dX^\alpha(\tau)}{d\tau}\frac{-1}{\sqrt{-|g(x)|}}\frac{\partial}{\partial X^\alpha(\tau)}\delta^4(x-X(\tau))m\frac{dX^\beta(\tau)}{d\tau}$$

$$=-\Gamma^\rho_{\rho\alpha}(x)T_m^{\alpha\beta}(x)+\frac{-1}{\sqrt{-|g(x)|}}\int_{-\infty}^{\infty} d\tau\frac{d}{d\tau}\delta^4(x-X(\tau))m\frac{dX^\beta(\tau)}{d\tau}$$

$$=-\Gamma^\rho_{\rho\alpha}(x)T_m^{\alpha\beta}(x)+\frac{-1}{\sqrt{-|g(x)|}}\int_{-\infty}^{\infty} d\tau\frac{d}{d\tau}\left[\delta^4(x-X(\tau))m\frac{dX^\beta(\tau)}{d\tau}\right]$$

$$+\int_{-\infty}^{\infty} d\tau\frac{\delta^4(x-X(\tau))}{\sqrt{-|g(x)|}}m\frac{d^2X^\beta(\tau)}{d\tau^2}$$

$$=-<2>+\int_{-\infty}^{\infty} d\tau\frac{\delta^4(x-X(\tau))}{\sqrt{-|g(x)|}}m\frac{d^2X^\beta(\tau)}{d\tau^2} \tag{A3}$$

$$\therefore \nabla_\alpha T_m^{\alpha\beta}(x)=\int_{-\infty}^{\infty} d\tau\frac{\delta^4(x-X(\tau))}{\sqrt{-|g(x)|}}m\frac{d^2X^\beta(\tau)}{d\tau^2}$$

$$+\int_{-\infty}^{\infty} d\tau\frac{\delta^4(x-X(\tau))}{\sqrt{-|g(x)|}}\Gamma^\beta_{\alpha\sigma}(x)m\frac{dX^\alpha(\tau)}{d\tau}\frac{dX^\sigma(\tau)}{d\tau}$$

$$=\int_{-\infty}^{\infty} d\tau\frac{\delta^4(x-X(\tau))}{\sqrt{-|g(x)|}}\ m\left[\frac{d^2X^\beta(\tau)}{d\tau^2}+\Gamma^\beta_{\alpha\sigma}(x)\frac{dX^\alpha(\tau)}{d\tau}\frac{dX^\sigma(\tau)}{d\tau}\right] \tag{A4}$$

The last equation tells us that if and only if the matter point's world line is a time-like geodesic, we have $\nabla_\alpha T_m^{\alpha\beta}(x)=0$.

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