Indeterminacy in Cosmology

(work with Vlad Tasic –
Foundations of Physics '23)

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes" (Laplace 1951: 4).

Determinism

What is determinism in physical theory?

To the best of our knowledge, physical laws are presented as differential equations:

Classical: Field eqns.
QM: Schrodinger eqn.

$$\boxed{\text{Determinism} \equiv \{\text{Initial state fixes unique future state}\}}$$

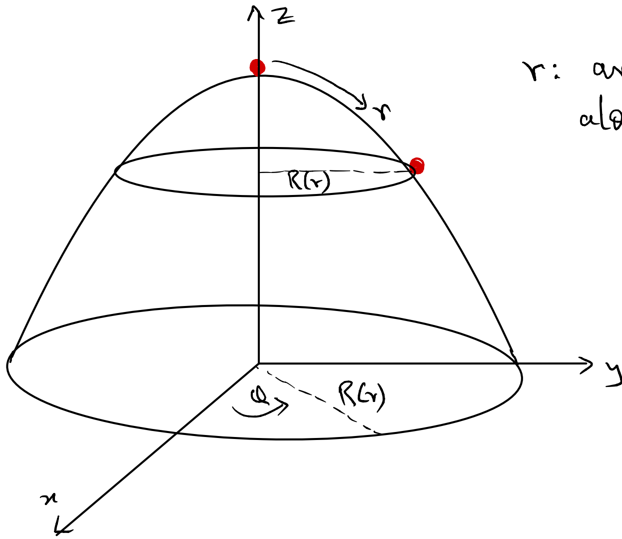
- An ontological doctrine about the temporal evolution of the world

(Ontological: relating concepts – in this case past and future states)

What does "$\equiv$" entail?

How fragile is determinism?

A Newtonian example: Norton's Dome (2008)



r: arc length along dome

Dome surface: $\sigma(r, \theta) = \left(R(r)\cos\theta,\ R(r)\sin\theta,\ z(r)\right)$

$$R(r) = \frac{2}{3}\left[1 - (1-r)^{3/2}\right] \qquad z(r) = z_0 - \frac{2}{3} r^{3/2}$$

Apex: $z = z_0$

Particle sliding without friction:

$$E = T + V = \frac{1}{2} m \dot{r}^2 - mgz = \frac{1}{2} m \dot{r}^2 - mg\left(z_0 - \frac{2}{3} r^{2/3}\right)$$

$$\dot{E} = 0 \quad \longrightarrow \quad \boxed{\ddot{r} = \sqrt{r}\,.}$$

* apparently, a harmless looking dome.

Indeterminacy $\ddot{r} = \sqrt{r}$

Consider initial data $r(0) = \dot{r}(0) = 0$

There are two solutions:

(i) $r(t) = 0$ — state of infinite rest

(ii) $r(t) = \frac{1}{144} t^4$

* For any T these can be combined to obtain

$$r(t) = \begin{cases} 0, & t < T \\ \frac{1}{144}(t-T)^4, & t \geq T \end{cases}$$

Particle sits at rest until an arbitrary time T and then starts moving spontaneously!

* A table top counter example to determinism in Newtonian mechanics.

* Violation of Newton's First Law of Motion.

What is going on?

Induced metric on Dome:

$$ds^2 = dr^2 + R^2(r)\, d\varphi^2$$

Gaussian curvature of Dome:

$$K(r) = \frac{4}{3}\left[\sqrt{1-r} - (1-r)^2\right]^{-1}$$

$\lim_{r \to 0^+} R(r) = 0$ — metric degenerate at apex.

$\lim_{r \to 0^+} K(r) = \infty$ — Gaussian Curvature divergent at apex.

Curvature singularity at $r = 0$.

Mathematical reason / "resolution"

For ODE IVP

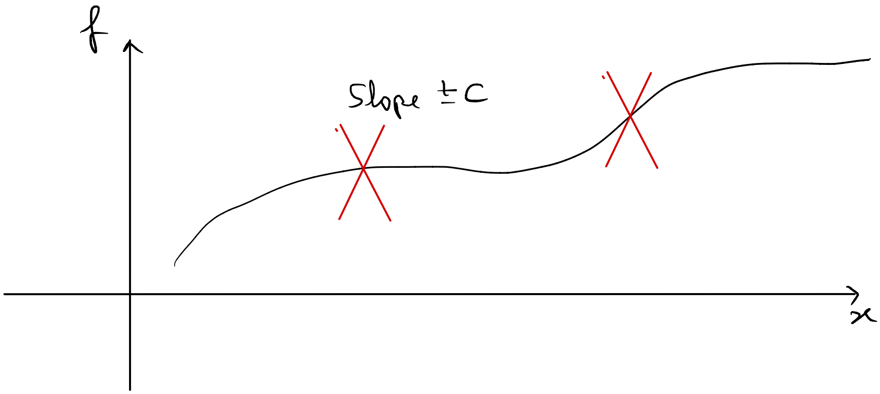
Existence of solutions: all functions in IVP continuous

Uniqueness of solutions: all functions in IVP satisfy Lipshitz continuity

Lipshitz continuity: For $f: \mathbb{R} \to \mathbb{R}$, there is a $C > 0$ such that

$$\frac{|f(x_2) - f(x_1)|}{|x_2 - x_1|} \le C \qquad \text{for all } x_1, x_2$$

"Functions do not change too fast"



Function curve never enters the slope "light cone".

For the dome, $\sqrt{r}$ violates Lipschitz continuity at $r = 0$.

Is there such an example in GR?

$$S[g,\varphi] = \frac{1}{8\pi G}\int d^4x \sqrt{-g}\, R(g) - \int d^4x \sqrt{-g}\, \mathcal{L}_M(g,\varphi).$$

$$\delta S = \int d^4x \left[\sqrt{-g}\,(G_{ab} - 8\pi T_{ab})\,\delta g_{ab} + \sqrt{-g}\,\frac{\delta \mathcal{L}_M}{\delta\varphi}\,\delta\varphi \right]$$

→ Densitized Einstein equns:

$$\sqrt{-g}\, G_{ab} = 8\pi G \sqrt{-g}\, T_{ab}$$

If $\sqrt{-g} \neq 0$ then $G_{ab} = 8\pi T_{ab}$.

Consider densitized FRW equns. for

$$ds^2 = -dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2).$$

$$a\dot{a}^2 = 2C\tilde{\rho}$$

for perfect fluid
$P = w\rho$

$$\dot{a}\ddot{a} = C(\beta+2)\tilde{\rho}$$

$$\left[\; C = \frac{4\pi G}{3}, \quad \tilde{\rho} = a^3\rho, \quad \beta = -3(1+w) \;\right]$$

Eliminating $\tilde{\rho}$ in 2nd eqn.

$$\boxed{a^2\ddot{a} - \left(1+\frac{\beta}{2}\right)a\dot{a}^2 = 0}$$

Indeterminate solutions for $\beta \in (-1,0)$

for $a(0) = \dot{a}(0) = 0$.

$$a_T(t) = \begin{cases} 0, & t < T \\ \alpha(t-T)^{-2/\beta} & t \geq T. \end{cases}$$

Universe sits at the point of the Big Bang up to an arbitrary time coordinate value T, and then spontaneously starts expanding.

* One parameter family of "cosmological domes"
 – indeterminacy dependend on equ. of state

$$P = w\rho \,, \quad w \in \left(-1, -\tfrac{2}{3}\right)$$

(– excludes cosmological constant).

* This is an indeterminacy w.r.t. a coord. time.

Is there indeterminacy with respect to relational observables?

Universe volume (Hubble time)

$$V(t_H).$$

$$V = a^3 \qquad t_H = \frac{a}{\dot{a}}$$

The indeterminate solution

$$a_T(t) = \begin{cases} 0, & t < T \\ \alpha(t-T)^{-2/\beta} & t \geq T. \end{cases}$$

becomes

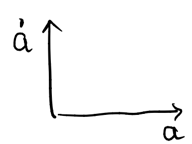
$$V(t_H) = 0$$

or $$V(t_H) = \alpha^3\left(-\frac{2}{\beta}t_H\right)^{-6/\beta}$$

* Arbitrary T eliminated.

* Solutions still not unique but cannot be combined into a solution that spontaneously starts expanding: t_H cannot be shifted, unlike coord. t

* $V(t_H)$ is a relational phase space curve like



So what?

* Simple systems from GR exhibit indeterminacy.

* Eqn. of state connected with Lifschitz continuity.

* In terms of relational observables spontaneous change in motion does not occur despite non-uniqueness.

Talking about (in)determinism requires care