Distorting black hole solutions of the vacuum Einstein equations

Matin Tavayef
Supervisors: Prof. Ivan Booth, Dr. Hari Kunduri

Theoretical Physics
Memorial University

June , 2023

MEMORIAL UNIVERSITY

Outline

1. The Weyl solutions
   - The Weyl solution in four dimensions
   - The Weyl solution in Higher Dimensions
     - The Black ring
   - Black hole with a “bubble”
2. Distorted black ring
   - Conical singularity
3. Future work

The Weyl solution in four dimensions

- The Weyl form is used to describe any axisymmetric, static, and vacuum spacetime.
- g_{ab} is a diagonal matrix.

The Weyl metric in 4-D

$$ds^2 = -e^{2U}dt^2 + e^{-2U}\left(e^{2\gamma}(dr^2 + dz^2) + r^2 d\phi^2\right)$$

- It is described by two orthogonal Killing vector fields that commute with each other.
- U(r, z) is an arbitrary axisymmetric solution of Laplace's equation in a three-dimensional flat space in cylindrical coordinates (r, z, ϕ).

The Laplace equation for an axisymmetric function

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0$$

- While $\gamma(r, z)$ satisfies the auxiliary equations which are integrable because U(r,z) is harmonic.

$$\frac{\partial \gamma}{\partial r} = r\left(\left(\frac{\partial U}{\partial r}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right)$$

$$\frac{\partial \gamma}{\partial z} = 2r\left(\frac{\partial U}{\partial r}\right)\left(\frac{\partial U}{\partial z}\right).$$

- Since U is harmonic, it can be regarded as a Newtonian potential produced by certain (axisymmetric) sources.

Weyl Spacetime in Higher Dimensions

- Emparan and Reall generalized the Weyl construction to all $D > 4$. The generalized Weyl metrics take the form

D dimensional Weyl metric

$$ds^2 = \sum_{i=1}^{D-2} \epsilon_i e^{2U_i} (dx^i)^2 + e^{2\nu} (dr^2 + dz^2) \quad .$$

where x_i are directions corresponding to the Killing vectors and $\epsilon_i = 1, -1$. In this initial case we will choose $\epsilon_1 = -1$ and $\epsilon_i = 1$.

- Each U_i is a harmonic axisymmetric solution to the three-dimensional Laplace's equation

$$\frac{\partial^2 U_i}{\partial r^2} + \frac{1}{r}\frac{\partial U_i}{\partial r} + \frac{\partial^2 U_i}{\partial z^2} = 0$$

This is equivalent to $\nabla^2 U_i = 0$.

Due to Einstein equations

$$\partial_r \nu = -\frac{1}{2r} + \frac{r}{2}\sum_{i=1}^{3}\left((\partial_r U_i)^2 - (\partial_z U_i)^2\right)$$

$$\partial_z \nu = r\sum_{i=1}^{3}\left((\partial_r U_i)(\partial_z U_i)\right) \quad .$$

The integribility condition on $\nu(r,z)$ is $\partial_z\partial_r\nu = \partial_r\partial_z\nu$.
The vacuum Einstein equations, $R_{\mu\nu} = 0$ impose the following constraint on U_i

The constraint on U_i

$$\sum_i U_i = \log r + constant \quad .$$

The solutions for U_i can be thought of as Newtonian potentials produced by certain sources, so the above constraint states that these sources must add up to give an infinite rod.

The Schwarzschild solution

- The exterior Schwartzschild metric in 4-dimensions is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \quad .$$

where $t \in \mathbb{R}, r > 2m$, and (θ, ϕ) are standard coordinates on S^2.
- If we write it in the Weyl form we will have

$$U = -\frac{1}{2} \log\left[\frac{M - z + \sqrt{(M-z)^2 + r^2}}{-M - z + \sqrt{(M+z)^2 + r^2}}\right] \quad .$$

Based on the constraint, the function U_2 must be the potential produced by semi-infinite rods for $z \geq M$ and $z \leq -M$.

Sources for a four dimensional Schwarzschild solution.

The Black ring

The static black ring is a vacuum solution that describes an asymptotically flat black hole with horizon $S^1 \times S^2$. The general metric of a black ring is given by

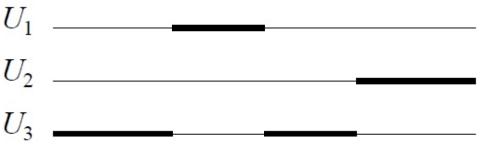
The Metric of a black ring

$$dS^2 = -\frac{F(x)}{F(y)}dt^2 + \frac{1}{A^2(x-y)^2}\left(F(x)\left((y^2-1)d\psi^2 + \frac{F(y)}{y^2-1}dy^2\right)\right.$$
$$\left. + F(y)^2\left(\frac{dx^2}{1-x^2}. + \frac{1-x^2}{F(x)}d\phi^2\right)\right)$$

$F(\xi) = 1 - \mu\xi$, $0 \leq \mu \leq 1$, $A > 0$, $-1 \leq x \leq 1$, $y \leq -1$. This metric will have three orthogonal commuting Killing vector fields, making it a Weyl solution.The functions U_i will be described as

$$e^{2U_1} = \frac{F(x)}{F(y)}, \quad e^{2U_2} = \frac{F(x)(y^2-1)}{A^2(x-y)^2}, \quad e^{2U_3} = \frac{F(y)^2(1-x^2)}{A^2(x-y)^2 F(x)}$$

The black ring



Sources for a black ring

Black hole with a “bubble”

- Well known black hole solutions have a simple exterior region. This means there are no interesting topological features outside the horizon. this is a consequence of the topological censorship theorem of Friedman, Scheich, and Witt (gr-qc/9305017 [gr-qc])
- In five and higher dimensions, this theorem allow for more complicated topology in the exterior region. For example, we can have **’bubbles’** (i.e. two-dimensional ‘holes’ that do not collapse to a point) outside the event horizon. These kinds of solutions are relatively new , see for example the paper of Horowitz, Kunduri and Lucietti (e-Print: 1704.04071 [hep-th])
- We can construct a simple example of this kind within the Weyl class.

The metric functions

$$e^{2U_1} = \frac{\mu_a}{\mu_b}, \quad e^{2U_2} = \frac{r^2 \mu_c}{\mu_a \mu_d}, \quad e^{2U_3} = \frac{\mu_d \mu_b}{\mu_c}$$

where

μ_k

$$\mu_k = \sqrt{r^2 + (z-k)^2} - (z-k)$$

The rod points obey $a < b < c < d$.

Black hole with a "bubble"

Specifically, in a constant time slice of the spacetime, the finite rod $c < z < d$ becomes an S^2 bubble , while the finite rod $b < z < c$ becomes a non-contractible 2-disk. Like the black ring, this solution also has a non-removable conical singularity.

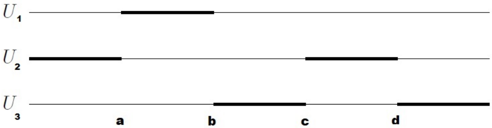


Figure: The rod structure for a black hole with a bubble. Here the finite rod along (a < z<b) corresponds to a static black hole horizon

Calculation of $\nu(r, z)$ for a Black hole with a "bubble"

- To check that the solution we are constructing is actually smooth everywhere, we need to know the function ν explicitly. This calculation is based on the method introduced by Hideo Iguchi and Takashi Mishima, (2007) (J. Phys. , Conf. Ser. 66 01205.)

If we define

$$\bar{U_c} = \frac{1}{2}\ln(R_c + (z - c))$$

$$\gamma_{cd} = \frac{1}{2}\bar{U_c} + \frac{1}{2}\bar{U_d} - \frac{1}{4}\ln Y_{cd}$$

$$Y_{cd} = R_c R_d + (z - c)(z - d) + r^2$$

$$R_c = \sqrt{r^2 + (z - c)^2}$$

We can rewrite $U_1 = \bar{U_2} - \bar{U_1}$, $U_2 = \ln r - \bar{U_2} - \bar{U_4} + \bar{U_3}$ and $U_3 = \bar{U_1} + \bar{U_4} - \bar{U_3}$. By using below equations

$$\partial_r \nu = -\frac{1}{2r} + \frac{r}{2}\sum_{i=1}^{3}\left((\partial_r U_i)^2 - (\partial_z U_i)^2\right), \quad \partial_z \nu = r\sum_{i=1}^{3}\left((\partial_r U_i)(\partial_z U_i)\right) \quad .$$

we will obtain

$$\nu(r,z) = \bar{U_3} - \bar{U_2} - \bar{U_4} + \gamma_{11} + \gamma_{22} + \gamma_{33} + \gamma_{44}$$
$$- \gamma_{12} - \gamma_{32} - 2\gamma_{34} + \gamma_{24} + \gamma_{14} - \gamma_{13} + \frac{1}{4}\ln\left(\frac{1}{2}\right)$$

The mass of the horizon
The series expansion of the g_{tt} term

$$g_{tt} = -1 + \frac{8M}{3\pi R^2} + O(R^{-4})$$

Where M is the mass. By considering the coordinate transformation $r = R^2/2\sin^2\theta, \quad z = R^2/2\cos 2\theta$, for a black hole with a bubble

$$g_{tt}(R,\theta) = -1 + \frac{2(b-a)}{R^2} + O(R^{-4})$$

The mass of the horizon

$$M = \frac{3\pi(b-a)}{4}$$

Distorted black ring:Abdolrahimi, Mann, Tzoumis (2020)

Distorted black hole solutions describe a black hole that exists within the gravitational field of external sources, even though the matter sources are not explicitly included in the solution. A distorted black ring's metric can be written in the following form

$$ds^2 = -e^{2(\widehat{U}+\widehat{W})}\frac{F(x)}{F(y)}dt^2 + \frac{1}{A^2(x-y)^2}\left[e^{2(\widehat{U}+\widehat{W}+\widehat{V})}\left(\frac{F(y)^2}{1-x^2}dx^2\right.\right.$$
$$\left.\left. + \frac{F(x)F(y)}{y^2-1}dy^2 + e^{-2\widehat{W}}F(x)(y^2-1)d\psi^2 + e^{-2\widehat{U}}\frac{F(y)^2}{F(x)}(1-x^2)d\phi^2\right.\right.$$

In the cylindrical coordinates the solution of Laplace equation is

$$\widehat{X}(r,z) = \sum_{n\geq 0}\left[A_n R^n + B_n R^{-(n+1)}\right]P_n(\cos\vartheta)$$

where

$$R = \frac{\sqrt{r^2+z^2}}{m}, \quad \cos\vartheta = \frac{z}{R}$$

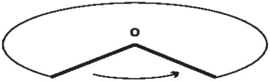
- $P_n(\cos\vartheta)$ represent the Legendre polynomials of the first kind.
- We only consider A_n coefficients, which describe the local distortion of a black ring due to external fields and will be called multipole moments.
- After some calculation we can obtain $\widehat{V} = \widehat{V} = \widehat{V}_1 + \widehat{V}_2$

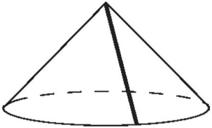
$$\widehat{V}_1 = \sum_{n,k\geq 1} \frac{nk}{n+k} \left(a_n a_k + a_n b_k + b_n b_k\right) R^{n+k} \left[P_n P_k - P_{n-1} P_{k-1}\right]$$

For the multiple moment n=1 , $\widehat{V}_2$ can be derived as

$$\widehat{V}_2 = \frac{1}{2z} \left[a_1 \left(R_1 - 2R_2 + 2R_3\right) - b_1 \left(R_1 + R_2 - R_3\right) \right.$$
$$\left. -3z(a_1 + b_1)\right] R P_1, \quad n = 1$$

Conical singularity





Conical singularity, M. Kenmoku, S. Uchida, T. Matsuyama, Int.J.Mod.Phys. D12 (2003) 677-687

In order to remove such coordinate singularities, we must fix the period of an angle ϕ corresponding to the Killing field $\frac{\partial}{\partial \phi}$ to be Δ_ϕ, which is given by

$$\Delta_\phi = 2\pi \lim_{r \to 0} \sqrt{\frac{r^2 \exp(2\nu)}{g_{ij} v_i v_j}} = 2\pi \quad (1)$$

Conical regularity condition

A black ring without any distortion has a conical singularity located at one of its poles. We consider the $x - \phi$ components of the conformal metric and $x = -\cos\theta, \quad 0 \le \theta \le \pi$ then

$$ds^2_{x\phi} = e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}\frac{dx^2}{1-x^2} + \frac{1-x^2}{F(x)}d\phi^2 \quad .$$

$$ds^2_{x\phi} = e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}d\theta^2 + \frac{\sin\theta^2}{1+\mu\cos\theta}d\phi^2 \quad .$$

There will be no conical singularity on the semiaxis $\theta = 0$, if Δ_ϕ, the period of ϕ satisfies below condition

$$\Delta_\phi = 2\pi\sqrt{1+\mu}e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}|_{\theta=0} \quad .$$

The regularity condition for $\theta = \pi$ is

$$\Delta_\phi = 2\pi\sqrt{1-\mu}e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}|_{\theta=\pi} \quad .$$

Conical regularity condition

The combination of them

$$\frac{e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}|_{\theta=\pi}}{e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}|_{\theta=0}} = \frac{\sqrt{1+\mu}}{\sqrt{1-\mu}}$$

The presence of a conical singularity in a distorted black ring cannot be eliminated by adjusting the multiple moments.
However, If we set $W = -\widehat{U}/2$ the left-hand side of the above equation becomes independent of y

Conical regularity condition

$$\frac{e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}|_{\theta=\pi}}{e^{2(\widehat{V}+2\widehat{U}+\widehat{W})}|_{\theta=0}} = \exp\left(-\frac{3}{2}(1-\mu)\sum_{n=1}^{4}\sum_{k=n}^{4}\frac{a_k}{\mu^n}\right) = \frac{\sqrt{1+\mu}}{\sqrt{1-\mu}} \quad .$$

When the dipole distortion $(a_{n>1} = 0)$ occurs

$$a_1 = \frac{\mu}{3(1-\mu)} \ln\left(\frac{1-\mu}{1+\mu}\right) \quad .$$

Conical regularity condition

When the quadrupole distortion $a_1 = 0, a_2 \neq 0, a_{n>2} = 0$ occurs we have

$$a_2 = \frac{\mu^2}{3(1-\mu^2)} \ln\left(\frac{1-\mu}{1+\mu}\right) \quad .$$

- The black ring without any distortion has conical singularities.
- We can eliminate the conical singularity present in the undistorted black ring solution by adjusting at least one of the multipole moments.

Regularity condition for a black hole with a bubble

If we write the regularity condition for U_2 finite region we have

$$1 = \frac{(d-c)^2}{d(d-b)}$$

$$c^2 - 2cd + db = 0$$

If we solve it

$$c = d - \sqrt{d^2 - db}$$

We know $b < c < d$. so $b < d - \sqrt{d^2 - db} < d$. It leads to $b > d$. This is a contradiction. This shows that there is no way to correct for this conical singularity.

Future work

- In our future work, we intend to investigate distortions of a vacuum five-dimensional black hole with a bubble (the black hole exterior has nontrivial topology).
- In particular, we will study how the geometry of the horizons is modified by the distortion of the full spacetime.
- There are two finite axis rods and each of them is associated to a conical singularity. Our focus will be on the conical singularity that is associated with the finite axis rod I_4.
- In this project, we will add a static, axisymmetric distortions to the black hole with a bubble solution and use the same method as for the black ring to remove the conical singularities and produce a smooth solution.

References

- F. Tomlinson, *Five-dimensional electrostatic black holes in a background field*, Class. Quantum Grav, 39 135008 (2022)
- Emparan, R., and Reall, H.S., *Generalized Weyl solutions*, Phys. Rev. D, 65, 084025, (2002)
- H. Weyl, Ann. Phys. (Leipzig) 54 (1917) 117.
- R. Emparan and H. S. Reall, Black Holes in Higher Dimensions , Living Rev. Rel. 11 (2008) 6.
- S. Abdolrahimi, A. A. Shoom and D. N. Page, *Distorted 5-dimensional vacuum black hole*, Phys. Rev. D 82, 124039 (2010).
- S. Abdolrahimi, R. B. Mann, and C. Tzounis, “Distorted black ring”, Phys. Rev. D 101, 10, 104002 (2020).

THANK YOU!