Characterisation of qutrit universal gates

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Theory Canada 15, June 16, 2023

Outline

- How to estimate the performance of quantum gates?
- Why is randomised benchmarking chosen?
- Universal qutrit randomised benchmarking.

Qutrits

Definition

- Three-level system: span($|0\rangle, |1\rangle, |2\rangle$).

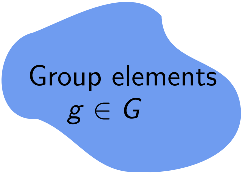
Advantages with respect to qubits

- Larger Hilbert space.
- More natural: avoid the truncation of a native higher level system.

Qutrit experiments [3, 6, 14, 7, 4, 15, 5, 11, 2, 9]

- Superconductors.
- Ion traps.
- Photons (interferometer).
- More.

Gates and groups



Gates and groups

Group elements
$g \in G$

Unitary representation of G
U_g

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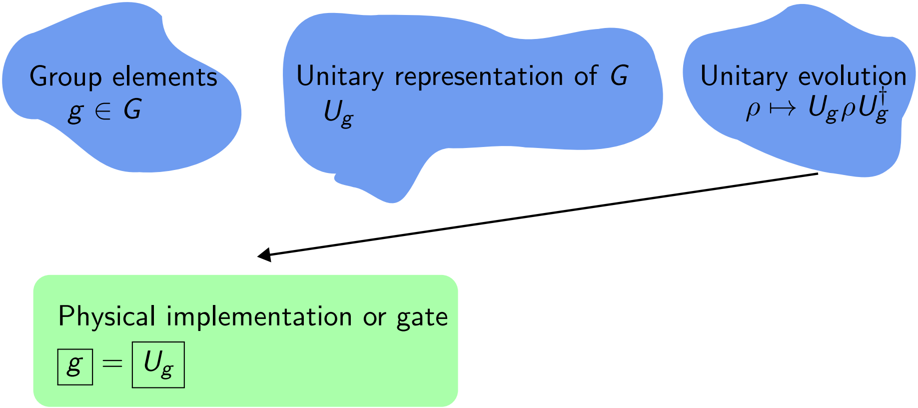
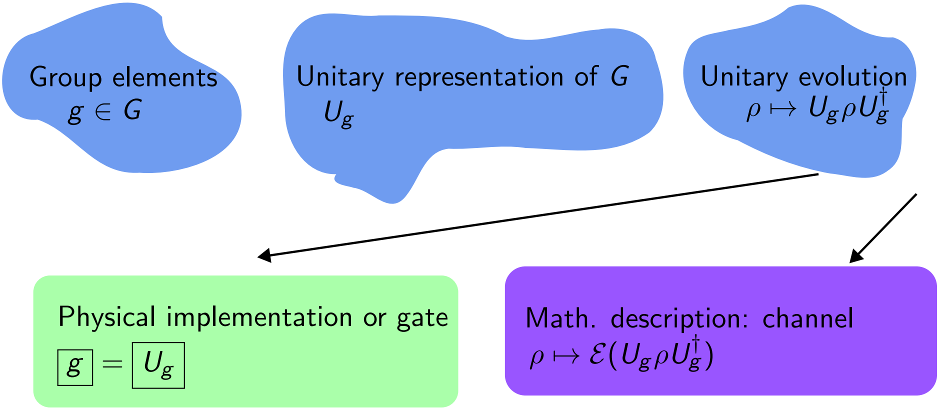
Unitary representation of G
U_g

Unitary evolution
$\rho \mapsto U_g \rho U_g^\dagger$

Gates and groups



Gates and groups



For instance, $\mathcal{E}$ (read caligraphic capital E $\neq$ epsilon) a depolarising channel.

Estimating performance = characterising

Why?

- Estimate the performance to improve the gates.
- Know how many operations we can perform before the computation is purely random.

Why is characterising not easy?

- Stochastic nature of quantum operations.
- Tomography (reconstruction) is expensive.

Fidelity

Fidelity between states

- How similar are two states ρ and σ.
- Defined by $F(\rho, \sigma) := \mathrm{Tr}\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$.

Average gate fidelity of a channel $\mathcal{E}$

- How close is the output of a noisy implementation $\rho \mapsto \mathcal{E}(U\rho U^\dagger)$ to the $\rho \mapsto U\rho U^\dagger$ on average.
- The average is computed over a set of "pure" states.
- Defined as follows. Let X be the set of pure states with probability measure μ.

$$\bar{F} = \int \mathrm{d}\mu F(U\rho U^\dagger, \mathcal{E}(\rho)) = \int \mathrm{d}\mu F(\rho, U^\dagger \mathcal{E}(\rho) U).$$

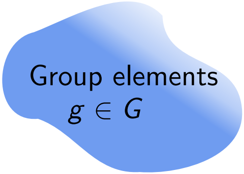
Introduction to Randomised Benchmarking

Summary of randomised benchmarking

- Goal of randomised benchmarking: estimate $\bar{F}$.
- RB requires: a gate set $\mathcal{G}$, preparation of the state $|0\rangle$, a measurement $\langle 0|$.

Group properties

- Composition of two gates is another gate from the gate set.
- Each gate has an inverse operation.



Physical implementation or gate

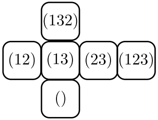
$$\boxed{g} = \boxed{U_g}$$

Process of Randomised Benchmarking

Algorithm [10]

1. For k in set of numbers of gates to compose (circuit depth) (commonly $k \in \{0, \cdots, 200\}$).
2. For N in number of repetitions per k (N commonly ~ 20).
    1. Prepare $|0\rangle$.
    2. Randomly sample k gates and apply them to $|0\rangle$.
    3. Generate the inversion gate (for the sequence) and apply to the state from the previous step.
    4. Measure $\langle 0|$.

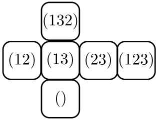
Algorithm: toy example S_3



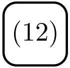
Prepare the state $|0\rangle$

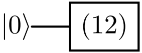
$|0\rangle$

Algorithm: toy example S_3



Roll the dice and apply the corresponding gate





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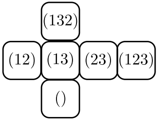
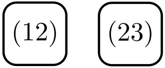
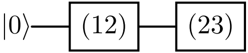


Roll a second time and apply the corresponding gate





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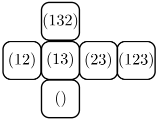
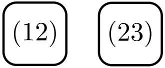
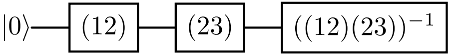
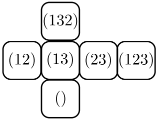


Apply the inversion gate

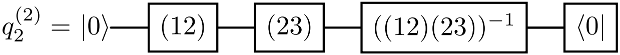




Algorithm: toy example S_3

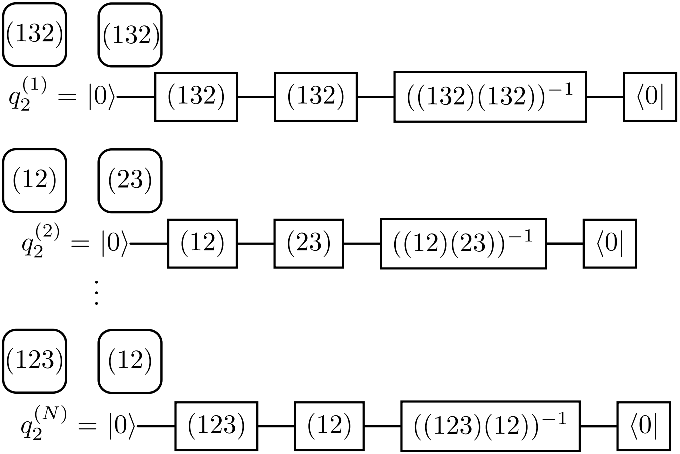


Measure with respect to the ground state $\langle 0|$

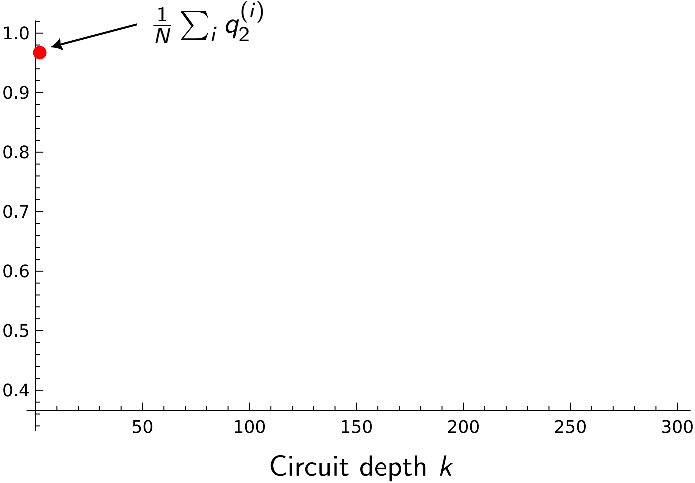


Algorithm: toy example S_3

We repeat the process above N times: sample N sequences of 2 gates



Survival probability curve; Clifford



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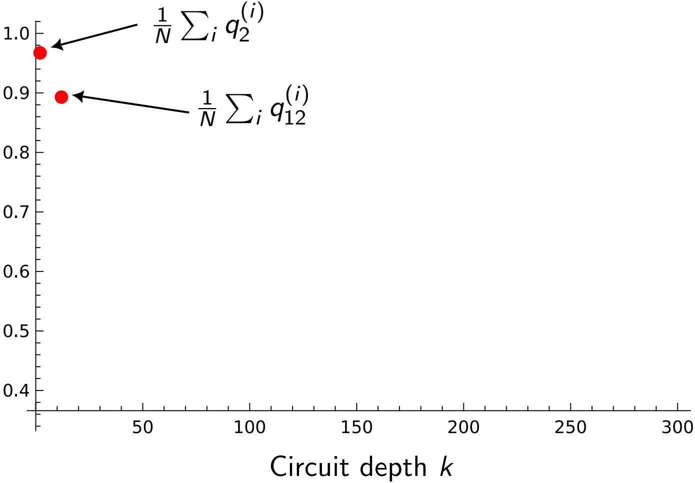

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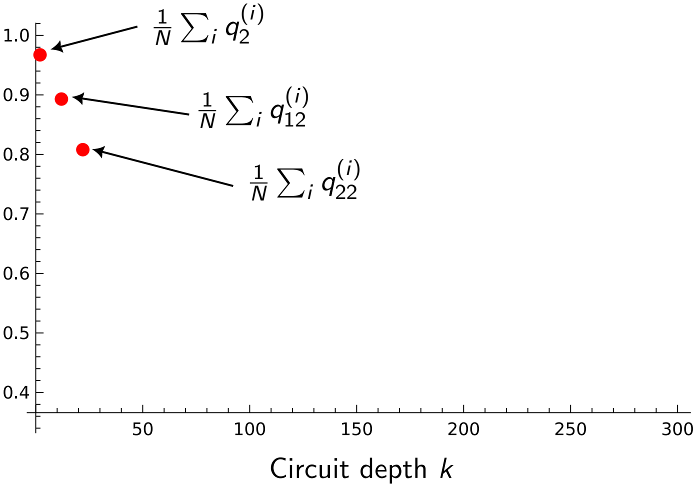
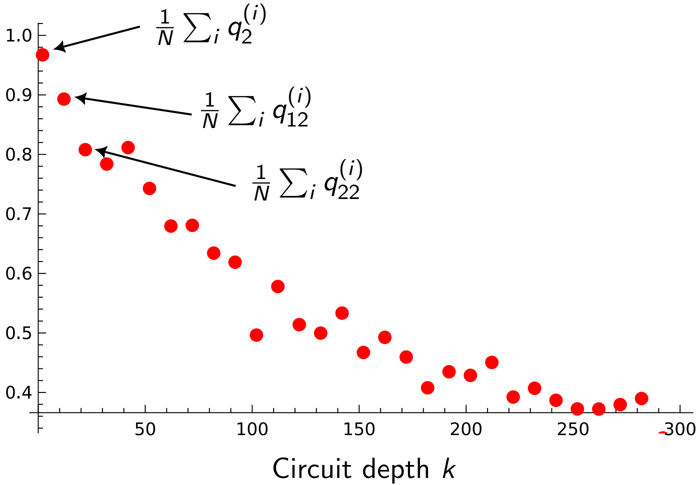


Survival probability curve; Clifford



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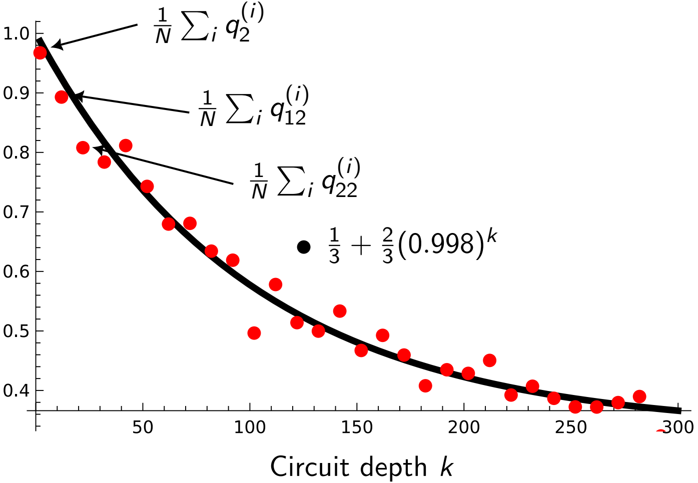
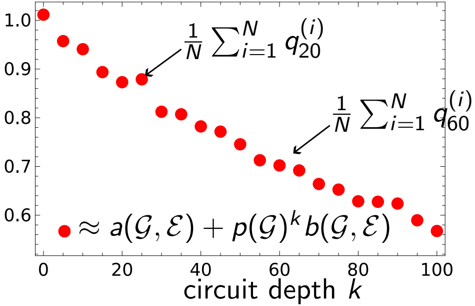


Data-analysis

- Survival probability curve.



- It can be proven with some representation theory results that:

$$\bar{F}(\mathcal{G}) = \frac{3p(\mathcal{G}) - 1}{2}. \tag{1}$$

- Recall that $\bar{F}$ is the average fidelity over a gate set.

Representation theory

PL representation [8]

- Given an orthonormal basis for the set of 3×3 matrices: $\{W_i\}$.
- $\Gamma(\mathcal{E})_{i,j} := 3^{-1/2}\,\mathrm{Tr}(W_i^\dagger \mathcal{E}(W_j))$.
- Γ is a representation.

Irreps

For a unitary irrep γ, $\Gamma \cong \gamma \otimes \bar{\gamma}$, where $\bar{}$ denotes complex conjugate.

Average gate fidelity [12]

For a given channel $\mathcal{E}$,

$$\bar{F}(\mathcal{E}, \mathbb{I}) = \frac{3\,\mathrm{Tr}(\Gamma) + 9}{36}. \tag{2}$$

If the noise $\mathcal{E}$ is the same for each gate-set member $\bar{F}(\mathcal{E}, \mathbb{I}) = \bar{F}(\mathcal{G})$.

Survival probability theoretical curve

It can be shown that:

$$\sum_{\mathbf{g}\in G^k} \langle 0|\, \tilde{U}_{\mathrm{inv}(\mathbf{g})} \tilde{U}_{g_1} \cdots \tilde{U}_{g_k} \,|0\rangle = \langle\!\langle 0|\mathcal{T}(\mathcal{E})^k|0\rangle\!\rangle, \tag{3}$$

where $\tilde{U} := \rho \mapsto \mathcal{E}(U\rho U^\dagger)$ and

$$\mathcal{T}(\mathcal{E}) := \sum_{g\in G} \Gamma(U_g)^\dagger \Gamma(\mathcal{E}) \Gamma(U_g). \tag{4}$$

The matrix $\mathcal{T}$ is known as the twirl of $\mathcal{E}$.

Pros and cons for randomised benchmarking

Pros

- Resistant against state preparation and measurement (SPAM) errors.
- Scales(*) with the number of qutrits.
- Simple data analysis.

Cons

- Info provided by $\bar{F}$ is sometimes incomplete.
- SPAM independence only under special cases.
- Independent sampling of gates is difficult/unrealistic.
- Limited (explicit case) to the Clifford case.

Clifford

Normaliser

Let N and G groups such that $N \leq G$. G normalises N if $\forall g \in G, h \in N$, $ghg^{-1} \in N$.

Pauli group for qutrits

X and Z matrices and phases $\omega := \exp(2\pi i/3)$.

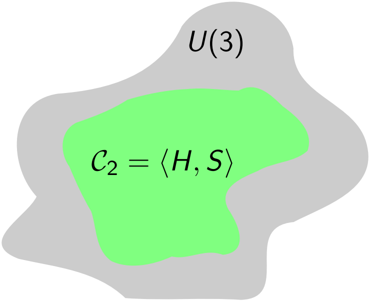
Characteristics

- Normaliser of the Pauli group.
- Can be efficiently simulated on a classical computer.
- Cannot efficiently approximate an arbitrary unitary.

Clifford is not universal

Characteristics

- Can be efficiently simulated on a classical computer.
- Cannot efficiently approximate an arbitrary unitary.

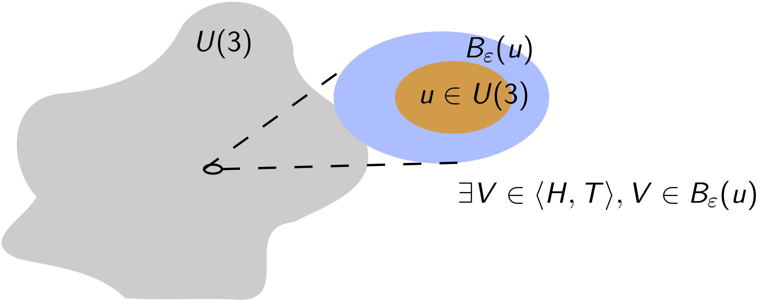


$$H := 3^{-1/2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, S := \begin{bmatrix} 1 & & \\ & 1 & \\ & & \omega^2 \end{bmatrix}. \quad (5)$$

Universal gates

Universal gates

- Gates with up to three qutrits.
- Finite set of gates approximate up to arbitrary accuracy any evolution.



Universal gates

Universal gates

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Efficient

- Polylog with respect to the inverse of the accuracy ε.
- Meaning, increasing one digit of precision in accuracy increases polynomially.

Solovay-Kitaev theorem [13]

- Efficient computation of the circuit.
- Efficient circuit depth.

T gates

T gate

- A T gate is any gate that is a generator of a universal gate set which is non-Clifford.
- Conjugating Pauli matrices by a T gate produces a Clifford gate.
- For instance,

$$T = \begin{bmatrix} 1 & & \\ & \exp(2\pi i/9) & \\ & & \exp(2\pi i/9)^8 \end{bmatrix}; \quad T^9 = \mathbb{I}. \tag{6}$$

The HDG: characterisation of T (my contribution)

HDG stands for Hyperdihedral group.

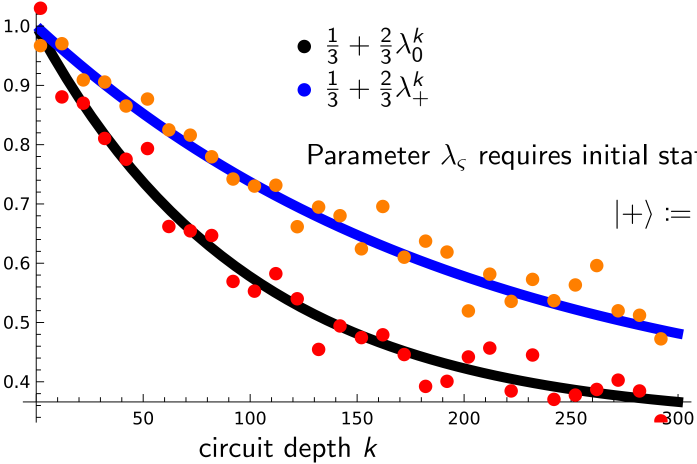
Motivation

- Generalise dihedral benchmarking [1].
- Fewer group elements than the Clifford group.
- Smallest number of parameters in the expression for $\bar{F}$.

Construction

- Every product of X and T.
- $$X := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & & \\ & \exp(2\pi/3) & \\ & & \exp(2\pi/3)^8 \end{bmatrix}. \quad (7)$$
- $X^x T^\alpha T'^\beta$, $x \in \{0, 1, 2\}$ and $\alpha, \beta \in \{0, \cdots, 8\}$.
- $T' := XTX^\dagger$.

The average gate fidelity (for the hdg) is $\bar{F} = \frac{9+3(1+2\lambda_0+6\lambda_+)}{36}$.



Parameter λ_ς requires initial state $|\varsigma\rangle$.

$|+\rangle := H\,|0\rangle$.

Resulting scheme

Order

- Has order 81 (Clifford has order 216) [16].

Satisfies imposed conditions

- Fewer gates than Clifford.
- Only two parameters: λ_0 and λ_+.

Outlook

- The HDG is the group $C_3 \ltimes C_9 \times C_9$.
- (Semidirect product structure) efficient sampling.
- Allows the use of induced representation results.
- Can be extended to prime level systems.

Conclusions

- Estimating the performance of quantum gates using the average gate fidelity.
- Randomised benchmarking is the Academy and industry standard for the performance quantification of quantum gates.
- I presented the generalisation, mainly done by the introduction of a group HDG, of the randomised benchmarking scheme to characterise universal qutrits gates.

Alberta NSERC CRSNG

Collaborators

- Dr. Hubert de Guise (Lakehead University).
- Dr. Barry C. Sanders (University of Calgary).

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Why two parameters?

- $\Gamma(\mathrm{HDG}) = \Sigma_I + \Sigma_0 + \Sigma_0^* + \Sigma_+ + \Sigma_+^*$.
- Each irrep has associated a parameters in the average gate fidelity.
- For high-fidelity gates, the phase is negligible.