Field Theory and the Electroweak Standard Model

— lecture 1 —

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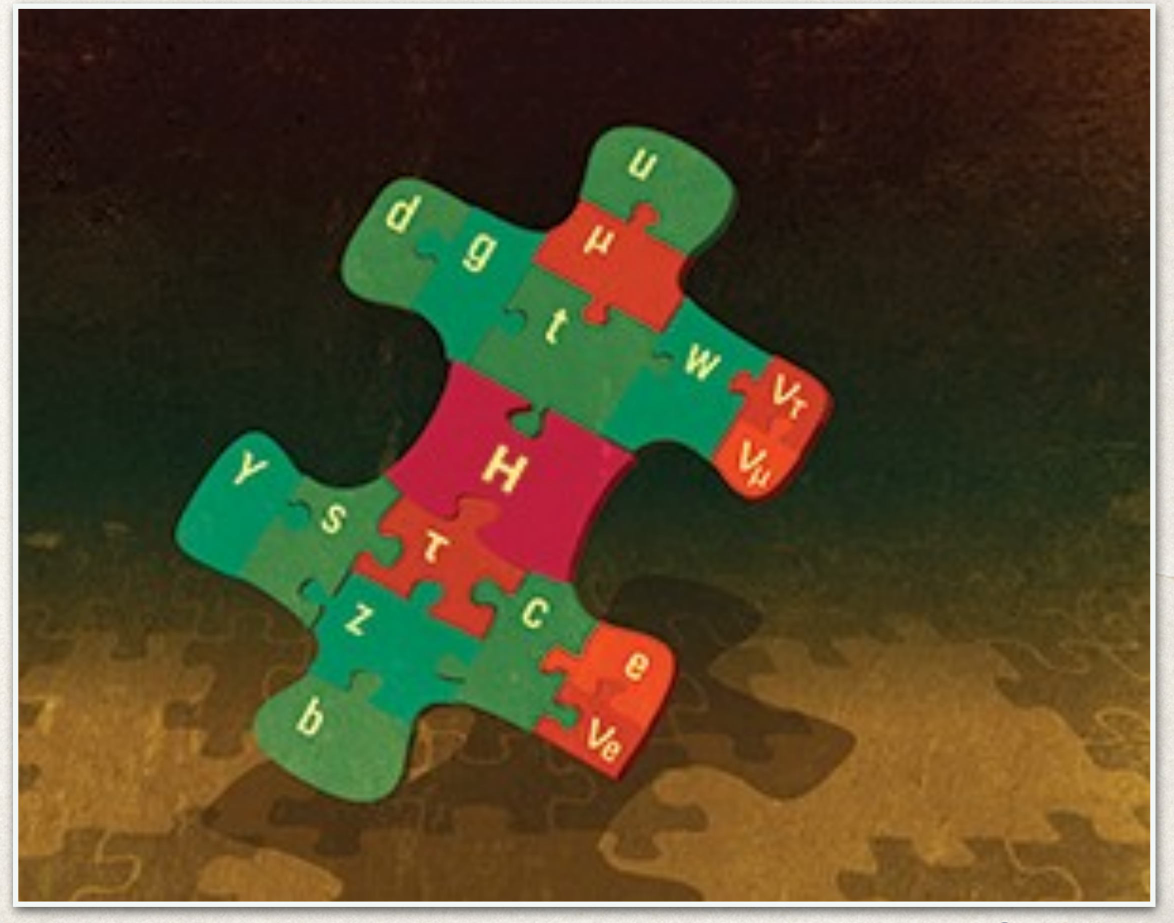
Prelude

Standard Model of particle physics

current state-of-the-art understanding of the fundamental particles of Nature and their interactions

- result of over 60+ years of research in experimental and theoretical particle physics
- extremely successful in description of experimental data
- with enormous predictive power
- its success culminated in the discovery of the Higgs boson 11 years ago



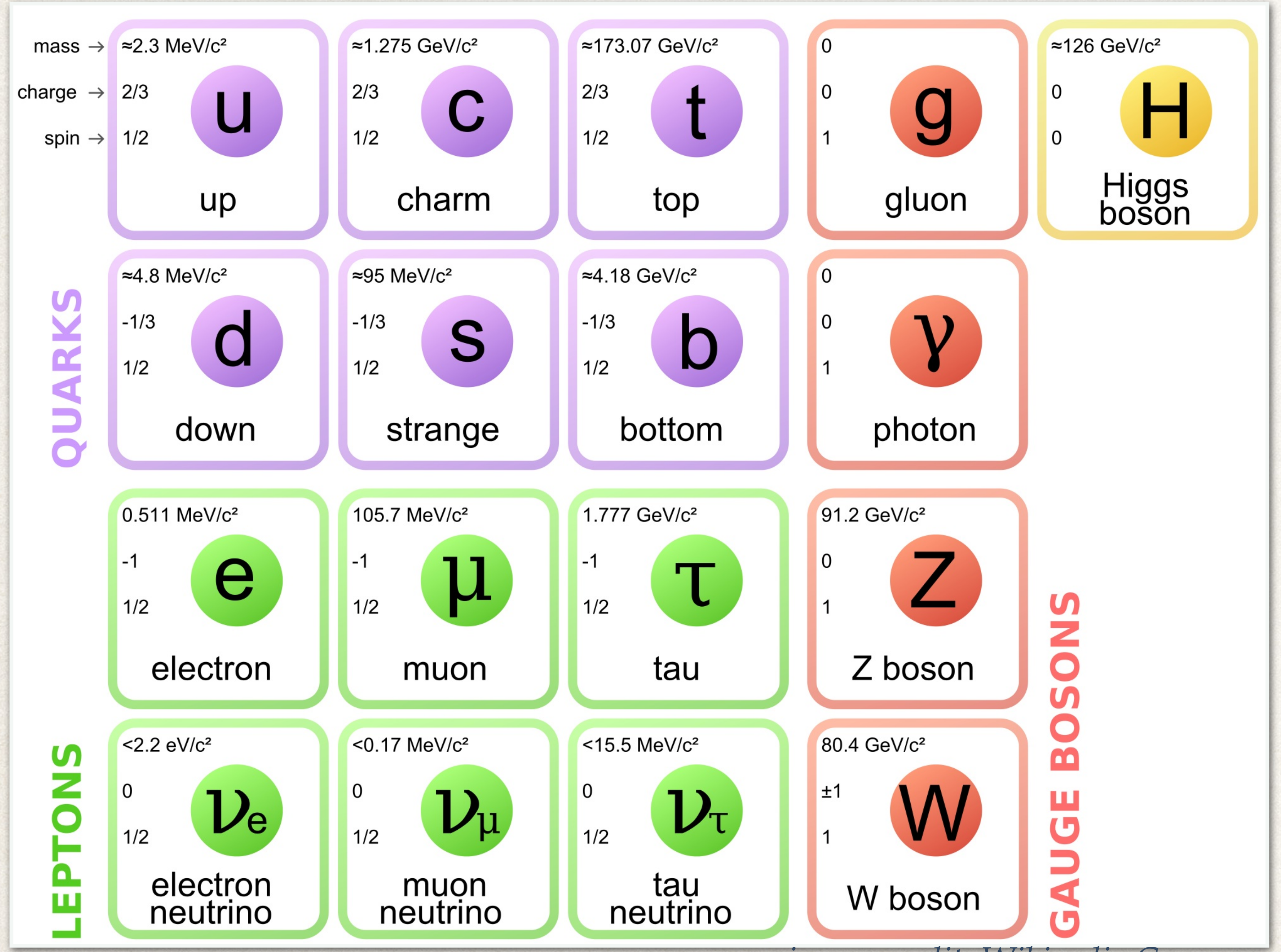
picture credit: Swedish Royal Academy of Sciences

Pinnacle of human thought

(image credit: P. Hernandez)

SM for pedestrians

- Consistent theoretical description of known fundamental particles and their interactions



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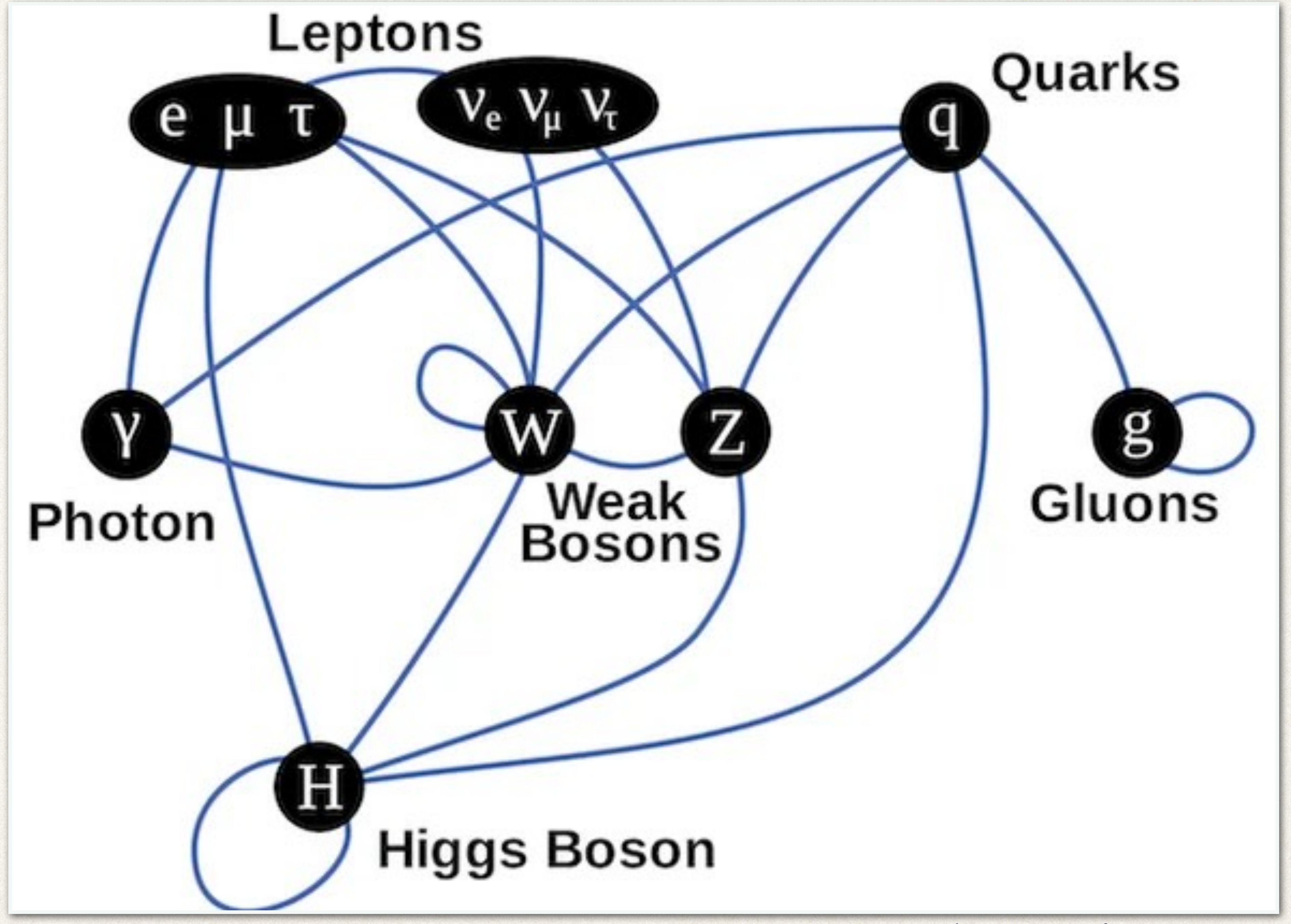


image credit: Scientific American

matter particles

force carriers

Prelude ctnd.

More precisely:

relativistic Quantum Field Theory

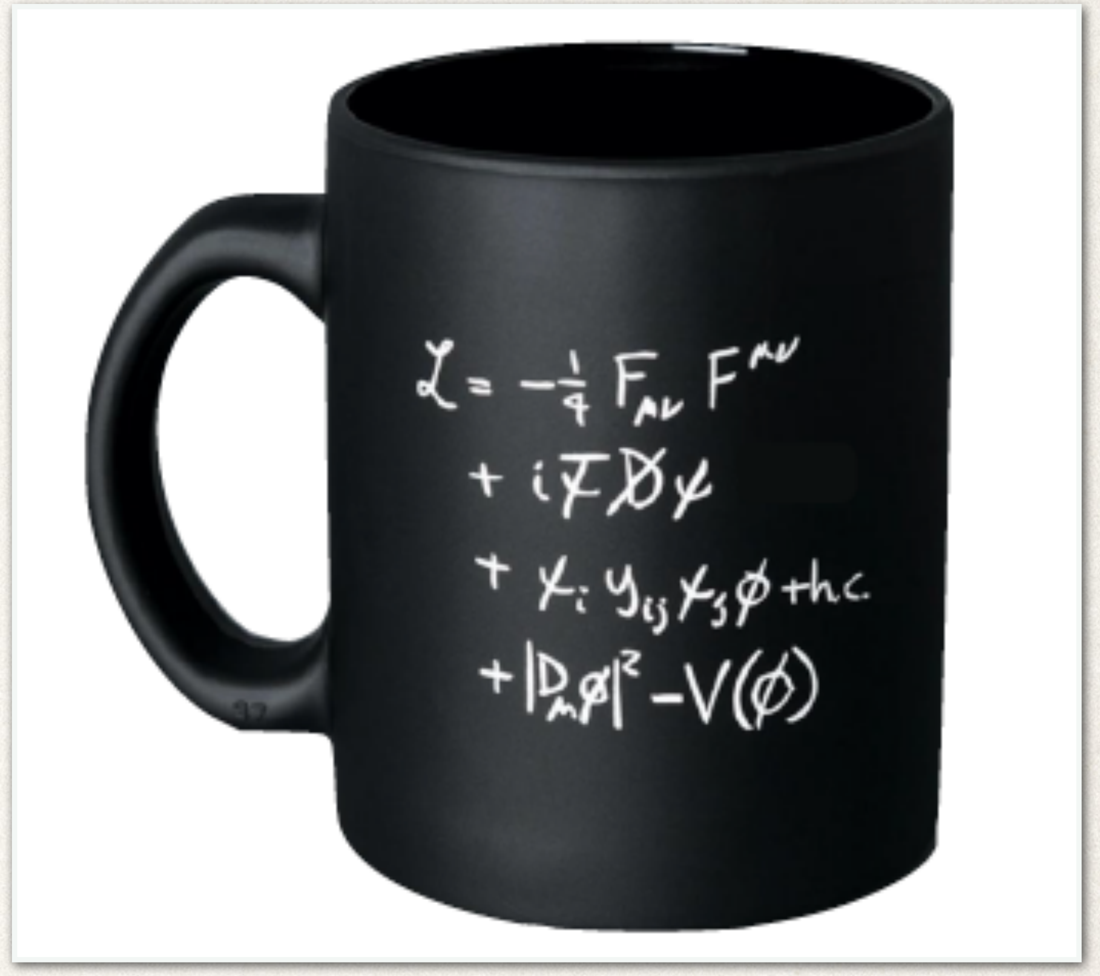
based on principle of local gauge symmetry with the symmetry group given by

$$SU(3)_c \times \; SU(2)_L \times U(1)_Y$$

Quantum Chromodynamics (QCD)
theory of strong interactions
exact symmetry

➜ see lectures by G. Heinrich

(famously fitting on a mug)



Prelude ctnd.

More precisely: Electroweak Standard Model =

relativistic Quantum Field Theory

based on principle of local gauge symmetry with the symmetry group given by

$$SU(3)_c \times SU(2)_L \times U(1)_Y$$

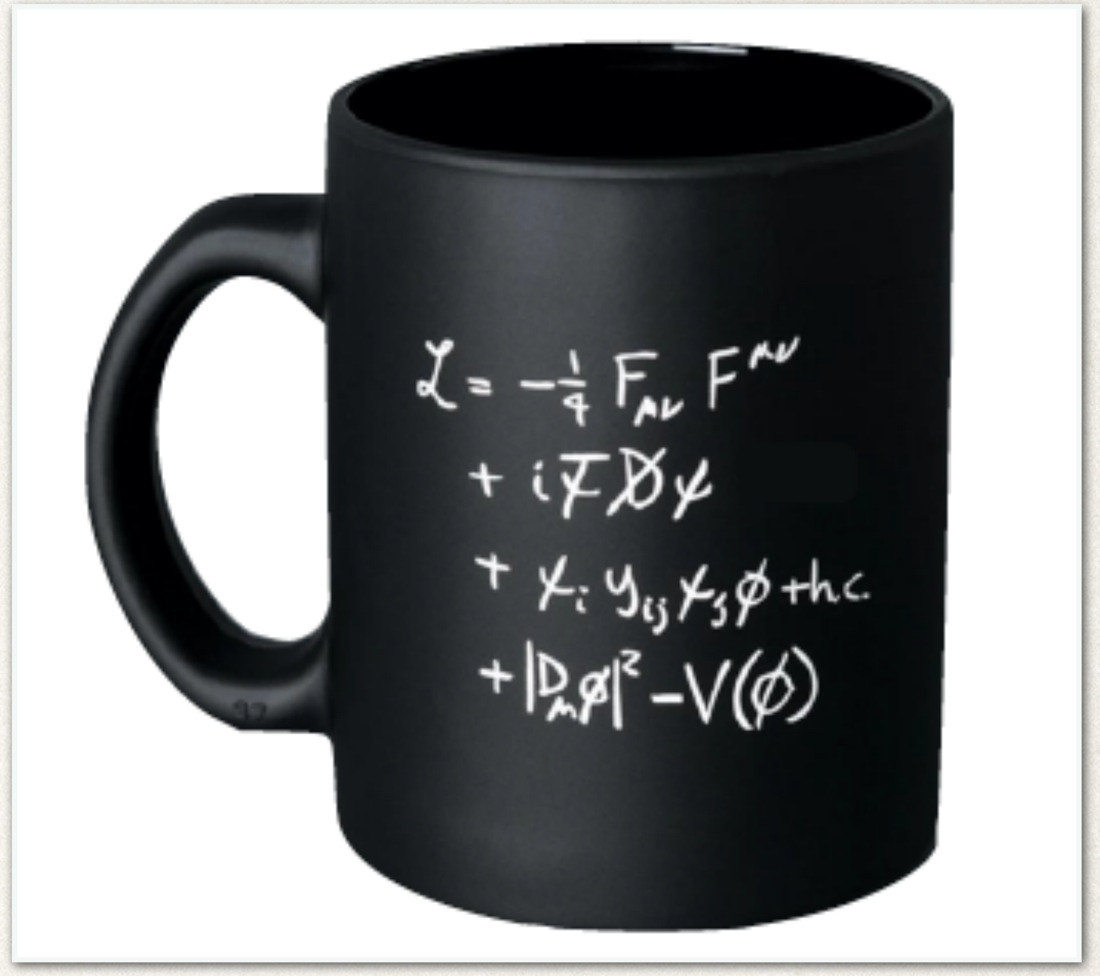
Electroweak (EW) theory

unified theory of weak and electromagnetic interactions

broken to $U(1)_Q$ of electromagnetism

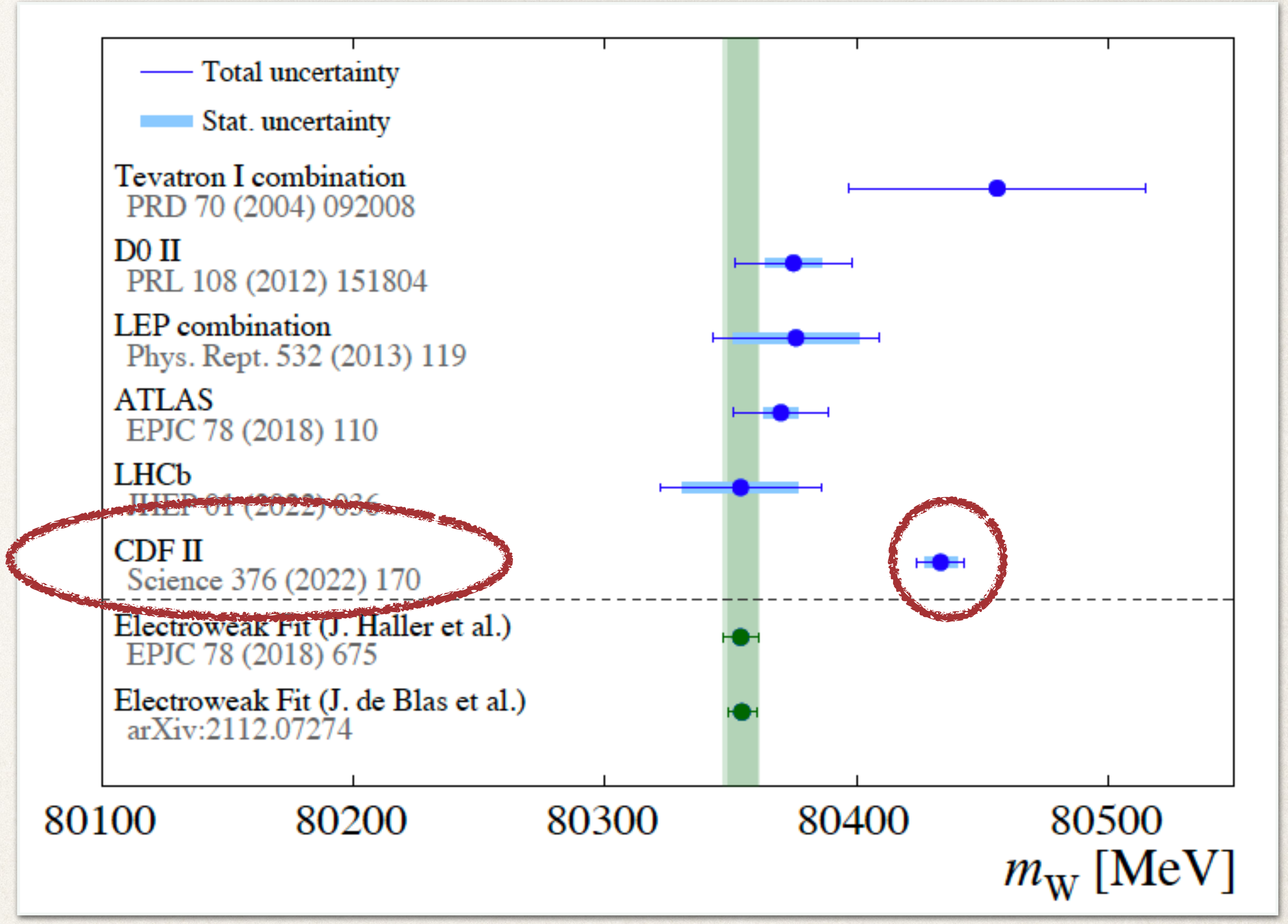
→ these lectures

(famously fitting on a mug)



Prelude, or motivation

- Standard Model (EW+ QCD) is a key to future discoveries in particle physics — any new phenomena will be seen as deviation from SM predictions
- The Higgs sector of the Standard Model is not yet established
- Time and again, new results appear which call for very deep understanding of the underlying Standard Model physics



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Literature

- There are plenty of resources on the subject, including:
    - Textbooks, for example:
        - M.D. Schwartz, *Quantum Field Theory and the Standard Model*
        - M. Maggiore, *A Modern Introduction to Quantum Field Theory*
        - I. Aitchison, A. Hey, *Gauge Theories in Particle Physics*
        - M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory*
        - S. Weinberg, *The Quantum Theory of Fields*, vol. 1 & 2
        - …
    - Write-ups and slides of excellent lectures given at previous editions of ESHEP!

Convention, notation

- Natural units: $\hbar = c = 1$
- Metric tensor in Minkowski space $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$
- 4-vectors

contravariant	covariant
$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$	$x_\mu = g_{\mu\nu}x^\nu$
$p^\mu = (p^0, p^1, p^2, p^3) = (E, \mathbf{p})$	$p_\mu = g_{\mu\nu}p^\nu$
$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = (\partial_0, \nabla)$	$\partial^\mu = (\partial_0, -\nabla)$

- Scalar product $A \cdot B = A^\mu B_\mu = A^0 B^0 - \mathbf{AB} = A_\mu B^\mu = g_{\mu\nu}A^\mu B^\nu = g^{\mu\nu}A_\mu B_\nu$ invariant under Lorentz transformation

Examples: $x^2 = x^\mu x_\mu = t^2 - \mathbf{x^2}$, $p^2 = p^\mu p_\mu = E^2 - \mathbf{p^2}$, $\Box = \partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla$

- For a free particle $p^2 = m^2 = E^2 - \mathbf{p^2}$

Fields, classically

- Fields = functions of space-time $\phi_i(x)$ with definite transformation properties under Lorentz transformations

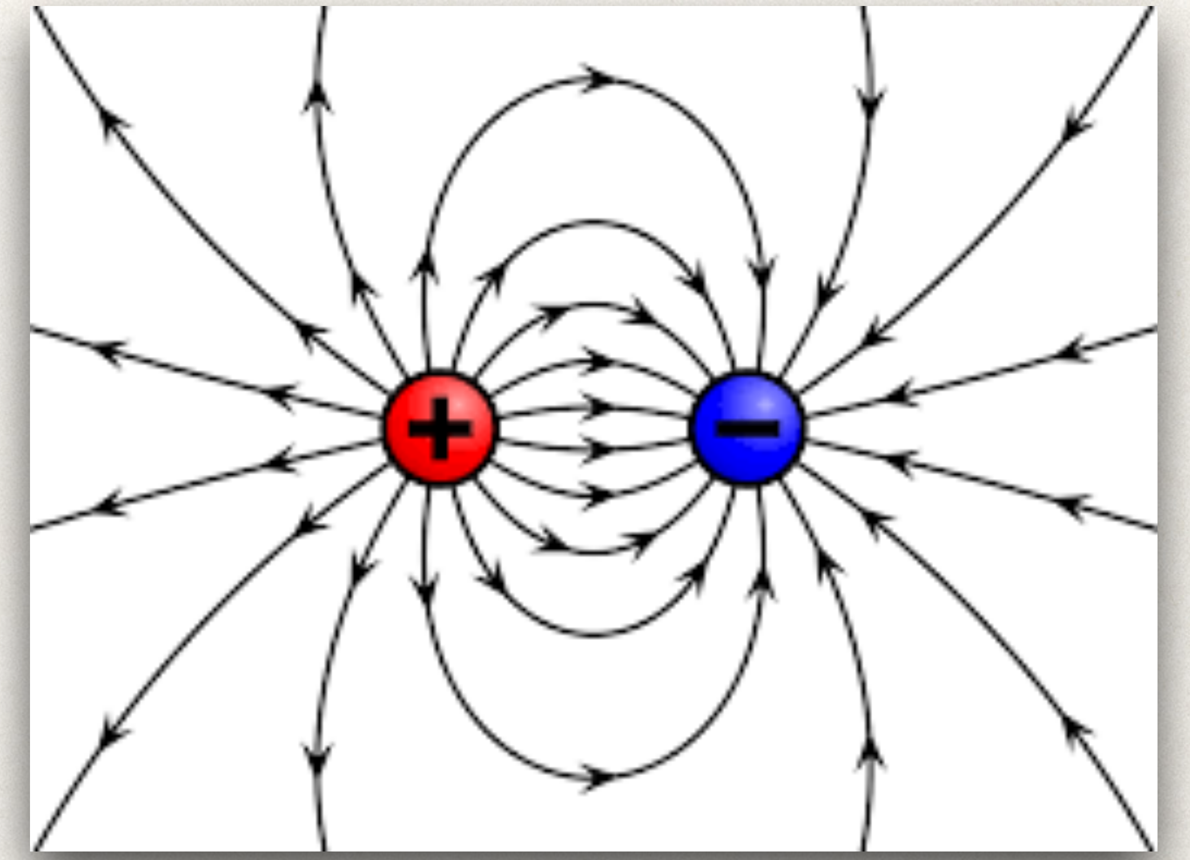
- In Lagrangian formalism, dynamics of the physical system involving a set of fields $\phi(x)$ determined by $L = \int d^3x \, \mathscr{L}(\phi, \partial_\mu \phi)$, yielding the action

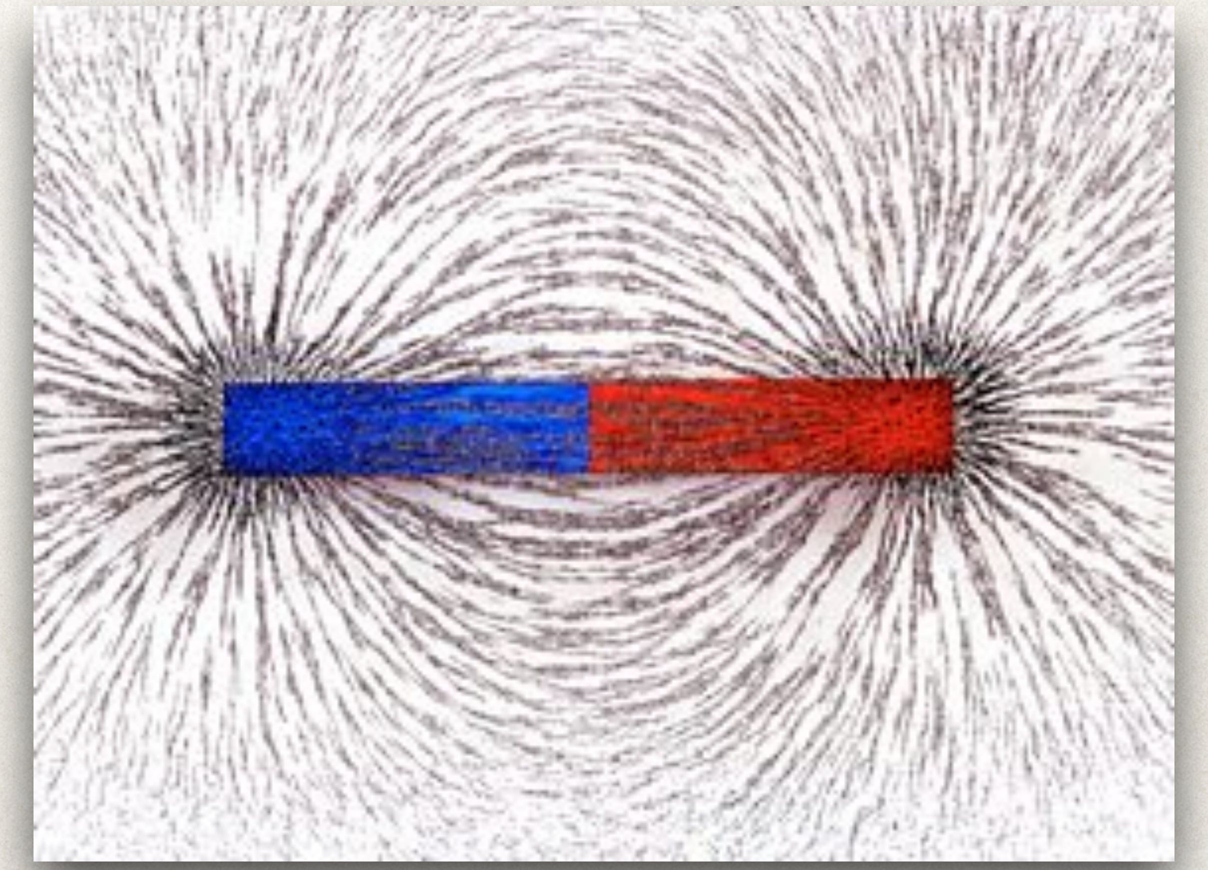
$$S[\phi] = \int dt \, L = \int d^4x \, \mathscr{L}(\phi, \partial_\mu \phi)$$

- Equation of motions, or Euler-Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_i)} = 0$$

follow from the principle of stationary action $\delta S = 0$





Field quantisation

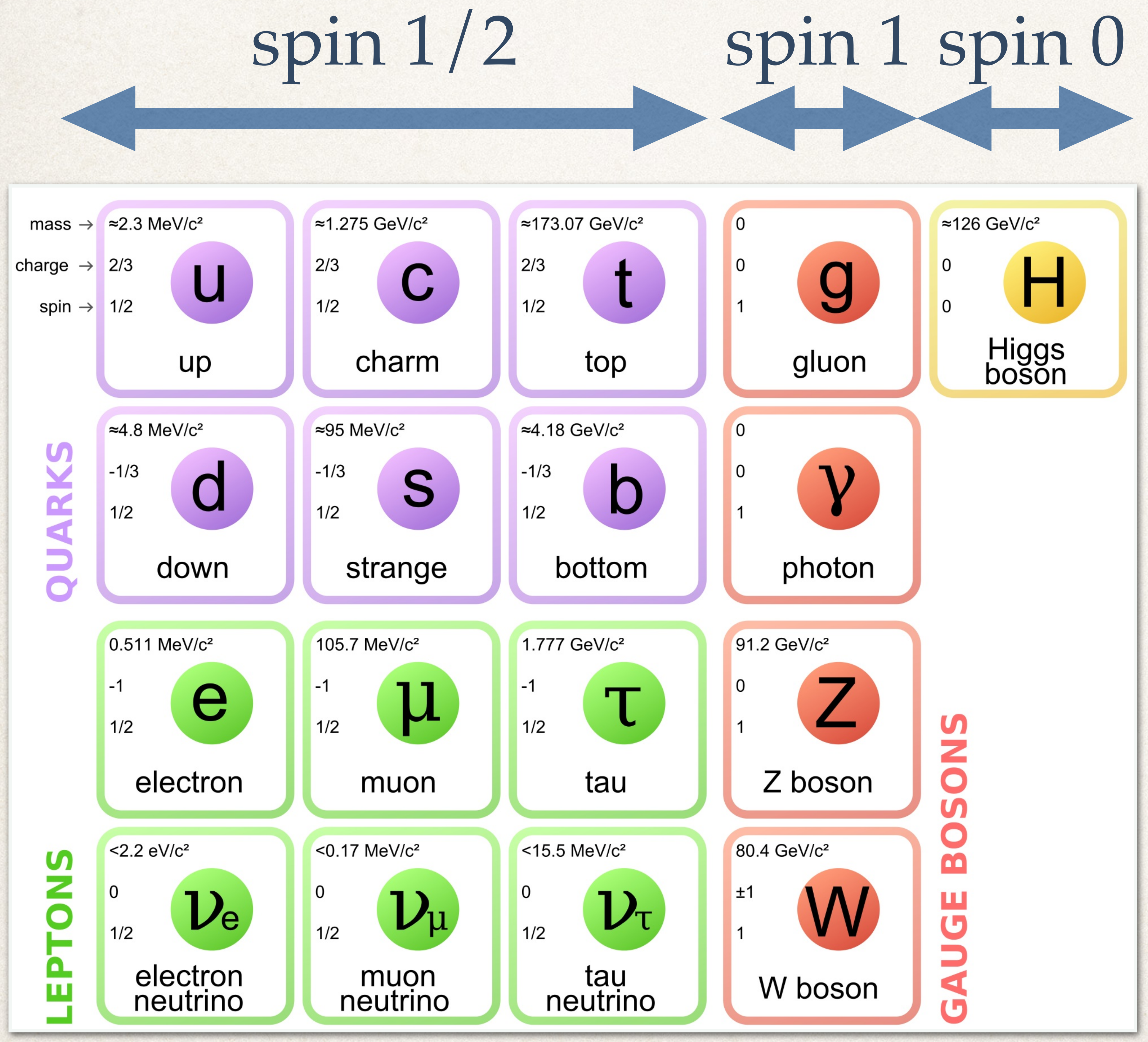
- Canonical quantisation: operator formulation
  - promote the field $\phi(x)$ and its conjugate momenta $\Pi(x) = \dfrac{\partial \mathscr{L}}{\partial(\partial_0 \phi(x))}$ to operators, impose quantisation conditions in the form of equal-time (anti)commutation relations (Heisenberg picture)
  - Analogy with quantisation in QM, where coordinates q_i and momenta p_i become operators $\hat{q}_i$, $\hat{p}_i$ that obey $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$ → "first" and "second" quantisation
  - creation and annihilation operators (again in analogy to QM)
  - results in intrinsically perturbative QFT

- Path integral quantisation
  - Transition amplitude between field configurations $\phi_i(x)$ at time t_i and $\phi_f(x)$ at time t_f given by sum over all possible field configurations, i.e. the quantum field "explores" all possible configurations

$$\int_{\phi_i(x)}^{\phi_j(x)} \mathscr{D}\phi \exp\left(i \int_{t_i}^{t_f} d^4x \, \mathscr{L} \right)$$

  - provides non-perturbative definition of the theory
  - Actual computations often simpler that in the operator formalism

The fields we need



- Scalar fields $\phi(x)$: spin 0
- Spinor fields $\psi_\alpha(x)$: spin 1/2
- Vector fields $A^\mu(x)$: spin 1

→ In QFT, particles correspond to excitation modes of the fields

Scalar field

- Consider free real scalar field with $\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 \leftrightarrow$ neutral spinless particle with mass m

- Euler-Lagrange equation of motion (e.o.m) is the Klein-Gordon equation $(\Box + m^2)\phi = 0$

- The most general solution of e.o.m. is a superposition of plane waves $e^{\pm ikx}$: $\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_\mathbf{k}}\left[a(\mathbf{k})e^{-ikx} + a^*(\mathbf{k})e^{ikx}\right]$

- Quantisation: $\left[\phi(t,\mathbf{x}),\Pi(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$, $\left[\phi(t,\mathbf{x}),\phi(t,\mathbf{y})\right] = 0$, $\left[\Pi(t,\mathbf{x}),\Pi(t,\mathbf{y})\right] = 0$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_\mathbf{k}}\left[a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right] \quad \Rightarrow \quad \left[a(\mathbf{p}),a^\dagger(\mathbf{q})\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \quad \left[a(\mathbf{p}),a(\mathbf{q})\right] = 0 \quad \left[a^\dagger(\mathbf{p}),a^\dagger(\mathbf{q})\right] = 0$$

- analogy to creation and annihilation operators of the harmonic oscillator in QM with one oscillator per each value of k, here relates to particle with $E_\mathbf{k} = (\mathbf{k}^2 + m^2)^{1/2}$

- Fock space of states: sum of an infinite set of Hilbert spaces, each representing an n-particle state
  - vacuum state defined by $a(\mathbf{p})|0\rangle = 0$, $\langle 0|0\rangle = 1$
  - generic n-particle state obtained by acting on vacuum with creation operators $|\mathbf{k_1}\ldots\mathbf{k_n}\rangle = (2E_{\mathbf{k_1}})^{(1/2)}\ldots(2E_{\mathbf{k_n}})^{(1/2)}a^\dagger(\mathbf{k_1})\ldots a^\dagger(\mathbf{k_n})|\mathbf{0}\rangle$

Scalar field

- Consider free real scalar field with $\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 \leftrightarrow$ neutral spinless particle with mass m

- Euler-Lagrange equation of motion (e.o.m) is the Klein-Gordon equation $(\Box + m^2)\phi = 0$

- The most general solution of e.o.m. is a superp

- Quantisation: $\left[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\ [\phi$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\mathbf{k}}}\left[a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right] \quad \Rightarrow \quad [a(\mathbf{p}), a^\dagger(\mathbf{q})] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \quad [a(\mathbf{p}), a(\mathbf{q})] = 0 \quad [a^\dagger(\mathbf{p}), a^\dagger(\mathbf{q})] = 0$$

- analogy to creation and annihilation operators o
particle with $E_{\mathbf{k}} = (\mathbf{k}^2 + m^2)^{1/2}$

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Hamiltonian

$$H = \int d^3x(\Pi\dot{\phi} - \mathscr{L}) \quad \Rightarrow \quad H = \int \frac{d^3k}{(2\pi)^3} E_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k})$$

$$H\, a^\dagger(\mathbf{k})|0\rangle = E_{\mathbf{k}} a^\dagger(\mathbf{k})|0\rangle$$

- Since $|\mathbf{k_1 k_2}\rangle = (2E_{\mathbf{k}_1})^{(1/2)}(2E_{\mathbf{k}_2})^{(1/2)}a^\dagger(\mathbf{k_1})a^\dagger(\mathbf{k_2})|\mathbf{0}\rangle$ and $\left[a^\dagger(\mathbf{k_1}), a^\dagger(\mathbf{k_2})\right] = 0$, it follows

$$|\mathbf{k_2 k_1}\rangle = |\mathbf{k_1 k_2}\rangle$$

i.e. scalar field quanta obey Bose-Einstein statistics → bosons

Scalar field

- Consider free real scalar field with $\mathscr{L} = \frac{1}{2}\partial_\mu$
- Euler-Lagrange equation of motion (e.o.m)
- The most general solution of e.o.m. is a supe
- Quantisation: $\left[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_\mathbf{k}} \left[a(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right]$$

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Complex scalar field: $\mathscr{L} = \partial_\mu\phi^\dagger\,\partial^\mu\phi - m^2\phi^\dagger\phi$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E_\mathbf{k}} \left[a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx}\right]$$

$$H = \int \frac{d^3k}{(2\pi)^3} E_\mathbf{k}[a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k})]$$

$$Q = \int \frac{d^3k}{(2\pi)^3}[a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k})]$$

$$Q\,a^\dagger(\mathbf{k})|0\rangle = (+1)\,a^\dagger(\mathbf{k})|0\rangle \qquad Q\,b^\dagger(\mathbf{k})|0\rangle = (-1)\,b^\dagger(\mathbf{k})|0\rangle$$

$a^\dagger$ creates particles , $b^\dagger$ creates antiparticles

$|\mathbf{k}\rangle$ is a one-particle state with definite momentum. In order to have localised particles one needs to build wave packets

$$|\chi\rangle = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_\mathbf{k}}} f_\chi(\mathbf{k})a^\dagger(\mathbf{k})|0\rangle$$

with $f_\chi(\mathbf{k})$ square-integrable (peaked around some $\mathbf{k_0}$ such that $\langle 0|\phi(x)|\chi\rangle$ is localised

Spinor fields: Dirac

- SM fermions described by 4-component spinor fields

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

- Their e.o.m. is given by the Dirac equation $(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0$
  which can be derived from the Dirac Lagrangian $\mathscr{L} = \bar{\psi}\,(i\gamma^\mu \partial_\mu - m)\,\psi$
  with $\bar{\psi} = \psi^\dagger \gamma^0$ and 4x4 Dirac matrices γ^μ ($\mu = 0,1,2,3$), obeying the algebra $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$

- Explicit form of the Dirac matrices not unique, an example is the Dirac representation $\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ (with Pauli matrices σ^i)

- Canonical quantisation relies on imposing anticommutation relations:

$$\left\{\psi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\right\} = i\delta_{\alpha,\beta}\,\delta^{(3)}(\mathbf{x} - \mathbf{y}) \qquad \left\{\psi_\alpha(\mathbf{x}, t), \psi_\beta(\mathbf{y}, t)\right\} = 0 \qquad \left\{\Pi_\alpha(\mathbf{x}, t), \Pi_\beta(\mathbf{y}, t)\right\} = 0$$

- The general solution of the Dirac equation is a superposition of plane waves $u(p)\,e^{-ipx}$ and $v(p)\,e^{ipx}$ with 4-component spinors $u(p)$ and $v(p)$ fulfilling $(p^\mu\gamma_\mu - m)\,u(p) = 0$ $\quad (p^\mu\gamma_\mu + m)\,v(p) = 0$

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_\mathbf{k}}} \sum_{s=1,2} \left(a_s(\mathbf{k})u^{(s)}(k)e^{-ikx} + b_s^\dagger(\mathbf{k})\bar{v}^{(s)}(k)e^{ikx}\right)$$

Spinor fields: Dirac ctnd.

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3\sqrt{2E_{\mathbf{k}}}} \sum_{s=1,2} \left(a_s(\mathbf{k}) u^{(s)}(k) e^{-ikx} + b_s^\dagger(\mathbf{k}) \bar{v}^{(s)}(k) e^{ikx} \right)$$

- Classically, $u(p)$ corresponds to positive energy solutions $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 + m^2}$, whereas $v(p)$ corresponds to negative energy solutions $E_{\mathbf{p}} = -\sqrt{\mathbf{p}^2 + m^2}$
- For each energy solution, two-fold degeneracy, i.e. $(p^\mu\gamma_\mu - m)\, u(p) = 0$ $(p^\mu\gamma_\mu + m)\, v(p) = 0$ have two solutions each
- They can be identified as helicity eigenstates, $\frac{1}{2}\frac{\mathbf{\Sigma}\,\mathbf{p}}{|\mathbf{p}|} u^{(1,2)} = \pm \frac{1}{2} u^{(1,2)}$ $\frac{1}{2}\frac{\mathbf{\Sigma}\,\mathbf{p}}{|\mathbf{p}|} v^{(1,2)} = \mp \frac{1}{2} v^{(1,2)}$
- After quantisation, interpretation of operators:
  - $a_s^\dagger(\mathbf{k})$ creates fermions, $a_s(\mathbf{k})$ annihilates fermions
  - $b_s^\dagger(\mathbf{k})$ creates antifermions, $b_s(\mathbf{k})$ annihilates antifermions

Spinor fields: Dirac ctnd.

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- $|\mathbf{k}, s; \mathbf{k}, s\rangle \propto a_s^\dagger(\mathbf{k}) a_s^\dagger(\mathbf{k}) |\mathbf{0}\rangle \propto \{a_s^\dagger(\mathbf{k}), a_s^\dagger(\mathbf{k})\} |0\rangle$ and $\left\{a^\dagger(\mathbf{k_1}), a^\dagger(\mathbf{k_2})\right\} = 0$,
  $\Rightarrow |\mathbf{k}, s; \mathbf{k}, s\rangle = 0$ Pauli exclusion principle → Fermi-Dirac statistics

Vector fields

- Charged field, massive case:
  - From Lagrangian $\mathscr{L} = -\frac{1}{4} W^{\dagger}_{\mu\nu} W^{\mu\nu} - \frac{m^2}{2} W^{\dagger}_{\mu} W^{\mu}$ (with $W^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu$) follows the field equation (Proca equation) $\left[\left(\Box + m^2\right) g^{\mu\nu} - \partial^\mu \partial^\nu\right] W_\nu = 0$
  - Solutions given by plane waves of the form $\epsilon_\mu(\mathbf{k}, \lambda)\, e^{\pm ikx}$, $\lambda = 1,2,3$ with 3 independent polarisation vectors $\epsilon_\mu(\mathbf{k}, \lambda)$
    $\epsilon(\mathbf{k}, \lambda) \cdot k = 0, \quad \epsilon(\mathbf{k}, \lambda) \cdot \epsilon(\mathbf{k}, \lambda') = -\delta_{\lambda,\lambda'} \quad \sum_{\lambda=1}^{3} \epsilon^*_\mu(\mathbf{k}, \lambda) \epsilon_\nu(\mathbf{k}, \lambda) = g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}$
  - Quantised vector field $W_\mu(x) = \sum_{\lambda=1}^{3} \int \frac{d^3k}{(2\pi)^3 \sqrt{E_\mathbf{k}}} \left[\epsilon_\mu(\mathbf{k}, \lambda)\, a_\lambda(\mathbf{k}) e^{-ikx} + \epsilon^*_\mu(\mathbf{k}, \lambda)\, b^\dagger_\lambda(\mathbf{k}) e^{ikx}\right]$
- Neutral field, massless case (for m=0 Proca eq. turns in Maxwell eq. $\partial_\mu F^{\mu\nu} = 0$):

$$A_\mu(x) = \sum_{\lambda=0}^{3} \int \frac{d^3k}{(2\pi)^3 \sqrt{E_\mathbf{k}}} \left[\epsilon_\mu(\mathbf{k}, \lambda)\, a_\lambda(\mathbf{k}) e^{-ikx} + \epsilon^*_\mu(\mathbf{k}, \lambda)\, a^\dagger_\lambda(\mathbf{k}) e^{ikx}\right]$$

Vector fields

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Canonical quantisation non-trivial → only two physical polarisations in the massless case, yet 4 degrees of freedom

Recap: free fields

- Scalar fields $\qquad |k\rangle = \sqrt{2E_k}\, a^\dagger(\mathbf{k})\,|0\rangle$

$\langle 0|\,\phi(x)\,|k\rangle = e^{-ikx} \qquad \langle k|\,\phi(x)\,|0\rangle = e^{ikx}$

- Fermion fields $\qquad |k,s\rangle = \sqrt{2E_k}\, a_s^\dagger(\mathbf{k})\,|0\rangle$

$\langle 0|\,\psi(x)\,|k,s\rangle = u^{(s)}(k)\, e^{-ikx} \qquad \langle k,s|\,\bar\psi(x)\,|0\rangle = \bar u^{(s)}(k)\, e^{ikx}$

- Antifermion fields $\qquad |k,s\rangle = \sqrt{2E_k}\, b_s^\dagger(\mathbf{k})\,|0\rangle$

$\langle 0|\,\bar\psi(x)\,|k,s\rangle = \bar v^{(s)}(k)\, e^{-ikx} \qquad \langle k,s|\,\psi(x)\,|0\rangle = v^{(s)}(k)\, e^{ikx}$

- Vector fields $\qquad |k,\lambda\rangle = \sqrt{2E_k}\, a_\lambda^\dagger(\mathbf{k})\,|0\rangle$

$\langle 0|\,A_\mu(x)\,|k,\lambda\rangle = \epsilon_\mu(\mathbf{k},\lambda)\, e^{-ikx} \qquad \langle k,\lambda|\,A_\mu(x)\,|0\rangle = \epsilon_\mu^*(\mathbf{k},\lambda)\, e^{ikx}$

Recap: free fields

- Scalar fields $|k\rangle = a^\dagger(\mathbf{k})|0\rangle$

  $\langle 0|\phi(x)|k\rangle = e^{-ikx}$ $\langle k|\phi(x)|0\rangle = e^{ikx}$

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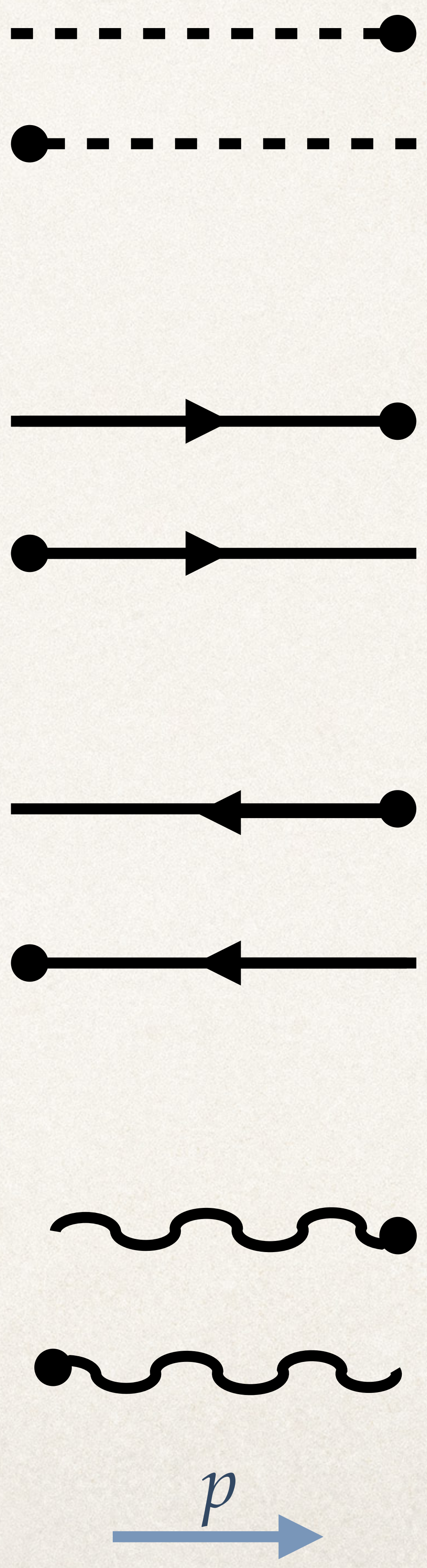
  $\langle 0|\psi(x)|k,s\rangle = u^{(s)}(k)\,e^{-ikx}$ $\langle k,s|\bar\psi(x)|0\rangle = \bar u^{(s)}(k)\,e^{ikx}$

- Antifermion fields $|k,s\rangle = b_s^\dagger(\mathbf{k})|0\rangle$

  $\langle 0|\bar\psi(x)|k,s\rangle = \bar v^{(s)}(k)\,e^{-ikx}$ $\langle k,s|\psi(x)|0\rangle = v^{(s)}(k)\,e^{ikx}$

- Vector fields $|k,\lambda\rangle = a_\lambda^\dagger(\mathbf{k})|0\rangle$

  $\langle 0|A_\mu(x)|k,\lambda\rangle = \epsilon_\mu(\mathbf{k},\lambda)\,e^{-ikx}$ $\langle k,\lambda|A_\mu(x)|0\rangle = \epsilon_\mu^*(\mathbf{k},\lambda)\,e^{ikx}$



1 incoming
1 outgoing

$u(k)$ incoming
$\bar u(k)$ outgoing

$\bar v(k)$ incoming
$v(k)$ outgoing

$\epsilon_\mu(\mathbf{k},\lambda)$ incoming
$\epsilon_\mu^*(\mathbf{k},\lambda)$ outgoing

Propagators

- So far: free particles. Eventually: interactions
- For simplicity, consider scalar fields. Interaction of the field $\phi(x)$with a source $J(x)$ will modify the Klein-Gordon eq.

$$(\partial_\mu \partial^\mu + m^2)\, \phi(x) = J(x)$$

  which can be obtained from the Lagrangian $\mathscr{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 + J\phi$

- An inhomogeneous equation of this sort can be solved provided the Green's function is known, i.e. the solution to the field equation with a delta function source, here

$$(\partial_\mu \partial^\mu + m^2) G(x-y) = -\,\delta^{(4)}(x-y)$$

- Fourier transformation $\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}$, $G(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} G(k)$ leads to $(k^2 - m^2)\, G(k) = 1$

- The solution $G_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$ is known as the Feynman propagator

Propagators ctnd.

$$G_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$$

- Using the field expansion expression and the properties of the $a^\dagger$, a operators, the amplitude for particle propagation from y to x is

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_{\mathbf{k}}} e^{-ik\cdot(x-y)}$$

- Integrating over k^0 in the Feynman propagator yields

$$iG_F(x-y) = \int \frac{d^3k}{(2\pi)^3 k^0} \left[e^{-ik\cdot(x-y)}\Theta(x^0-y^0) + e^{ik\cdot(x-y)}\Theta(y^0-x^0)\right]_{k^0=E_{\mathbf{k}}} = \langle 0|\phi(x)\phi(y)|0\rangle\Theta(x^0-y^0) + \langle 0|\phi(y)\phi(x)|0\rangle\Theta(y^0-x^0)$$

The appearance of the theta functions results from the $+i\epsilon$ term in the denominator, providing prescription how to treat the poles at $k^2 = m^2$

- Time-ordering operator T arranges operators in chronological order, from right to left: $iG_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle$
  - Propagation of a particle from y to x if $x^0 > y^0$
  - Propagation of a particle from x to y if $y^0 > x^0$, or propagation of an antiparticle for complex fields; $iG_F(x-y) = \langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle$

Feynman propagators

In position-space | **In momentum-space**

- Scalar field

$$\langle 0|T(\phi(x)\phi^\dagger(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)}$$

$$\frac{i}{k^2-m^2+i\epsilon}$$

- Fermion field

$$\langle 0|T(\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i(k_\mu\gamma^\mu+m)}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)}$$

$$\frac{i\left(k_\mu\gamma^\mu+m\right)}{k^2-m^2+i\epsilon}$$

- Massive vector field

$$\langle 0|T(W_\mu(x)\bar{W}_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i\left(-g_{\mu\nu}+k_\mu k_\nu/m^2\right)}{k^2-m^2+i\epsilon}e^{-ik\cdot(x-y)}$$

$$\frac{i\left(-g_{\mu\nu}+k_\mu k_\nu/m^2\right)}{k^2-m^2+i\epsilon}$$

- Massless vector field (Feynman gauge)

$$\langle 0|T(A_\mu(x)\bar{A}_\nu(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2+i\epsilon}e^{-ik\cdot(x-y)}$$

$$\frac{-ig_{\mu\nu}}{k^2+i\epsilon}$$

Gauge fixing

- EM wave has two degrees of freedom: two polarisation vectors for transverse polarisation $\boldsymbol{\epsilon}(\mathbf{k},\lambda)\mathbf{k} = 0,\ (\lambda = 1{,}2)$ but Lorentz covariant formulation of Maxwell eqs. uses on the 4-vector potential A^μ

- The Maxwell Lagrangian is invariant under the gauge transformation $A_\mu \to A_\mu - \partial_\mu \theta$ with θ an arbitrary regular function. The gauge transformation can be used to remove unphysical polarisations

- The equation for the propagator of the massless vector field $(-k^2 g^{\mu\nu} + k^\mu k^\nu) G_{\nu\rho} = g^\mu_\rho$ does not have a solution

- Canonical quantisation non-trivial (redundant d.o.f or non-covariant formulation)

- In covariant quantisation one adds a gauge-fixing term $\mathscr{L}_{GF}$ to the Maxwell Lagrangian (and imposes a Lorenz-condition-like restriction on the Fock space)

$$\mathscr{L}_{GF} = -\frac{1}{2\zeta}(\partial^\mu A^a_\mu)^2$$

ζ: arbitrary finite parameter ($\zeta = 1$ Feynman gauge, $\zeta = 0$ Landau gauge)

$$= \frac{-i\delta_{ab}}{p^2 + i\epsilon}\left(g^{\mu\nu} - (1-\zeta)p^\mu p^\nu / p^2\right)$$

- The procedure breaks gauge invariance, but physical results are independent of the gauge.

Interactions

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}}$$

free part   interaction part

- Use perturbation theory (→interaction as a small perturbation to the free theory) to calculate physical quantities such as cross sections etc.

- Interaction localised in a region of spacetime → treat particles as free at far away in the past and in the future (free asymptotic states)

$$|\psi(t=-\infty)\rangle = |p_1,\ldots,p_n;\text{in}\rangle \qquad |\psi(t=\infty)\rangle = |p'_1,\ldots,p'_m;\text{out}\rangle$$

- Transition amplitude for a scattering process defines the unitary S-matrix operator

$$\langle p'_1,\ldots,p'_m;\text{out}\,|\,p_1,\ldots,p_n;\text{in}\rangle = \langle\psi(t=\infty)\,|\,\psi(t=-\infty)\rangle = \langle f\,|\,S\,|\,i\rangle = S_{fi}$$ with $|\psi(t=-\infty)\rangle = |i\rangle$ and $|\psi(t=\infty)\rangle = S|i\rangle$

- $S^\dagger S = \mathbf{1} \quad \Rightarrow \quad \sum_k S^*_{kf} S_{ki} = \delta_{fi} \quad \Rightarrow \quad \sum_k |S_{ki}|^2 = 1$ probabilities over all $i \to k$ transitions sum up to 1

probability conservation

S-matrix and Feynman rules

- Dyson expansion of the S operator $S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots \int d^4x_n T\left(\mathcal{H}_{\text{int}}(x_1)\dots\mathcal{H}_{\text{int}}(x_n)\right)$

with $\mathcal{H}_{\text{int}}$ the interaction part of the Hamiltonian density in the interaction picture

$\Rightarrow$ calculation of $\langle p_1', \dots, p_m' | S | p_1, \dots, p_n \rangle$ involves time-ordered products of field operators

$\rightarrow$ consider e.g. $\langle 0 | a(\mathbf{p_1'})\dots a(\mathbf{p_m'}) | T(\phi(x_1)\dots\phi(x_l)) | a^\dagger(\mathbf{p_1})\dots a^\dagger(\mathbf{p_n}) | 0 \rangle$

- Wick's theorem enables decomposing generic $\langle 0 | T(\phi(x_1)\dots\phi(x_n) | 0 \rangle$ into products of propagators $\langle 0 | T(\phi(x_i)\phi(x_j)) | 0 \rangle$ e.g.

$\langle 0 | T(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) | 0 \rangle = G_F(x_1 - x_2)G_F(x_3 - x_4) + G_F(x_1 - x_3)G_F(x_2 - x_4) + G_F(x_1 - x_4)G_F(x_2 - x_3)$

- In reality, need to be more careful as e.g. vacuum of the theory also affected by interactions
    - $\rightarrow$ Lehmann-Symanzik-Zimmerman formula relates $\langle p_1', \dots, p_m' | S | p_1, \dots, p_n \rangle$ with $\langle 0 | T(\phi(x_1)\dots\phi(x_m)\phi(y_1)\phi\dots(y_n) | 0 \rangle$
    - The resulting expressions for the transition amplitudes can be given a graphical representation as building blocks of the diagrams depicting the process $\rightarrow$ Feynman rules

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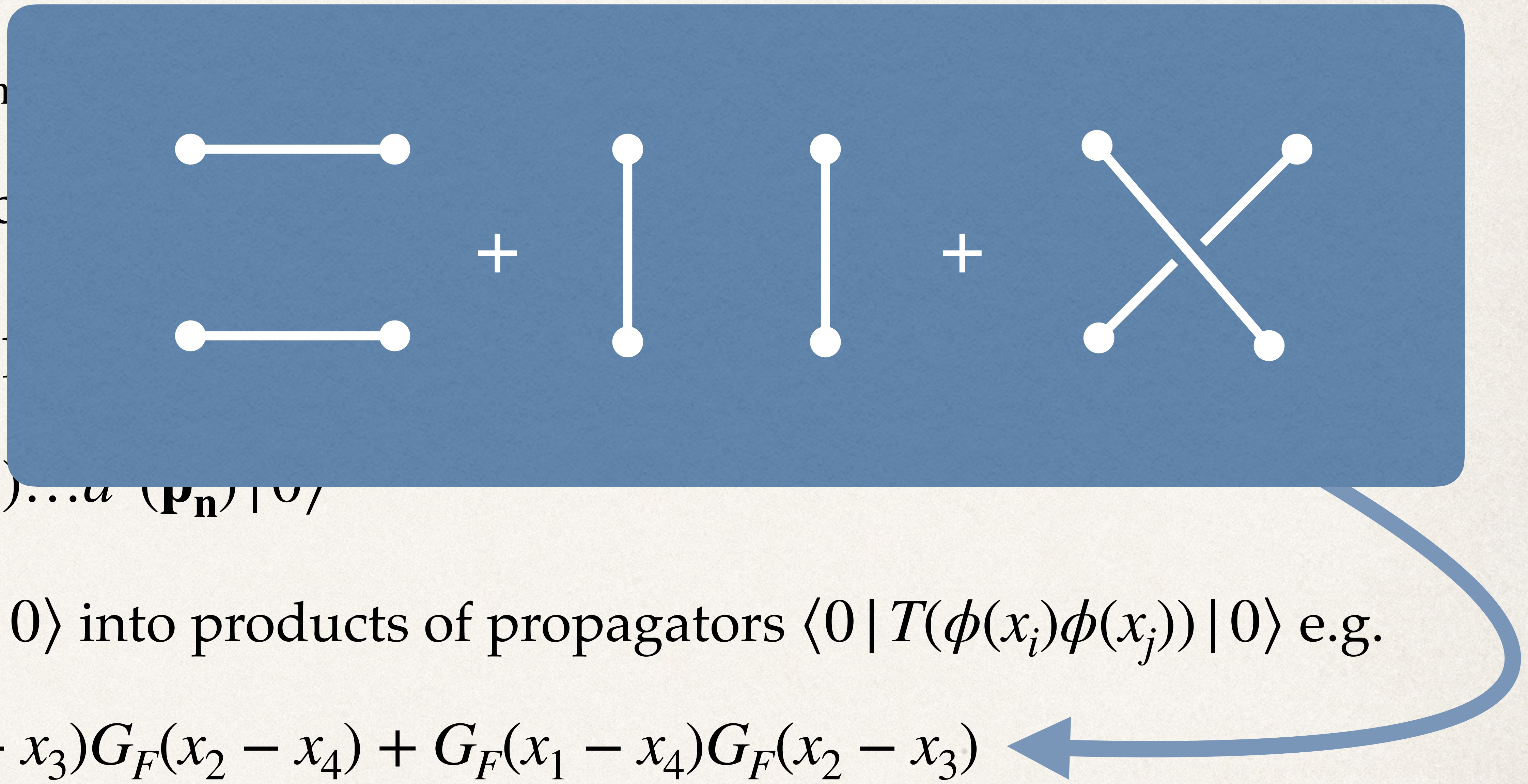

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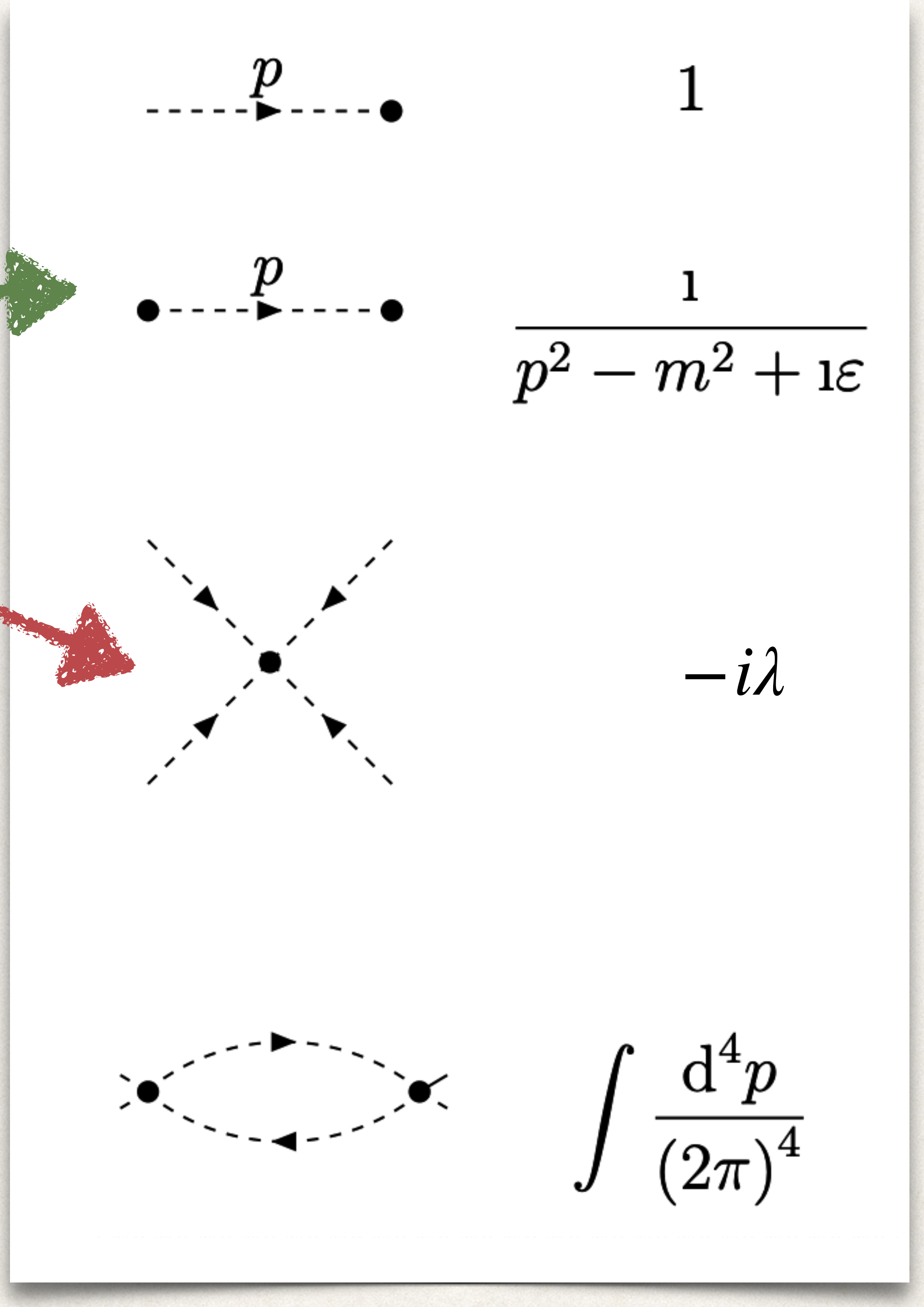
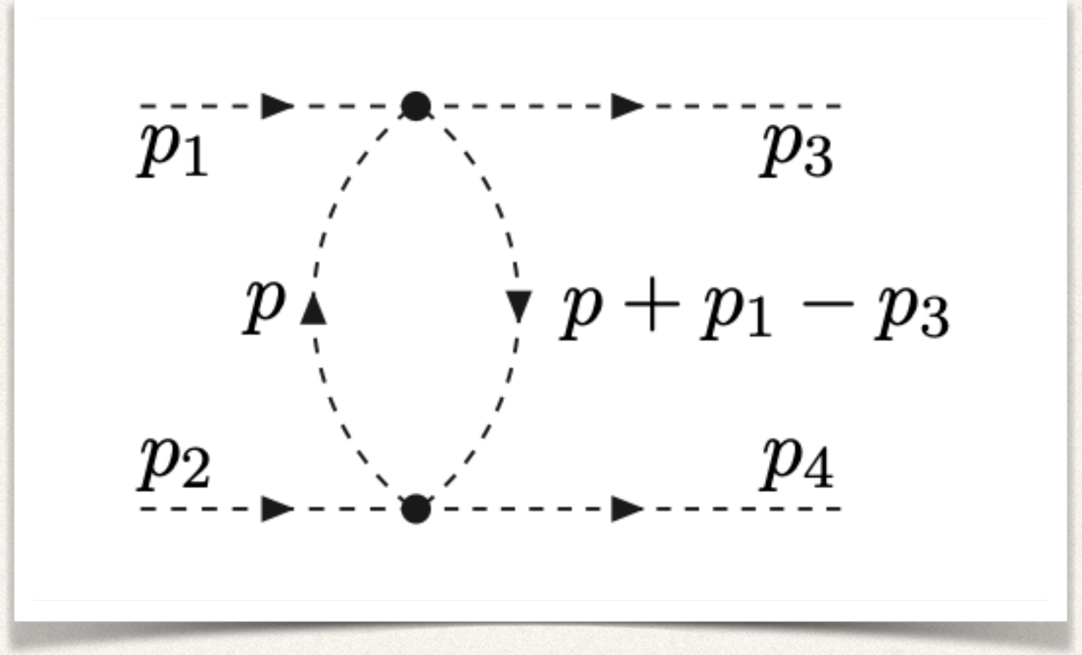
Feynman rules, ϕ^4 theory

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

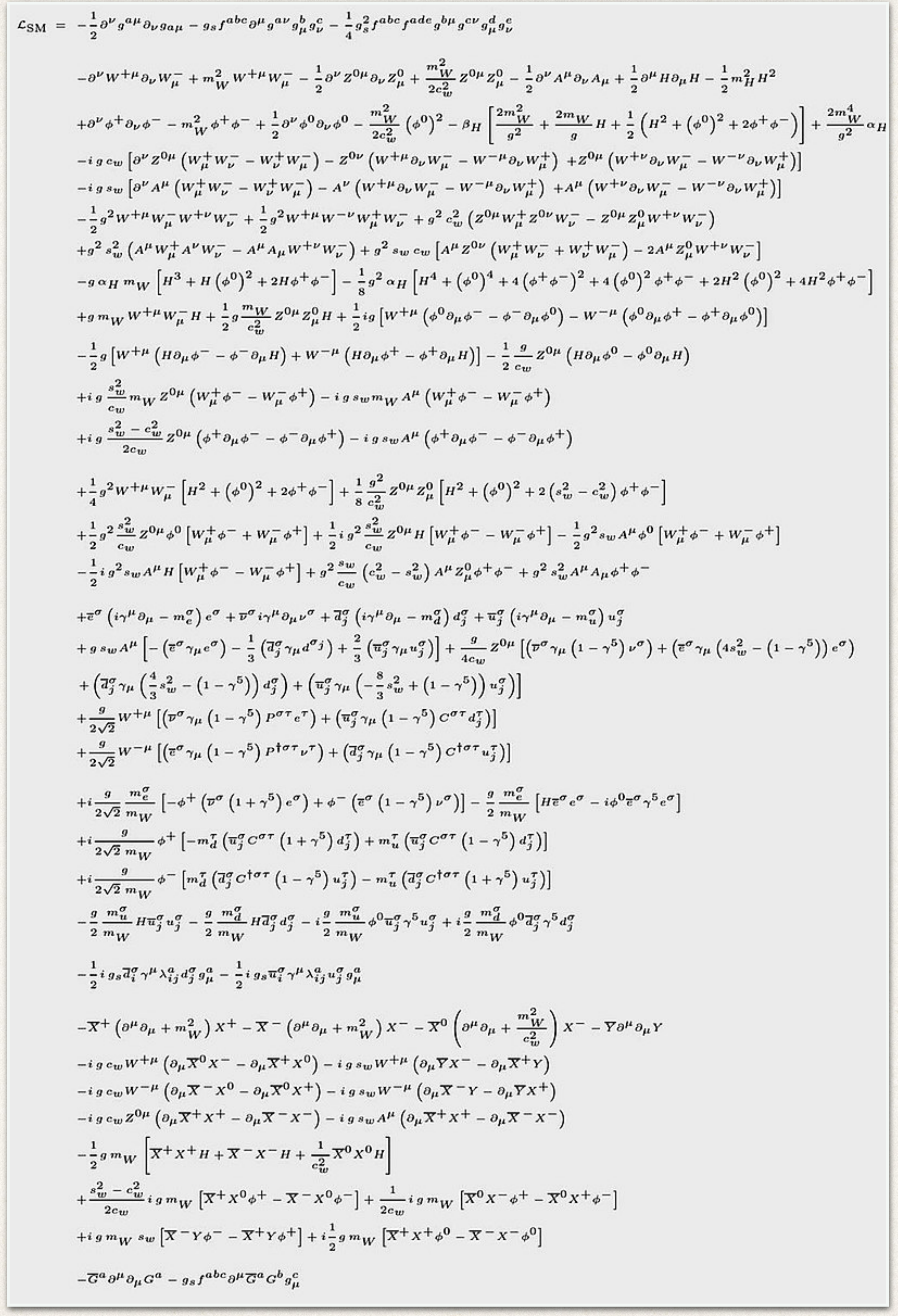




$$\frac{1}{2}(-i\lambda)^2 \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2} \frac{1}{(p + p_1 - p_3)^2 - m^2}$$

Guiding principles

- Symmetry principle
  - gauge invariance but also Lorentz and CPT invariance
- Unitarity (conservation of probability)
- Renormalisability (finite predictions)
- Correspondance to already existing, well-tested theories: QED, Fermi theory,..
- Minimality: no unnecessary fields or interactions other than those needed to explain observation



picture credit: T.D. Gutierrez

Construction tools: groups

- Mathematical language of symmetry is group theory
- A group G is a set of elements g_i with a multiplication law

$$g_j \cdot g_k \in G$$

  with a unity, an inverse and associativeness.
  - Example: U(N) consisting of of NxN unitary matrices $UU^\dagger = U^\dagger U = 1$
- Special group: elements are matrices with determinant = 1
  - Example: unitary special groups SU(N)
- Abelian groups: elements obey $g_j \cdot g_k = g_k \cdot g_j$
  - Example: unitary group U(1) consisting of a set of phase factors $e^{i\alpha}$
- Non-abelian groups: $g_j \cdot g_k \neq g_k \cdot g_j$
  - Example: U(N), SU(N), …
- Direct product $G \times H$ of two groups G and H, $[g_i, h_j] = 0$

  has a multiplication law for elements (g_i, h_j)

$$(g_k, h_l) \cdot (g_m, h_n) = (g_k \cdot g_m, h_l \cdot h_n)$$

Construction tools: Lie groups

- Representation of a group is a special realisation of the multiplication law. Set of matrices $\{R(g_i)\}$ such that
  if $g_i \cdot g_j = g_k$ then $R(g_i)R(g_j) = R(g_k)$

- A general gauge symmetry group G is a compact Lie group

$$g(\alpha^1, \ldots, \alpha^k, \ldots) \in G \qquad g(\boldsymbol{\alpha}) = \exp(i\alpha^k T^k)$$

$\alpha^k = \alpha^k(x) \in \mathbb{R}$ $\quad$ T^k= Hermitian generators of the group $\quad$ Lie algebra: $[T^i, T^j] = if^{ijk}T^k$

$\mathrm{Tr}[T^i T^j] \equiv \delta_{ij}/2$ $\quad$ structure constants: $f^{ijk} = 0$ for abelian groups, $f^{ijk} \neq 0$ for non-abelian groups

- Example: SU(2) $\quad g(\alpha^1, \alpha^2, \alpha^3) = \exp[i\alpha^k T^k] \quad k = 1,2,3$

$$f^{ijk} = \epsilon_{ijk} \qquad T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(Pauli matrices/2)

- SU(N) has $N^2 - 1$ linearly independent generators which are traceless hermitian matrices

Construction tools: group representations

- Fundamental representation with dimension N
  - unitary NxN matrices
  - N^2-1 generators T^k
  - fermion transformations in the SM

- Adjoint representation with dimension N^2-1
  - unitary (N^2-1)x (N^2-1) matrices
  - N^2-1 generators $\left(T^k_{adj}\right)_{ij} = -if_{kij}$
  - gauge boson transformations in the SM

Examples

- SU(2): 3 generators, $f^{ijk} = \epsilon_{ijk}$,

  fundamental rep: $T^k = \sigma^k/2$ (Pauli matrices/2)

  adjoint rep: $\left(T^k_{adj}\right)_{ij} = -if_{kij} = -i\epsilon_{kij}$

- SU(3): 8 generators

  fundamental rep: $T^k = \lambda^k/2$ (Gell-Mann matrices/2)

  adjoint rep: $\left(T^k_{adj}\right)_{ij} = -if_{kij}$

The gauge paradigm: QED

- The free Dirac field Lagrangian

$$\mathscr{L}_{\text{Dirac}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

is invariant under global phase U(1) transformations

$$\psi \to e^{i\alpha}\psi \qquad \bar{\psi} \to e^{-i\alpha}\bar{\psi} \qquad (\alpha = \text{ constant phase} \qquad \bar{\psi} = \psi^\dagger \gamma^0)$$

- Under local phase ("gauge") U(1) transformations

$$\psi \to e^{i\alpha(x)}\psi, \quad \bar{\psi} \to e^{-i\alpha(x)}\bar{\psi} \qquad \partial_\mu \psi(x) \to e^{i\alpha(x)}\partial_\mu \psi(x) + ie^{i\alpha(x)}\partial_\mu \alpha(x)\, \psi(x)$$

→ introduce covariant derivative with the transformation rule $D_\mu \psi(x) \to e^{i\alpha(x)} D_\mu \psi(x)$

so that $\mathscr{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x)$ is invariant

fulfilled by $D_\mu \equiv \partial_\mu + igA_\mu(x)$ with a new vector field $A_\mu(x)$ transforming as $A_\mu \to A_\mu - \frac{1}{g}\partial_\mu \alpha(x)$

The gauge paradigm: QED (2)

- $\mathscr{L} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x)$ is invariant with $D_\mu = \partial_\mu + igA_\mu(x)$

$$\mathscr{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) - g\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x)$$

interaction piece of the fermion field with a gauge vector (photon) field with g the electric charge of the electron

- Full QED Lagrangian obtained by adding the Maxwell Lagrangian for a vector field $A_\mu(x)$

$$\mathscr{L}_{\text{QED}} = \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi(x) - \frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is also invariant under the local phase transformation

- Since $A_\mu A^\mu$ not gauge invariant, the term is not allowed → massless photon

The gauge paradigm: QED (2)

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Gauge principle: invariance of theory under local symmetry
Promoting global symmetry to local leads to an interacting theory

Non-abelian gauge theories

- Consider now a general case when the local symmetry transformation of fields form a non-abelian group SU(N)

$$\psi(x) \to U(\alpha(x))\psi(x) \quad \text{with} \quad U(\alpha(x)) = \exp\left[ig\alpha^k(x)T^k\right] \qquad k = 1,\ldots,N^2-1$$

- T^k are the generators of the group SU(N) obeying the group algebra $\left[T^i, T^j\right] = if^{ijk}T^k$

- In analogy to QED $\partial_\mu\psi(x) \to \exp\left[ig\alpha^k(x)T^k\right]\partial_\mu\psi(x) + ig(\partial_\mu\alpha^k(x))T^k\exp\left[ig\alpha^k(x)T^k\right]\psi(x)$
  and the Lagrangian $\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is not invariant under the transformation

- Way out: introduce
  vector gauge fields $W^\mu = W^{\mu,1}T^1 + W^{\mu,2}T^2 + \ldots = W^{\mu,k}T^k$
  covariant derivative $D^\mu\psi \equiv (\partial^\mu + igW^\mu)\psi$

- Requesting gauge invariance of $\bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ means $D^\mu\psi \to UD^\mu\psi$ and $D^\mu \to UD^\mu U^{-1}$

- It follows

$$W^\mu \to UW^\mu U^{-1} - \frac{i}{g}U(\partial^\mu U^{-1})$$

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Non-abelian gauge theories (2)

- Transformations: $\psi(x) \to \exp\left[ig\alpha^k(x)T^k\right]\psi(x)$ $\quad D^\mu \to UD^\mu U^{-1}$ $\quad W^\mu \to UW^\mu U^{-1} - \frac{i}{g}U(\partial^\mu U^{-1})$

- Generalisation of the QED field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = -\frac{i}{e}[D^\mu, D^\nu]$ to $W^{\mu\nu} \equiv -\frac{i}{g}[D^\mu, D^\nu]$

  Since $D^\mu\psi = (\partial^\mu + igW^\mu)\psi$ it follows $W^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu + ig\left[W^\mu, W^\nu\right]$

  and from $W^\mu = W^{\mu,k}T^k$ $\quad \Rightarrow W^{\mu\nu,k} = \partial^\mu W^{\nu,k} - \partial^\nu W^{\mu,k} - gf^{ijk}W^{\mu,i}W^{\nu,j}$

- Transformation of the field tensor: $W^{\mu\nu} \to UW^{\mu\nu}U^{-1}$

- The kinetic term $-\frac{1}{4}W^k_{\mu\nu}W^{\mu\nu,k} = -\frac{1}{2}\mathrm{Tr}\left[W_{\mu\nu}W^{\mu\nu}\right]$ is then gauge invariant and hence the Lagrangian

$$\mathcal{L}_{YM} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2}\mathrm{Tr}\left[W_{\mu\nu}W^{\mu\nu}\right]$$

is also gauge invariant

General features of non-abelian gauge theories

- $N^2 - 1$ generators of the SU(N) symmetry group → $N^2 - 1$ gauge fields

- Similarly to QED, the interaction of gauge fields with fermion fields is given by the $-g\,\bar{\psi}\gamma^{\mu}T^{k}W^{k}_{\mu}\psi$ term in the Lagrangian

- New types of interaction in comparison with an abelian theory: from $-\frac{1}{4}W^{k}_{\mu\nu}W^{\mu\nu,k}$ with $W^{\mu\nu,k} = \partial^{\mu}W^{\nu,k} - \partial^{\nu}W^{\mu,k} - gf^{ijk}W^{\mu,i}W^{\nu,j}$ follow terms that are cubic and quartic in gauge boson fields → gauge bosons interact with each other

- Gauge bosons are massless since the term $W^{k}_{\mu}W^{\mu,k}$ is not invariant under local gauge transformations

- Gauge invariance fixes the strength of the gauge boson self-interactions and interactions with the fermion fields in terms of a single parameter g

QCD Lagrangian

- The kinetic part for the gluon field

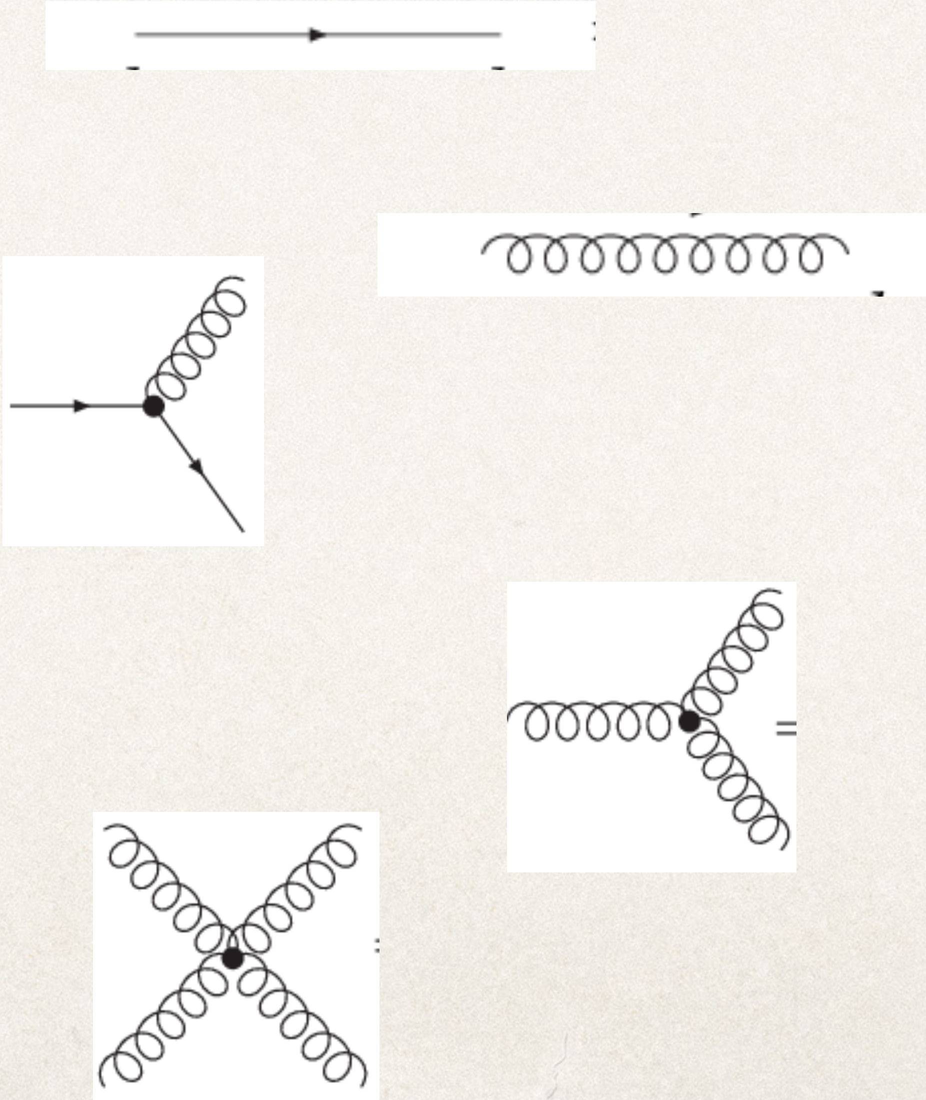
$$\mathscr{L}_G = -\frac{1}{4}F^k_{\mu\nu}F^{\mu\nu,k} \qquad\qquad F^{\mu\nu,k} = \partial^\mu A^{\nu,k} - \partial^\nu A^{\mu,k} - g_s f^{ijk} A^{\mu,i} A^{\nu,j}$$

carries information about triple and quartic gluon self-interactions.

- Altogether, summing over flavours

$$\begin{aligned}\mathscr{L}_{QCD} = \sum_f \bar{\psi}^{(f)}\,(i\gamma^\mu\partial_\mu - m_f)\,\psi^{(f)} \\ -(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\ -g_s\,\bar{\psi}(f)\gamma^\mu T^a A^a_\mu \psi(f) \\ -\frac{1}{2}g_s\,(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f_{abc} A^{\mu,b} A^{\nu,c} \\ -\frac{1}{4}g_s^2\, f_{abc}\, A^{\mu,b} A^{\nu,c} f_{ade}\, A^d_\mu A^e_\nu\end{aligned}$$

Feynman rules



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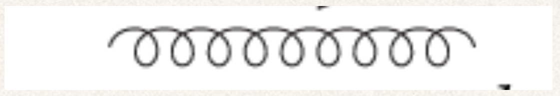
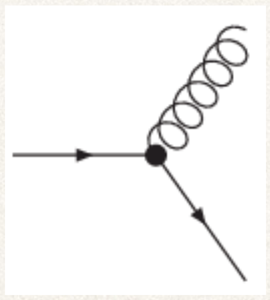
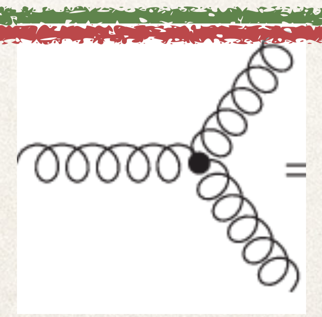
Feynman rules

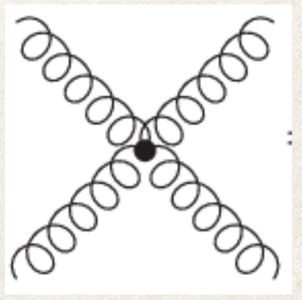
$$\begin{aligned}\mathscr{L}_{QCD} = & \sum_f \bar{\psi}^{(f)} \left(i\gamma^\mu \partial_\mu - m_f\right) \psi^{(f)} \\ & -(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\ & -g_s\, \bar{\psi}(f) \gamma^\mu T^a A^a_\mu \psi(f) \\ & -\frac{1}{2} g_s\, (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f_{abc} A^{\mu,b} A^{\nu,c} \\ & -\frac{1}{4} g_s^2\, f_{abc}\, A^{\mu,b} A^{\nu,c} f_{ade}\, A^d_\mu A^e_\nu\end{aligned}$$



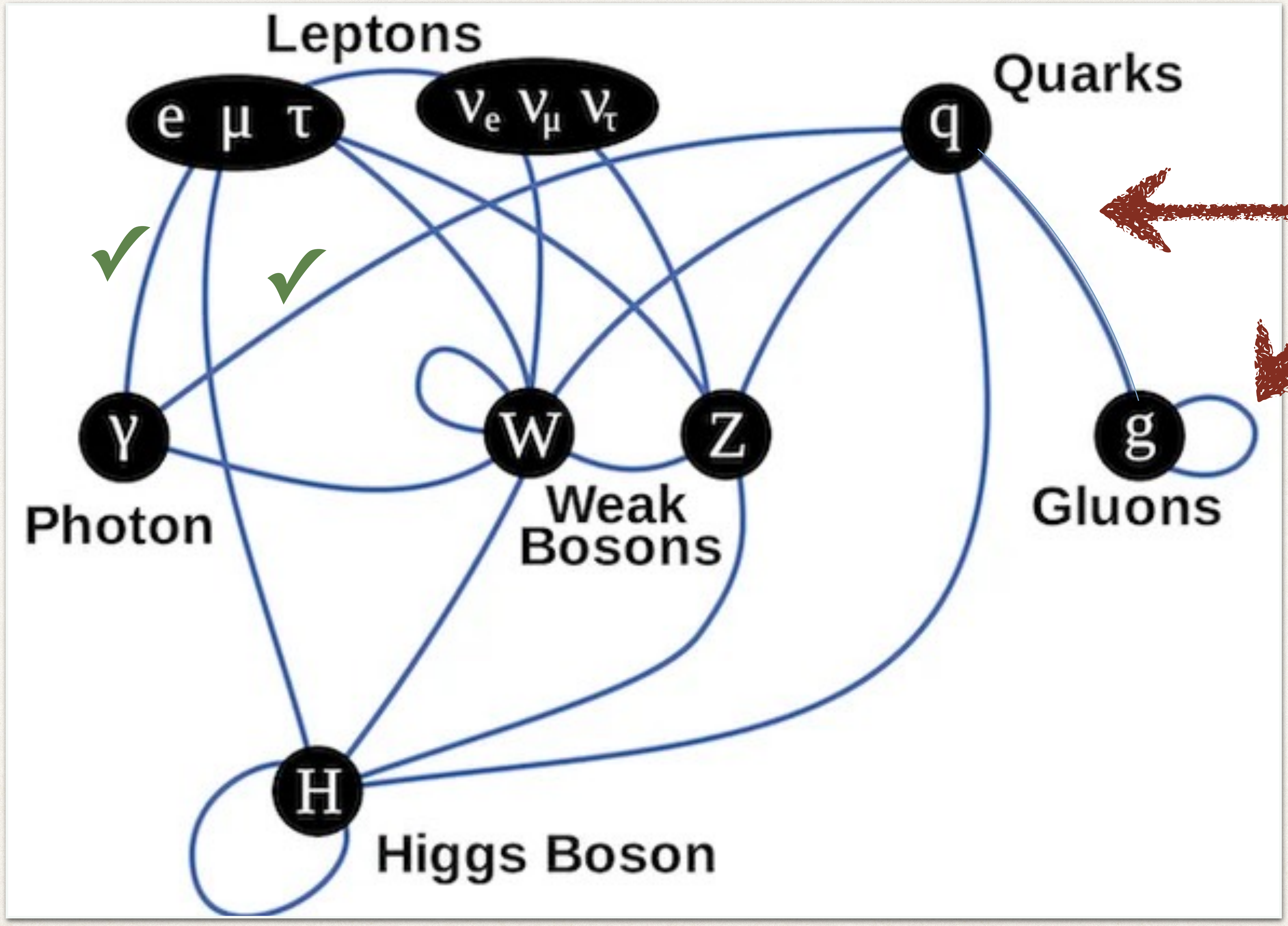


QED-like





non-abelian



after the coffee break!