Lecture 1: basics of QCD

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Outline

- Introduction
- Properties of the strong interactions
- Experimental evidence for colour
- QCD Lagrangian
- Colour algebra
- Feynman rules
Quantum Chromodynamics is a very rich field!

We will focus on perturbative QCD.
Importance of QCD corrections

perturbation theory in the strong coupling $\alpha_s$

$$\sigma = \sigma^{\text{LO}} + \alpha_s \sigma^{\text{NLO}} + \alpha_s^2 \sigma^{\text{NNLO}} + \mathcal{O}(\alpha_s^3)$$

leading order

next-to-leading order

next-to-next-to-leading order

$\alpha_s(M_Z) \simeq 0.118 \Rightarrow \text{NLO corrections } \sim \mathcal{O}(10\%)$

NNLO corrections typically a few %

but there are prominent exceptions, e.g. Higgs production in gluon fusion:

NLO corr. $\sim$100%, NNLO $\sim$30%

From M. Grazzini, D. de Florian
Importance of QCD corrections

perturbation theory in the strong coupling $\alpha_s$

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but there are prominent exceptions,

\text{e.g. Higgs production in gluon fusion: } 
\text{NLO corr. } \sim 100\%, \text{ NNLO } \sim 30\%
QCD

• QCD is the theory of one of the 4 fundamental forces in nature: the strong interactions

• it describes the interactions between quarks and gluons, also called partons

• what does “strong” interaction mean? about $10^{38}$ times larger than the gravitational force (at length scales of the size of a nucleon, ~1fm)

• however the strong coupling is not a constant, it depends on the energy

• at high energies, the coupling is small: $\rightarrow$ asymptotic freedom

• at small energies: coupling large, no free quarks and gluons $\rightarrow$ confinement (hadrons)
Hadronic collisions

• at hadron colliders (e.g. CERN LHC), QCD is everywhere

• need to factorise perturbative from non-perturbative part

important concepts:

• factorisation

• asymptotic freedom
non-perturbative stuff

https://indico.cern.ch/event/505613/contributions/2230824/
visualisation of pile-up (multiple soft collisions in each bunch crossing) in the ATLAS tracker (Run I)

separation of perturbative from non-perturbative parts highly non-trivial!
Stages of an event

- parton shower
- hadronisation
- parton distribution functions (PDFs)
- underlying event
- fixed order calculations (production and decay)
- resummation

- reduce scale uncertainties
- $\mu_R, \mu_F$ -dependence
- reduce parametric uncertainties (couplings, masses)
- reduce uncertainties in particular kinematic regions

artwork by G. Luisoni
Asymptotic freedom

- strong coupling is not constant
  - becomes weaker as energy scale increases
  - at very large energies quarks and gluons are almost free particles

will be discussed in more detail later
Factorisation

\[ d\sigma_{pp \rightarrow B+X} = \sum_{i,j} \int_0^1 dx_1 f_{i/p_a}(x_1, \alpha_s, \mu_f) \int_0^1 dx_2 f_{j/p_b}(x_2, \alpha_s, \mu_f) \times d\hat{\sigma}_{ij \rightarrow B+X}(\{p\}, x_1, x_2, \alpha_s(\mu_r), \mu_r, \mu_f) J(\{p\}) + \mathcal{O}\left(\frac{\Lambda}{Q}\right)^p \]

high energy scattering, calculable as a series in perturbation theory can be (mostly) separated from non-perturbative components
Properties of quarks

- quarks are fermions (spin 1/2)
- they are the constituents of hadrons
- they come in 6 flavours, forming 3 generations of up-type and down-type quarks:
  \[
  \begin{pmatrix}
  u \\ d
  \end{pmatrix}, \begin{pmatrix}
  c \\ s
  \end{pmatrix}, \begin{pmatrix}
  t \\ b
  \end{pmatrix}
  \] (doublets under weak interactions and strong isospin)
- charges: up-type: 2/3, down-type: -1/3 (opposite for antiquarks)
- additional quantum number: colour charge: SU(3) local gauge theory
- the masses of the different quark flavours are very different, we do not know why
**Introduction to QCD**

Gudrun Heinrich

**Experimenteller Nachweis**

- **1860**
  - **1 eV/c²**
  - **10 eV/c²**
  - **100 eV/c²**
  - **1 MeV/c²**
  - **10 MeV/c²**
  - **100 MeV/c²**
  - **1 GeV/c²**
  - **10 GeV/c²**
  - **100 GeV/c²**
- **1900**
  - **Ruhemasse**
- **1962**
  - **< 2,2 eV** (νₑ)
- **1969**
  - **< 0,2 MeV** (νₑ/µ⁻)
- **1969**
  - **2,3 MeV** (νₓ/τ⁻)
- **1974**
  - **1,3 GeV** (νₓ/τ⁻)
- **1983**
  - **80,4 GeV** (W⁻/B⁺)
- **1995**
  - **173,5 GeV** (Z⁻/B⁻)
- **2012**
  - **< 15,5 MeV** (νₓ)
- **1936**
  - **< 105,7 MeV** (ν₇)
- **1979**
  - **0,0 GeV** (Gluon)
- **1981**
  - **0,0 GeV** (Photon)

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Evidence for colour

how do we know the colour quantum number exists?

evidence:

- jet measurements
- hadronic R-ratio
- existence of doubly charged baryons, e.g. $\Delta^{++} = |u uu\rangle$
  would violate Pauli’s exclusion principle without additional quantum number

- Pion decay $\Gamma(\pi^0 \rightarrow \gamma\gamma) \sim \alpha^2 \frac{m^3_{\pi}}{f^2_{\pi}} \left( e_u^2 - e_d^2 \right)^2 N_c^2$

  note however that with $e_u = \frac{1}{2} \left( \frac{1}{N_c} + 1 \right)$, $e_d = \frac{1}{2} \left( \frac{1}{N_c} - 1 \right)$ the width would be independent of $N_c$
Evidence for Colour

hadronic R-ratio:

\[ R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \]

\[ R(s) = N_c \sum_{f=u,d,s,c,\ldots} e_f^2 \theta(s - 4m_f^2) \]

\[ R = 3 \left( \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \ldots \right) \]

\[ \text{threshold energy to produce quark-antiquark-pair of mass } m_f \]
R-ratio

resonances removed, higher order corrections included

Harlander, Steinhauser, hep-ph/0212294

\[ m_c \approx 1.3 \text{ GeV} \]
\[ m_b \approx 4.5 \text{ GeV} \]

\[ N_c \sum_{f=u,d,s,c,b} e_f^2 \theta(s - 4m_f^2) = 3 \left( \frac{4}{9} + \frac{1}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9} \right) = \frac{11}{3} \]
QCD Lagrangian

strong interactions: $SU(N_c)$ gauge theory, $N_c = 3$ “colours” of quarks

fermionic part: quark fields for flavour $f$: $q_f^i(x)$ $i = 1, 2, 3$ colour index

for free quark fields:

$$\mathcal{L}_q^{(0)}(q_f, m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x) (i \gamma_\mu \partial^\mu - m_f)_{jk} q_f^k(x)$$

apply SU(N) group transformation: $q_i \rightarrow q_i' = U_{ij} q^j$, $\bar{q}_i \rightarrow \bar{q}_i' = \bar{q}^j U^{-1}_{ji}$

$$U_{ij} = \exp \left\{ i \sum_{a=1}^{N_c-1} t^a \theta^a \right\}_{ij} = \delta_{ij} + i \sum_{a=1}^{N_c-1} t^a \theta^a + \mathcal{O}(\theta^2)$$
QCD Lagrangian

\[ t^\alpha_{i,j} = \lambda^\alpha_{i,j} / 2 \]  generators of SU(3) in fundamental representation (3x3 matrices)

\[ \lambda^\alpha_{i,j} : \text{Gell-Mann matrices (traceless, hermitian)} \]

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\]

more about colour algebra coming soon
QCD Lagrangian

QCD is a *local* gauge theory

*local* means that the gauge transformation parameter *depends on the space-time point*:

\[ \theta = \theta(x) \quad \text{therefore} \quad U = U(x) \quad \text{and} \]

\[ \partial_\mu q'(x) = \partial_\mu (U(x)q(x)) = U(x)\partial_\mu q(x) + (\partial_\mu U(x)) q(x) \]

how can we keep \( \mathcal{L}_q \) invariant despite this additional term?

introduce coupling to a gauge field \( A_\mu^a \) (gluon) through covariant derivative

\[
(D^\mu [A])_{ij} = \delta_{ij} \partial^\mu + ig_s t_i^a A_\mu^a
\]
QCD Lagrangian

Define \( A^\mu = \sum_{a=1}^{N_c^2-1} t^a A^\mu_a \) then in index-free notation \( D^\mu[A] = \partial^\mu + ig_s A^\mu \)

Quark Lagrangian with “minimal coupling” of gluon:

\[
\mathcal{L}_q(q_f, m_f) = \sum_{j,k=1}^{N_c} \bar{q}^j_f(x) (i \gamma_\mu D^\mu[A] - m_f)_{jk} q^k_f(x)
\]

Invariant under local gauge transformations if we have

\[
D^\mu[A]q(x) \overset{!}{\to} U\left(D^\mu[A]q(x)\right)
\]
SU(3) gauge invariance

This gives a condition on the transformed gluon field $A'_\mu$

$$D^\mu[A'] = U \left( D^\mu[A] \right) U^{-1} \Rightarrow \partial_\mu + ig_s A'_\mu = \frac{i}{g_s} \left( \partial_\mu U(x) \right) U^{-1}$$

$$A'_\mu = U(x) A_\mu U^{-1}(x) + \frac{i}{g_s} \left( \partial_\mu U(x) \right) U^{-1}(x)$$
Yang-Mills Lagrangian

purely gluonic part: Yang-Mills Lagrangian \((\text{C.N.}\text{Yang, R.}\text{Mills, 1954})\)

\[
\mathcal{L}_{\text{YM}} = -\frac{1}{4} F_{\mu \nu}^a F^{a,\mu \nu}
\]

field strength tensor \(F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_s f^{abc} A_\mu^b A_\nu^c\)

\(f^{abc}\) : structure constants of SU(3) (totally antisymmetric)

\([T^a, T^b] = i f^{abc} T^c\quad a, b, c = 1, \ldots, N_c^2 - 1\)

\(\Rightarrow\) 8 gluons (in the adjoint representation of SU(3))

non-Abelian structure of SU(3) is related to gluon-self-interactions
so far we have \( \mathcal{L} = \mathcal{L}_q + \mathcal{L}_{YM} \)

\[
\mathcal{L}_q(q_f, m_f) = \sum_{j,k=1}^{N_c} \bar{q}_f^j(x) \left( i \gamma_\mu D^\mu[A] - m_f \right)_{jk} q_f^k(x)
\]

\[
\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}
\]

\( \mathcal{L}_q + \mathcal{L}_{YM} \) is gauge invariant, but:

path integral contains physically equivalent configurations (by gauge transformations)

\[ \Rightarrow \text{ path integral over the action is not well defined due to this redundancy} \]
Gauge fixing

- the gluon propagator $\Delta_{\mu\nu}^{ab}(p) = \Delta_{\mu\nu}(p)\delta^{ab}$ is constructed from the inverse of the bilinear term in the gluon fields $\sim A^a_{\mu} A^b_{\nu}$ in the action

$$S_{YM} = i \int d^4x \left\{ -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu} \right\} \supset \frac{i}{2} \int d^4x A^a_{\mu}(x) \left( \partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A^b_{\nu}(x) \delta_{ab}$$

- in momentum space:

$$\sim \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{A}^a_{\mu}(p) \left( p^2 g^{\mu\nu} - p^\mu p^\nu \right) \tilde{A}^b_{\nu}(p) \delta_{ab}$$
Gauge fixing

in momentum space the propagator should fulfill (cf QED Green's function to solve e.o.m.)

\[ i \Delta_{\mu\rho}(p) \left[ p^2 g^{\rho\nu} - p^\rho p^\nu \right] = g_\mu^\nu \quad (1) \]

however as \[ \left[ p^2 g^{\rho\nu} - p^\rho p^\nu \right] p_\nu = 0 \]

Eq. (1) has zero modes, so the matrix in the square brackets is not invertible

reason: \( \mathcal{L}_q + \mathcal{L}_{YM} \) contains redundant, physically equivalent configurations

\[ \Rightarrow \text{need gauge fixing: add constraint on gluon fields with a Lagrange multiplier} \]
QCD Lagrangian

covariant gauges: add condition \( \partial_\mu A^\mu (x) = 0 \)

\[ \Rightarrow \mathcal{L}_{\text{GF}} = -\frac{1}{2\lambda} (\partial_\mu A^\mu)^2, \quad \lambda \in \mathbb{R} \]

leads to bilinear term of the form

\[ i \left( p^2 g^{\mu\nu} - \left( 1 - \frac{1}{\lambda} \right) p^\mu p^\nu \right) \]

with inverse

\[ \Delta_{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon} \left[ g_{\mu\nu} - (1 - \lambda) \frac{p_\mu p_\nu}{p^2} \right] \quad \text{(colour part } \delta^{ab} \text{ omitted)} \]

\( \lambda = 1 \) : propagator in Feynman gauge

\( \lambda = 0 \) : propagator in Landau gauge

\( i\varepsilon \) shifts the poles of the propagator slightly away from the real axis
QCD Lagrangian

- however in covariant gauges unphysical (non-transverse) degrees of freedom can propagate

- their effect is cancelled by ghost fields $\eta^a$ [L. Faddeev, V. Popov 1967]

(coloured complex scalars obeying Fermi statistics, do not occur as external states)

$$\mathcal{L}_{FP} = \eta_a^{\dagger} M^{ab} \eta_b$$

Faddeev-Popov matrix in covariant gauge:

$$M^{ab} = \delta^{ab} \partial_\mu \partial^\mu + g_s f^{abc} A^c_\mu \partial^\mu$$

complete QCD Lagrangian:

$$\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_q + \mathcal{L}_{GF} + \mathcal{L}_{FP}$$
Physical gauges

- unphysical degrees of freedom and the ghost fields can be avoided by choosing **axial (physical) gauges:** condition $n_\mu A^\mu = 0$; $n^\mu$ vector with $p \cdot n \neq 0$

in axial gauges: $$\mathcal{L}_{GF} = -\frac{1}{2\alpha} \left( n^\mu A^\mu \right)^2$$

gluon propagator: $$\Delta_{\mu\nu}(p, n) = \frac{-i}{p^2 + i\varepsilon} \left( g_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n} + \frac{n^2 p_\mu p_\nu}{(p \cdot n)^2} \right)$$

**special case** $n^2 = 0$: **light-cone gauge**

no propagating ghost fields: $$M^{ab}_{\text{axial}} = \delta^{ab} n_\mu \partial^\mu + g_s f^{abc} n_\mu A^\mu \text{zero}$$
Physical gauges

in light-cone gauge we have \[ \Delta_{\mu\nu}(p, n) = \frac{i}{p^2 + i\varepsilon} d_{\mu\nu}(p, n) \]

\[ d_{\mu\nu}(p, n) = -g_{\mu\nu} + \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n} = \sum_{\lambda=1,2} \epsilon^\lambda_\mu(p) (\epsilon^\lambda_\nu(p))^* \]

\[ \epsilon^\lambda_\nu(p) \) : polarisation vector \] \[ \epsilon_1 = (0, 1, 0, 0), \epsilon_2 = (0, 0, 1, 0) \] or \[ \epsilon_{L,R} = (0, 1, \pm i, 0)/\sqrt{2} \]

\[ \Rightarrow \text{ only the two physical polarisations propagate} \]

e.g. choose \( p = (p^0, 0, 0, p^0), n = (p^0, 0, 0, -p^0) \) then

\[ -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p \cdot n} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]
Colour algebra
Colour Algebra

SU(N): Lie group \((\text{elements depend on a finite number of continuous parameters})\)

U: unitary \(UU^\dagger = 1\); S: “special” \((\det U=1)\)

representation: mapping of group elements onto matrices, such that group operations translate to matrix operations

associated algebra: generators fulfill

\[
[T^a, T^b] = if^{abc}T^c \quad a, b, c = 1, \ldots, N^2 - 1
\]

independent of the representation

number of generators = \textit{dimension} of the group
Colour Algebra

important representations:

• fundamental representation: generators are $N \times N$ matrices
  
  $t_{i,j}^a = \frac{\lambda_{i,j}^a}{2}$ \quad $i, j = 1 \ldots N$

  $\lambda_{i,j}^a$ : Gell-Mann matrices

• adjoint representation: generators are $(N^2 - 1) \times (N^2 - 1)$ matrices
  
  i.e. indices run over dimension of the group

  generators $(F^{a})_{bc}$ with $(F^{a})_{bc} = -i f^{abc}$

quarks are in the fundamental representation, gluons in the adjoint

$\Rightarrow$ 8 gluons, each quark flavour comes in three colours
Colour Algebra

explicit example for a quark-gluon vertex

\[ -i g_s \bar{\psi}_i \frac{\lambda^a_{ij}}{2} \psi_j A^a_\mu \]

\( i = 1, j = 2, a = 1 \):

\[ \alpha = -\frac{i}{2} g_s \quad \bar{\psi}_{qR} \quad \begin{pmatrix} \lambda_1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \psi_{qG} = -\frac{i}{2} g_s \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \]

Gluons can be represented as double lines of colour-anticolour combinations, e.g.

\[ \lambda^1 \rightarrow \frac{1}{\sqrt{2}} \bar{r}g \quad \lambda^8 \rightarrow \frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b}) \]

Note that the combination \[ \frac{1}{\sqrt{3}}(r\bar{r} + g\bar{g} + b\bar{b}) \] does not occur for the gluon because this would correspond to a colour singlet.
Gell-Mann matrices (again)

\[ t^a_{ij} = \frac{\lambda^a_{ij}}{2} \quad \text{generators of SU}(3) \text{ in fundamental representation} \]

\[ \lambda^a_{ij} : \text{Gell-Mann matrices (traceless, hermitian)} \]

\[
\begin{align*}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , &
\lambda^2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , &
\lambda^3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , &
\lambda^5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , &
\lambda^6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} , &
\lambda^8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} .
\end{align*}
\]
**Colour Algebra pictorially**

\[ i \rightarrow j \quad \text{colour} \quad \delta_{ij} \]

\[ i \rightarrow a \rightarrow j \quad \text{colour} \quad t^a_{ij} \]

\[ a \rightarrow b \rightarrow c \quad \text{colour} \quad \delta_{ab} \]

\[ a \rightarrow b \rightarrow c \quad \text{colour} \equiv if^{abc} \]

\[ [t^a, t^b] = if^{abc} t^c \]

\[ \text{colour} \equiv \delta_{ij} \delta^{ij} = N_c \]

\[ \text{colour} \equiv \delta_{ab} \delta^{ab} = N_c^2 - 1 \]
**Colour “Casimirs”**

**group invariants:**

\[
\sum_{j,a} t^a_{ij} t^a_{jk} = C_F \delta_{ik}, \quad \sum_{a,d} F^a_{ba} F^a_{dc} = C_A \delta_{bc}
\]

\[C_F, C_A\] : eigenvalues of Casimir operators in fundamental/adjoint representation

(Casimir operators commute with any element of the Lie algebra)

\[
C_F = T_R \frac{N_c^2 - 1}{N_c}, \quad C_A = 2 T_R N_c
\]

\[
\text{Trace}[T^a T^b] = T_R \delta^{ab}
\]

usually \(T_R = \frac{1}{2}\) (convention)
Colour “Fierz identities”

the double line representation for the gluons also allows us to derive some identities, for example

\[ t^a_{ij} t^a_{kl} = T_R \left( \delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \]

\[ \begin{array}{cc}
  j & i \\
  k & l
\end{array} = T_R \left( \begin{array}{cc}
  j & i \\
  k & l
\end{array} \right) \]

second term is the case where quarks have the same colour → no gluon is exchanged;

normalisation can be seen e.g. by contracting with \( \delta_{jk} \)

\[ \text{lhs: } t^a_{ij} t^a_{kl} \delta_{jk} = t^a_{ij} t^a_{jl} = C_F \delta_{il} \quad \text{ rhs: } T_R \left( \delta_{il} N_c - \frac{1}{N_c} \delta_{il} \right) = T_R \frac{N_c^2 - 1}{N_c} \delta_{il} = C_F \delta_{il} \]
Colour Algebra

note that contracting with $\delta_{jk}$ pictorially corresponds to

$$\delta_{jk} = \begin{array}{c}
\text{(quark self energy diagram)}
\end{array}$$
colour decomposition

we can express gluon amplitudes entirely in terms of generators $t_{ij}^a$

based on

$$\begin{align*}
\text{Trace}(t^a t^b t^c) - \text{Trace}(t^c t^b t^a) &= i T_R f^{abc} \\
\end{align*}$$

we can eliminate $f^{abc}$

similarly

$$\begin{align*}
q\bar{q}gggg \ldots &\rightarrow \text{Tr}(t^{a_1} t^{a_2} \ldots t^{a_n})_{ij} + \text{permutations}
\end{align*}$$
colour decomposition

idea: split amplitude into a colour part and a kinematic part

\[ A_n^{\text{tree}} = g_s^{n-2} \sum_{\sigma \in S_{n-2}} (t^{a(3)} \ldots t^{a(n)})_{j_1 i_2} A_n^{\text{tree}}(1^{\lambda_1}, 2^{\lambda_2}, \sigma(3^{\lambda_3}), \ldots, \sigma(n^{\lambda_n})) \]

also eliminates the \( t^a \), based on

\[ t_{ij}^{a} t_{kl}^{a} = T_R \left( \delta_{il} \delta_{jk} - \frac{1}{N_c} \delta_{ij} \delta_{kl} \right) \]

leads to a large reduction of complexity and more manifest IR singularity structure

• there are various ways to do a colour decomposition

• the “colour flow decomposition” also eliminates the \( t^a \), based on

\[ \text{table from arXiv:hep-ph/9910563} \]

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QCD Feynman rules

Propagators:

- **gluon** \( \Delta_{\mu\nu}^{ab}(p) = \delta^{ab} \Delta_{\mu\nu}(p) \)
- **quark** \( \Delta^{ij}(p) = \delta^{ij} \frac{i(p + m)}{p^2 - m^2 + i\varepsilon} \)
- **ghost** \( \Delta^{ab}(p) = \delta^{ab} \frac{i}{p^2 + i\varepsilon} \)

\[
\Delta_{\mu\nu}(p) = \frac{-i}{p^2 + i\varepsilon} \left[ g_{\mu\nu} - (1 - \lambda) \frac{p_\mu p_\nu}{p^2} \right] \quad \text{covariant gauge} \\
\lambda = 1 : \text{Feynman gauge}
\]

\[
\Delta_{\mu\nu}(p, n) = \frac{-i}{p^2 + i\varepsilon} \left[ g_{\mu\nu} - \frac{p_\mu n_\nu + n_\mu p_\nu}{p \cdot n} \right] \quad \text{light-cone gauge} \\
n^2 = 0
\]
Feynman rules

Vertices:

quark-gluon

$$\Gamma_{gq\bar{q}}^{\mu, a} = -i \ g_s \ (t^a)_{ij} \gamma^\mu$$

ghost-gluon

$$\Gamma_{g\eta\bar{\eta}}^{\mu, a} = -i \ g_s \ (F^a)_{bc} \not{p}^\mu$$

$$= -g_s \ f^{abc} \ p^\mu$$
Feynman rules

3-gluon

$$\Gamma_{\alpha\beta\gamma}^{abc}(p, q, r) = -i g_s (F^a)_{bc} V_{\alpha\beta\gamma}(p, q, r)$$

$$V_{\alpha\beta\gamma}(p, q, r) = (p - q)_{\gamma} g_{\alpha\beta} + (q - r)_{\alpha} g_{\beta\gamma} + (r - p)_{\beta} g_{\alpha\gamma}$$

4-gluon

$$\Gamma_{\alpha\beta\gamma\delta}^{abcd} = -i g_s^2 \left[ + f^{xac} f^{xbd} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma}) 
+ f^{xad} f^{xbc} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}) 
+ f^{xab} f^{xdc} (g_{\alpha\delta} g_{\beta\gamma} - g_{\alpha\gamma} g_{\beta\delta}) \right]$$
4-gluon vertex

- for the 4-gluon vertex the colour and the kinematic part do not factorise

- however one can achieve a factorised form with an auxiliary field carrying two Lorentz indices, with propagator

\[ a^\alpha \gamma = \delta^\beta \delta^\gamma b = -\frac{i}{2} \delta^{ab} (g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}) \]

and coupling to the gluons with the rule

\[ \xi = i \sqrt{2} g_s f x^{ac} g^{\alpha\xi} g^{\gamma\zeta} \]

- the 4-gluon vertex then can be written as the sum of 3 diagrams where colour and Lorentz structure factorise
Feynman rules

**spinors:**

- incoming fermion: $u(p, s)$
- incoming anti-fermion: $\bar{v}(p, s)$
- outgoing fermion: $\bar{u}(p, s)$
- outgoing anti-fermion: $v(p, s)$

**polarisation vectors:**

- incoming vector boson: $\varepsilon_\mu(k, \lambda)$
- outgoing vector boson: $\varepsilon^*_\mu(k, \lambda)$
Feynman rules

**Further rules:**

- Momentum conservation at each vertex
- Factor (-1) for each closed fermion loop
- Factor (-1) for switching identical external fermions
- Integrate over loop momenta with \( \int \frac{d^4l}{(2\pi)^4} \)
Summary

• Factorisation and asymptotic freedom are essential to separate short-distance from long-distance (non-perturbative) dynamics

• Without QCD corrections, (most of) the data are not well described

• Description of QCD as SU(3) local gauge theory has important consequences, for example self-interactions between the gluons

• Colour algebra: can be separated from kinematics

• Next: cross sections, running coupling, scale uncertainties
Appendix
Quiz

• How many different quarks do we have in the SM?
• Does factorisation always hold?
• How can we represent the colour charge of gluons?
• What is characteristic for physical gauges?
• Why are ghost fields not relevant in QED?
• Which interaction is stronger at $\sqrt{s} \approx M_Z$:
  the gluon self-interaction or the Higgs boson self-interaction?
• What are the red and pink curves in the plot above?
Conventions

we will use so-called “natural units”: \( \hbar = c = 1 \)

in these conventions, energy, mass and momentum have the same units

momentum vector: \( p^\mu = (E, \vec{p}) = (p^0, \vec{p}) \)

“on-shell” four-momentum: \( p^2 = m^2 = E^2 - \vec{p}^2 \)

\( (c \neq 1 : E^2 = m^2 c^4 + \vec{p}^2 c^2) \)

proton mass: \( m_p \simeq 1 \text{ GeV} = 10^9 \text{ eV} \)

remember Heisenberg: \( \Delta p \Delta x \geq \frac{\hbar}{2} \)

therefore with \( \hbar = c = 1 \) large energies mean small distances
useful spinor relations

\[(\not\!p - m) u(p, s) = 0 \quad \overline{u}(p, s)(\not\!p - m) = 0\]
\[(\not\!p + m) v(p, s) = 0 \quad \overline{v}(p, s)(\not\!p + m) = 0\]

(Dirac equation)

\[\overline{v}(p, r)u(p, s) = 0 \quad \overline{u}(p, r)u(p, s) = 2m \delta_{rs}\]
\[\overline{u}(p, r)v(p, s) = 0 \quad \overline{v}(p, r)v(p, s) = -2m \delta_{rs}\]

(orthogonality)

\[\sum_s u(p, s)\overline{u}(p, s) = \not\!p + m\]
\[\sum_s v(p, s)\overline{v}(p, s) = \not\!p - m\]

(completeness)