Karlsruhe Institute of Technology

Collaborative Research Center TRR 257

Particle Physics Phenomenology after the Higgs Discovery

Lecture 1: basics of QCD

Gudrun Heinrich

Karlsruhe Institute of Technology

gudrun.heinrich@kit.edu

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Outline

- Introduction
- Properties of the strong interactions
- Experimental evidence for colour
- QCD Lagrangian
- Colour algebra
- Feynman rules

QCD

Quantum ChromoDynamics is a very rich field!

asymptotic freedom

confinement

strong CP-problem

QCD at finite temperature

spectroscopy

lattice gauge theory

quark-gluon-plasma

flavour puzzles

We will focus on perturbative QCD

Importance of QCD corrections

perturbation theory in the strong coupling α_s

$$\sigma = \sigma^{\mathrm{LO}} + \alpha_s \sigma^{\mathrm{NLO}} + \alpha_s^2 \sigma^{\mathrm{NNLO}} + \mathcal{O}(\alpha_s^3)$$

leading order

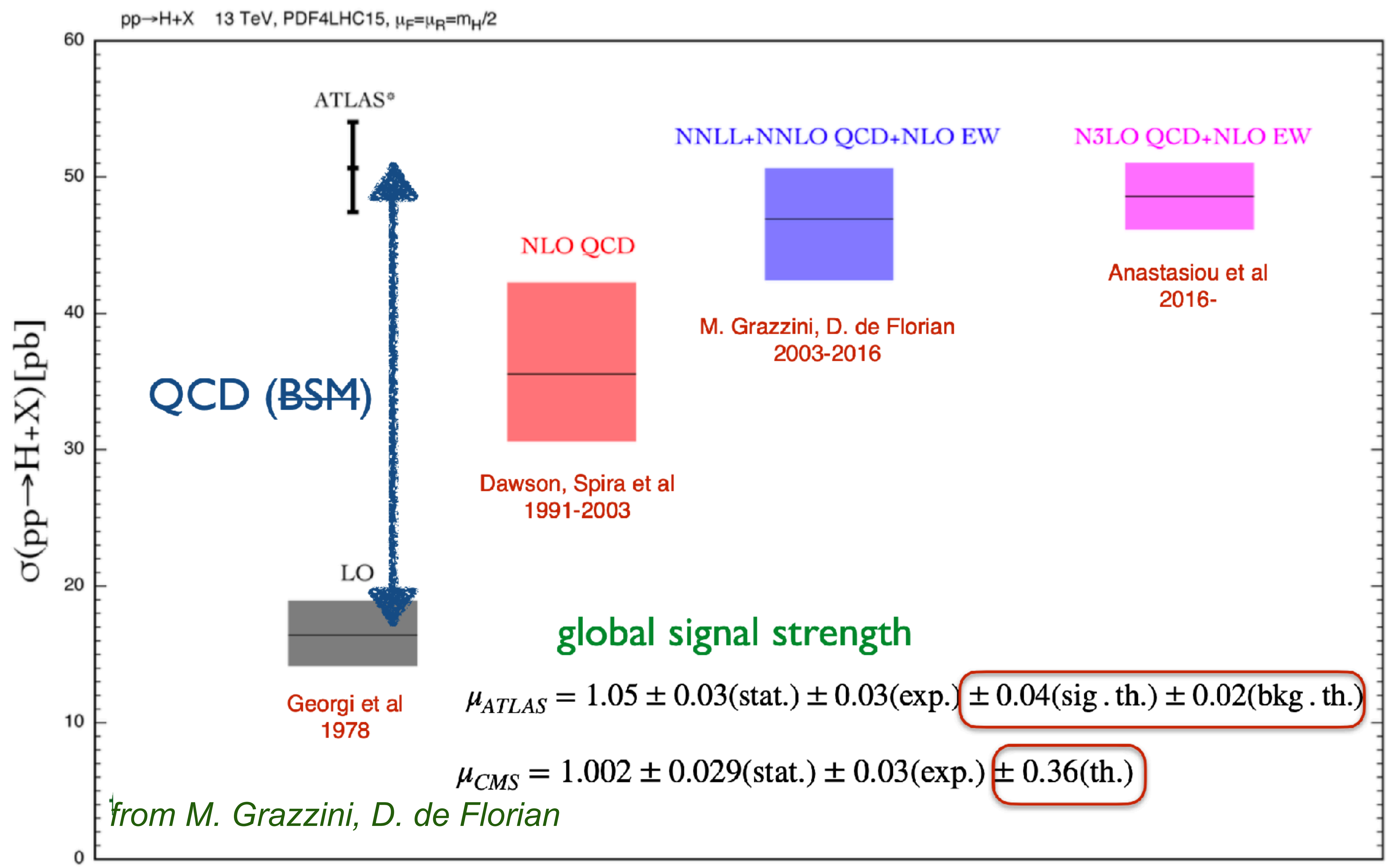
next-to-leading order

next-to-next-to-leading order

$\alpha_s(M_Z) \simeq 0.118 \Rightarrow$ NLO corrections ~ $\mathcal{O}(10\%)$

NNLO corrections typically a few %

but there are prominent exceptions,
e.g. Higgs production in gluon fusion:
NLO corr. ~100%, NNLO ~30%



global signal strength

$$\mu_{ATLAS} = 1.05 \pm 0.03(\mathrm{stat.}) \pm 0.03(\mathrm{exp.}) \boxed{\pm 0.04(\mathrm{sig\,.\,th.}) \pm 0.02(\mathrm{bkg\,.\,th.})}$$

$$\mu_{CMS} = 1.002 \pm 0.029(\mathrm{stat.}) \pm 0.03(\mathrm{exp.}) \boxed{\pm 0.36(\mathrm{th.})}$$

from M. Grazzini, D. de Florian

Importance of QCD corrections

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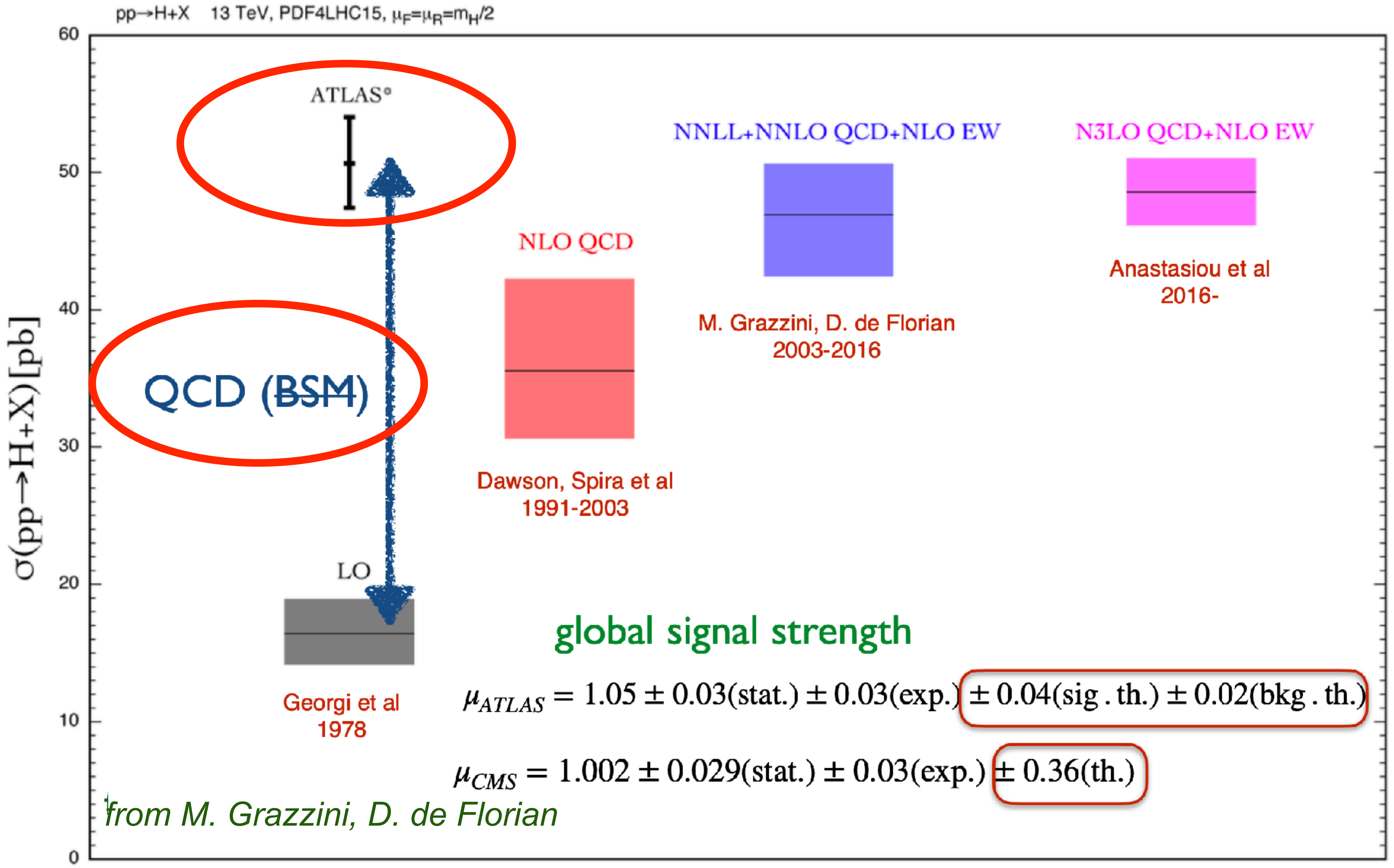

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QCD

- QCD is the theory of one of the 4 fundamental forces in nature:
  the strong interactions

- it describes the interactions between quarks and gluons, also called *partons*

- what does “strong” interaction mean?
  about 10^{38} times larger than the gravitational force (at length scales of the size of a nucleon, ~1fm)

- however the strong coupling is not a constant, it depends on the energy

- at high energies, the coupling is small: ⟶ asymptotic freedom

- at small energies: coupling large, no free quarks and gluons
  ⟶ confinement (hadrons)

Hadronic collisions

- at hadron colliders (e.g. CERN LHC), QCD is everywhere

- need to factorise perturbative from non-perturbative part

important concepts:

- factorisation
- asymptotic freedom

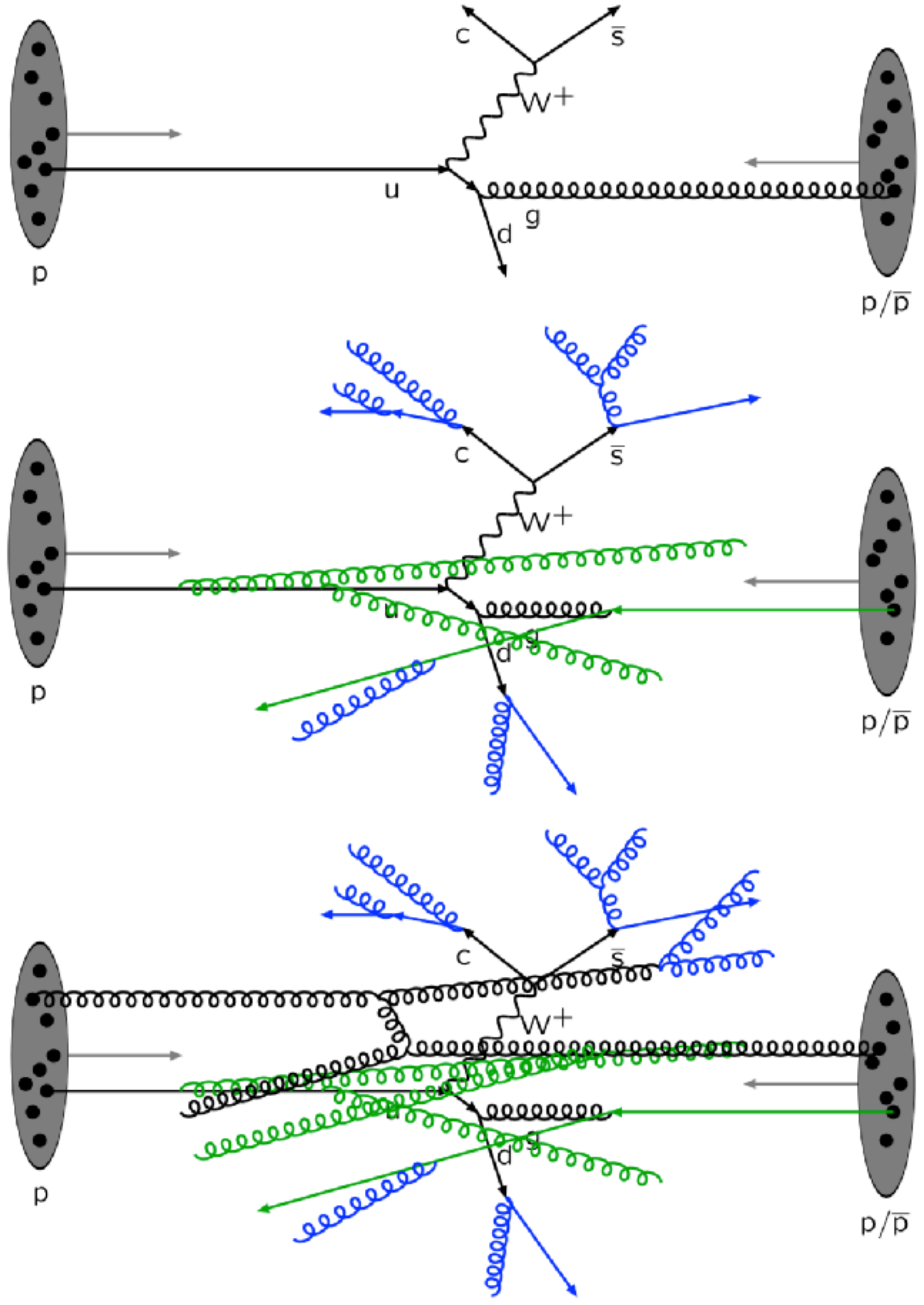
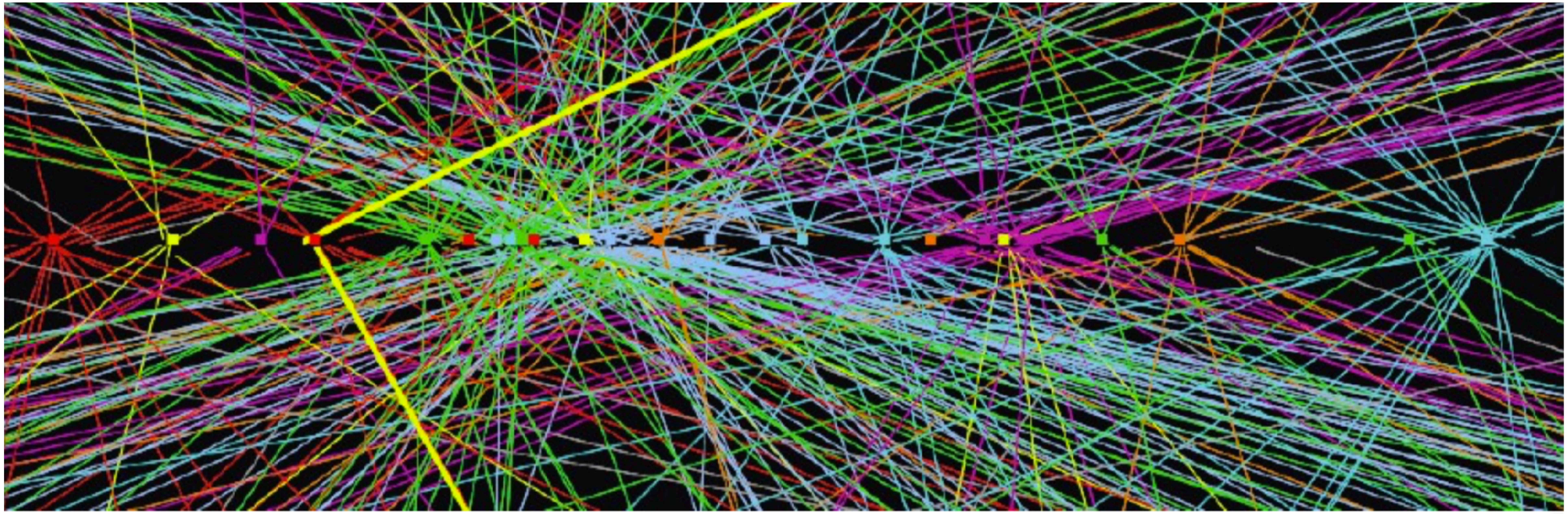


figure: T. Sjøstrand

non-perturbative stuff

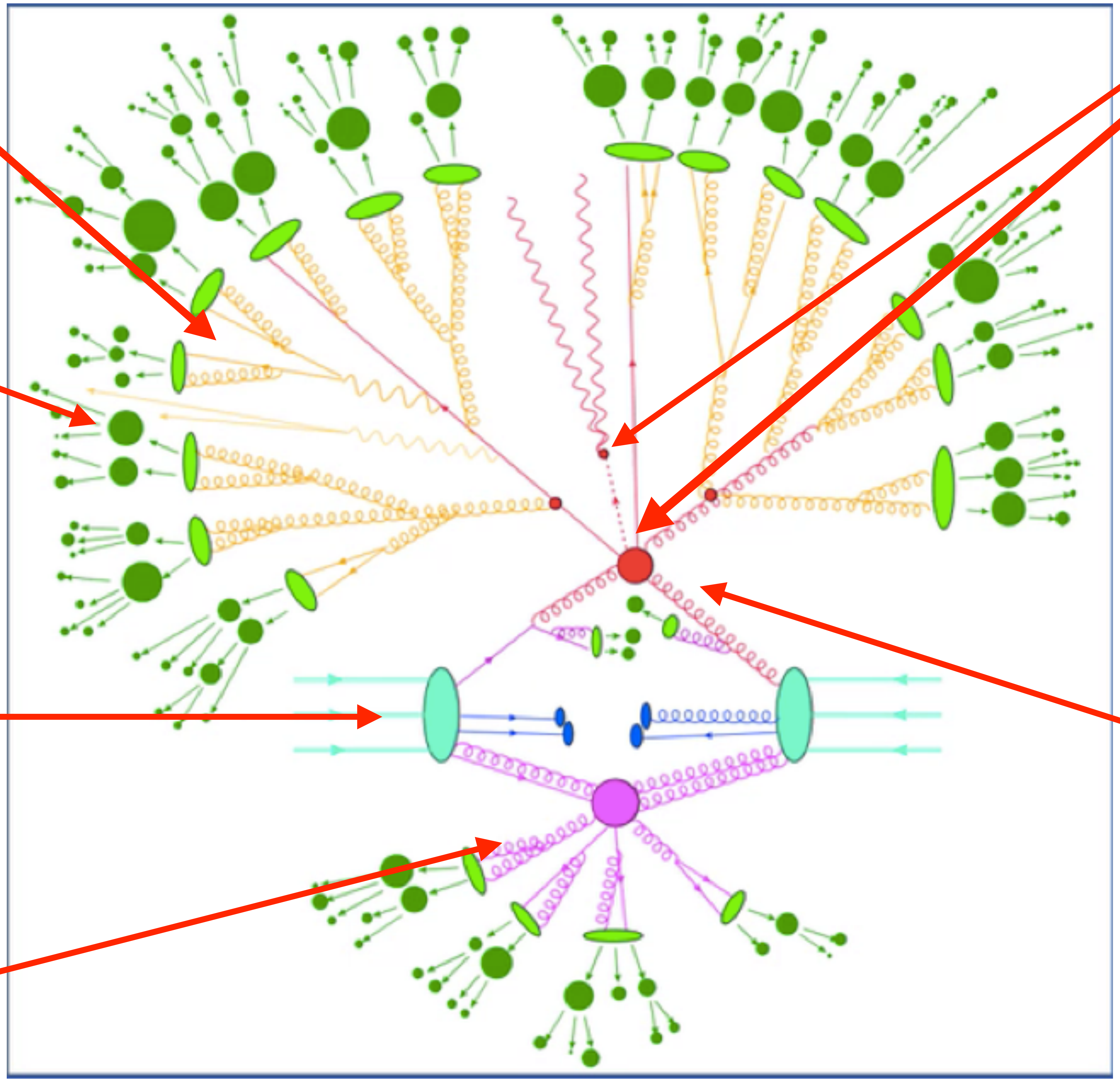


https://indico.cern.ch/event/505613/contributions/2230824/

visualisation of pile-up (multiple soft collisions in each bunch crossing) in the ATLAS tracker (Run I)

separation of perturbative from non-perturbative parts highly non-trivial!

Stages of an event



parton shower

hadronisation

parton distribution functions (PDFs)

underlying event

fixed order calculations (production and decay)

reduce scale uncertainties

μ_r, μ_f -dependence

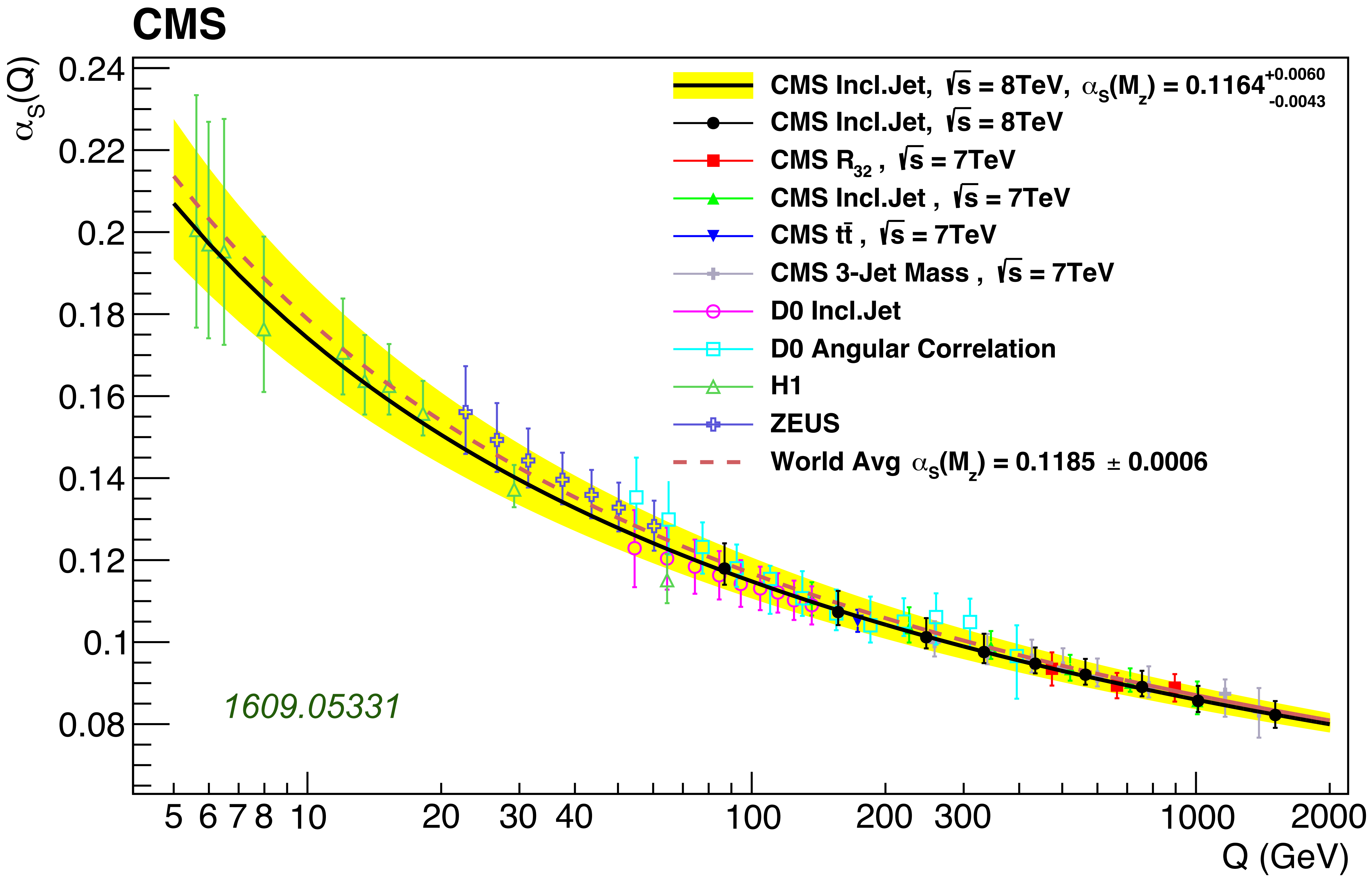
reduce parametric uncertainties (couplings, masses)

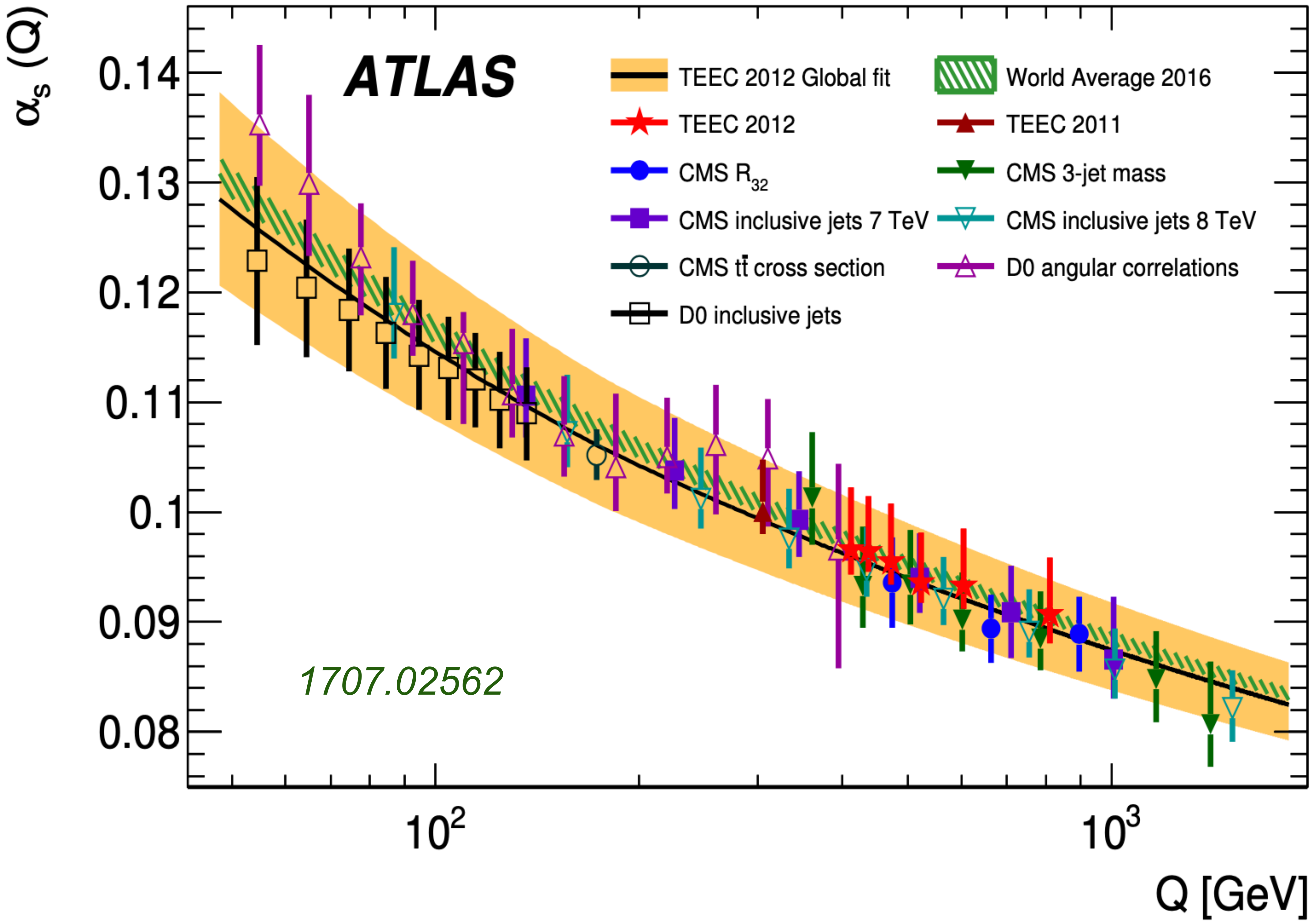
resummation

reduce uncertainties in particular kinematic regions

artwork by G.Luisoni

Asymptotic freedom



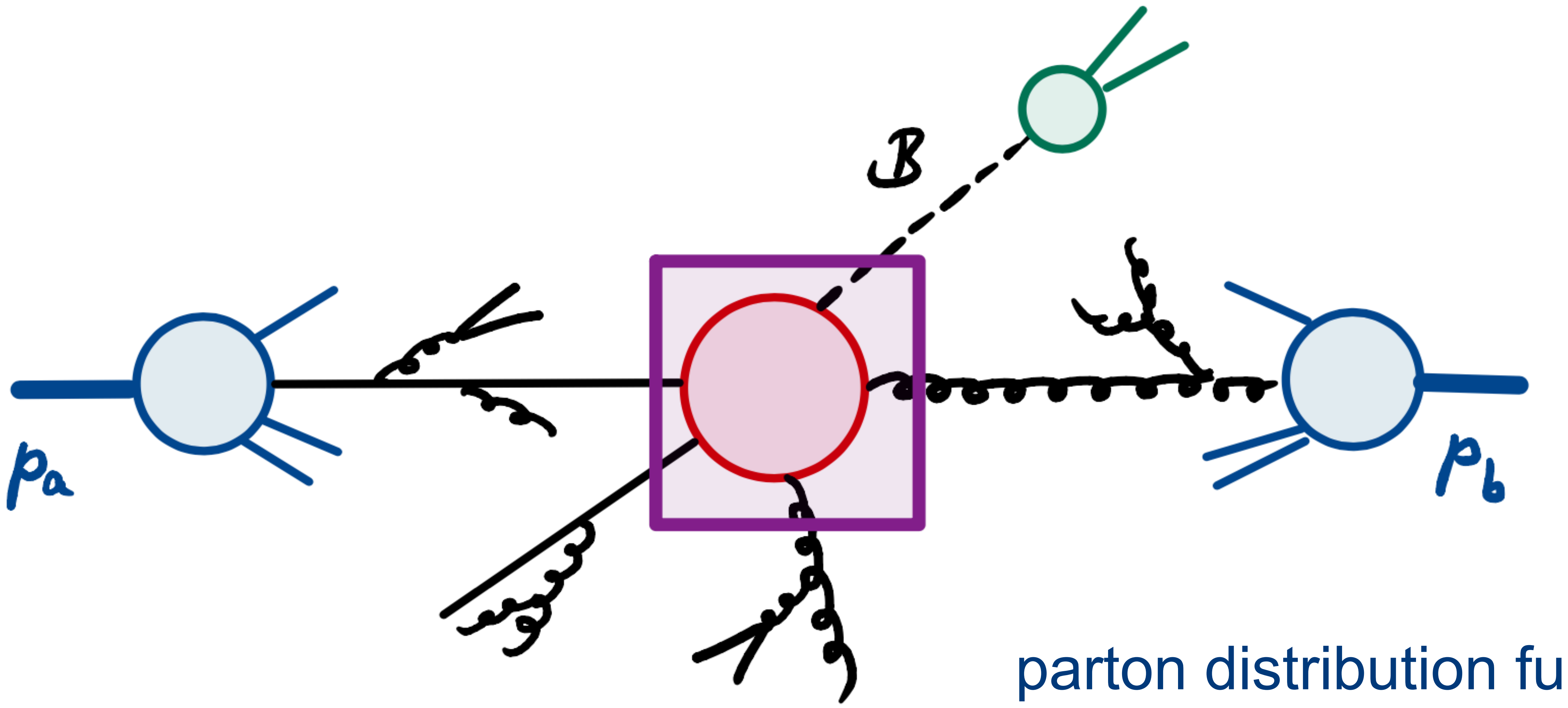


strong coupling is not constant

- becomes weaker as energy scale increases
- at very large energies quarks and gluons are almost free particles

will be discussed in more detail later

Factorisation



parton distribution functions (PDFs)

factorisation scale

$$d\sigma_{pp\to B+X} = \sum_{i,j}\int_0^1 dx_1\, f_{i/p_a}(x_1,\alpha_s,\mu_f)\int_0^1 dx_2\, f_{j/p_b}(x_2,\alpha_s,\mu_f)$$

$$\times d\hat{\sigma}_{ij\to B+X}(\{p\},x_1,x_2,\alpha_s(\mu_r),\mu_r,\mu_f)\,J(\{p\})\;+\;\mathcal{O}\left(\frac{\Lambda}{Q}\right)^p$$

power corrections

high energy scattering, calculable as a series in perturbation theory

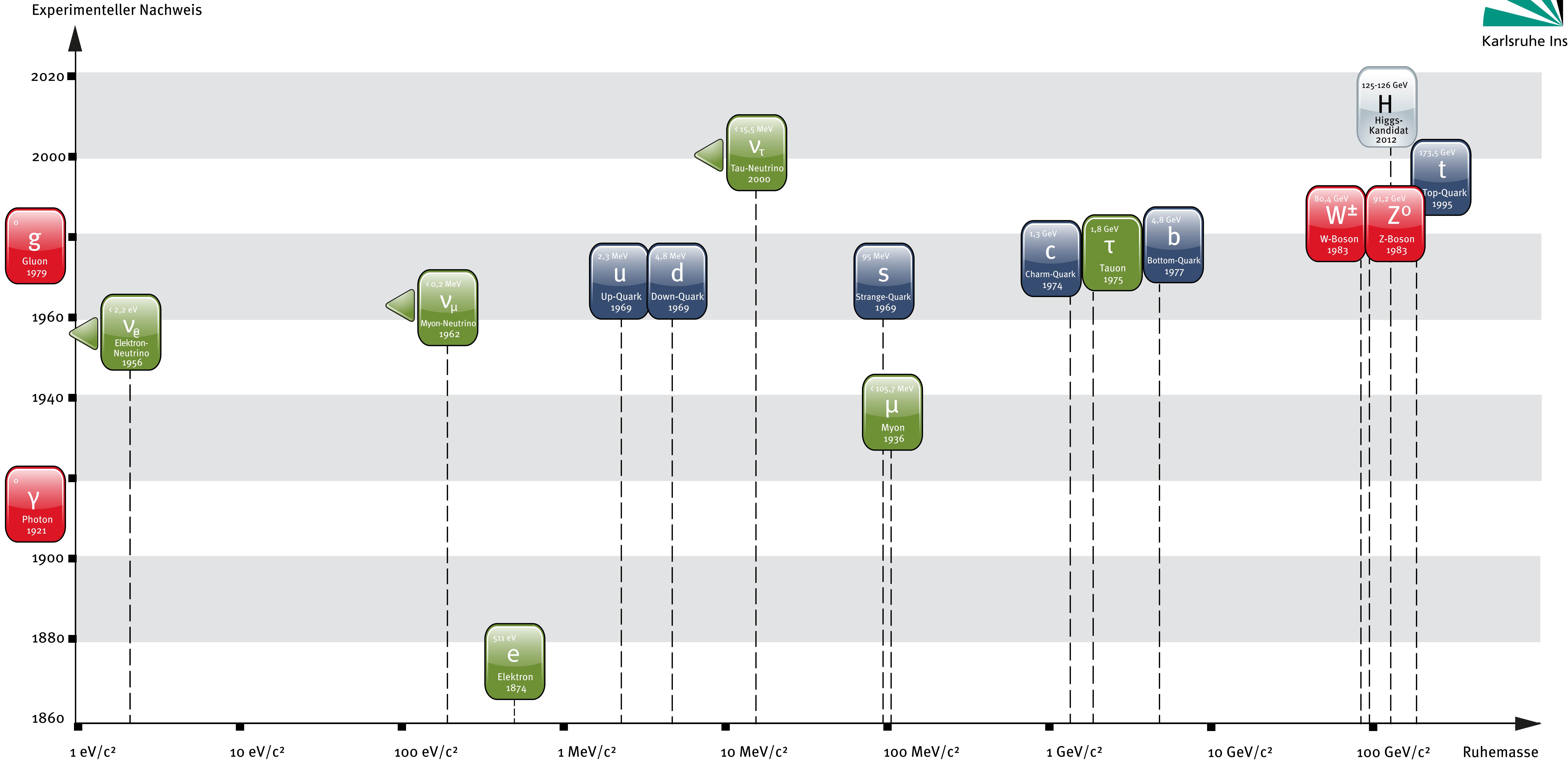
can be (mostly) separated from non-perturbative components

Properties of quarks

- quarks are fermions (spin 1/2)
- they are the constituents of hadrons
- they come in 6 flavours, forming 3 generations of up-type and down-type quarks

$$\binom{u}{d}, \binom{c}{s}, \begin{pmatrix} t \\ b \end{pmatrix}$$ (doublets under weak interactions and strong isospin)

- charges: up-type: 2/3, down-type: -1/3 (opposite for antiquarks)
- additional quantum number: colour charge: SU(3) local gauge theory
- the masses of the different quark flavours are very different, we do not know why

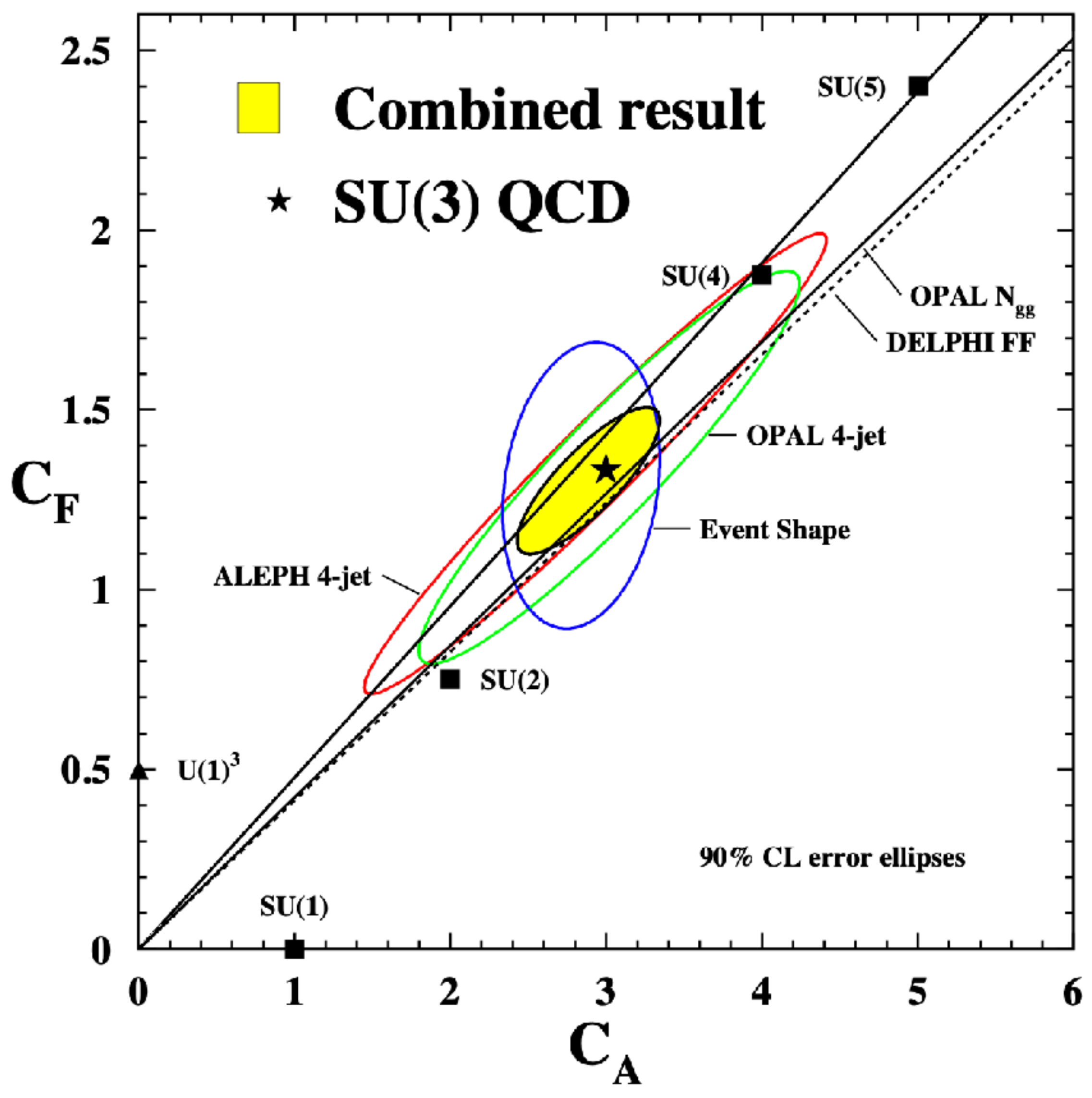

Experimenteller Nachweis
2020
2000
1960
1940
1900
1880
1860
1 eV/c²
10 eV/c²
100 eV/c²
1 MeV/c²
10 MeV/c²
100 MeV/c²
1 GeV/c²
10 GeV/c²
100 GeV/c²
Ruhemasse
0
g
Gluon
1979
0
γ
Photon
1921
‹ 2,2 eV
ν_e
Elektron-Neutrino
1956
‹ 0,2 MeV
ν_μ
Myon-Neutrino
1962
511 eV
e
Elektron
1874
2,3 MeV
u
Up-Quark
1969
4,8 MeV
d
Down-Quark
1969
‹ 15,5 MeV
ν_τ
Tau-Neutrino
2000
95 MeV
s
Strange-Quark
1969
‹ 105,7 MeV
μ
Myon
1936
1,3 GeV
c
Charm-Quark
1974
1,8 GeV
τ
Tauon
1975
4,8 GeV
b
Bottom-Quark
1977
80,4 GeV
W±
W-Boson
1983
91,2 GeV
Z0
Z-Boson
1983
125-126 GeV
H
Higgs-Kandidat
2012
173,5 GeV
t
Top-Quark
1995

Evidence for colour

how do we know the colour quantum number exists ?

examples:

- jet measurements



hep-ex/0603011

- hadronic R-ratio
- existence of doubly charged baryons, e.g. $\Delta^{++} = |uuu\rangle$

  would violate Pauli's exclusion principle without additional quantum number

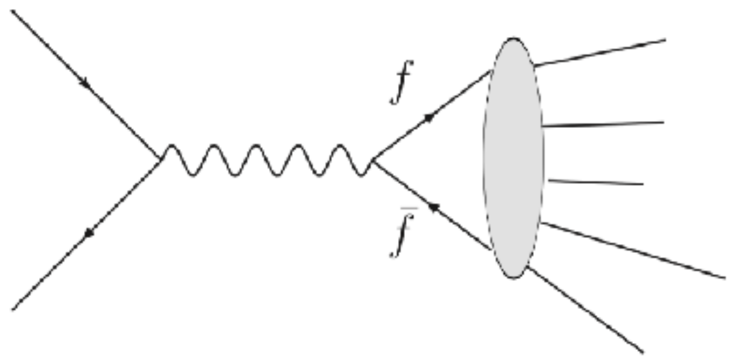
- Pion decay $\Gamma(\pi^0 \to \gamma\gamma) \sim \alpha^2 \frac{m_\pi^3}{f_\pi^2} (e_u^2 - e_d^2)^2 \boxed{N_c^2}$ ← number of colours

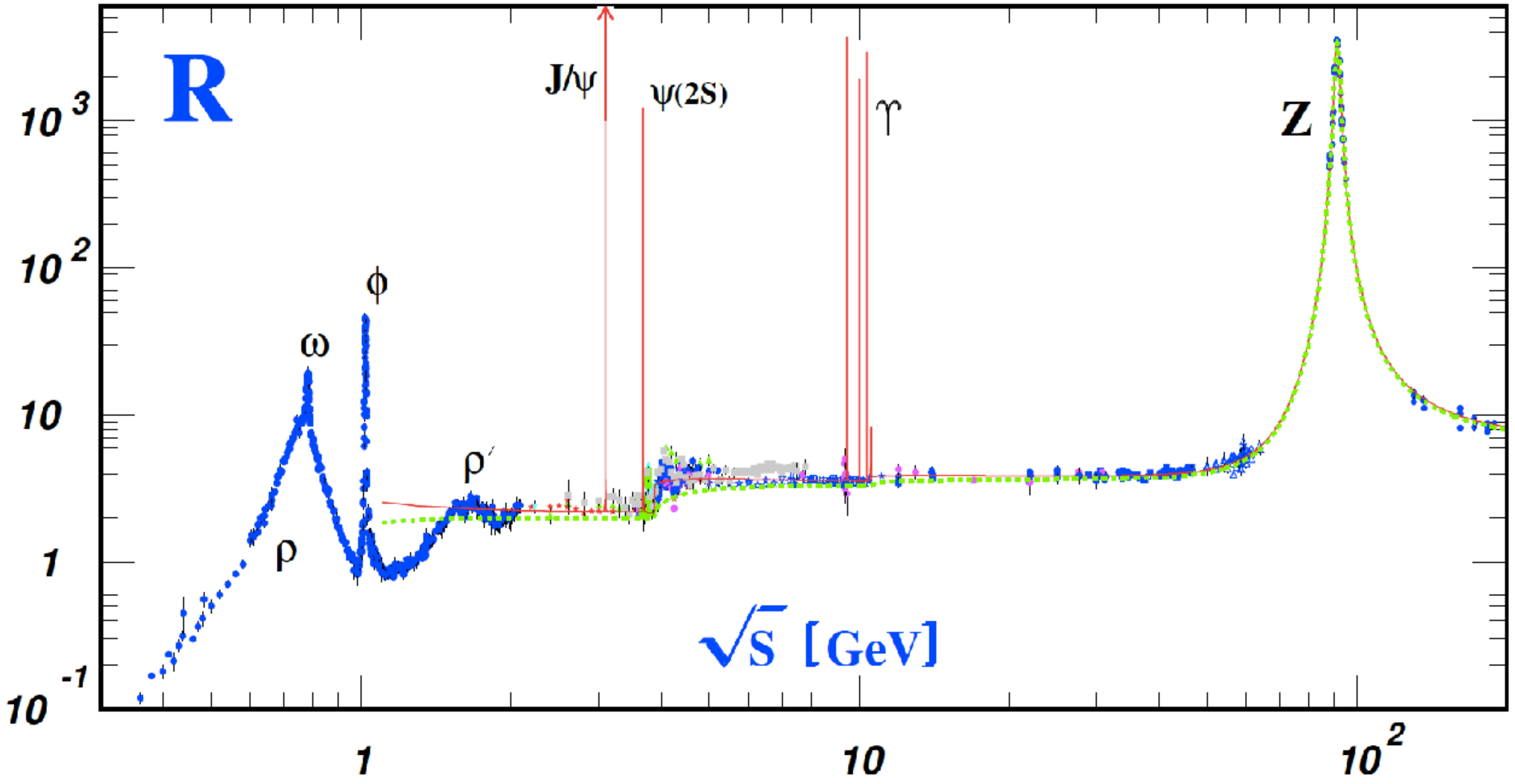
note however that with $e_u = \frac{1}{2}(\frac{1}{N_c}+1), e_d = \frac{1}{2}(\frac{1}{N_c}-1)$ the width would be independent of N_c

Evidence for Colour

hadronic R-ratio:

$$R(s) = \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$





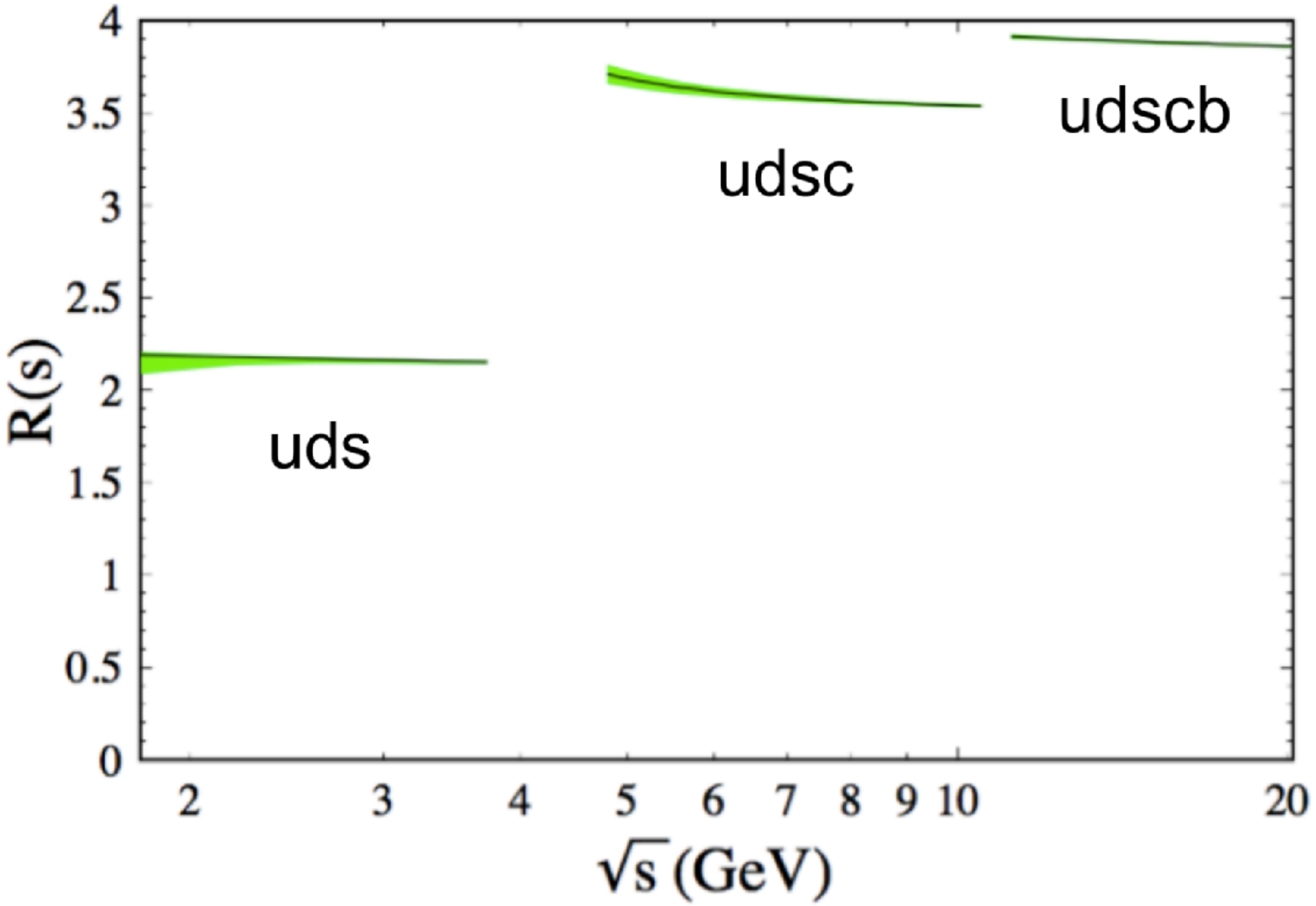
$$R(s) = \boxed{N_c} \sum_{f=u,d,s,c,\dots} e_f^2\, \theta(s - 4m_f^2)$$

threshold energy
to produce quark-antiquark-pair
of mass m_f

$$R = 3\left((\frac{2}{3})^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})^2 + \dots\right)$$

u d s

R-ratio



resonances removed,
higher order corrections included

Harlander, Steinhauser, hep-ph/0212294

$$m_c \simeq 1.3\,\text{GeV}$$

$$m_b \simeq 4.5\,\text{GeV}$$

$$N_c \sum_{f=u,d,s,c,b} e_f^2\,\theta(s-4m_f^2) = 3\left(\tfrac{4}{9}+\tfrac{1}{9}+\tfrac{1}{9}+\tfrac{4}{9}+\tfrac{1}{9}\right) = \frac{11}{3}$$

u d s c b

QCD Lagrangian

strong interactions: $SU(N_c)$ gauge theory, $N_c = 3$ “colours” of quarks

fermionic part: quark fields for flavour *f:* $q_f^i(x)$ $i = 1, 2, 3$ colour index

for free quark fields:

$$\mathcal{L}_q^{(0)}(q_f, m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x)\,(i\,\gamma_\mu \partial^\mu - m_f)_{jk}\, q_f^k(x)$$

apply SU(N) group transformation: $q_i \to q_i' = U_{ij}\, q^j \;,\; \bar{q}_i \to \bar{q}_i' = \bar{q}^{\,j} U_{ji}^{-1}$

$$U_{ij} = \exp\left\{ i \sum_{a=1}^{N_c^2-1} t^a\, \theta^a \right\}_{ij} = \delta_{ij} + i \sum_{a=1}^{N_c^2-1} t^a\, \theta^a + \mathcal{O}(\theta^2)$$

generator

group transformation parameter

QCD Lagrangian

$t^a_{ij} = \lambda^a_{ij}/2$ generators of SU(3) in fundamental representation (3x3 matrices)

λ^a_{ij} : Gell-Mann matrices (traceless, hermitian)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

more about colour algebra coming soon

QCD Lagrangian

QCD is a *local* gauge theory

local means that the gauge transformation parameter *depends on the space-time point*:

$\boxed{\theta = \theta(x)}$ therefore $U = U(x)$ and

$$\partial_\mu q'(x) = \partial_\mu \left(U(x) q(x)\right) = U(x) \partial_\mu q(x) + \left(\partial_\mu U(x)\right) q(x)$$

how can we keep $\mathcal{L}_q$ invariant despite this additional term?

introduce coupling to a gauge field A_a^μ (gluon) through covariant derivative

$$(D^\mu[A])_{ij} = \delta_{ij} \partial^\mu + i\, g_s\, t^a_{ij} A^\mu_a$$

QCD Lagrangian

define $\mathbf{A}^\mu = \sum_{a=1}^{N_c^2-1} t^a A_a^\mu$ then in index-free notation $\mathbf{D}^\mu[\mathbf{A}] = \partial^\mu + i\, g_s\, \mathbf{A}^\mu$

quark Lagrangian with “minimal coupling” of gluon:

$$\mathcal{L}_q(q_f\,,\, m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x)\; (i\,\gamma_\mu \mathbf{D}^\mu[\mathbf{A}] - m_f)_{jk}\; q_f^k(x)$$

invariant under local gauge transformations if we have

$$\mathbf{D}^\mu[\mathbf{A}]q(x) \overset{!}{\to} U\Big(\mathbf{D}^\mu[\mathbf{A}]\, q(x)\Big)$$

SU(3) gauge invariance

This gives a condition on the transformed gluon field $\mathbf{A}'_\mu$

$$\mathbf{D}^\mu[\mathbf{A}'] \overset{!}{=} U\Big(\mathbf{D}^\mu[\mathbf{A}]\Big)U^{-1} \Rightarrow \partial_\mu + ig_s\mathbf{A}'_\mu \overset{!}{=} U\left(\partial_\mu + ig_s\mathbf{A}_\mu\right)U^{-1}$$

$$\boxed{\mathbf{A}'_\mu = U(x)\mathbf{A}_\mu U^{-1}(x) + \frac{i}{g_s}(\partial_\mu U(x))U^{-1}(x)}$$

Yang-Mills Lagrangian

purely gluonic part: Yang-Mills Lagrangian (C.N.Yang, R.Mills, 1954)

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a,\mu\nu}$$

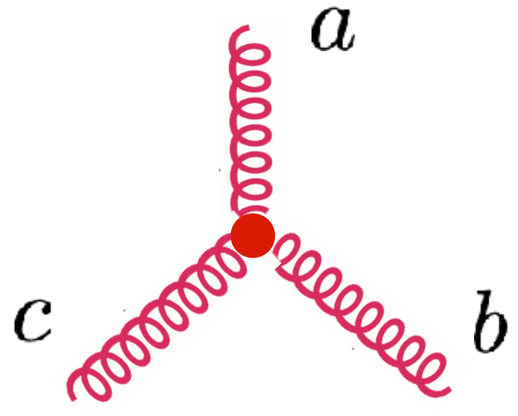
field strength tensor $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s\, f^{abc} A^b_\mu A^c_\nu$

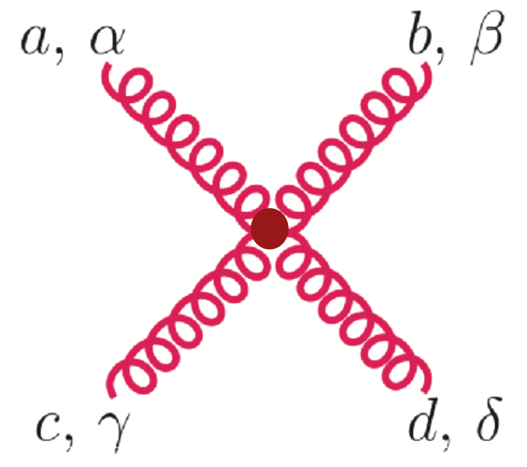
f^{abc} : structure constants of SU(3) (totally antisymmetric)

$[T^a, T^b] = i f^{abc} T^c \qquad a, b, c = 1, \ldots, N_c^2 - 1$

$\Rightarrow$ 8 gluons (in the adjoint representation of SU(3))

non-Abelian structure of SU(3) is related to gluon-self-interactions





QCD Lagrangian

so far we have $\mathcal{L} = \mathcal{L}_q + \mathcal{L}_{\mathrm{YM}}$

$$\mathcal{L}_q(q_f,\, m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x)\, (i\,\gamma_\mu \mathbf{D}^\mu[\mathbf{A}] - m_f)_{jk}\, q_f^k(x)$$

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4}\, F^a_{\mu\nu}\, F^{a,\mu\nu}$$

$\mathcal{L}_q + \mathcal{L}_{YM}$ is gauge invariant, but:

path integral contains physically equivalent configurations (by gauge transformations)

$\Rightarrow$ path integral over the action is not well defined due to this redundancy

Gauge fixing

- the gluon propagator $\Delta^{ab}_{\mu\nu}(p) = \Delta_{\mu\nu}(p)\delta^{ab}$ is constructed from the

  inverse of the bilinear term in the gluon fields $\sim A^a_\mu A^b_\nu$ in the action

$$S_{\rm YM} = i\int \mathrm{d}^4x \left\{-\frac{1}{4}\,F^a_{\mu\nu}\,F^{a,\mu\nu}\right\} \supset \frac{i}{2}\int \mathrm{d}^4x\, A^a_\mu(x)\,(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\,A^b_\nu(x)\delta_{ab}$$

- in momentum space: $\sim \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\tilde{A}^a_\mu(p)\,(p^2 g^{\mu\nu} - p^\mu p^\nu)\,\tilde{A}^b_\nu(p)\,\delta_{ab}$

Gauge fixing

in momentum space the propagator should fulfill *(cf QED Green's function to solve e.o.m.)*

$$i\,\Delta_{\mu\rho}(p)\,\left[p^2 g^{\rho\,\nu}\;-\;p^\rho p^\nu\right]=g_\mu^\nu \qquad (1)$$

however as $\left[p^2 g^{\rho\,\nu}-p^\rho p^\nu\right]p_\nu=0$

Eq. (1) has *zero modes*, so the matrix in the square brackets is *not invertible*

reason: $\mathcal{L}_q+\mathcal{L}_{\mathrm{YM}}$ contains redundant, physically equivalent configurations

$\Rightarrow$ need *gauge fixing:* add constraint on gluon fields with a Lagrange multiplier

QCD Lagrangian

covariant gauges: add condition $\partial_\mu A^\mu(x) = 0$

$$\Rightarrow \quad \mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\lambda}\left(\partial_\mu A^\mu\right)^2, \quad \lambda \in \mathbb{R}$$

leads to bilinear term of the form $i\left(p^2 g^{\mu\nu} - \left(1-\frac{1}{\lambda}\right)p^\mu p^\nu\right)$

with inverse $\Delta_{\mu\nu}(p) = \dfrac{-i}{p^2 + i\,\varepsilon}\left[g_{\mu\nu} - (1-\lambda)\dfrac{p_\mu p_\nu}{p^2}\right]$ (colour part δ^{ab} omitted)

$\lambda = 1$: propagator in Feynman gauge

$\lambda = 0$: propagator in Landau gauge

$i\varepsilon$ shifts the poles of the propagator slightly away from the real axis

QCD Lagrangian

- however in covariant gauges unphysical (non-transverse) degrees of freedom can propagate

- their effect is cancelled by *ghost fields* η^a *[L. Faddeev, V. Popov 1967]*

(coloured complex scalars obeying Fermi statistics, do not occur as external states)

$$\mathcal{L}_{FP} = \eta_a^\dagger \, M^{ab} \, \eta_b$$

Faddeev-Popov matrix in covariant gauge: $M^{ab} = \delta^{ab} \, \partial_\mu \partial^\mu + g_s \, f^{abc} A_\mu^c \partial^\mu$

complete QCD Lagrangian: $\boxed{\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_q + \mathcal{L}_{GF} + \mathcal{L}_{FP}}$

Physical gauges

- unphysical degrees of freedom and the ghost fields can be avoided by choosing

axial (physical) gauges: condition $n_\mu A^\mu = 0\,;$ n^μ vector with $p\cdot n \neq 0$

in axial gauges: $$\mathcal{L}_{GF} = -\frac{1}{2\alpha}\left(n^\mu A_\mu\right)^2$$

gluon propagator: $$\Delta_{\mu\nu}(p,n) = \frac{-i}{p^2 + i\,\varepsilon}\left(g_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{p\cdot n} + \frac{n^2\,p_\mu p_\nu}{(p\cdot n)^2}\right)$$

special case $n^2 = 0:$ *light-cone gauge*

no propagating ghost fields: $$M^{ab}_{axial} = \delta^{ab}\,n_\mu\partial^\mu + g_s\,f^{abc}\,\underbrace{n_\mu A^\mu_c}_{\text{zero}}$$

Physical gauges

in light-cone gauge we have $\Delta_{\mu\nu}(p,n) = \dfrac{i}{p^2 + i\,\varepsilon}\, d_{\mu\nu}(p,n)$

$$d_{\mu\nu}(p,n) = -g_{\mu\nu} + \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n} = \sum_{\lambda=1,2} \epsilon_\mu^\lambda(p) \left(\epsilon_\nu^\lambda(p)\right)^*$$

$\epsilon_\nu^\lambda(p)$: polarisation vector $\epsilon_1 = (0,1,0,0), \epsilon_2 = (0,0,1,0)$ or $\epsilon_{L,R} = (0,1,\pm i,0)/\sqrt{2}$

$\Rightarrow$ only the two physical polarisations propagate

e.g. choose $p = (p^0,0,0,p^0),\; n = (p^0,0,0,-p^0)$ then

$$-g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Colour algebra

Colour Algebra

SU(N): Lie group *(elements depend on a finite number of continuous parameters)*

U: unitary $U\,U^\dagger = \mathbf{1}$; S: "special" (det U=1)

representation: mapping of group elements onto matrices, such that group operations translate to matrix operations

associated algebra: generators fulfill

$$\boxed{[T^a, T^b] = i f^{abc} T^c \quad a, b, c = 1, \ldots, N^2 - 1}$$

independent of the representation

number of generators = *dimension* of the group

Colour Algebra

important representations:

- fundamental representation: generators are $N \times N$ matrices

$$t^a_{ij} = \lambda^a_{ij}/2 \qquad i,j = 1 \ldots N$$

λ^a_{ij} : Gell-Mann matrices

- adjoint representation: generators are $(N^2 - 1) \times (N^2 - 1)$ matrices

i.e. indices run over dimension of the group

generators $(F^a)_{bc}$ with $(F^a)_{bc} = -i\, f^{abc}$

quarks are in the fundamental representation, gluons in the adjoint

$\Rightarrow$ 8 gluons, each quark flavour comes in three colours

Colour Algebra

explicit example for a quark-gluon vertex $-ig_s\,\bar\psi_i\,\frac{\lambda^a_{ij}}{2}\,\psi_j\,A^a_\mu$

$$i=1, j=2, a=1:$$

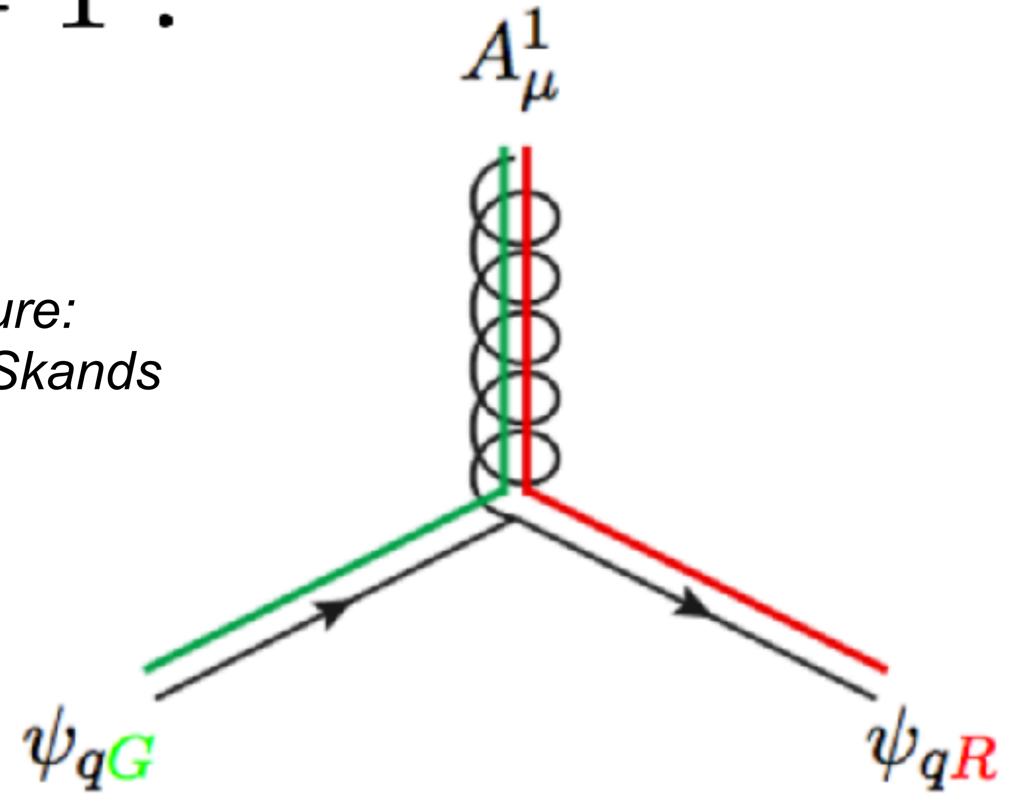


figure:
Peter Skands

$$\propto -\tfrac{i}{2}g_s \qquad \bar\psi_{qR} \qquad \lambda^1 \qquad \psi_{qG}$$

$$= -\tfrac{i}{2}g_s \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

gluons can be represented as double lines of colour-anticolour combinations, e.g.

$$\lambda^1 \rightarrow \frac{1}{\sqrt{2}}\bar{r}g \quad , \quad \lambda^8 \rightarrow \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$$

note that the combination $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$ does not occur for the gluon

because this would correspond to a colour singlet

Gell-Mann matrices (again)

$t^a_{ij} = \lambda^a_{ij}/2$ generators of SU(3) in fundamental representation

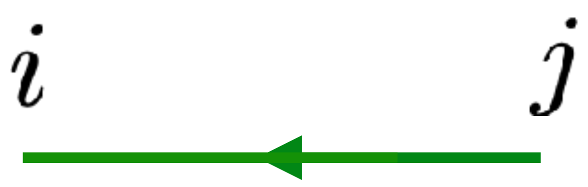
λ^a_{ij} **:** Gell-Mann matrices (traceless, hermitian)

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

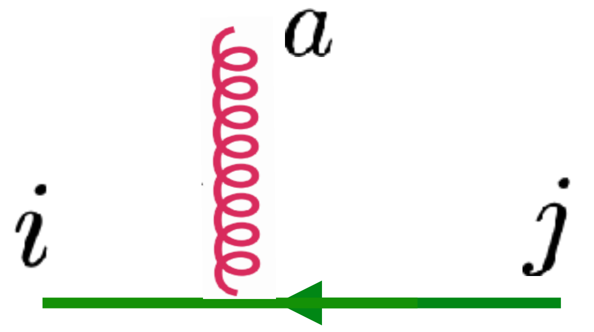
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Colour Algebra pictorially

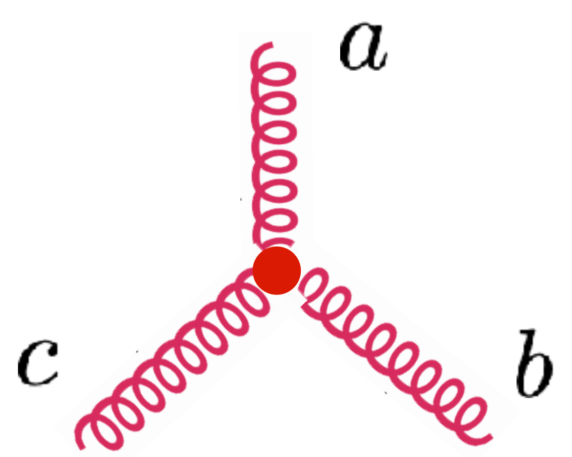


$\overset{\text{colour}}{=} \delta_{ij}$

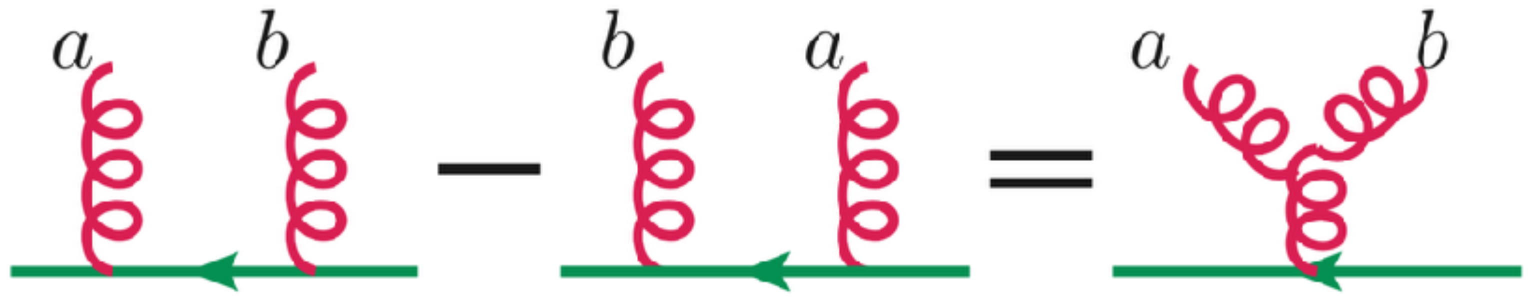
$\overset{\text{colour}}{=} \delta_{ab}$



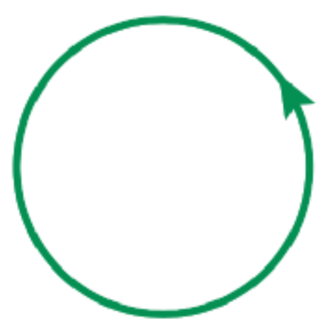
$\overset{\text{colour}}{=} t^a_{ij}$



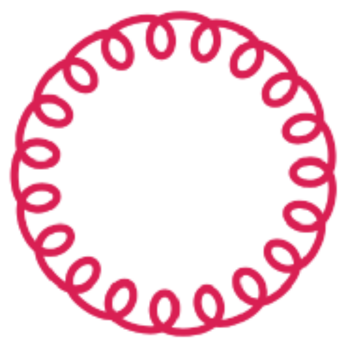
$\overset{\text{colour}}{=} i f^{abc}$



$$[t^a, t^b] = i f^{abc} t^c$$



$\overset{\text{colour}}{=} \delta_{ij}\delta^{ij} = N_c$



$\overset{\text{colour}}{=} \delta_{ab}\delta^{ab} = N_c^2 - 1$

Colour "Casimirs"

group invariants:

$$\sum_{j,a} t^a_{ij} t^a_{jk} = C_\mathrm{F}\, \delta_{ik}\,, \qquad \sum_{a,d} F^a_{bd} F^a_{dc} = C_\mathrm{A}\, \delta_{bc}$$

C_F, C_A : eigenvalues of Casimir operators in fundamental/adjoint representation

(Casimir operators commute with any element of the Lie algebra)

$$\boxed{C_F = T_R \frac{N_c^2 - 1}{N_c}\,,\; C_\mathrm{A} = 2\, T_R\, N_c}$$

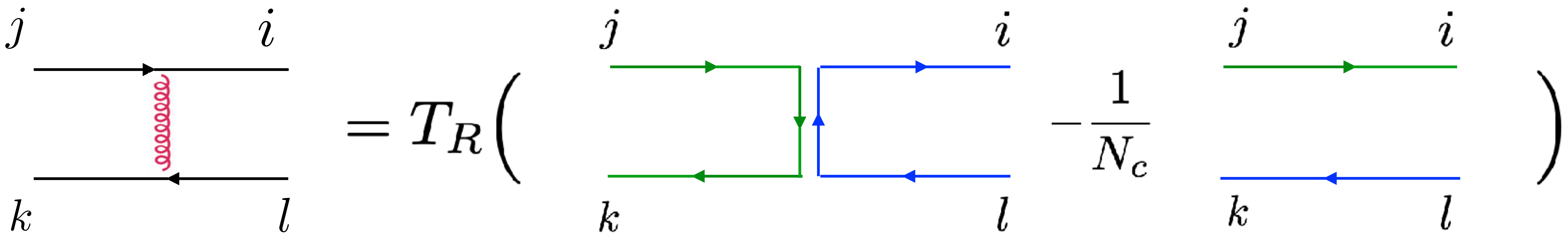
$$\mathrm{Trace}[T^a T^b] = T_R\, \delta^{ab}$$

usually $T_R = \frac{1}{2}$ (convention)

Colour "Fierz identities"

the double line representation for the gluons also allows us to derive some identities, for example

$$t^a_{ij}t^a_{kl} = T_R\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)$$



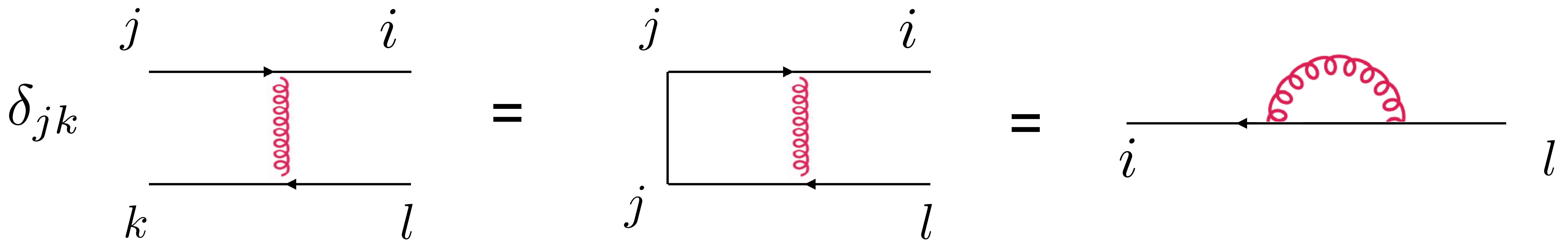
second term is the case where quarks have the same colour → no gluon is exchanged;

normalisation can be seen e.g. by contracting with δ_{jk}

lhs: $t^a_{ij}t^a_{kl}\delta_{jk} = t^a_{ij}t^a_{jl} = C_F\delta_{il}$ rhs: $T_R\left(\delta_{il}N_c - \frac{1}{N_c}\delta_{il}\right) = T_R\frac{N_c^2-1}{N_c}\delta_{il} = C_F\delta_{il}$

Colour Algebra

note that contracting with δ_{jk} pictorially corresponds to

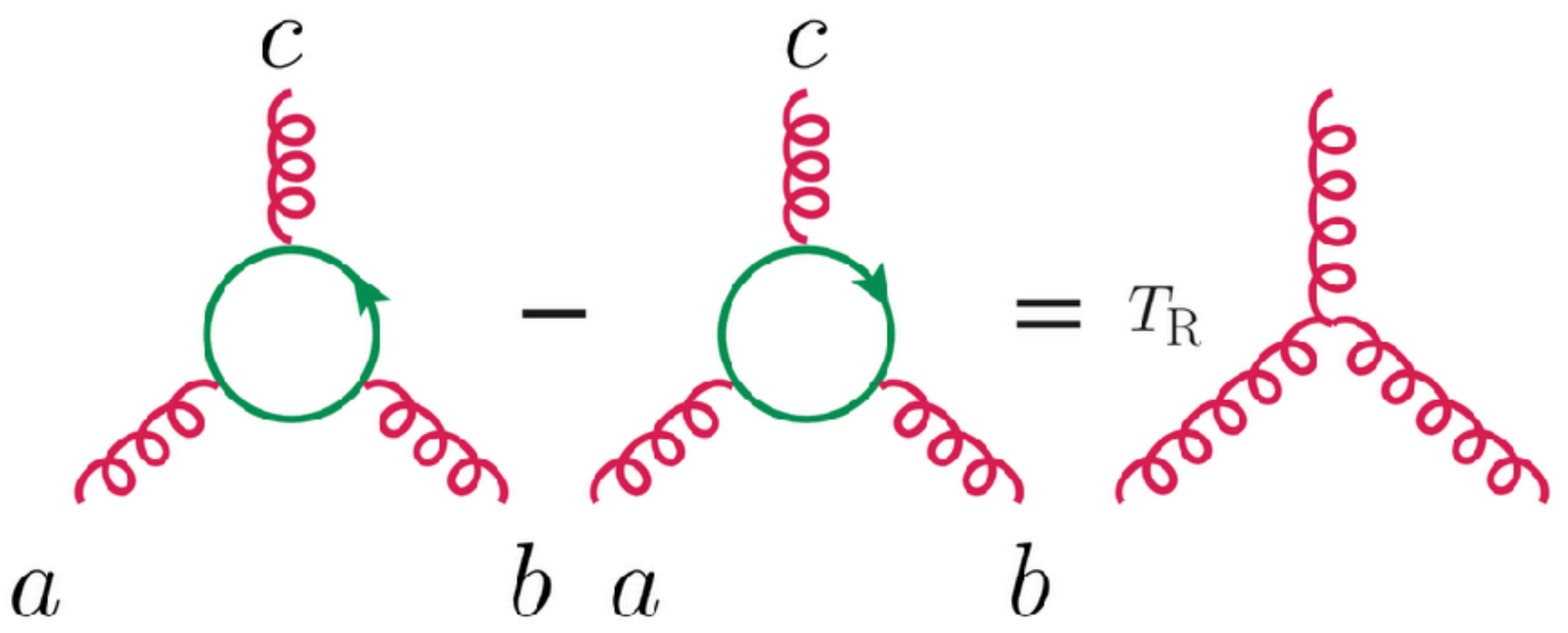


(quark self energy diagram)

colour decomposition

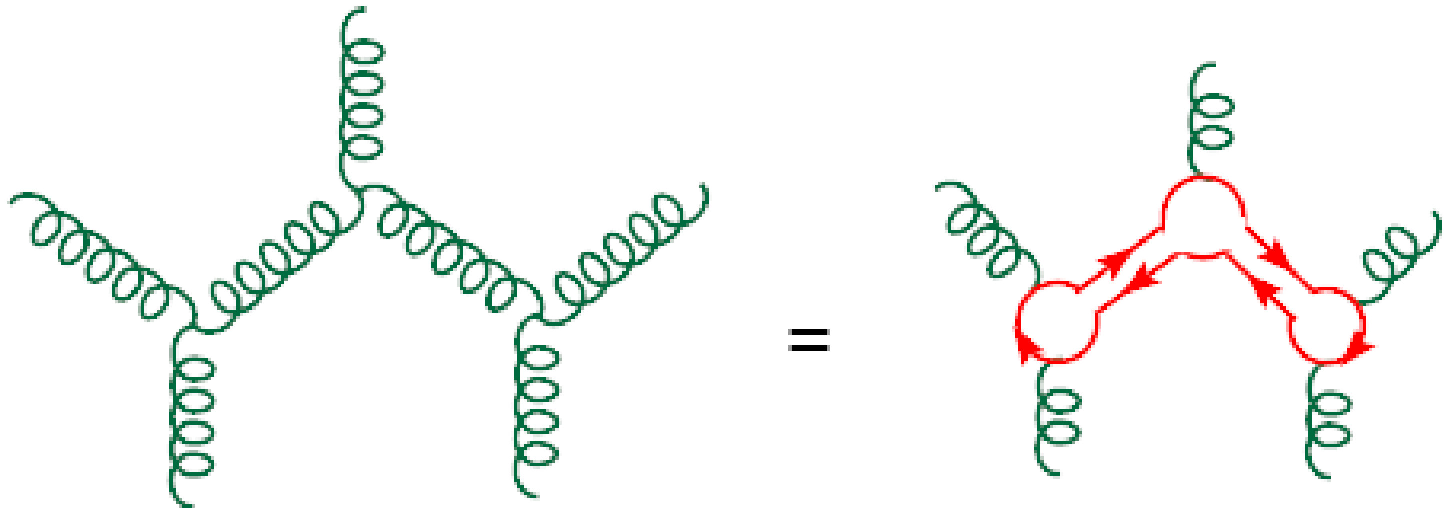
we can express gluon amplitudes entirely in terms of generators t^a_{ij}

based on



we can eliminate f^{abc}

$$\mathrm{Trace}(t^a t^b t^c) - \mathrm{Trace}(t^c t^b t^a) = i\, T_R f^{abc}$$



+ *permutations*

$\mathrm{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n})$ + all non-cyclic permutations (with corresponding signs)

similarly $q\bar{q}gggg\ldots \to \mathrm{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n})_{ij}$ + permutations

colour decomposition

idea: split amplitude into a colour part and a *kinematic* part

$$\mathcal{A}_n^{\text{tree}} = g_s^{n-2} \sum_{\sigma\in S_{n-2}} \underbrace{(t^{a_{\sigma(3)}}\cdots t^{a_{\sigma(n)}})_{j_1 i_2}}_{\text{colour}} \, \underbrace{A_n^{\text{tree}}(1_{\bar q}^{\lambda_1}, 2_q^{\lambda_2}, \sigma(3^{\lambda_3}),\ldots,\sigma(n^{\lambda_n}))}_{\text{“partial amplitude”}},$$

colour

“partial amplitude”
(kinematics only, permutation of colour labels)

leads to a large reduction of complexity and more manifest IR singularity structure

- there are various ways to do a colour decomposition
- the “colour flow decomposition” also eliminates the t^a, based on

$$t^a_{ij} t^a_{kl} = T_R \left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl} \right)$$

n-gluon amplitudes

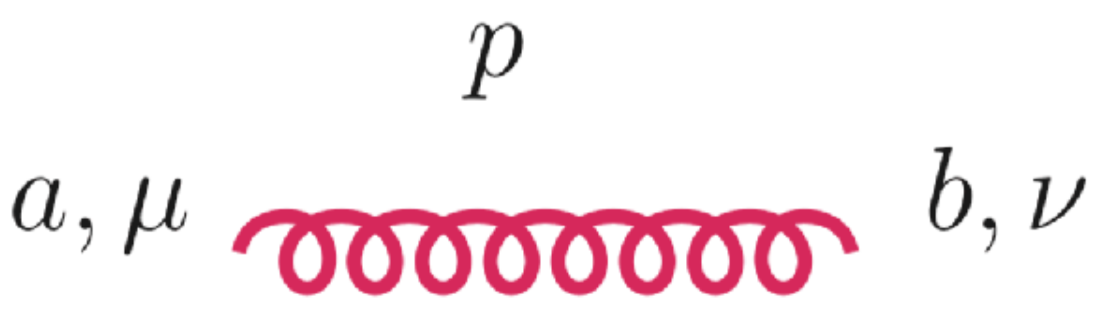
	# diagrams	
n	partial amplitude	full amplitude
4	3	4
5	10	25
6	36	220
7	133	2485
8	501	34300
9	1991	559405
10	7335	10525900
11	28199	224449225
12	108281	5348843500

table from arXiv:hep-ph/9910563

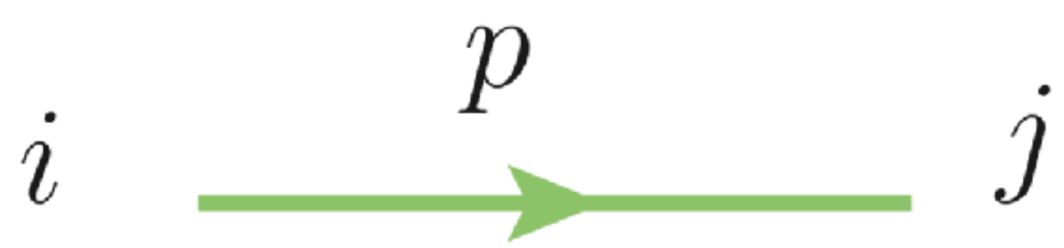
QCD Feynman rules

Propagators:

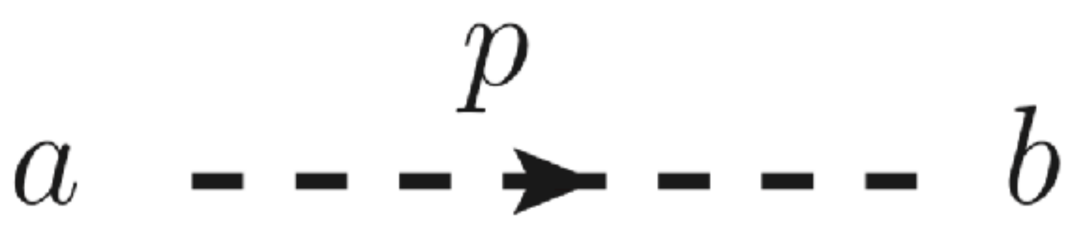
gluon $\Delta^{ab}_{\mu\nu}(p) = \delta^{ab}\Delta_{\mu\nu}(p)$



quark $\Delta^{ij}(p) = \delta^{ij}\dfrac{i(\not{p}+m)}{p^2 - m^2 + i\,\varepsilon}$



ghost $\Delta^{ab}(p) = \delta^{ab}\dfrac{i}{p^2 + i\,\varepsilon}$



$$\Delta_{\mu\nu}(p) = \frac{-i}{p^2 + i\,\varepsilon}\left[g_{\mu\nu} - (1-\lambda)\frac{p_\mu p_\nu}{p^2}\right]$$

covariant gauge

$\lambda = 1$: Feynman gauge

$$\Delta_{\mu\nu}(p,n) = \frac{-i}{p^2 + i\,\varepsilon}\left[g_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n}\right]$$

light-cone gauge

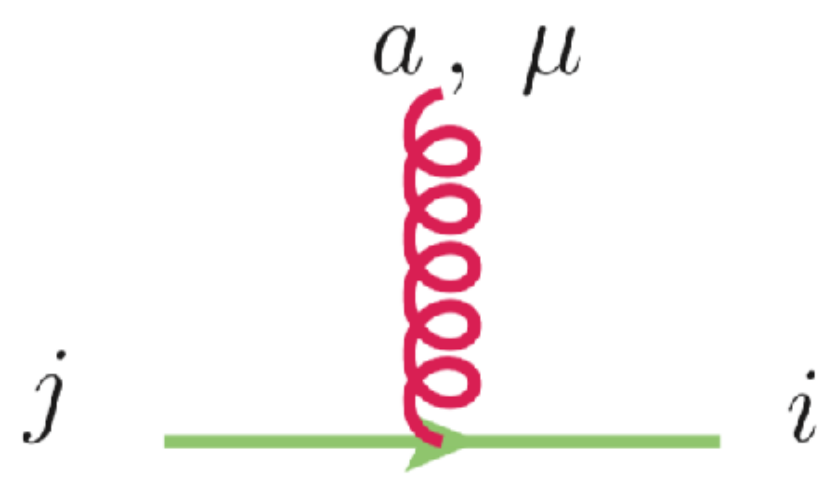
$n^2 = 0$

Feynman rules

Vertices:

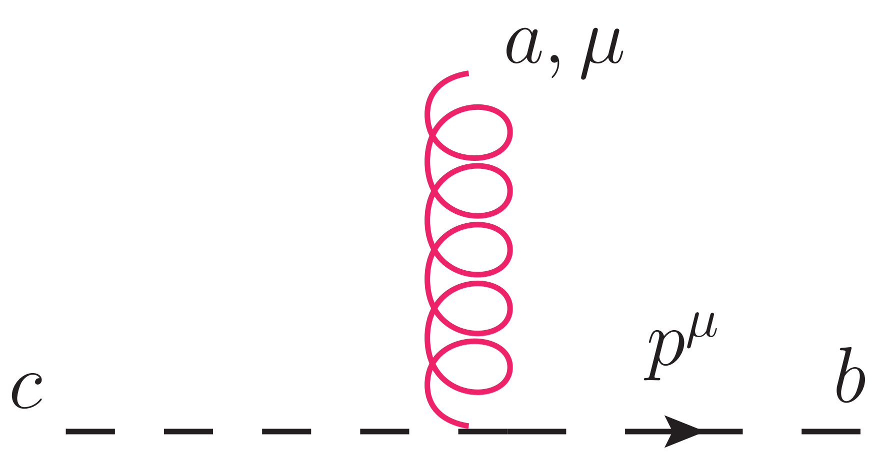
quark-gluon

$$\Gamma^{\mu,\,a}_{gq\bar{q}} = -i\,g_s\,(t^a)_{ij}\gamma^\mu$$



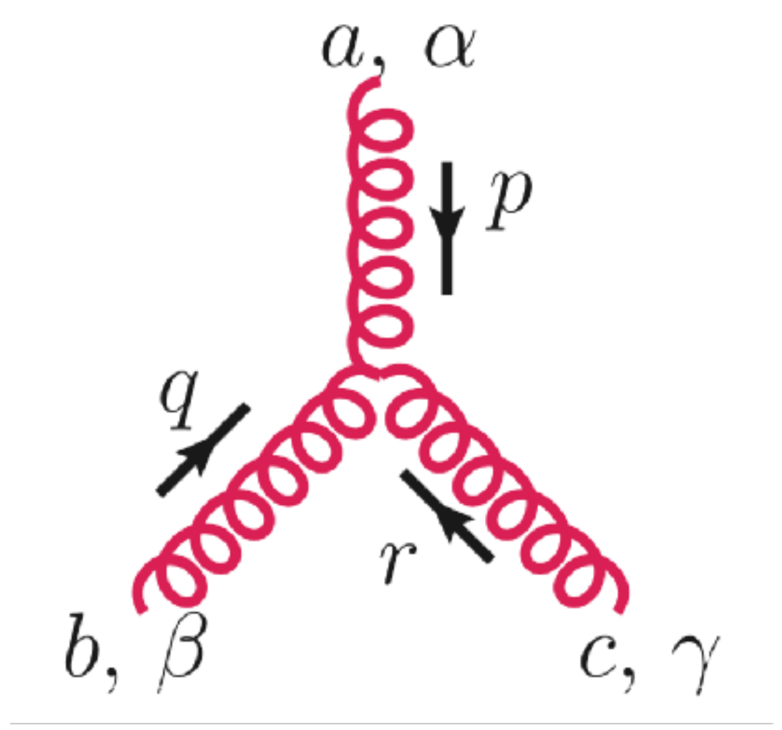
ghost-gluon

$$\Gamma^{\mu,\,a}_{g\eta\bar{\eta}} = -i\,g_s\,(F^a)_{bc}\,p^\mu$$
$$= -g_s\,f^{abc}\,p^\mu$$



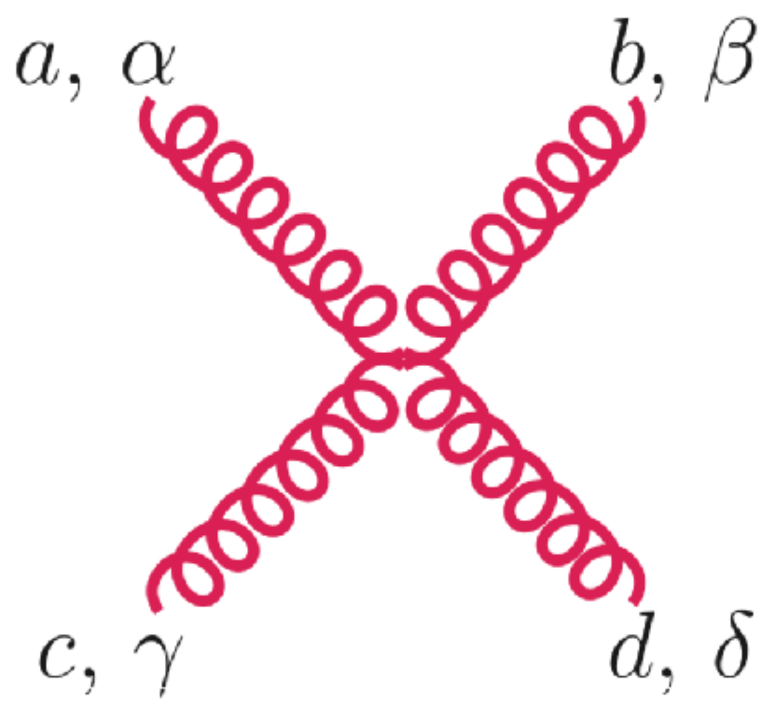
Feynman rules

3-gluon $$\Gamma^{abc}_{\alpha\beta\gamma}(p,q,r) = -i\, g_s\,(F^a)_{bc}\, V_{\alpha\beta\gamma}(p,q,r)$$



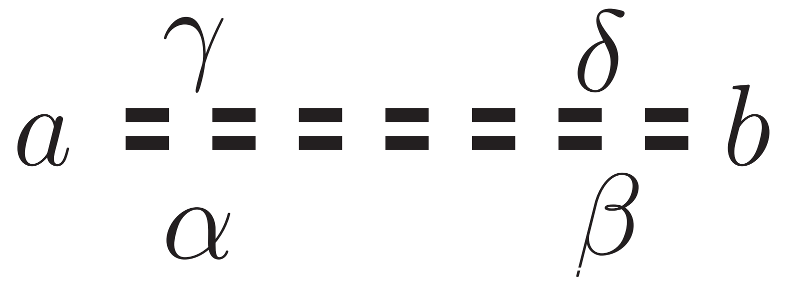
$$V_{\alpha\beta\gamma}(p,q,r) = (p-q)_\gamma g_{\alpha\beta} + (q-r)_\alpha g_{\beta\gamma} + (r-p)_\beta g_{\alpha\gamma}$$

4-gluon $$\Gamma^{abcd}_{\alpha\beta\gamma\delta} = -i\, g_s^2 \begin{bmatrix} +f^{xac}\, f^{xbd}\,(g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\delta}g_{\beta\gamma}) \\ +f^{xad}\, f^{xcb}\,(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\beta}g_{\gamma\delta}) \\ +f^{xab}\, f^{xdc}\,(g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta}) \end{bmatrix}$$



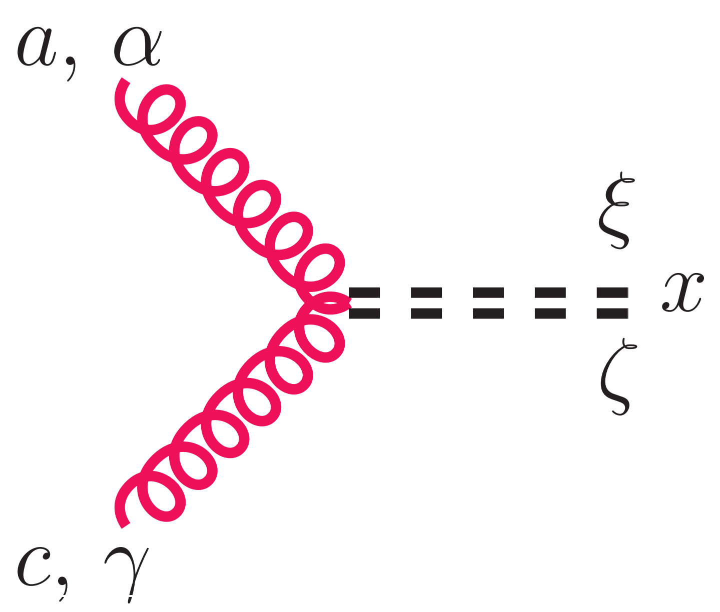
4-gluon vertex

- for the 4-gluon vertex the colour and the kinematic part do not factorise
- however one can achieve a factorised form with an auxiliary field carrying two Lorentz indices, with propagator



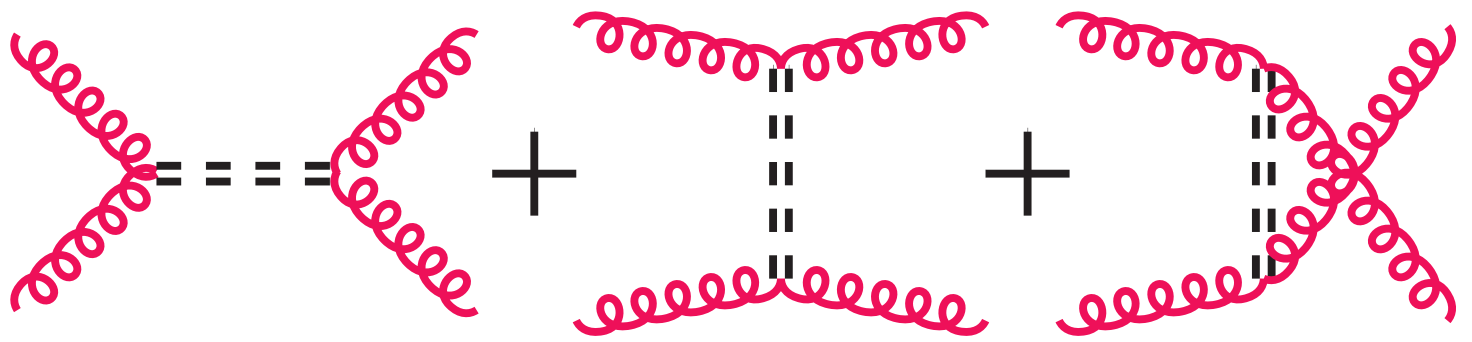
$$= -\frac{i}{2}\delta^{ab}(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\delta}g^{\beta\gamma})$$

and coupling to the gluons with the rule



$$= i\sqrt{2}g_s\, f^{xac}g^{\alpha\xi}g^{\gamma\zeta}$$

- the 4-gluon vertex then can be written as the sum of 3 diagrams where colour and Lorentz structure factorise



Feynman rules

spinors:

incoming fermion	$u(p,s)$	
incoming anti-fermion	$\bar{v}(p,s)$	
outgoing fermion	$\bar{u}(p,s)$	
outgoing anti-fermion	$v(p,s)$	

polarisation vectors:

incoming vector boson	$\varepsilon_\mu(k,\lambda)$	
outgoing vector boson	$\varepsilon^*_\mu(k,\lambda)$	

Feynman rules

further rules:

- momentum conservation at each vertex
- factor (-1) for each closed fermion loop
- factor (-1) for switching identical external fermions
- integrate over loop momenta with $\int \frac{d^4 l}{(2\pi)^4}$

Summary

- Factorisation and asymptotic freedom are essential to separate short-distance from long-distance (non-perturbative) dynamics

- Without QCD corrections, (most of) the data are not well described

- Description of QCD as SU(3) local gauge theory has important consequences, for example self-interactions between the gluons

- Colour algebra: can be separated from kinematics

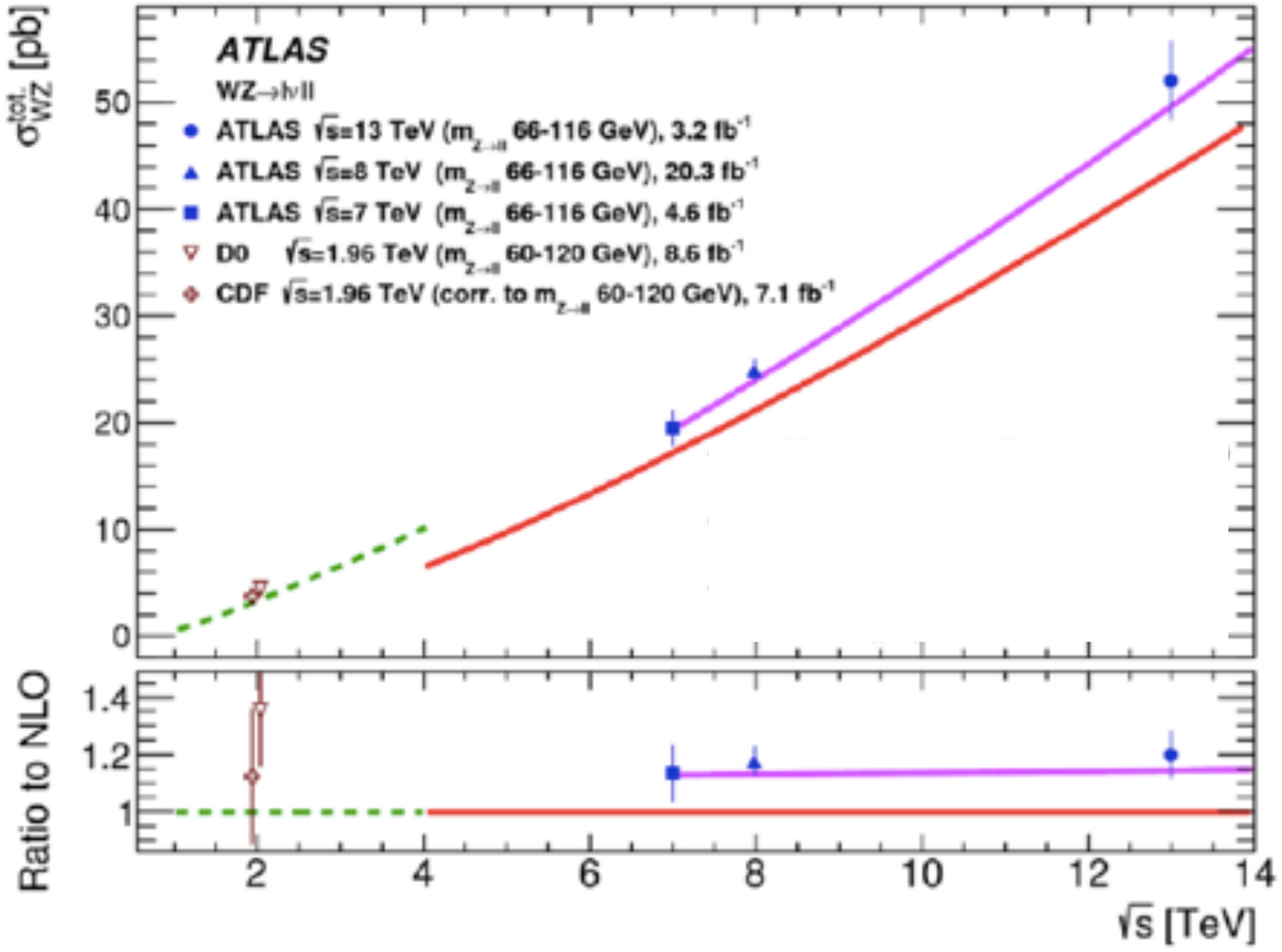
- Next: cross sections, running coupling, scale uncertainties

Appendix

Quiz

- How many different quarks do we have in the SM?
- Does factorisation always hold?
- How can we represent the colour charge of gluons?
- What is characteristic for physical gauges?
- Why are ghost fields not relevant in QED?
- Which interaction is stronger at $\sqrt{s} \approx M_Z$:

the gluon self-interaction or the Higgs boson self-interaction?



Wiesemann, Grazzini, Kallweit, Rathlev ‘17

- What are the red and pink curves in the plot above?

Conventions

we will use so-called “**natural units**”: $\boxed{\hbar = c = 1}$

in these conventions, energy, mass and momentumhave the same units

momentum vector: $p^\mu = (E, \vec{p}) = (p^0, \vec{p})$

“on-shell” four-momentum: $p^2 = m^2 = E^2 - \vec{p}^{\,2}$

$(c \neq 1 : E^2 = m^2c^4 + \vec{p}^{\,2}c^2)$

proton mass: $m_p \simeq 1\, GeV = 10^9\, eV$

remember Heisenberg: $\Delta p \Delta x \geq \frac{\hbar}{2}$

therefore with $\hbar = c = 1$ large energies means small distances

useful spinor relations

$$(\not{p} - m)\, u(p,s) = 0 \qquad \bar{u}(p,s)(\not{p} - m) = 0$$
$$(\not{p} + m)\, v(p,s) = 0 \qquad \bar{v}(p,s)(\not{p} + m) = 0$$

(Dirac equation)

$$\bar{v}(p,r)u(p,s) = 0 \qquad \bar{u}(p,r)u(p,s) = 2m\, \delta_{rs}$$
$$\bar{u}(p,r)v(p,s) = 0 \qquad \bar{v}(p,r)v(p,s) = -2m\, \delta_{rs}$$

(orthogonality)

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m$$
$$\sum_s v(p,s)\bar{v}(p,s) = \not{p} - m$$

(completeness)