Ising-like models on Euclidean black holes
arXiv:2306.08547

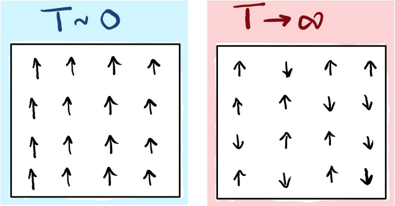
Mustafa Saeed

University of New Brunswick

June 16, 2023

Introduction

- The Ising model is a lattice of spins on a plane. Temperature is an external parameter. The effect of temperature on the order of spins is known.



- The transition between perfect order to disorder occurs at the critical temperature, $T_c = 2.27$.

- "General relativity is a theory of space, time and gravitation formulated by Einstein in 1915." [1]
- The gravitational field of a black hole (aka Lorentzian Schwarzschild spacetime) is a solution to Einstein's equations. The gravitational field of a black hole (with mass M and surface area a) satisfies [2]

$$dM = \left(\frac{1}{8\pi M}\right)\left(\frac{da}{4}\right).$$

- This resembles the first law of thermodynamics

$$dE = T\,dS$$

where E is energy, T is temperature and S is entropy.

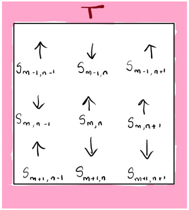
- This correspondence motivated Gibbons and Hawking to treat black hole spacetime as a statistical mechanics system. They succeeded by using Euclidean Schwarzschild geometry [3].
- We asked the question: "Is there a way to construct an Ising-like model on this thermal background? If there is, then how do the spins behave as M is varied?"

Outline

1. Ising model
2. Euclidean black holes
3. Ising-like models on Euclidean black holes

Ising model

- Lattice of spins on a 2d plane in contact with a heat bath.



- Hamiltonian

$$H = -\sum_{m,n=1}^{N} s_{m,n}\left(s_{m+1,n} + s_{m,n+1}\right).$$

- A configuration of spins x has weight

$$p_x = \frac{\exp\left(-\frac{H_x}{T}\right)}{Z}, \quad Z(T) = \sum_x \exp\left(-\frac{H_x}{T}\right).$$

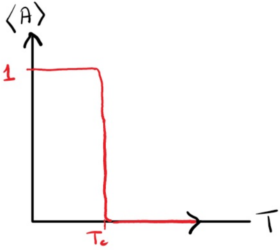
- Alignment

$$A = \frac{1}{N^2} \left| \sum_{m,n=1}^{N} s_{m,n} \right|.$$

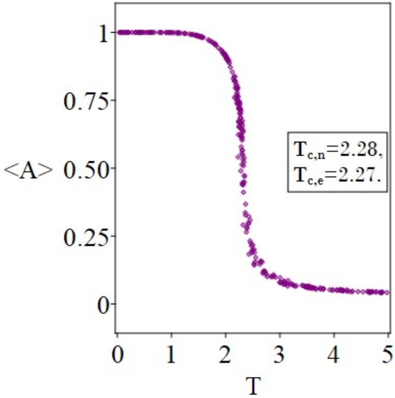
- Statistical average of alignment

$$< A >= \sum_x A_x p_x.$$

- Phase transition of the Ising model occurs at $T_c = 2.27$. This is an exact result due to Onsager [4].



Used the Monte Carlo (MC) algorithm to calculate $< A >$ for temperatures selected randomly from $T \in (0, 5]$.



Summary: The 2d Ising model exhibits a phase transition. We reproduced the known result using numerical methods.

Euclidean black holes

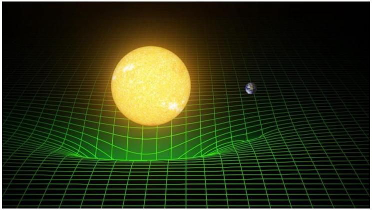
- Spacetime event $\equiv (t, r, \theta, \phi)$.

- Distances in spacetime:

$$ds^2 = g_{ab}(t, \vec{x}) dx^a dx^b.$$

$g_{ab}(t, \vec{x})$ is the metric.

- Einstein's theory of general relativity (GR) describes the interactions of matter and spacetime.
- Tenets of the theory:
    - Gravity is the curvature of spacetime.
    - John Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve."



- Schwarzschild spacetime:

$$ds^2 = -\left(1 - 2Mr^{-1}\right) dt^2 + \left(1 - 2Mr^{-1}\right)^{-1} dr^2 + r^2 d\Omega_{(2)}{}^2.$$

Here M is the mass of the black hole, $d\Omega_{(2)}{}^2$ is the metric describing a two-sphere, $t \in \mathbb{R}$, $r \in \mathbb{R}^+$; the spacetime has coordinate and physical singularities at $r = 2M$ (the horizon) and $r = 0$ respectively.

- The spacetime has temperature

$$T = (8\pi M)^{-1}.$$

- Motivated by the thermodynamic properties of the black hole spacetime, Gibbons and Hawking tried to treat it as a statistical mechanics system. They succeeded by using Euclidean Schwarzschild background [3].
- To derive the Euclidean Schwarzschild space transform to Euclidean time $\tau = it$ and a new radial coordinate $\rho(r, M)$ and enforce Euclidean time to have periodicity $8\pi M$ [1]. The $\tau - \rho$ part of the metric:

$$ds^2 = \rho^2 \left(\frac{d\tau}{4M}\right)^2 + \left[\frac{r(\rho)}{2M}\right]^4 d\rho^2, \quad \tau = [0, 8\pi M), \quad \rho = [0, 4M).$$

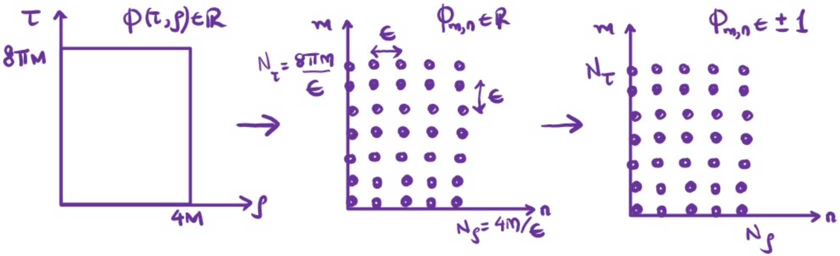
Summary: Euclidean black hole geometry is used to understand black hole thermodynamics; it can be derived from Lorentzian Schwarzschild spacetime through the application of a procedure which requires Euclidean time to have periodicity $8\pi M$ [1].

Ising-like models on Euclidean black holes

- Action of a scalar field on Euclidean background:

$$S = \frac{1}{2}\int d^4x\sqrt{g}\left(g^{ab}\partial_a\Phi\partial_b\Phi\right).$$

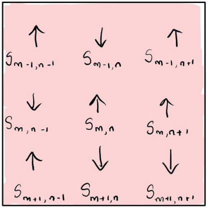
  Choose Euclidean Scwarzschild background and $\Phi \equiv \Phi(\tau,\rho)$. M determines coordinate ranges and is present in the integrand.
- Discretize background into a $N_\tau \times N_\rho$ lattice with lattice spacing ε (first arrow) and restrict the value of the field to ± 1 (second arrow).



- The action is the Ising-like model on Euclidean Schwarzschild background:

$$S = -4\pi \sum_{m,n=1}^{N_\tau, N_\rho} \left\{ \Phi_{m,n} \left[\frac{16M^3}{\rho_n \left(1 - \left(\frac{\rho_n}{4M}\right)^2\right)^4} \Phi_{m+1,n} + (\rho_n M)\, \Phi_{m,n+1} \right] \right\}.$$

- Features of this inhomogeneous Ising model:
  - Each interaction is a function of space.
  - M
    - Influences the vertical and horizontal interactions differently,
    - Determines lattice size.
- The spins are at temperature $T = (8\pi M)^{-1}$. But there is no heat bath.

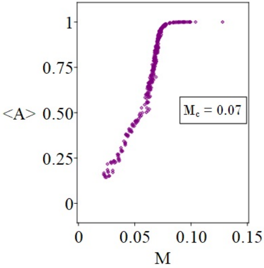


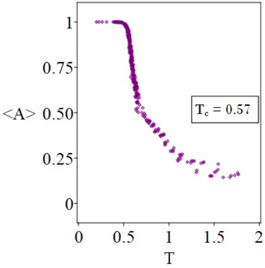
The background has temperature.

- For fixed M a lattice configuration - x - has weight

$$P(x) = \frac{\exp(-S(x))}{Z}, \quad Z = \sum_x \exp(-S(x)).$$

- Used Monte Carlo simulations (with $\varepsilon = 0.01$) to study $< A >$ by varying M. Numerical results indicate a second order phase transition.





Summary: We constructed an Ising-like model on Euclidean black hole background, investigated the thermodynamic properties of spins by varying M and found evidence of a phase transition at sub-Planckian mass.

References

[1] Robert M. Wald.
General Relativity.
Chicago Univ. Pr., Chicago, USA, 1984.

[2] J. M. Bardeen, B. Carter, and S. W. Hawking.
The four laws of black hole mechanics.
Communications in Mathematical Physics, 31, 1973.

[3] G. W. Gibbons and S. W. Hawking.
Action Integrals and Partition Functions in Quantum Gravity.
Phys. Rev. D, 15:2752–2756, 1977.

[4] Lars Onsager.
Crystal statistics. i. a two-dimensional model with an order-disorder transition.
Phys. Rev., 65:117–149, Feb 1944.