Karlsruhe Institute of Technology

Collaborative Research Center TRR 257

Particle Physics Phenomenology after the Higgs Discovery

Lecture 2: running coupling, scale uncertainties, NLO

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Outline

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- strong CP problem
- from amplitudes to cross sections
- tree level amplitudes
- polarisation sums
- running coupling
- scale uncertainties
- basics of NLO

The strong CP problem

complete QCD Lagrangian: $\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_q + \mathcal{L}_{GF} + \mathcal{L}_{FP}$

The strong CP problem

complete QCD Lagrangian: $\mathcal{L}_{QCD} = \mathcal{L}_{YM} + \mathcal{L}_q + \mathcal{L}_{GF} + \mathcal{L}_{FP}$

complete?

- from a theory point of view, another term would be allowed because it is gauge invariant and renormalizable
- we can form a dual field strength tensor: $\tilde{F}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta}$

$\epsilon^{\mu\nu\alpha\beta}$: 1 for even permutation, -1 for odd permutation, 0 otherwise

and a Lagrangian

$$\mathcal{L}_\Theta = \Theta\,\frac{g_s}{32\pi^2}\sum_a F^a_{\mu\nu}\tilde{F}^{a,\mu\nu}$$

The strong CP problem

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$$\mathcal{L}_\Theta = \Theta \frac{g_s}{32\pi^2} \sum_a F^a_{\mu\nu} \tilde{F}^{a,\mu\nu}$$

- this Lagrangian would violate CP invariance and contribute to the electric dipole moment of the neutron
- measurements lead to $\Theta < 10^{-10}$
- often in physics a symmetry is behind if a parameter is very small
- Peccei and Quinn (1977) suggested a spontaneously broken U(1) gauge symmetry
- the corresponding Goldstone boson is called ***axion***
- the axion would also be a very good dark matter candidate
- searches for axions are ongoing

The strong CP problem

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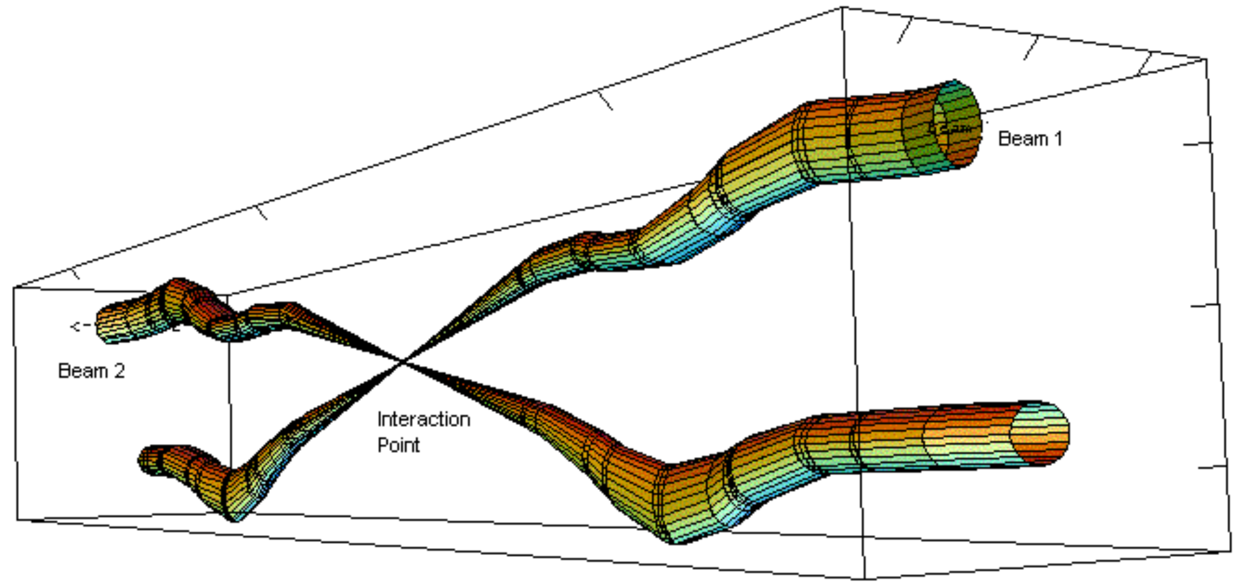
more in the lectures by Mikhail Shaposhnikov

Cross sections

n bunches , f: bunch frequency, F: bunch crossing area

N_a, N_b : number of particles per bunch

luminosity $L = f \cdot n \cdot \frac{N_a N_b}{F}$



Relative beam sizes around IP1 (Atlas) in collision

reaction rate R: $R = L \cdot \sigma$

LHC: *n* = 2808 bunches, $f \simeq 11$ kHz

$N_a = N_b \simeq 10^{11}$, $F = \pi r^2, r \sim 30 \mu m$ (at the collision point) $\Rightarrow L \simeq 10^{34} \frac{1}{cm^2 s}$

units: $1 \text{ barn} = 10^{-24} \text{cm}^2$

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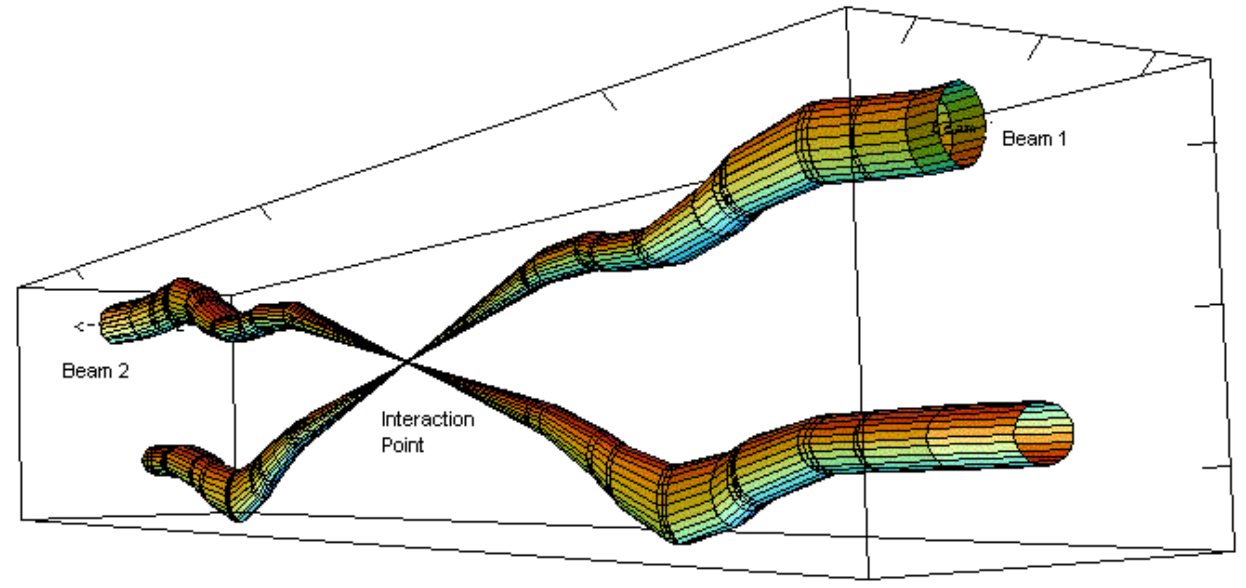

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example: $@\sqrt{s} = 14\,\text{TeV}$

$\sigma_{t\bar{t}} \simeq 1\, nb = 10^{-33} cm^2$

$R = L \cdot \sigma$ = 10 top quark pairs per sec.

From amplitudes to cross sections

- for a scattering process $q_a + q_b \to p_1 + \ldots + p_N$

$$d\sigma = \frac{J}{\text{flux}} \cdot |\mathcal{M}|^2 \cdot d\Phi_N$$

$\mathcal{M}$: matrix element (derived via Feynman rules)

$d\Phi_N$: phase space of *N* final state particles

J : statistical factor, $J = 1/j\,!$ for each group of identical particles in the final state

$$\text{flux} = 4\sqrt{(q_a \cdot q_b)^2 - m_a^2 m_b^2} \longrightarrow 4 q_a \cdot q_b = 2\hat{s} \qquad (m = 0, \text{cms})$$

unpolarised: $|\mathcal{M}|^2 \to |\overline{\mathcal{M}}|^2 = \prod_{\text{initial}} \frac{1}{N_{\text{pol}} N_{\text{col}}} \sum_{\text{pol,col}} |\mathcal{M}|^2$ average over initial state, sum over final state pol., col.

Sum over spins/polarisations of external particles

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gluons: $$\sum_{\text{phys. pol. }\lambda} \epsilon^{\mu}_{\lambda}(k)\epsilon^{\nu,\star}_{\lambda}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{k\cdot n}$$

for photons, due to $k_1^{\mu_1}\dots k_n^{\mu_n}\mathcal{M}_{\mu_1\dots\mu_n} = 0$

we can replace the above sum by $-g^{\mu\nu}$

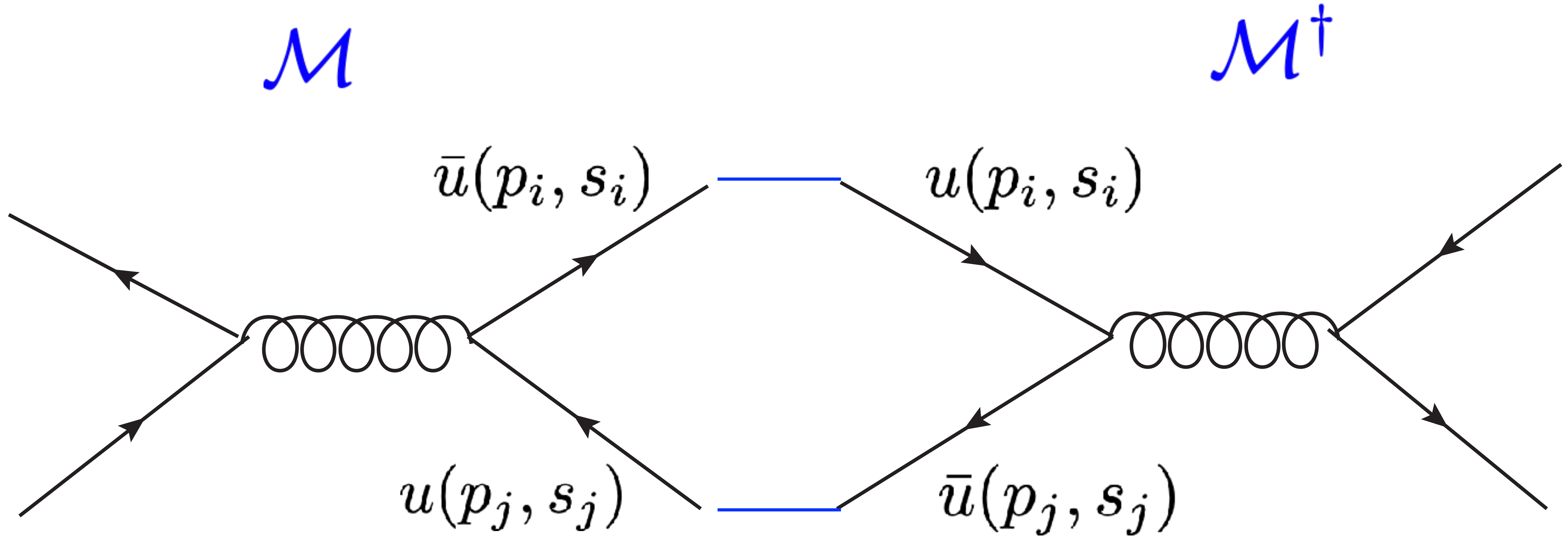
fermions: Γ_1, Γ_2 strings of γ-matrices $\bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$

$$\sum_{s_i,s_j} \left(\bar{u}(p_i,s_i)\Gamma_1 u(p_j,s_j)\right)\left(\bar{u}(p_i,s_i)\Gamma_2 u(p_j,s_j)\right)^\dagger$$
$$= \text{Trace}[\Gamma_1(\not{p}_j + m_j)\bar{\Gamma}_2(\not{p}_i + m_i)]$$

Sum over spins/polarisations of external particles

$$\sum_{s_i,s_j} \left(\bar{u}(p_i,s_i)\Gamma_1 u(p_j,s_j)\right)\left(\bar{u}(p_i,s_i)\Gamma_2 u(p_j,s_j)\right)^\dagger$$
$$= \text{Trace}[\Gamma_1(\not{p}_j + m_j)\bar{\Gamma}_2(\not{p}_i + m_i)]$$

graphical representation:



$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m$$
$$\sum_s v(p,s)\bar{v}(p,s) = \not{p} - m$$

$$\gamma_0^\dagger = \gamma_0$$
$$\gamma_0\gamma_i\gamma_0 = \gamma_i$$

closed fermion loop

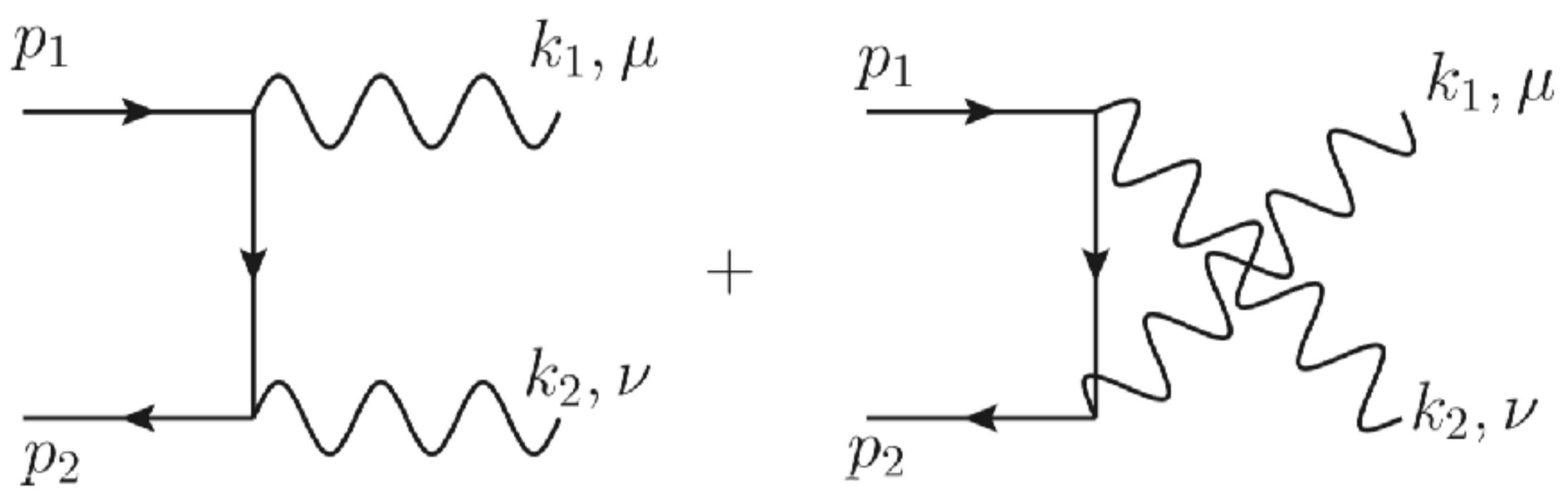
$\Rightarrow$ Trace

see exercise in the appendix

Tree level amplitudes

the non-Abelian structure of QCD leads to important differences compared to QED (unphysical polarisations, beta-function, …)

consider first a simple QED process: $e^+e^- \to \gamma\gamma$



$$\mathcal{M} = -i\, e^2\, \epsilon_1^\mu(k_1)\epsilon_2^\nu(k_2)\, M_{\mu\nu} \quad , \qquad M_{\mu\nu} = M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu}$$

$$M^{(1)}_{\mu\nu} = \bar{v}(p_2)\,\gamma_\nu \frac{\not{p}_1 - \not{k}_1}{(p_1-k_1)^2}\gamma_\mu\, u(p_1)$$

$$M^{(2)}_{\mu\nu} = \bar{v}(p_2)\,\gamma_\mu \frac{\not{p}_1 - \not{k}_2}{(p_1-k_2)^2}\gamma_\nu\, u(p_1)$$

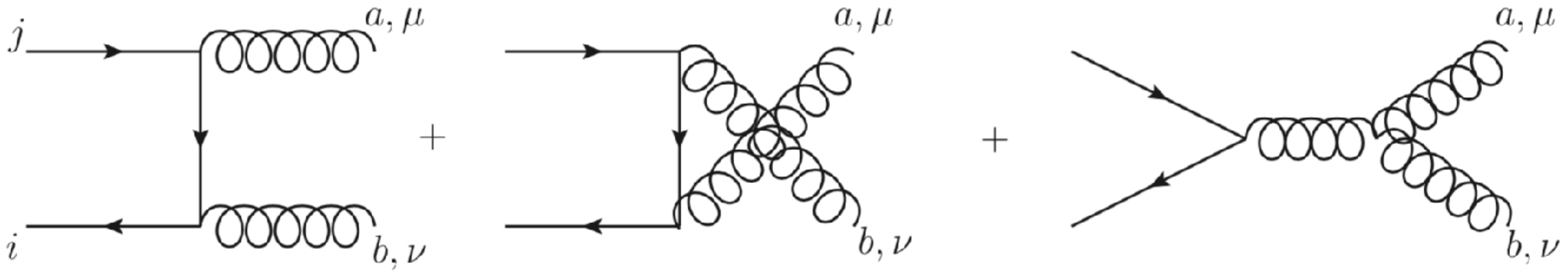
we find:

$$k_i^\mu \left[M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu}\right] = 0$$

QED Ward Identity

Tree level amplitudes

QCD analogue: $q\bar{q} \to gg$



$$\mathcal{M} = -i\, g_s^2\, \epsilon_1^{\mu}(k_1)\epsilon_2^{\nu}(k_2)\, M_{\mu\nu}^{\rm QCD}$$

$$M_{\mu\nu}^{\rm QCD} = (t^b t^a)_{ij}\, M_{\mu\nu}^{(1)} + (t^a t^b)_{ij}\, M_{\mu\nu}^{(2)} + M_{\mu\nu}^{(3)}$$

Tree level amplitudes

use $(t^b t^a)_{ij} = (t^a t^b)_{ij} - i\, f^{abc} t^c_{ij}$

$$M^{\rm QCD}_{\mu\nu} = (t^a t^b)_{ij} \left[M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu} \right] - i\, f^{abc} t^c_{ij}\, M^{(1)}_{\mu\nu} + M^{(3)}_{\mu\nu}$$

term in square brackets is the same as in QED, so $k_i^\mu \left[M^{(1)}_{\mu\nu} + M^{(2)}_{\mu\nu} \right] = 0$

for the remaining terms we find $k_1^\mu M^{(1)}_{\mu\nu} = -\bar{v}(p_2)\, \gamma_\nu\, u(p_1)$

$$k_1^\mu M^{(3)}_{\mu\nu} = \underbrace{-i\, f^{abc} t^c_{ij} \bar{v}(p_2)\, \gamma_\nu\, u(p_1)} + i\, f^{abc} t^c_{ij}\, \bar{v}(p_2)\ \not{k}_1\, u(p_1)\, \frac{k_{2,\nu}}{2 k_1 \cdot k_2}$$

cancels with contribution from $M^{(1)}_{\mu\nu}$

vanishes **only** when contracted with the polarisation vector of a **physical** gluon, i.e. if $\epsilon^\nu(k_2) \cdot k_2 = 0$

Difference QED vs QCD

QCD: $k_1^{\mu}\epsilon^{\nu}(k_2)M_{\mu\nu} \sim \epsilon(k_2)\cdot k_2 \;\Rightarrow$ vanishes only for physical gluons

QED: $k_1^{\mu_1}\ldots k_n^{\mu_n}\mathcal{M}_{\mu_1\ldots\mu_n} = 0$ regardless whether $\epsilon(k_j)\cdot k_j = 0$ or not

for cross sections we need $|\overline{\mathcal{M}}|^2$ built as follows:

$$\sum_{\text{pol}\,\lambda_1,\lambda_2} \epsilon_{\mu_1,\lambda_1}(k_1)\epsilon_{\mu_2,\lambda_2}(k_2)\mathcal{M}^{\mu_1\mu_2}\epsilon^{\star}_{\nu_1,\lambda_1}(k_1)\epsilon^{\star}_{\nu_2,\lambda_2}(k_2)\left(\mathcal{M}^{\nu_1\nu_2}\right)^{\dagger}$$

Let us consider just the sum over $\lambda_1 \in \{0,1,2,3\}$ (all polarisations, also unphysical ones).

The second boson is treated analogously

Polarisation sums

In QED, we can make the replacement $\sum_{\lambda_1} \epsilon_{\mu_1,\lambda_1}(k_1)\epsilon^{\star}_{\nu_1,\lambda_1}(k_1) \to -g_{\mu_1\nu_1}$

In QCD, this will in general lead to the wrong result. Why?

sum over *physical* polarisations:

$$\sum_{i=L,R} \epsilon_i^{\mu}(k)\epsilon_i^{\nu,\star}(k) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -g^{\mu\nu} + \frac{k^{\mu}n^{\nu} + k^{\nu}n^{\mu}}{k \cdot n}$$

$$\epsilon_{L,R} = (0, 1, \pm i, 0)/\sqrt{2} \qquad k = (k^0, 0, 0, k^0) \qquad n = (k^0, 0, 0, -k^0)$$

In QED $k^{\mu}\mathcal{M}_{\mu\nu} = 0 \Rightarrow$ only $g^{\mu\nu}$ part of polarisation sum will contribute

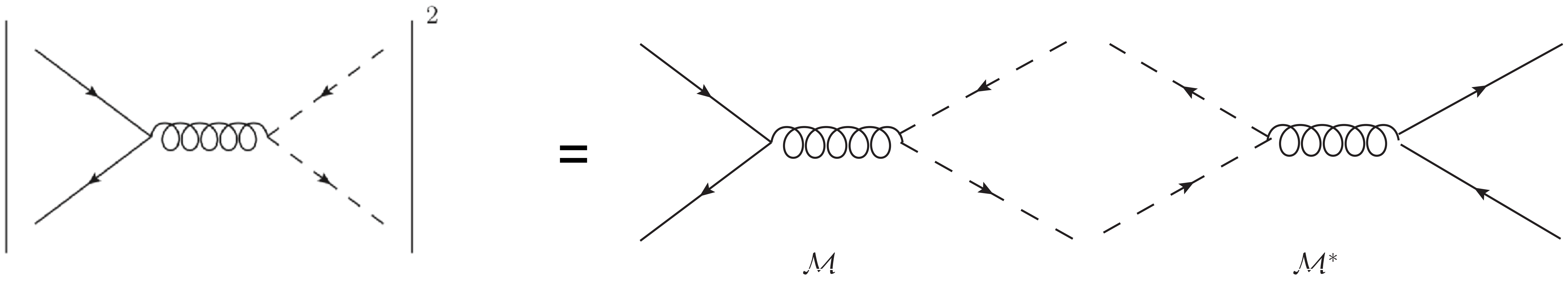
Polarisation sums

In QCD: $k_1^\mu \mathcal{M}_{\mu\nu} \epsilon^\nu(k_2) \sim \epsilon(k_2) \cdot k_2$

therefore, if $\epsilon(k_2) \cdot k_2 \neq 0$ we can **not** just use $-g^{\mu\nu}$ for the polarisation sum

- either we use $$\sum_{\text{phys. pol. } \lambda} \epsilon_\lambda^\mu(k) \epsilon_\lambda^{\nu,\star}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{k \cdot n}$$

- or we use $-g^{\mu\nu}$ and also include the ghost contributions in $|\mathcal{M}|^2$



Polarisation sums

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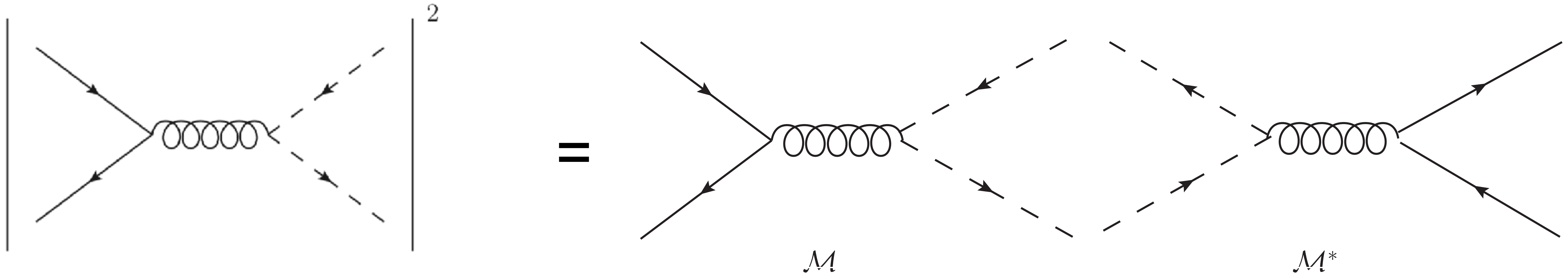
it can be shown that

$$S_{\text{unphys.}} \equiv \sum_{\text{unphysical pol.}} |\epsilon_\mu(k_1)\epsilon_\nu(k_2)\mathcal{M}^{\mu\nu}|^2 = \left| i\, g_s^2 f^{abc} t^c \bar{v}(p_2) \frac{\not{k}_1}{(k_1+k_2)^2} u(p_1) \right|^2$$

calculating the ghost contribution



results in $-S_{\text{unphys.}}$ (minus sign from Feynman rules for closed fermion loop)

$\Rightarrow$ ghost degrees of freedom cancel the unphysical gluon polarisations!

Running coupling

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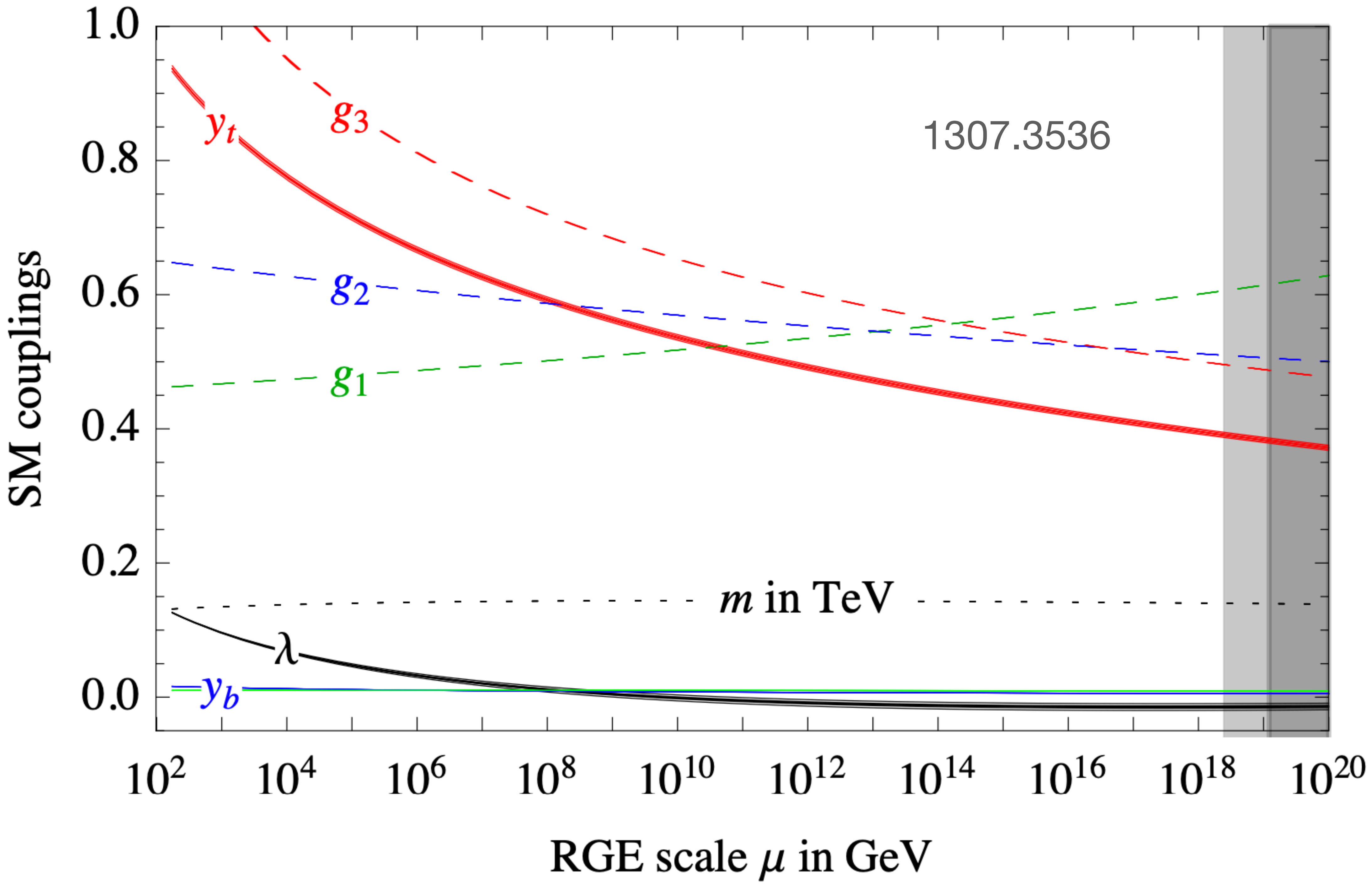
Mols Bjerge



QCD beta-function

- couplings are not constant, depend on the scale at which the interaction takes place

- strong coupling, running at leading order: $\alpha_s(Q^2) = \dfrac{1}{b_0 \log\left(Q^2/\Lambda^2_{QCD}\right)}$

$$b_0 = \frac{1}{4\pi}\left(\frac{11}{3}C_A - \frac{4}{3}T_R\,N_f\right)$$

Λ_{QCD} : scale where perturbative description breaks down

number of quark flavours

$b_0 > 0$ for $N_f < 11/2\,C_A$

- where does the running come from? ⟶ renormalisation introduces a scale μ

QCD beta-function

- to get an idea how this arises, consider the hadronic R-ratio

$$R(s) = \frac{\sigma(e^+e^- \to \text{ hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

perturbative expansion:

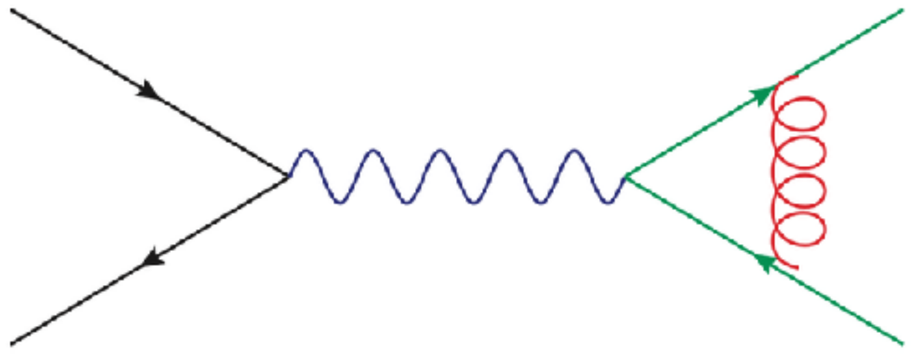
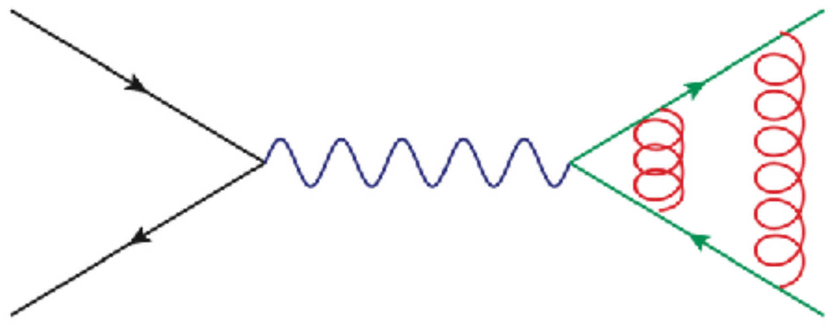
$$R(s) = K_{QCD}(s)\, R_0 \;, \qquad R_0 = N_c \sum_f Q_f^2\, \theta(s - 4m_f^2)$$

$$K_{QCD}(s) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n\geq 2} C_n \left(\frac{s}{\mu^2}\right) \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

known up to 5 loops

QCD beta-function

$$K_{QCD}(s) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \sum_{n\geq 2} C_n\left(\frac{s}{\mu^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n$$

n=1  n=2  + …

explicit calculation: integrate over loop momentum *k*, diverges for $k \to \infty$ ultraviolet divergence

for now regulate with cutoff Λ_{UV}: $\int_0^{\Lambda_{UV}} d|k|$

n=1: cutoff dependence cancels due to Ward Identity

n=2, i.e. up to order α_s^2:

$$K_{QCD}(s) = 1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2\left[c + b_0\pi\log\frac{\Lambda_{UV}^2}{s}\right] + \mathcal{O}(\alpha_s^3)$$ → result is infinite for $\Lambda_{UV} \to \infty$?

QCD beta-function

however α_s is not the measured coupling but the “bare” coupling α_s^0

redefine coupling: $\alpha_s(\mu) = \alpha_s^0 + b_0 \log \frac{\Lambda_{UV}^2}{\mu^2} \alpha_s^2$

$\alpha_s(\mu)$: **renormalised coupling**

insert into K_{QCD} , expand consistently to order α_s^2

$$K_{QCD}^{\mathrm{ren}}(\alpha_s(\mu), \mu^2/s) = 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[c + b_0 \pi \log \frac{\mu^2}{s}\right] + \mathcal{O}(\alpha_s^3)$$

finite, but now depends on μ, explicitly and implicitly through $\alpha_s(\mu)$

QCD beta-function

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physical quantity $R^{\rm ren} = R_0\, K^{\rm ren}_{QCD}$ cannot depend on unphysical scale

$$\Rightarrow \quad \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} R^{\rm ren}(\alpha_s(\mu), \mu^2/Q^2) = 0 = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_s}{\partial \mu^2} \frac{\partial}{\partial \alpha_s} \right) R^{\rm ren}(\alpha_s(\mu), \mu^2/Q^2)$$

define $t = \ln \frac{Q^2}{\mu^2}$, $\beta(\alpha_s) = \mu^2\, \partial\alpha_s / \partial\mu^2$ $\Rightarrow$ $\left(-\frac{\partial}{\partial t} + \beta(\alpha_s) \frac{\partial}{\partial \alpha_s} \right) R = 0$

renormalisation group equation

solve by ansatz $t = \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}$ with $\alpha_s \equiv \alpha_s(\mu^2)$

QCD beta-function

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$$t = \int_{\alpha_s}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}$$ differentiate both sides w.r.to t

$$1 = \frac{1}{\beta(\alpha_s(Q^2))} \frac{\partial \alpha_s(Q^2)}{\partial t} \quad \Rightarrow \quad \boxed{\frac{\partial \alpha_s(Q^2)}{\partial t} = \beta\left(\alpha_s(Q^2)\right)}$$

solve iteratively in perturbation theory $\beta(\alpha_s) = -b_0 \alpha_s^2 \left[1 + \sum_{n=1}^{\infty} b_n \, \alpha_s^n\right]$

leading order: $Q^2 \frac{\partial \alpha_s}{\partial Q^2} = \frac{\partial \alpha_s}{\partial t} = -b_0 \alpha_s^2 \Rightarrow -\frac{1}{\alpha_s(Q^2)} + \frac{1}{\alpha_s(\mu^2)} = -b_0 \, t$

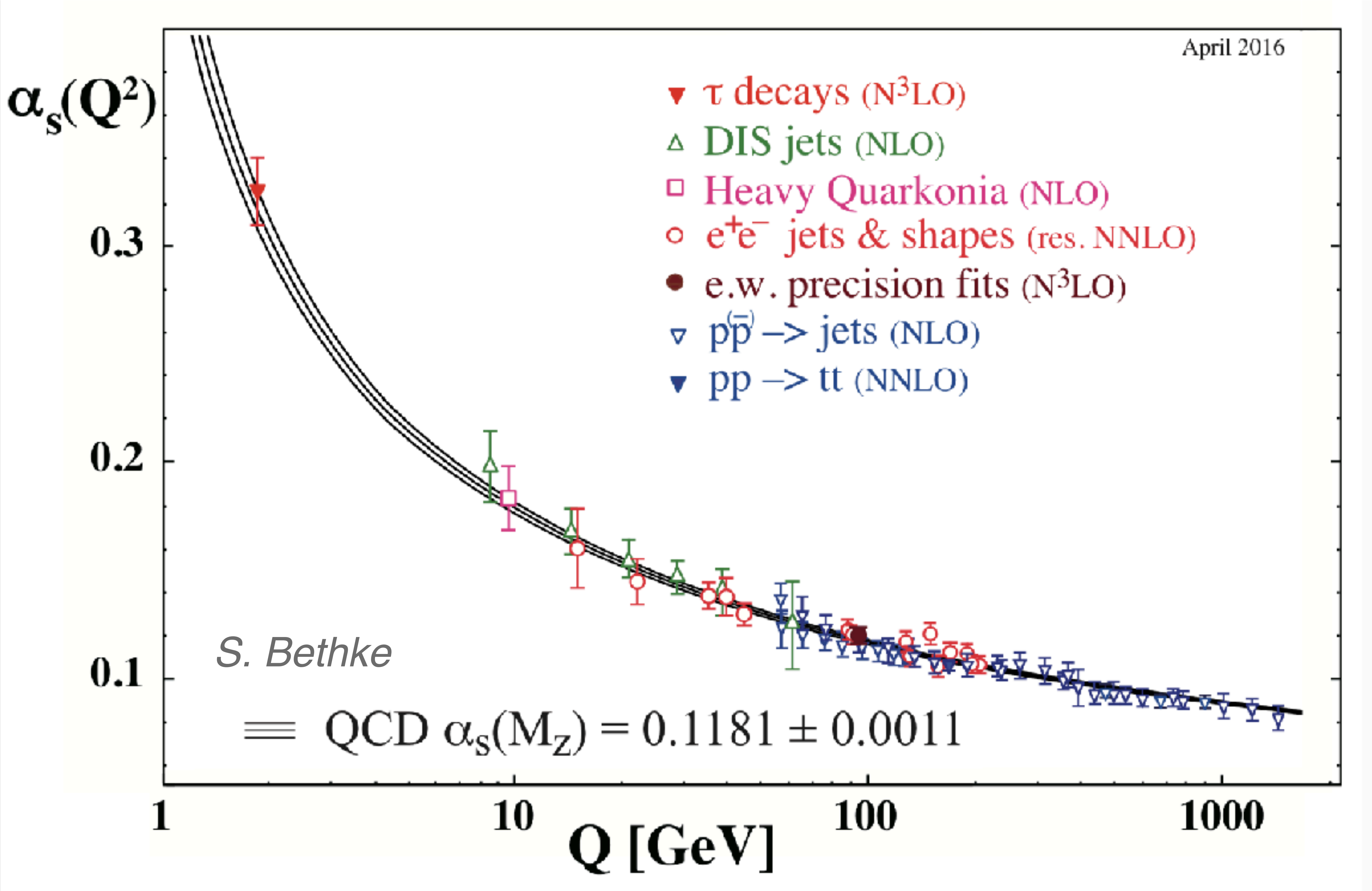
$$\Rightarrow \alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + b_0 \, t \, \alpha_s(\mu^2)} \, .$$

running coupling

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + b_0\, t\, \alpha_s(\mu^2)} \qquad t = \ln\frac{Q^2}{\mu^2}$$

$$\Rightarrow \quad \alpha_s(Q^2) \xrightarrow{Q^2\to\infty} \frac{1}{b_0 t} \xrightarrow{Q^2\to\infty} 0$$

asymptotic freedom



QCD lambda parameter

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It can be useful to define a dimensionful parameter Λ (integration constant)

$$\ln\left(\frac{Q^2}{\Lambda^2}\right) = -\int_{\alpha_s(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_s(Q)}^{\infty} \frac{dx}{b_0\, x^2\, (1 + b_1\, x + \ldots)}$$

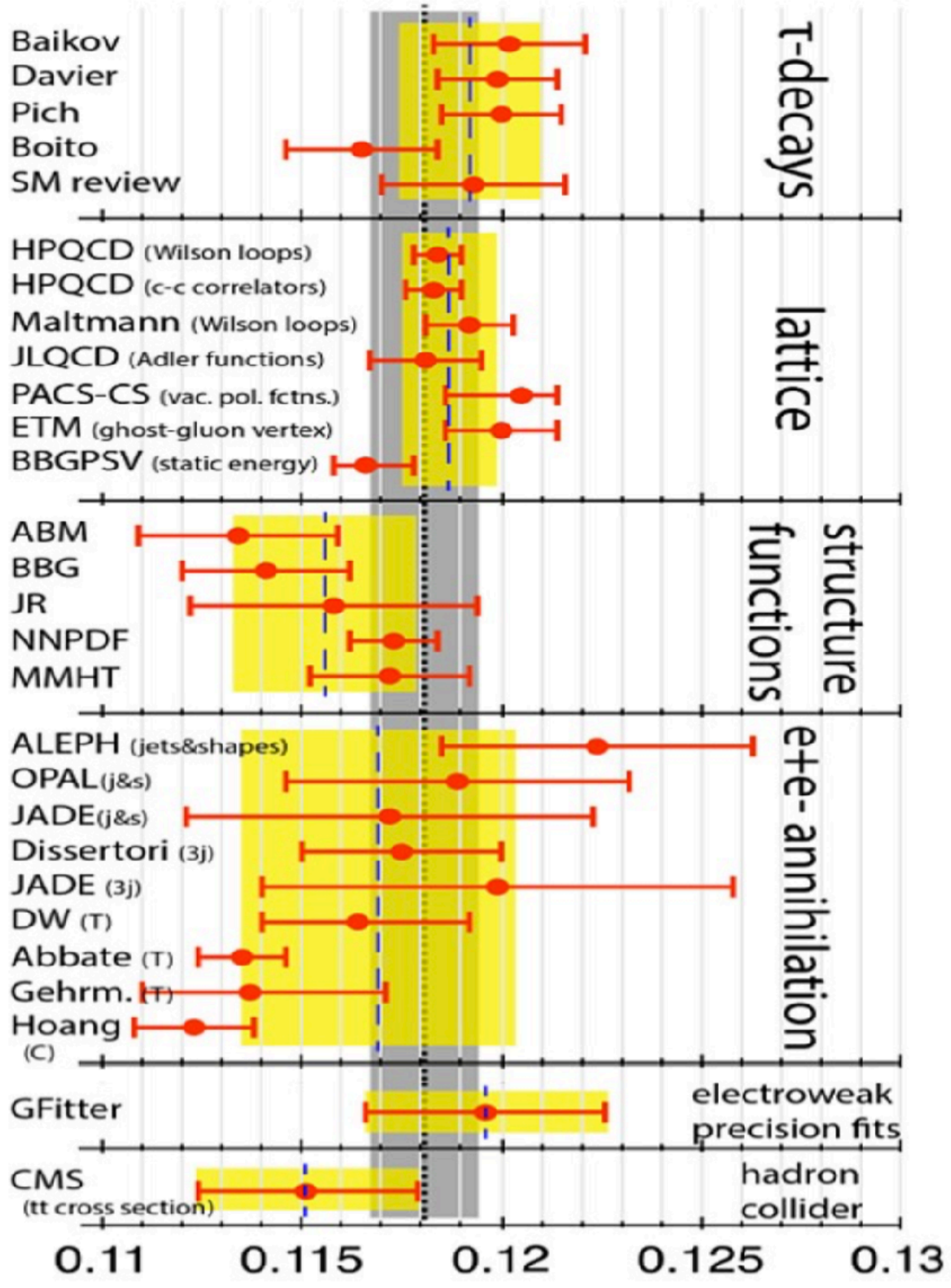
Keeping only $b_0(\mathrm{LO}), b_1(\mathrm{NLO})$

$$\alpha_s(Q) = \frac{1}{b_0\, \ln\left(\frac{Q^2}{\Lambda^2}\right)} \quad \text{(LO)}$$

$$\alpha_s(Q) = \frac{1}{b_0\, \ln\left(\frac{Q^2}{\Lambda^2}\right)} \left[1 - \frac{b_1\, \ln \ln\left(\frac{Q^2}{\Lambda^2}\right)}{b_0\, \ln\left(\frac{Q^2}{\Lambda^2}\right)}\right] \quad \text{(NLO)}$$

Note that Λ depends on the number of active flavours N_f

Below the scale Λ strong interactions become non-perturbative , $\Lambda \simeq 200\,\mathrm{MeV}$



class averages:

$\alpha_s(M_Z) = 0.1192 \pm 0.0018$ (±1.5%)

$\alpha_s(M_Z) = 0.1184 \pm 0.0012$ (±1.0%)

$\alpha_s(M_Z) = 0.1156 \pm 0.0021$ (±1.8%)

$\alpha_s(M_Z) = 0.1169 \pm 0.0034$ (±2.9%)

$\alpha_s(M_Z) = 0.1196 \pm 0.0030$ (±2.5%)

$\alpha_s(M_Z) = 0.1151 \pm 0.0028$ (±2.5%)

unweighted χ^2 average: $\alpha_s(M_Z) = 0.1181 \pm 0.0011$ (±0.9%)

World average of

$$\alpha_s(M_Z)$$

is based on observables at different energies and lattice QCD calculations

beta-functions

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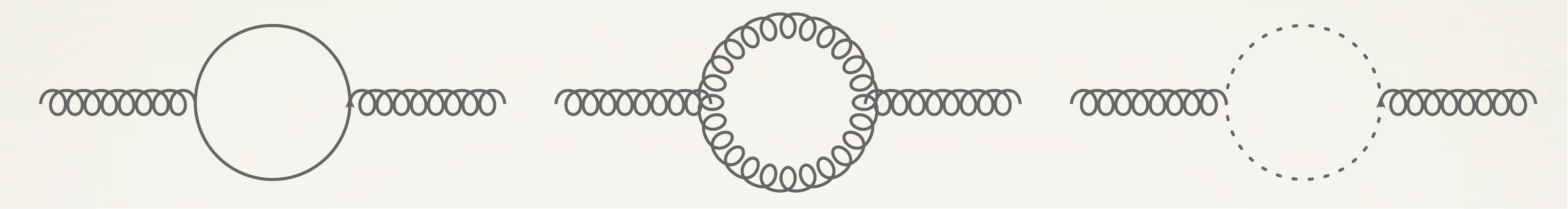
QCD: $$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b_0 \, \ln\left(\frac{Q^2}{\mu^2}\right)}$$

coupling decreases with energy

$$b_0 = \frac{1}{12\pi}(11 N_c - 2 N_f)$$

$b_0 > 0$ for $N_f < \frac{11}{2} N_c$



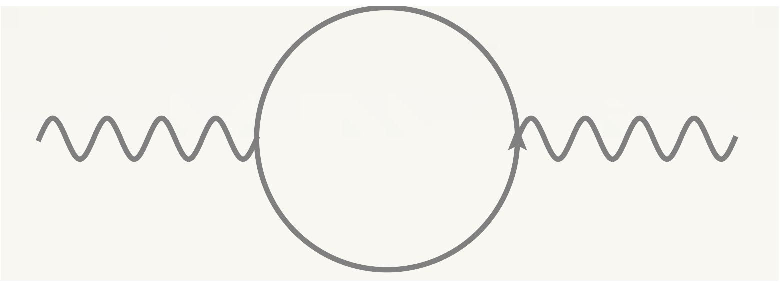
(a) (b) (ghost loop)

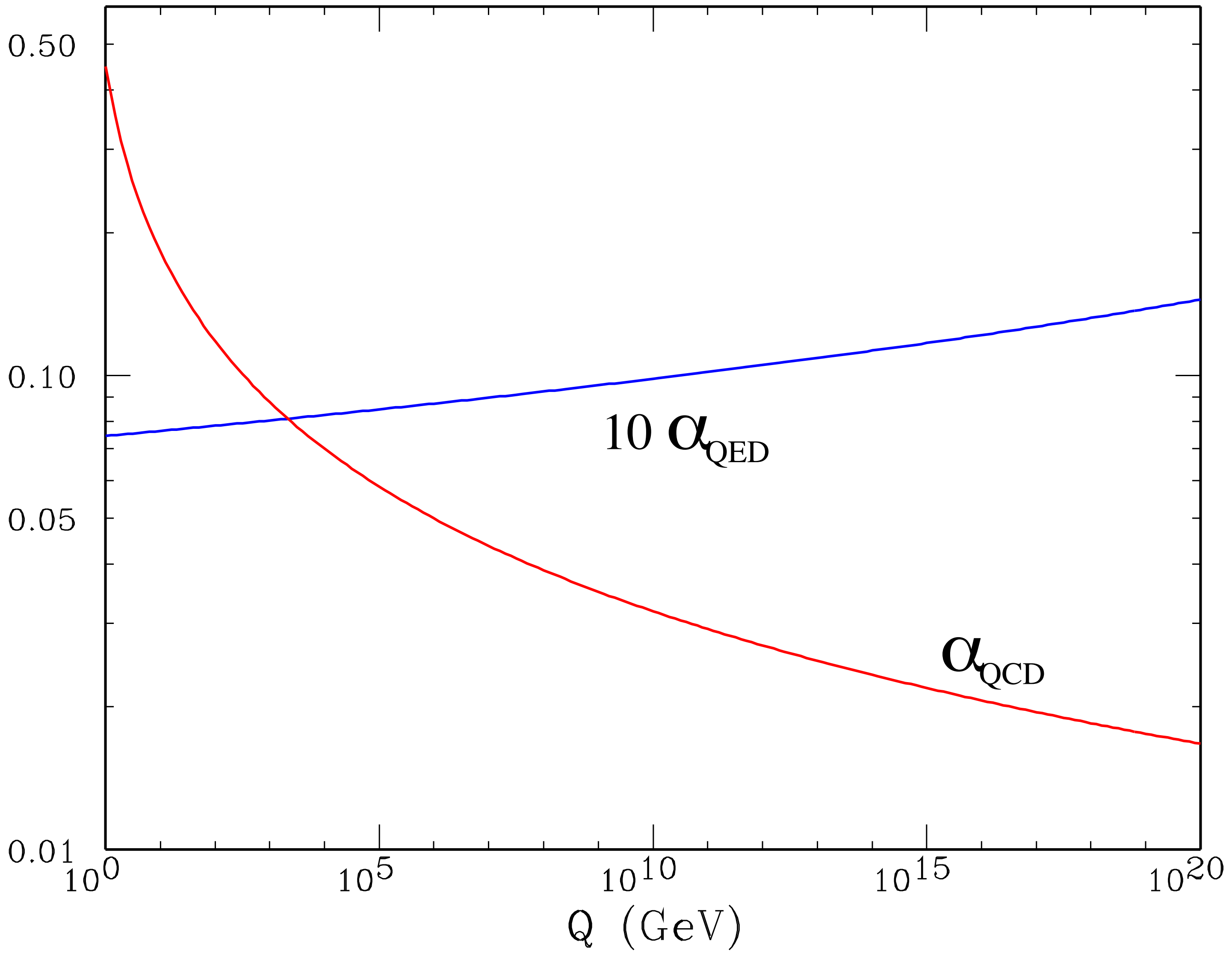
N_f N_c

QED: $$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi} \ln\left(\frac{Q^2}{m_e^2}\right)}$$

1/137 → α_0

coupling grows with energy





beta-functions

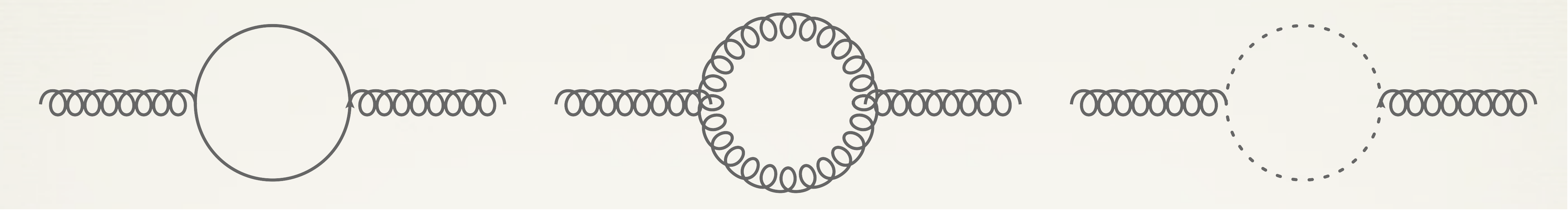
QCD: $$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \alpha_s(\mu^2)\, b_0 \, \ln\left(\frac{Q^2}{\mu^2}\right)}$$

coupling decreases with energy

$$b_0 = \frac{1}{12\pi}(11N_c - 2N_f)$$

$$b_0 > 0 \quad \text{for} \quad N_f < \frac{11}{2}N_c$$

only non-Abelian gauge theories can be asymptotically free (but don't have to)



(a) N_f

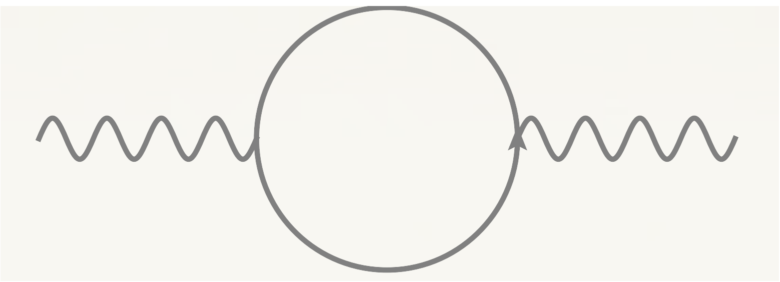
(b) N_c

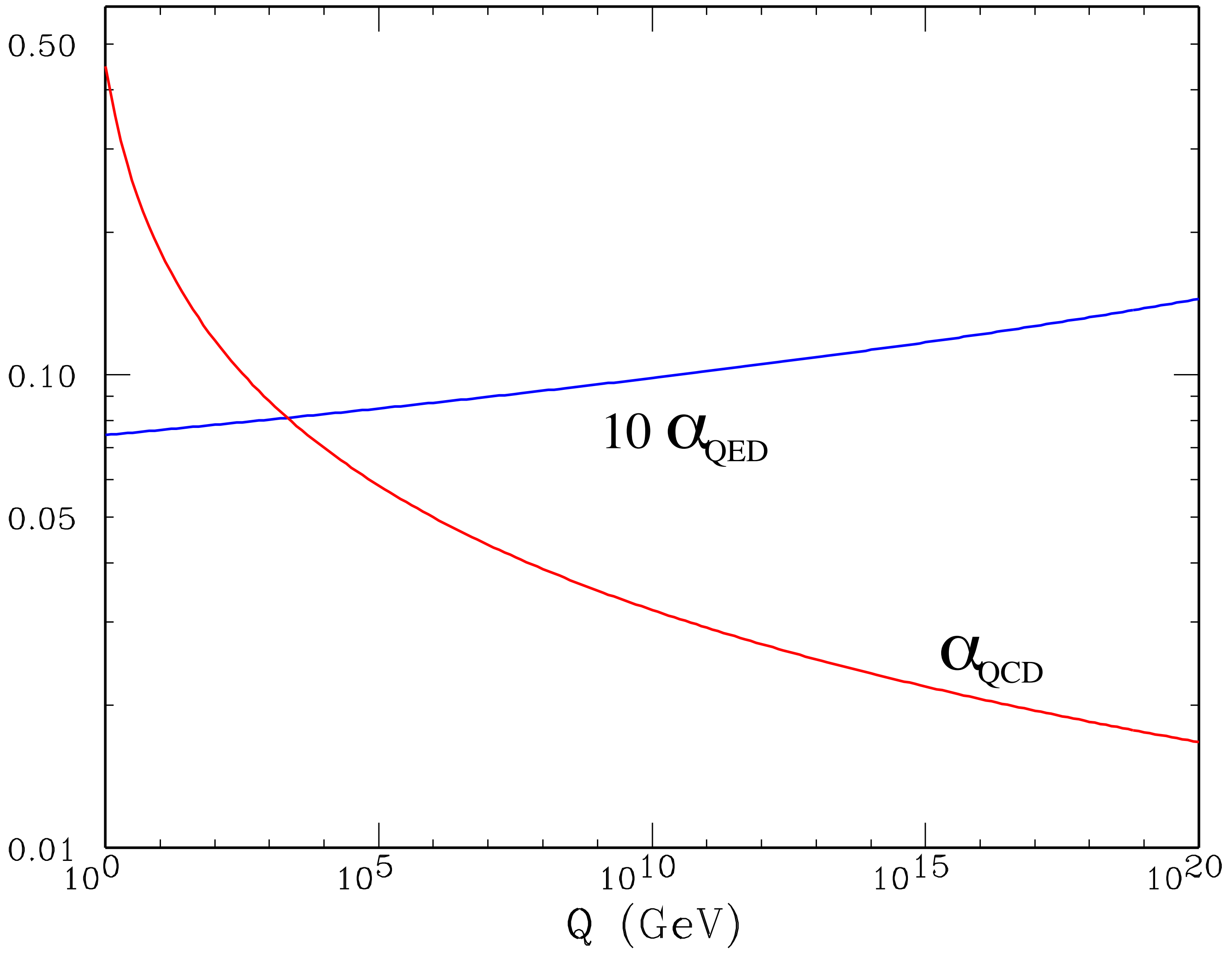
(ghost loop)

QED: $$\alpha(Q^2) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\, \ln\left(\frac{Q^2}{m_e^2}\right)}$$

1/137 → α_0

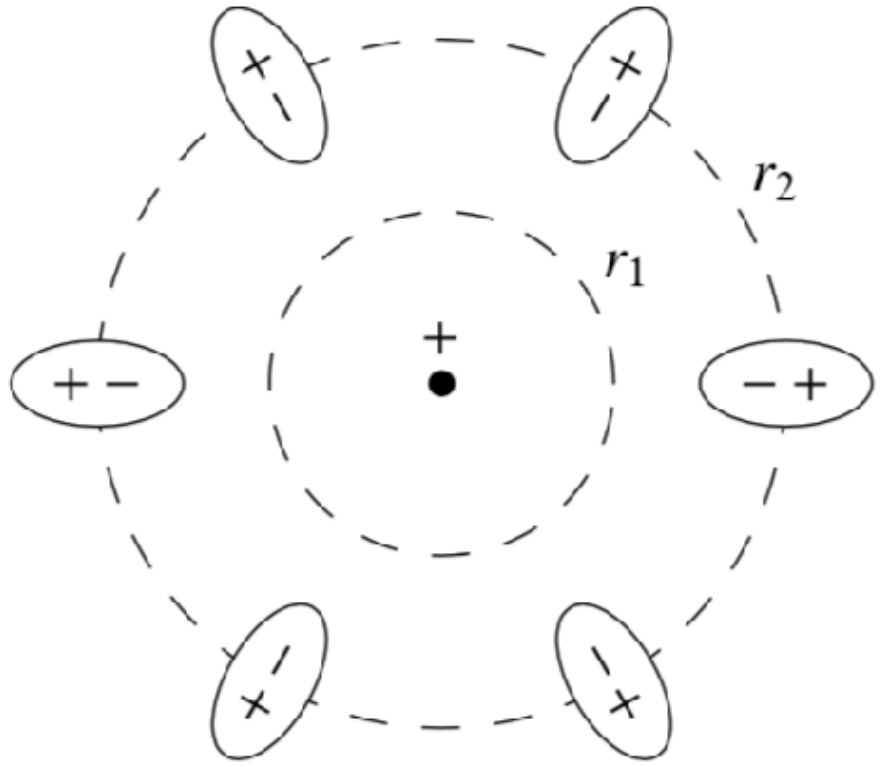
coupling grows with energy





screening/anti-screening

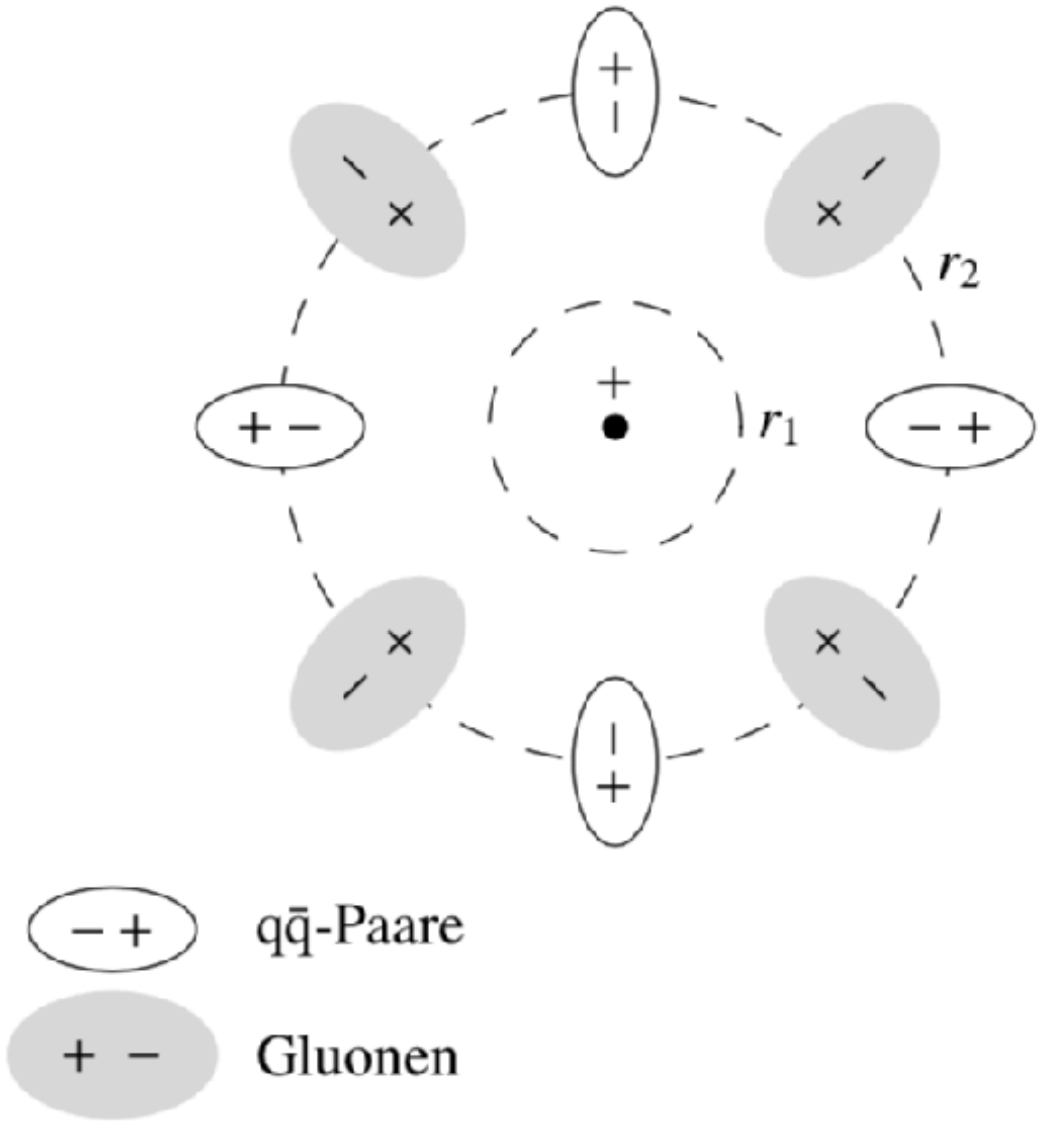
QED:



similar to screening in dielectric material

$$\alpha(r_2) < \alpha(r_1)$$

QCD:



quarks: screening

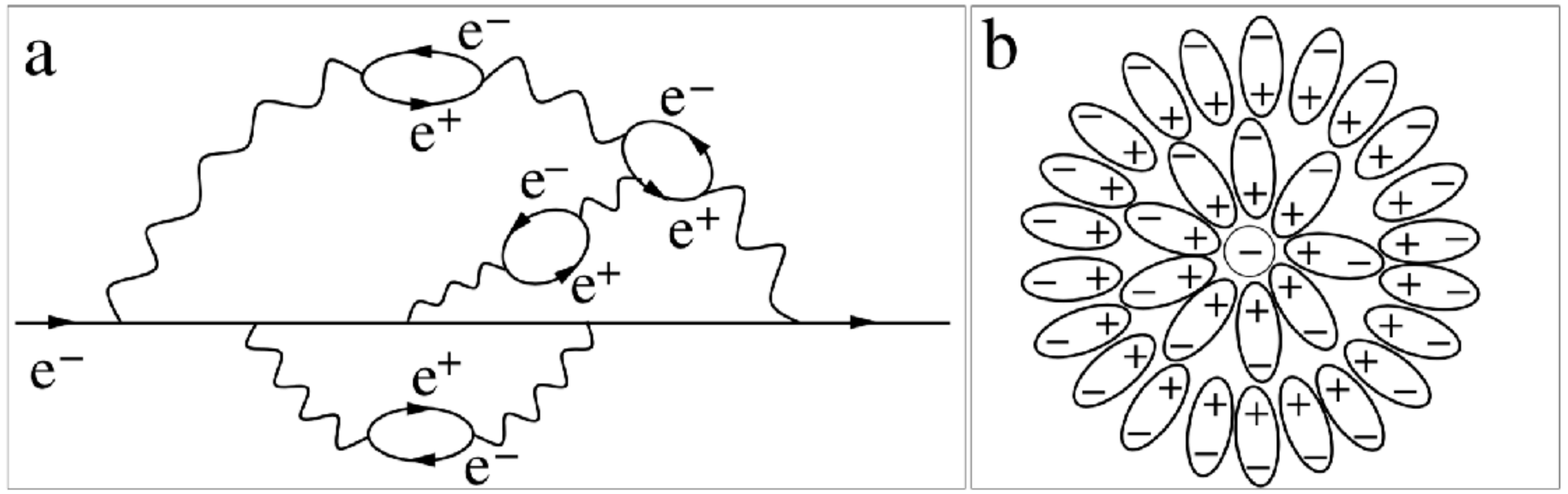
gluons: enforcement of colour charge

$$\alpha_s(r_2) > \alpha_s(r_1)$$

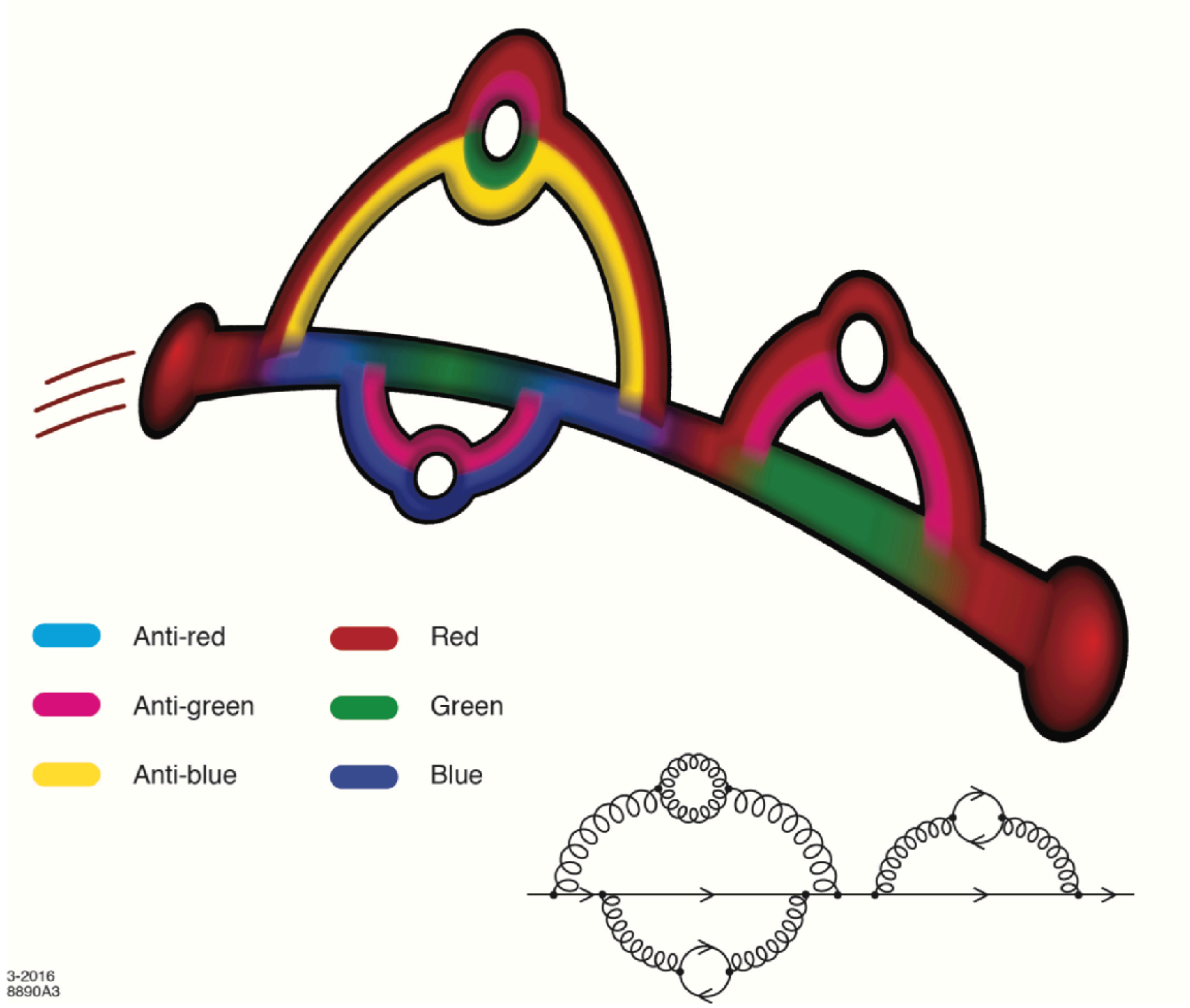
Figures: Paul, Kaiser, Wiese, TUM

screening/anti-screening

QED:



QCD:



$$b_0 = \frac{1}{12\pi}(11N_c - 2N_f) \longrightarrow$$ gluon term dominates

Deur, Brodsky, Teramond, arXiv:1604.08082

Scale uncertainties

$$\sigma = \alpha_s^k(\mu_r)\left(\sigma^{LO}(\mu_f) + \alpha_s(\mu_r)\sigma^{NLO}(\mu_r,\mu_f) + \alpha_s^2(\mu_r)\sigma^{NNLO}(\mu_r,\mu_f) + \ldots\right)$$

renormalisation scale (μ_r) factorisation scale (μ_f)

- scale dependence: due to truncation of perturbative series

  → use scale variations as a measure of missing higher orders

- for an observable *O* calculated up to order N in perturbation theory: $O^{(N)}(\mu) = \sum_n^N c_n(\mu)\alpha_s(\mu^2)^n$

$$\frac{d}{d\log(\mu^2)} O^{(N)}(\alpha_s(\mu)) = \beta(\alpha_s)\frac{\partial O^{(N)}}{\partial \alpha_s} \sim \mathcal{O}\left(\alpha_s(\mu)^{N+1}\right)$$

because $\beta(\alpha_s) = -b_0\alpha_s^2 + \mathcal{O}(\alpha_s^3)$

scale uncertainties

⇒ the more orders we calculate the smaller the scale uncertainties

However, there are exceptions!

for example,

- if new partonic channels open up beyond LO

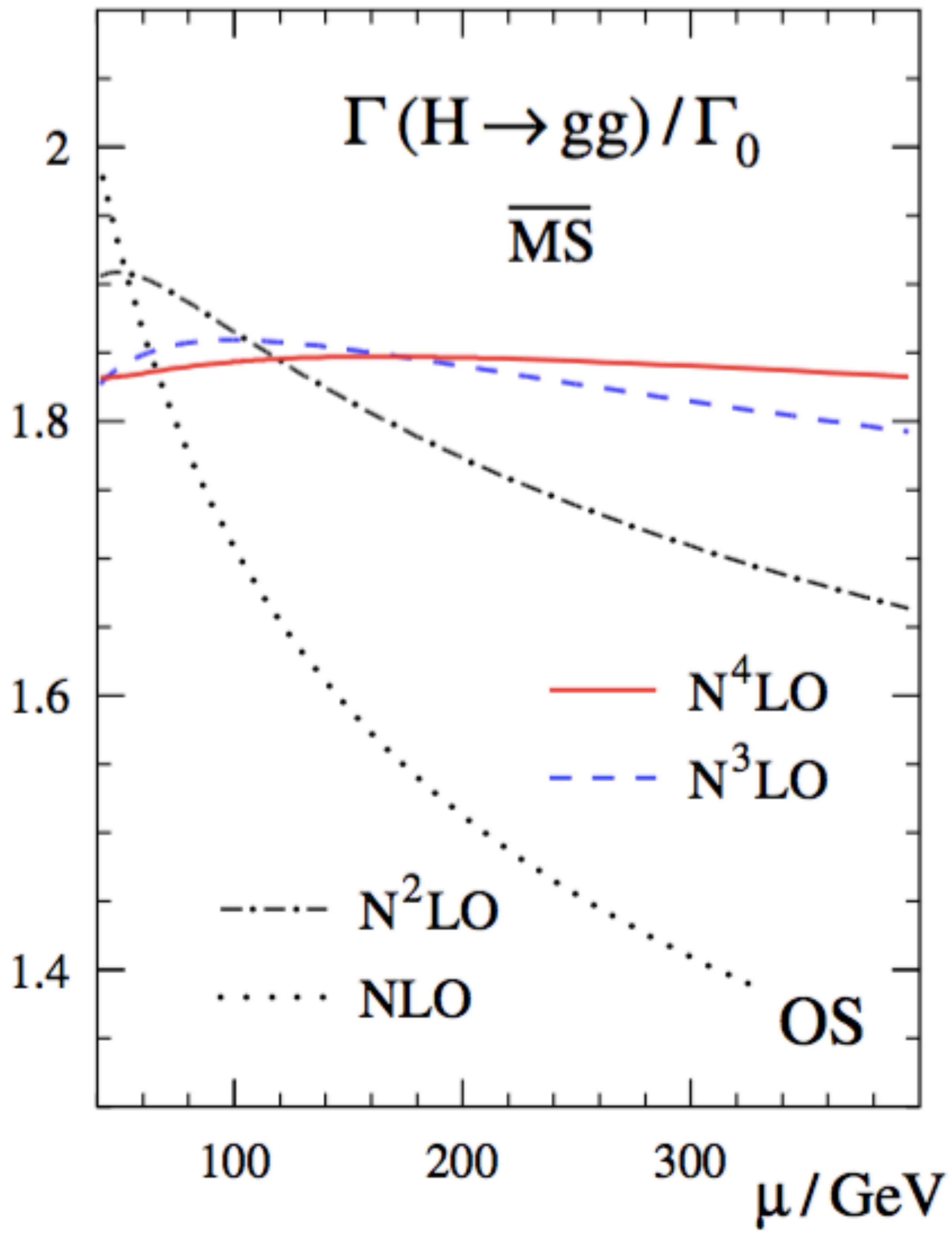
  e.g. Higgs production in gluon fusion

- if the central scale is chosen inconveniently

  see example from single jet inclusive cross sections and W+3jets

- if the observable is very sensitive to extra radiation

  see also jet+X, W+3jets; in many cases resummation may be required



F. Herzog et al. 1707.01044

- accidental cancellations between μ_r and μ_f dependence or between different partonic channels

  see Drell-Yan example

scale uncertainties: Higgs production in gluon fusion

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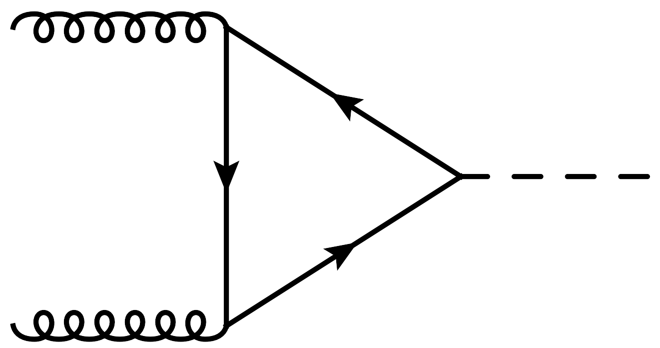
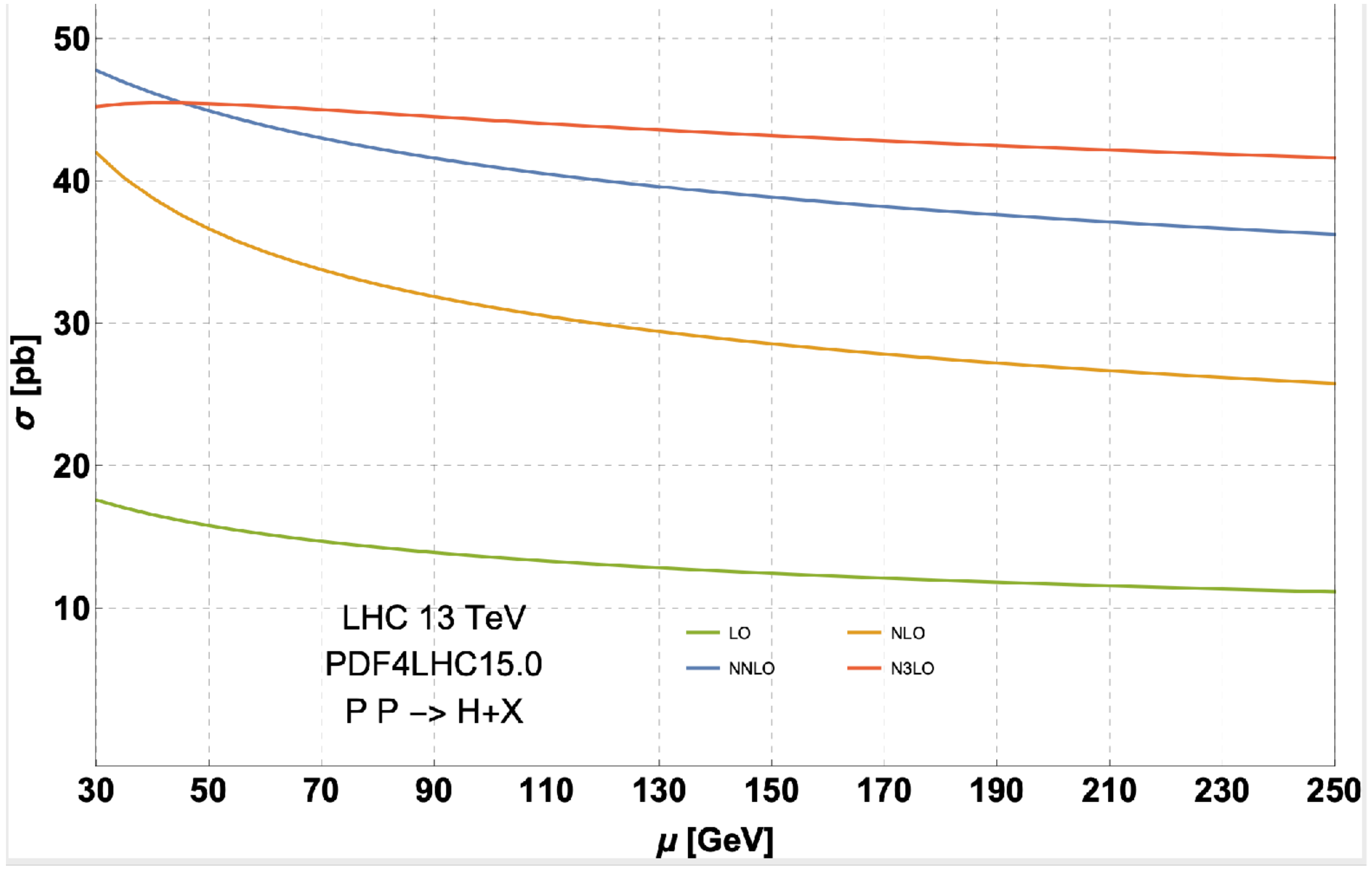
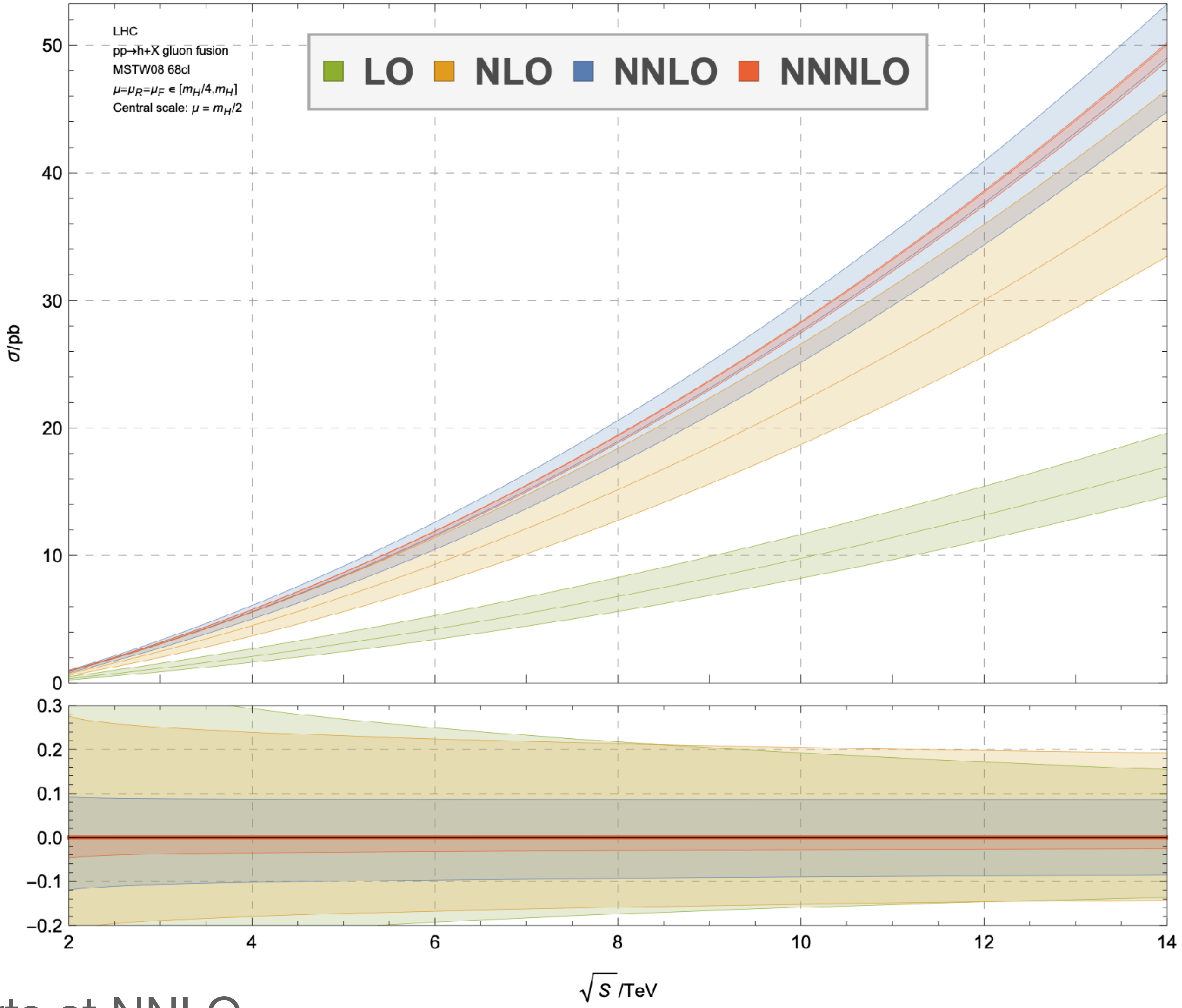


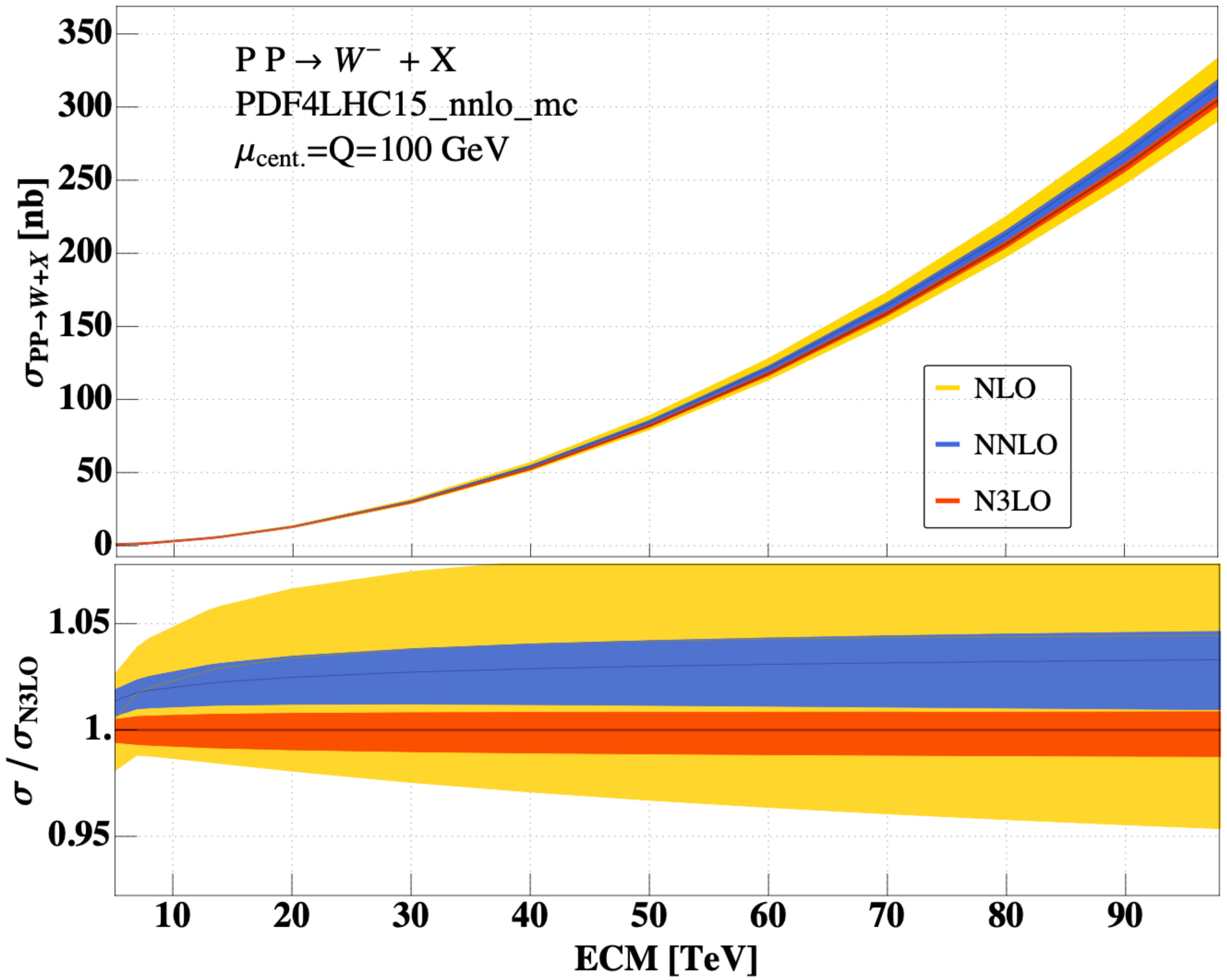


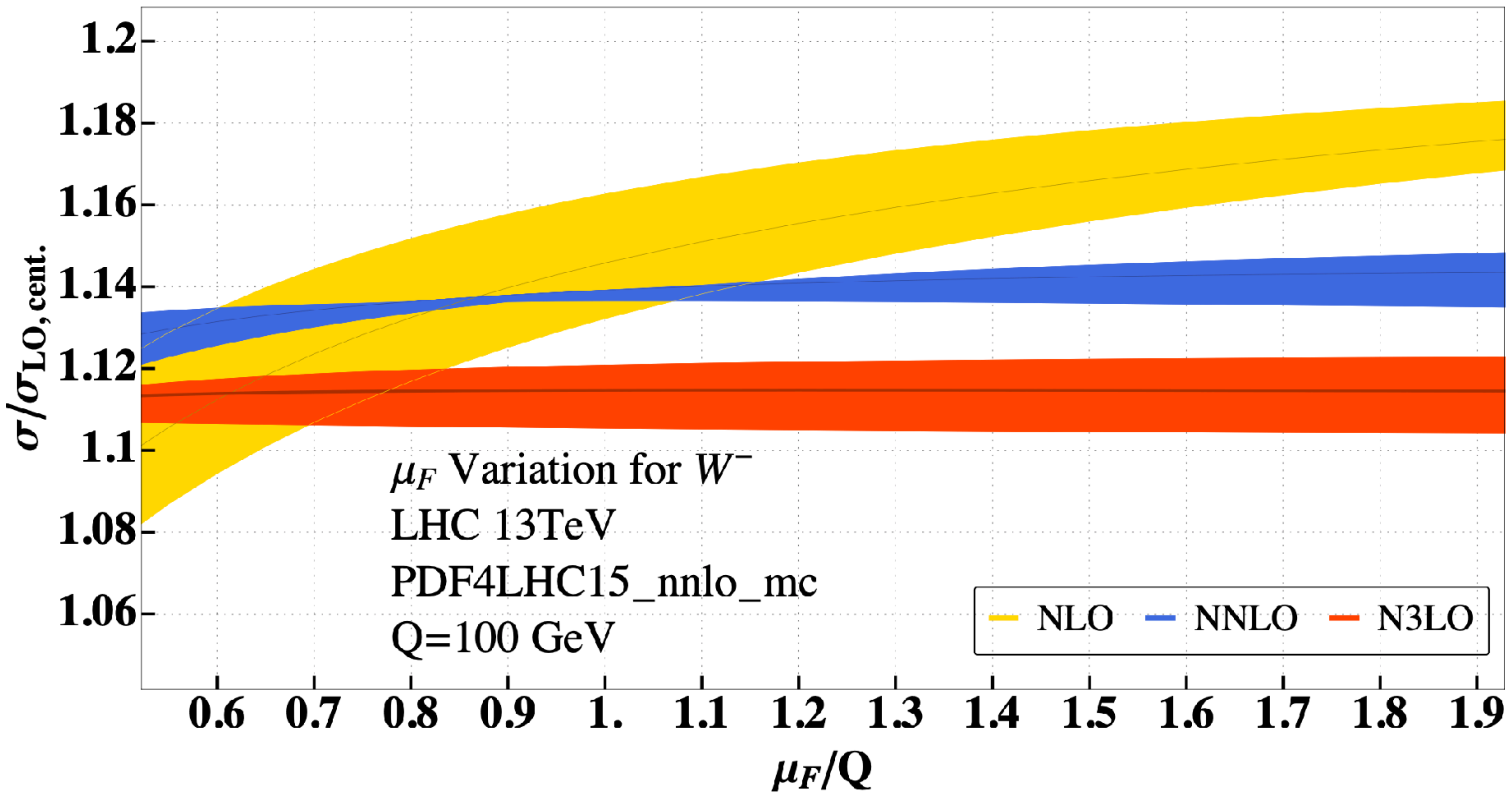


Anastasiou et al. 1503.06056
B. Mistlberger, 1802.00833

stabilisation starts at NNLO

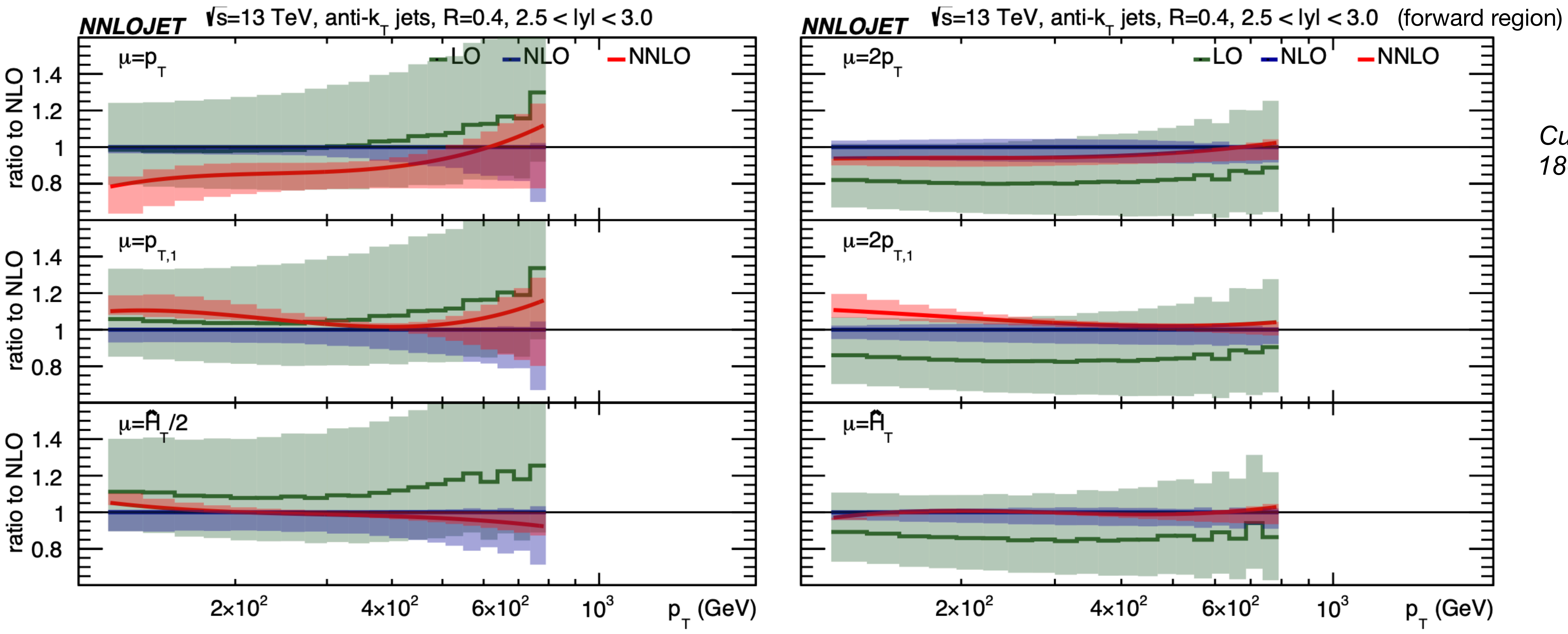
scale uncertainties: W-production (Drell-Yan)





Duhr, Dulat, Mistlberger
2007.13313

scale uncertainties: single jet inclusive xs

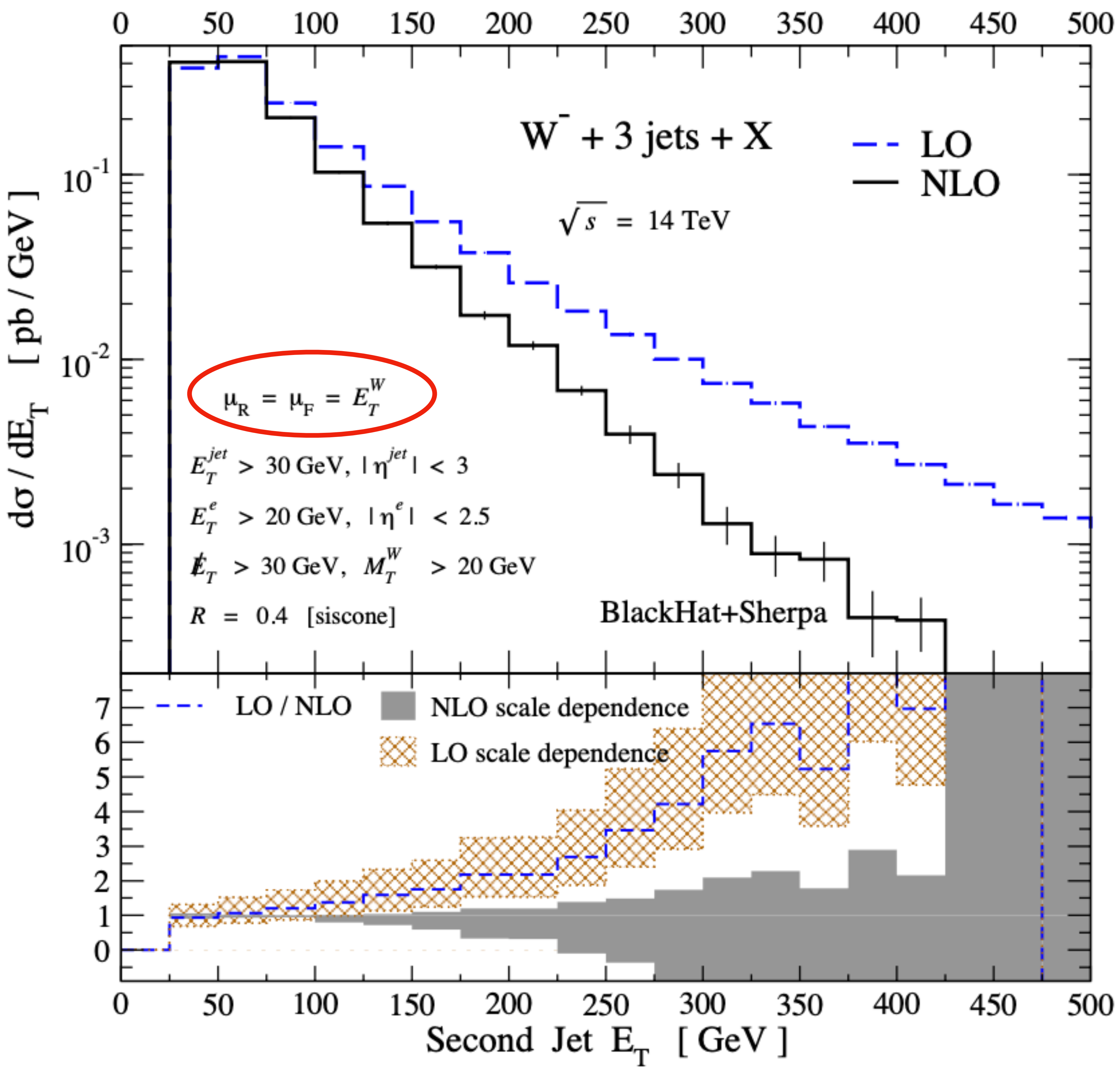


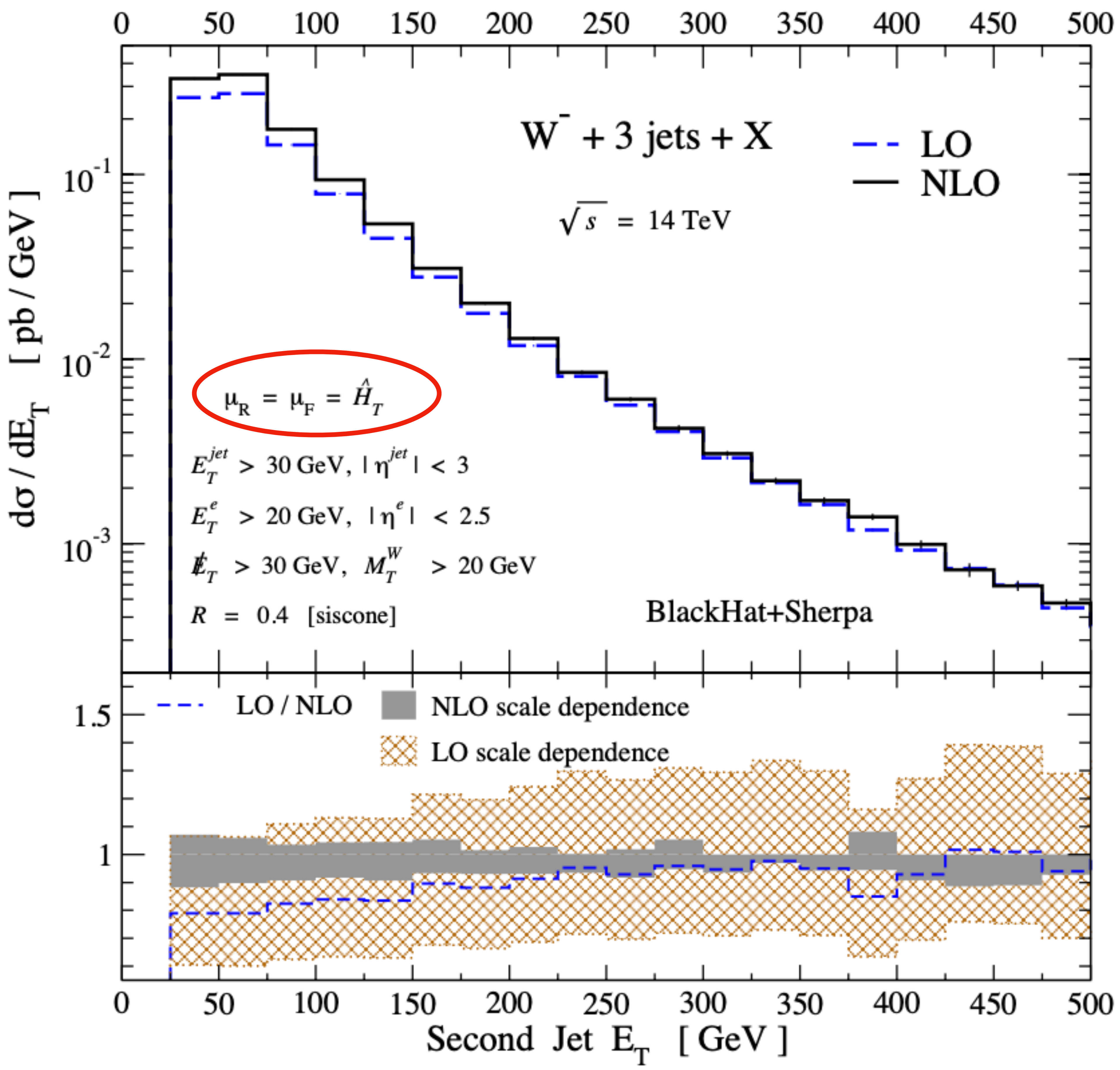
Currie et al. 1807.03692

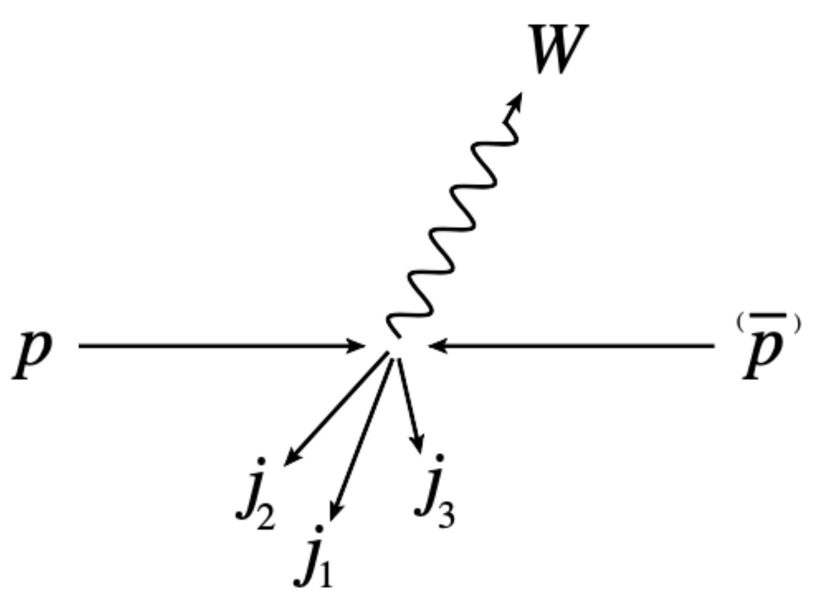
p_T : individual jet transverse momentum, $p_{T,1}$: leading jet transverse momentum, $\hat{H}_T$: sum of parton transverse momenta

scale uncertainties: W+3jets

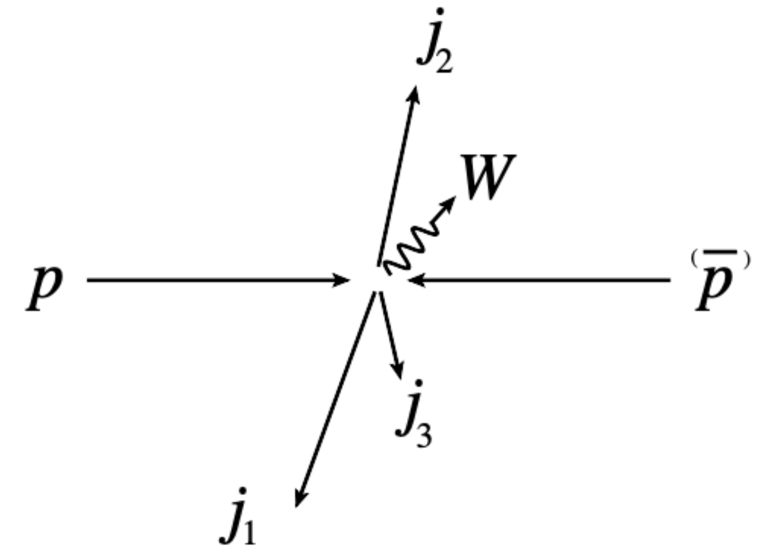
Berger et al. 0907.1984







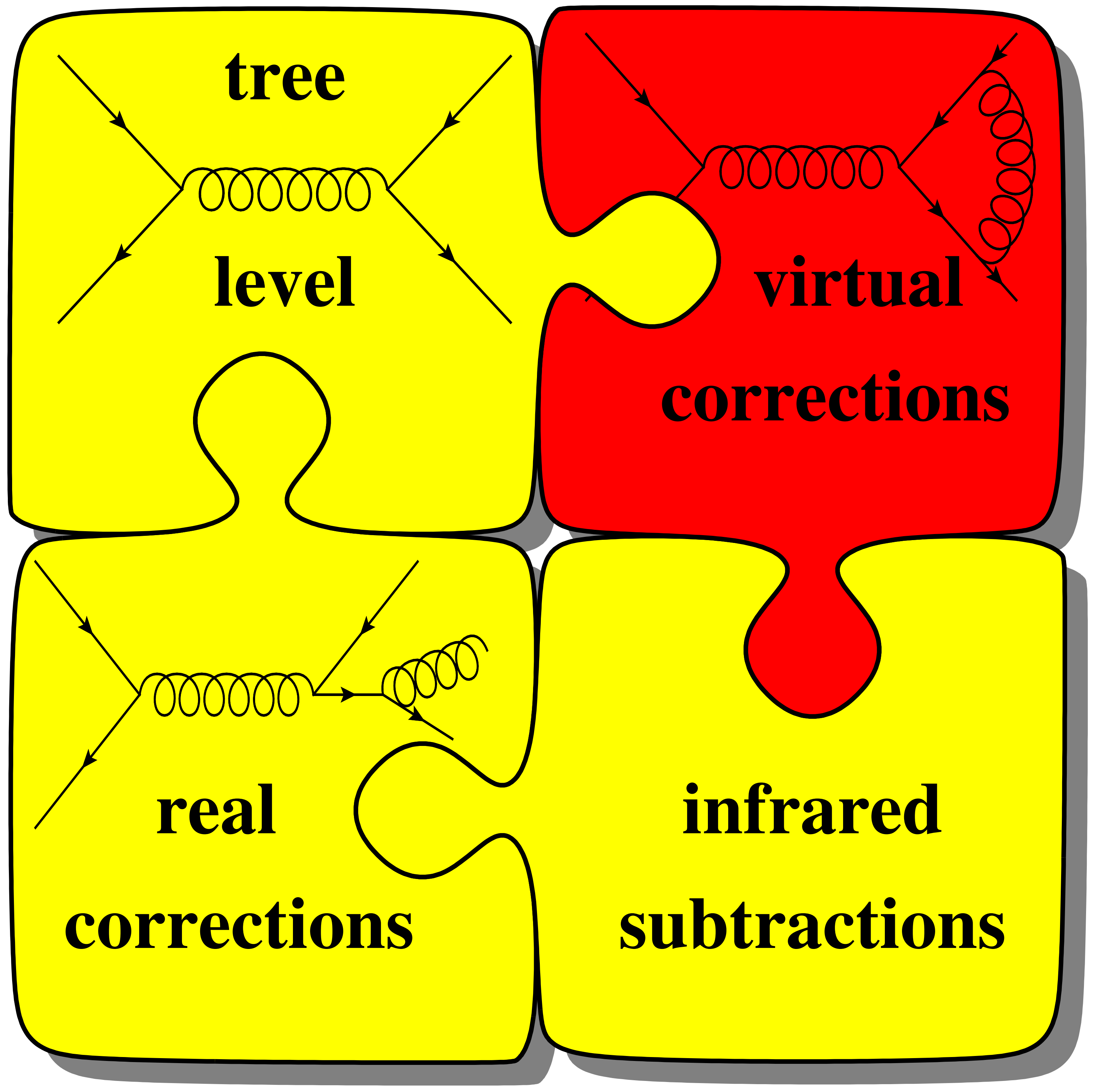
(a)



(b)

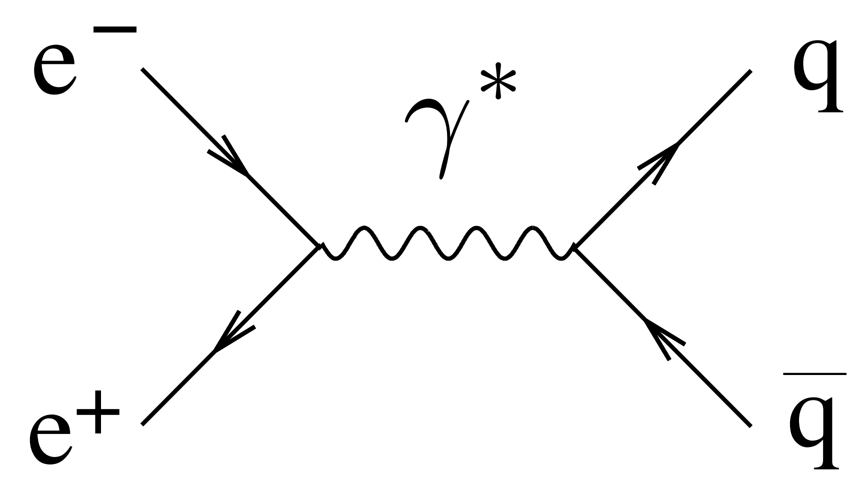
(b) dominates at large jet E_T

Basics of NLO calculations



NLO basics

start with simple example: e+e- annihilation

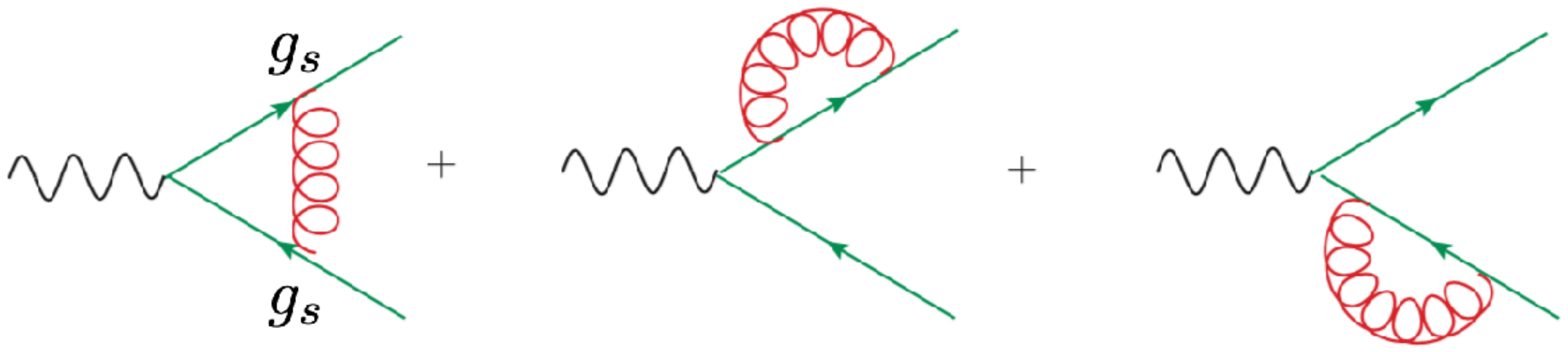


at leading order: $$\sigma^{LO} = \frac{4\pi\alpha^2}{3s}\, e_q^2\, N_c$$ (Z exchange not considered)

split off leptonic part and consider $\gamma^* \to q\bar{q}$

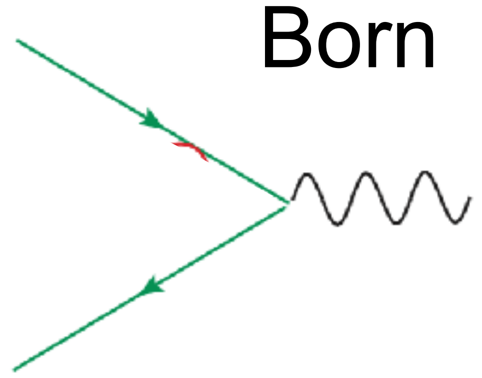
NLO: order α_s corrections at cross section level

will be interfered with



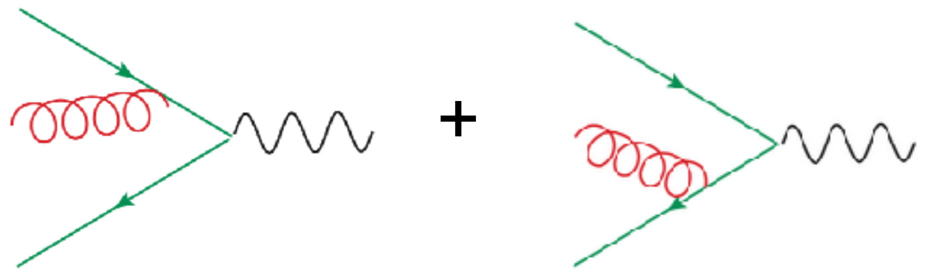
virtual

Born

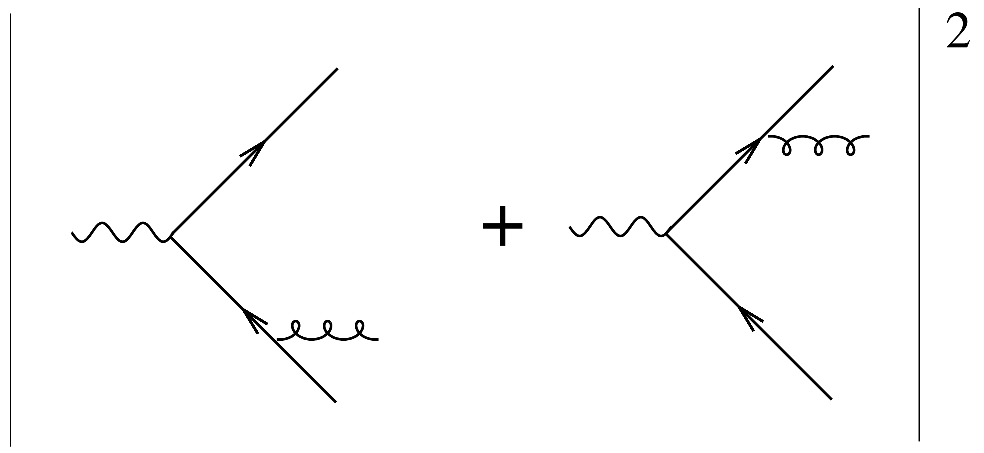


real

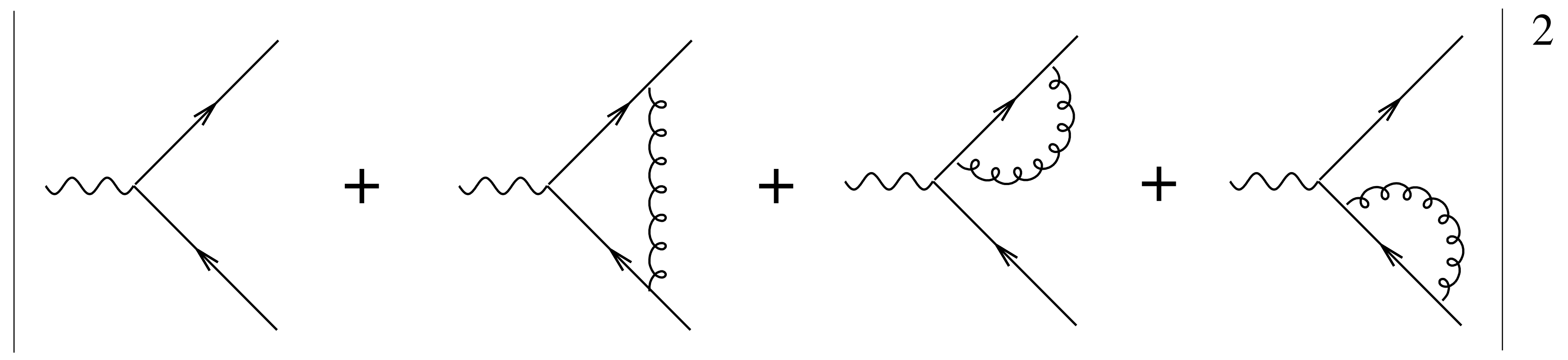
itself



NLO basics



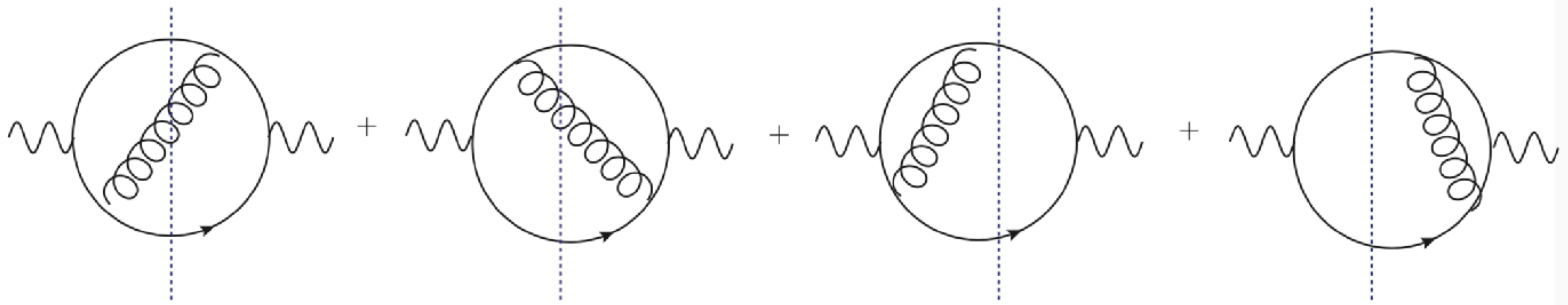
real radiation



virtual corrections

at order g_s^2

$|\mathcal{M}|^2$ pictorially: $\mathcal{M}$ left of the cut, $\mathcal{M}^\dagger$ right of the cut



claim: sum over all cuts above is finite ; individual diagrams contain infrared singularities

must be so due to KLN-Theorem

Cancellation of IR singularities

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KLN Theorem

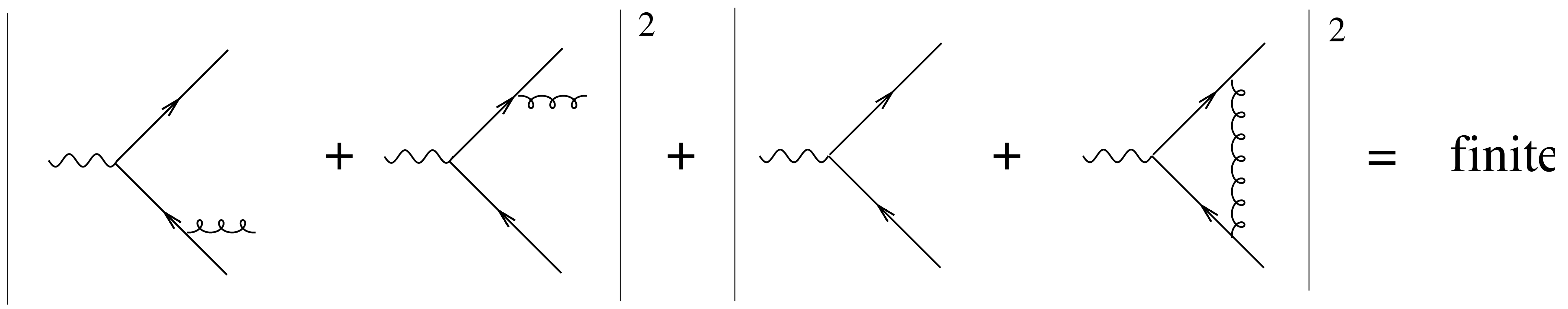
Kinoshita, Lee, Nauenberg, 1962, 1964

Soft and collinear singularities cancel in the sum over degenerate states

what are degenerate states ?

- a quark emitting a soft gluon, or a collinear quark-gluon system cannot be distinguished from simply a quark
- virtual corrections are not directly observable

$\Rightarrow$ in the considered inclusive cross section, singularities cancel between real and virtual corrections



note:
does not hold for initial state radiation in hadronic collisions

reason: exact initial states unknown for partons in the proton

(see later)

structure of NLO cross sections

$$\mathcal{B}_n = \int \mathrm{d}\phi_n \, |\mathcal{M}_0|^2 = \int \mathrm{d}\phi_n B_n$$

$$\mathcal{V}_n = \int \mathrm{d}\phi_n \, 2Re\left(\mathcal{M}_{\mathrm{virt}}\mathcal{M}_0^*\right) = \int \mathrm{d}\phi_n \frac{V_n}{\epsilon}$$

$$\mathcal{R}_n = \int \mathrm{d}\phi_{n+1} \, |\mathcal{M}_{\mathrm{real}}|^2 = \int \mathrm{d}\phi_n \int_0^1 \mathrm{d}x \, x^{-1-\epsilon} \, R_n(x)$$

$$\sigma^{NLO} = \int \mathrm{d}\phi_n \left\{ \left(B_n + \frac{V_n}{\epsilon} \right) J(p_1 \ldots p_n, 0) + \int_0^1 \mathrm{d}x \, x^{-1-\epsilon} \, R_n(x) \, J(p_1 \ldots p_n, x) \right\}$$

J is called *measurement function* and defines the observable

cancellation of IR singularities can only work if $\lim_{x \to 0} J(p_1 \ldots p_n, x) = J(p_1 \ldots p_n, 0)$

structure of NLO cross sections

$$\mathcal{B}_n = \int \mathrm{d}\phi_n \, |\mathcal{M}_0|^2 = \int \mathrm{d}\phi_n B_n$$

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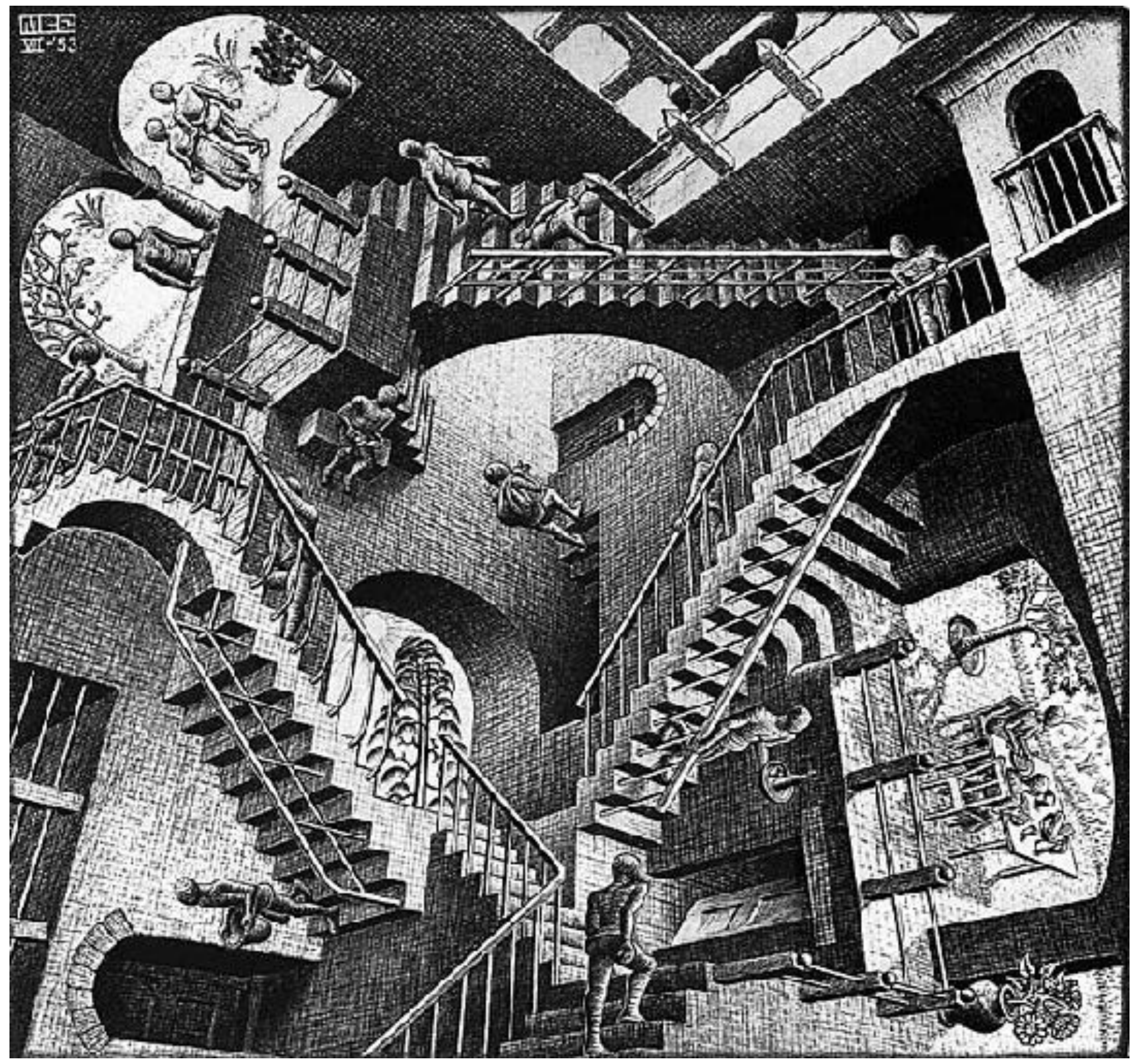
J is called *measurement function* and defines the observable

cancellation of IR singularities can only work if $\lim_{x \to 0} J(p_1 \ldots p_n, x) = J(p_1 \ldots p_n, 0)$ **infrared safety**

Dimensional regularisation 't Hooft, Veltman '72; Bollini, Giambiagi '72

A convenient way to isolate singularities:

continue space-time from 4 to $D = 4 - 2\epsilon$ dimensions



- regulates both UV and IR divergences, formally UV: $\epsilon > 0$, IR: $\epsilon < 0$
- does not violate gauge invariance
- poles can be isolated in terms of $1/\epsilon^b$

➡ need phase space integrals in D dimensions

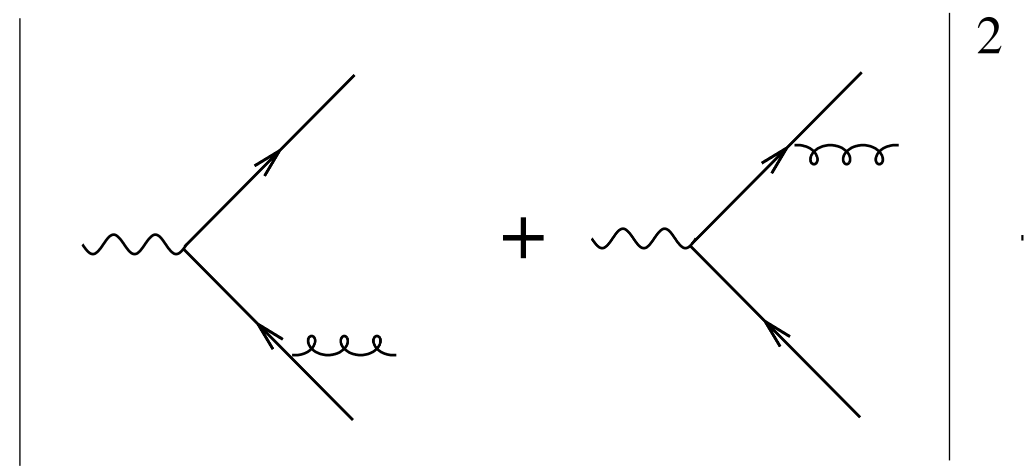
➡ need integration over virtual loop momenta in D dimensions

$$g^2 \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} \longrightarrow g^2 \mu^{2\epsilon} \int_{-\infty}^{\infty} \frac{d^Dk}{(2\pi)^D} \quad ,$$

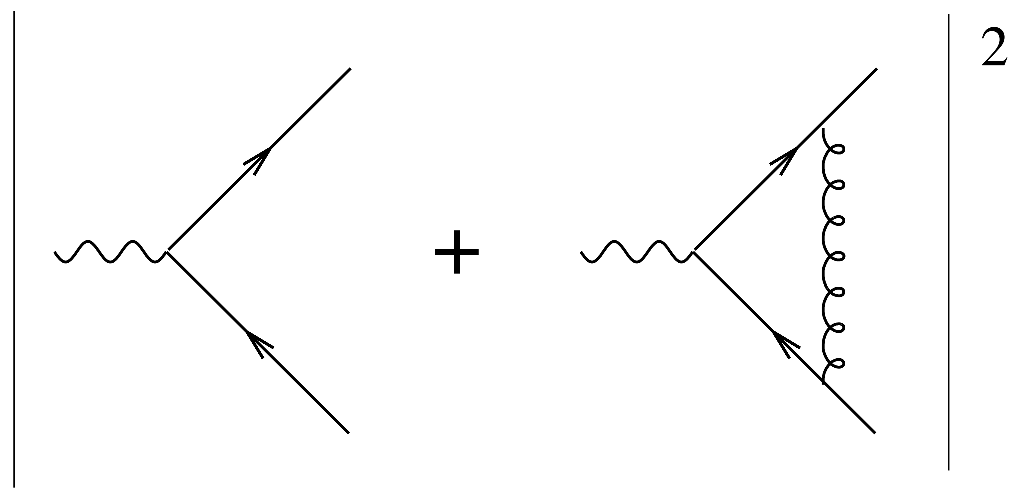
$\mu^{2\epsilon}$ to keep coupling (mass-)dimensionless in D dim.

Cancellation of IR singularities

real and virtual corrections live on different phase spaces



3-particle phase space



2-particle phase space

$$\sigma^{NLO} = \underbrace{\int \mathrm{d}\phi_2 \, |\mathcal{M}_0|^2}_{\sigma^{LO}} + \int_R \mathrm{d}\phi_3 \, |\mathcal{M}_{\mathrm{real}}|^2 + \int_V \mathrm{d}\phi_2 \, 2Re \left(\mathcal{M}_{\mathrm{virt}} \mathcal{M}_0^*\right)$$

real radiation matrix element

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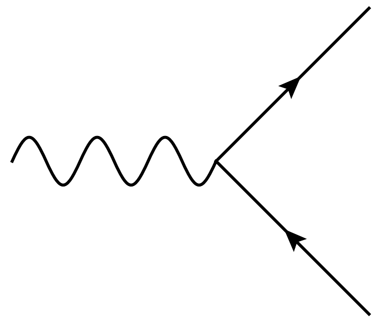
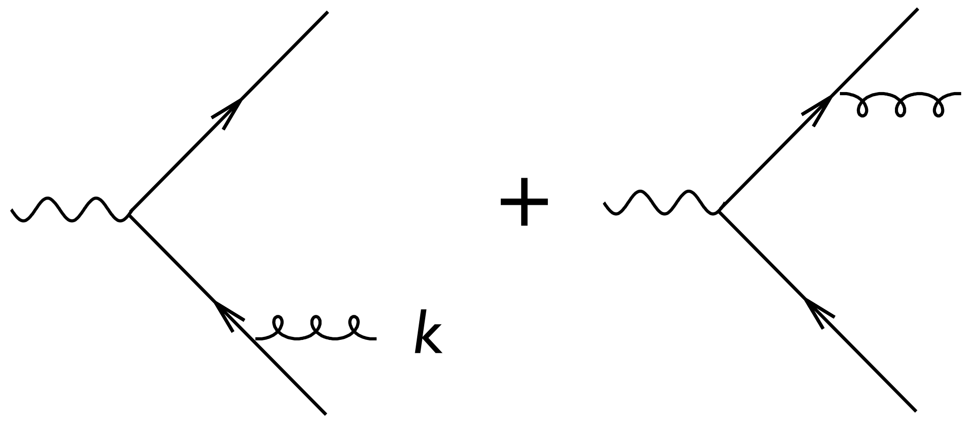
at LO: $|\overline{\mathcal{M}}_0|^2 = \frac{1}{3} 4e^2 Q_q^2 N_c\, s$ $\left| \vphantom{\int} \right.$  $\left. \vphantom{\int} \right|^2$

with extra gluon radiation: $\left| \vphantom{\int} \right.$  $\left. \vphantom{\int} \right|^2 .$

$$
\begin{aligned}
p^\gamma &= \sqrt{s}\,(1,0,0,0)\\
p_1 &= E_1\,(1,0,0,1)\\
p_2 &= E_2\,(1,0,\sin\theta,\cos\theta)\\
k &\equiv p_3 = p^\gamma - p_1 - p_2
\end{aligned}
$$

in 4 dimensions:

$$
|\overline{\mathcal{M}}_{\mathrm{real}}|^2 = |\overline{\mathcal{M}}_0|^2\, \frac{2g^2\, C_F}{s} \left(\frac{s_{13}}{s_{23}} + \frac{s_{23}}{s_{13}} + 2s\,\frac{s_{12}}{s_{13}s_{23}} \right)
$$

$$
s_{ij} = (p_i + p_j)^2
$$

singularity structure

Karlsruhe Institute of Technology

substitute $y_1 = s_{12}/Q^2, y_2 = s_{13}/Q^2, y_3 = s_{23}/Q^2$ and keep D dimensions

$$|\mathcal{M}|^2_{\rm real} = C_F e^2 Q_f^2 g_s^2\, 8\,(1-\epsilon)\left\{\frac{2}{y_2 y_3} + \frac{-2+(1-\epsilon)y_3}{y_2} + \frac{-2+(1-\epsilon)y_2}{y_3} - 2\epsilon\right\}$$

limits:

soft: $p_3 \to 0 \Rightarrow s_{13}, s_{23} \to 0 \quad \Rightarrow y_2 \text{ and } y_3 \to 0$

collinear: $p_3 \,||\, p_1 \Rightarrow y_2 \to 0$, $p_3 \,||\, p_2 \Rightarrow y_3 \to 0$

in these limits the matrix element is singular

- we know that the singularities should cancel with the virtual corrections
- however we first have to isolate them to make the cancellation manifest

phase space in D dimensions

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Example $Q \to p_1 + p_2 + p_3$

$$\mathrm{d}\Phi_{1\to 3} = \frac{1}{4}(2\pi)^{3-2\,D}\,\mathrm{d}E_1\mathrm{d}E_2\mathrm{d}\theta_1\,(E_1 E_2 \sin\theta)^{D-3}\,\mathrm{d}\Omega_{D-2}\,\mathrm{d}\Omega_{D-3}$$
$$\Theta(E_1)\,\Theta(E_2)\,\Theta(E - E_1 - E_2)\,\delta((Q - p_1 - p_2)^2)\ .$$

variable transformation: $E_1, E_2, \theta \to s_{12}, s_{23}, s_{13}$

dimensionless variables: $y_1 = s_{12}/Q^2, y_2 = s_{13}/Q^2, y_3 = s_{23}/Q^2$

$$d\Phi_{1\to 3} = (2\pi)^{3-2\,D}\,2^{-1-D}(Q^2)^{D-3}\,d\Omega_{D-2}\,d\Omega_{D-3}\,dy_1\,dy_2\,dy_3$$
$$(y_1\,y_2\,y_3)^{D/2-2}\,\Theta(y_1)\,\Theta(y_2)\,\Theta(y_3)\,\delta(1 - y_1 - y_2 - y_3)$$

$D/2 - 2 = -\epsilon$

real radiation in D dimensions

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$$d\Phi_{1\to 3} = (2\pi)^{3-2D}\, 2^{-1-D} (Q^2)^{D-3}\, d\Omega_{D-2}\, d\Omega_{D-3}\, dy_1\, dy_2\, dy_3$$
$$(y_1\, y_2\, y_3)^{D/2-2}\, \Theta(y_1)\, \Theta(y_2)\, \Theta(y_3)\, \delta(1-y_1-y_2-y_3)$$

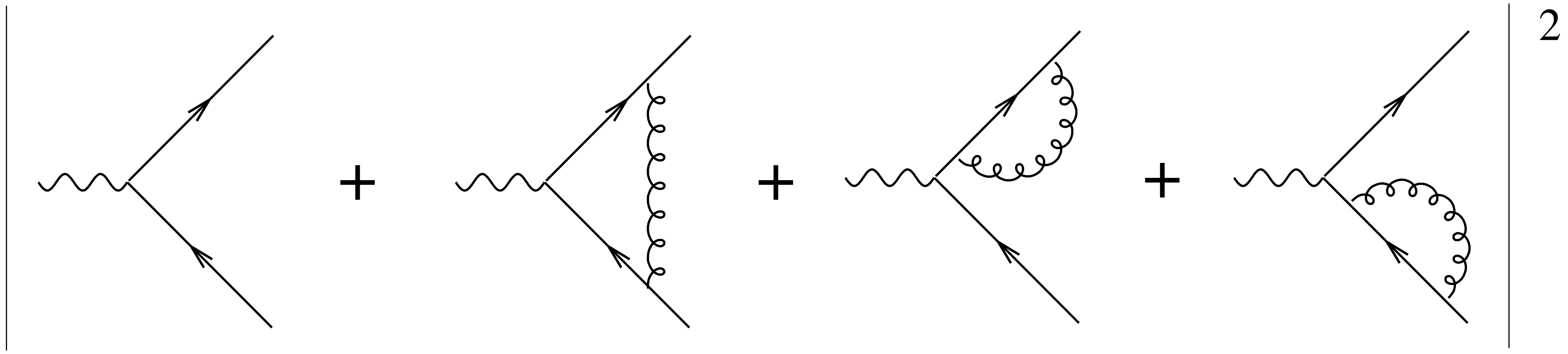
substitute $y_1 = 1 - z_1, y_2 = z_1 z_2, y_3 = z_1(1-z_2)$, $\det J = z_1$

$$\int \mathrm{d}\Phi_3 |\mathcal{M}|^2_{\text{real}} = \alpha\, C_F \frac{\alpha_s}{\pi}\, Q_f^2\, \tilde{H}(\epsilon)\, (Q^2)^{1-2\epsilon} \int_0^1 \mathrm{d}z_1 \int_0^1 \mathrm{d}z_2\, z_1^{-2\epsilon} \Big(z_2(1-z_1)(1-z_2) \Big)^{-\epsilon}$$
$$\left\{ \frac{2}{z_1 z_2 (1-z_2)} + \frac{-2+(1-\epsilon) z_1 (1-z_2)}{z_2} + \frac{-2+(1-\epsilon) z_1 z_2}{1-z_2} - 2\epsilon z_1 \right\}.$$

singularities regulated by ϵ

$\tilde{H}(\epsilon) = 1 + \mathcal{O}(\epsilon)$ (combination of Γ -functions)

virtual corrections



we will not go through the calculation but only quote the result:

$$R^{\text{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\}$$

obtained by calculating the loop integrals in *D* dimensions, $D = 4 - 2\epsilon$

combine real and virtual

$$R^{\mathrm{real}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-3\epsilon)} \left(\frac{s}{4\pi\mu^2}\right)^{-\epsilon} \left\{ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \mathcal{O}(\epsilon) \right\}$$

gluon both soft and collinear

$$R^{\mathrm{virt}} = R^{LO} \times \frac{\alpha_s}{2\pi} C_F \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \left(\frac{-s}{4\pi\mu^2}\right)^{-\epsilon} \left\{ -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \mathcal{O}(\epsilon) \right\}$$

KLN theorem at work!

$$R = R^{LO} \times \left\{ 1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2) \right\}$$

Quiz (instead of summary)

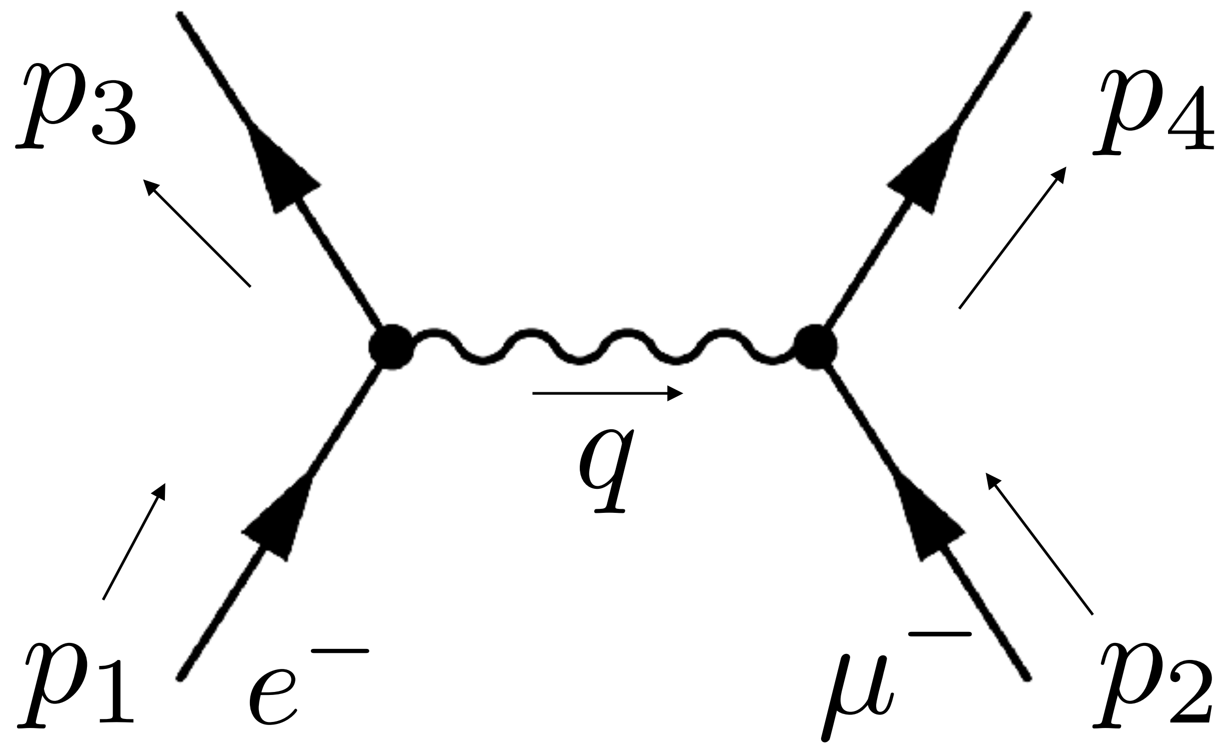
- What is the difference between QED and QCD when performing the sum over polarisations?
- What is the reason for the dependence of a theory prediction on an unphysical scale?
- What would happen if there were more than 16 fermion flavours (with masses near the EW scale)?
- If the scale uncertainties do not decrease significantly at the next order, what could be the reason?
- If IR singularities cancel between real and virtual corrections, why do we need to isolate them (e.g. as 1/eps poles in dimensional regularisation)?

Appendix 1: exercise $e\mu \to e\mu$

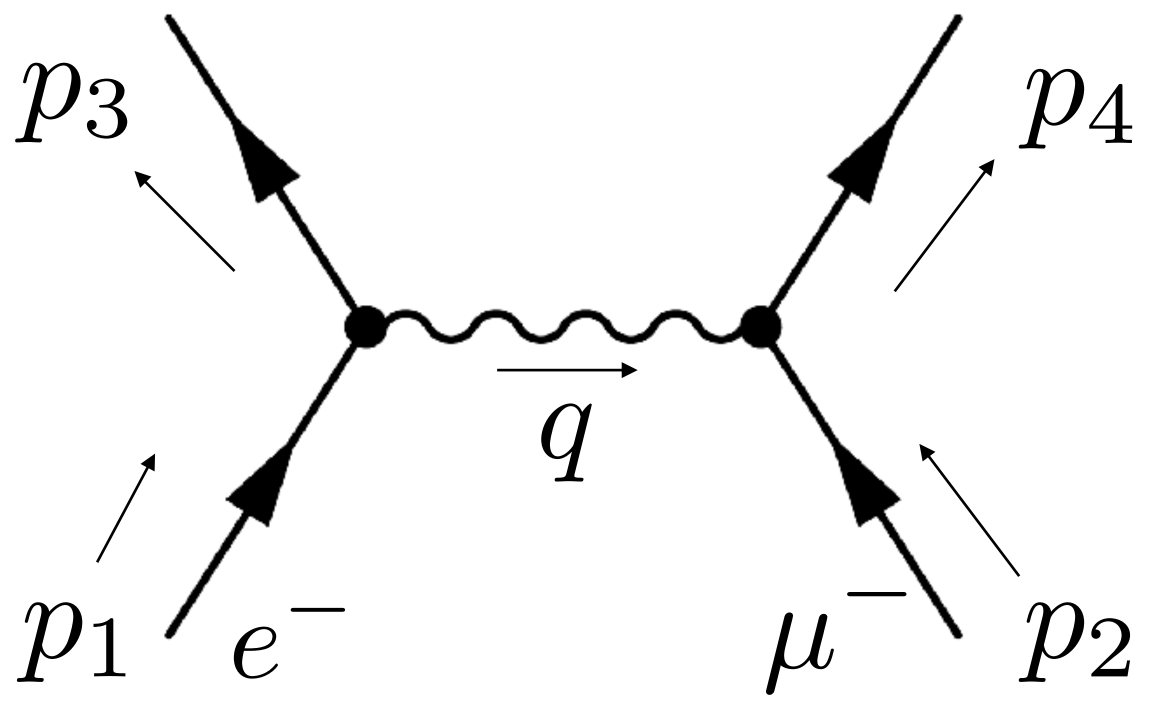
Example for a simple cross section calculation

calculate $d\sigma/d\Omega$ for electron-muon scattering

$$e^-(p_1) + \mu^-(p_2) \to e^-(p_3) + \mu^-(p_4)$$



example electron-muon-scattering



$$e^-(p_1) + \mu^-(p_2) \to e^-(p_3) + \mu^-(p_4)$$

$$q = p_1 - p_3 = p_4 - p_2$$

$$\mathcal{M} = \bar{u}(p_3, s_3)_\alpha (-ie\gamma_\mu)_{\alpha\beta}\, u(p_1, s_1)_\beta \frac{-i\, g^{\mu\nu}}{q^2 + i\epsilon}\, \bar{u}(p_4, s_4)_\rho (-ie\gamma_\nu)_{\rho\sigma}\, u(p_2, s_2)_\sigma$$

$$= -\frac{e^2}{q^2 + i\epsilon}\, \bar{u}(p_3, s_3)_\alpha (\gamma_\mu)_{\alpha\beta}\, u(p_1, s_1)_\beta\, \bar{u}(p_4, s_4)_\rho (\gamma^\mu)_{\rho\sigma}\, u(p_2, s_2)_\sigma$$

cross section: $\sigma \sim |\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^\dagger = \mathcal{M}(\mathcal{M}^*)^T$

unpolarised: sum over final state spins, average over initial state spins

$$|\overline{\mathcal{M}}|^2 = \frac{1}{n_{s_1} n_{s_2}} \sum_{s_1, s_2, s_3, s_4} |\mathcal{M}|^2$$, here $n_{s_1} = n_{s_2} = 2$

electron-muon-scattering

useful formula: Γ_1, Γ_2 some strings of γ -matrices, $\bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$

$$\sum_{s_i, s_j} \left(\bar{u}(p_i, s_i)\Gamma_1 u(p_j, s_j)\right) \left(\bar{u}(p_i, s_i)\Gamma_2 u(p_j, s_j)\right)^\dagger$$
$$= \text{Trace}[\Gamma_1(\not{p}_j + m_j)\bar{\Gamma}_2(\not{p}_i + m_i)]$$

proof: use

$$\sum_s u(p,s)\bar{u}(p,s) = \not{p} + m \qquad \gamma_0^\dagger = \gamma_0$$
$$\sum_s v(p,s)\bar{v}(p,s) = \not{p} - m \qquad \gamma_0\gamma_i\gamma_0 = \gamma_i$$

therefore:

$$\sum_{s_1, s_3} \underbrace{\left(\bar{u}(p_3, s_3)\gamma^\mu u(p_1, s_1)\right)}_{\mathcal{M}} \underbrace{\left(\bar{u}(p_3, s_3)\gamma^{\mu'} u(p_1, s_1)\right)^\dagger}_{\mathcal{M}^\dagger}$$

$$= \text{Trace}[\gamma^\mu(\not{p}_1 + m_1)\gamma^{\mu'}(\not{p}_3 + m_3)]\,,$$ analogous for $\sum_{s_2, s_4}$

$$\Rightarrow \quad \overline{|\mathcal{M}|^2} = \frac{e^4}{4q^4} \text{Trace}[\gamma_\mu(\not{p}_1 + m_e)\gamma_{\mu'}(\not{p}_3 + m_e)]\,\text{Trace}[\gamma^\mu(\not{p}_2 + m_\mu)\gamma^{\mu'}(\not{p}_4 + m_\mu)]$$

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$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{4q^4}\,\mathrm{Trace}[\gamma_\mu(\not p_1 + m_e)\gamma_{\mu'}(\not p_3 + m_e)]\,\mathrm{Trace}[\gamma^\mu(\not p_2 + m_\mu)\gamma^{\mu'}(\not p_4 + m_\mu)]$$

$$\mathrm{Trace}[\gamma_\mu(\not p_1 + m_e)\gamma_{\mu'}(\not p_3 + m_e)]$$

$$= \mathrm{Trace}[\gamma_\mu\,\not p_1\gamma_{\mu'}\,\not p_3] + m_e^2\,\mathrm{Trace}[\gamma_\mu\gamma_{\mu'}] + m_e\,\underbrace{\mathrm{Trace}[\gamma_\mu\,\not p_i\gamma_{\mu'}]}_{0} + m_e\,\underbrace{\mathrm{Trace}[\gamma_\mu\,\not p_3\gamma_{\mu'}]}_{0}$$

$$= 4\left(p_1^\mu p_3^{\mu'} + p_3^\mu p_1^{\mu'} - p_1\cdot p_3\,g^{\mu\mu'}\right) + 4m_e^2\,g^{\mu\mu'}$$

analogous for second trace $\Rightarrow$ contraction of Lorentz indices

Mandelstam-variables for 2-particle scattering (Lorentz invariant):

$$s = (p_1+p_2)^2$$
$$t = (p_1-p_3)^2$$
$$u = (p_2-p_3)^2$$

$s+t+u = \sum_{i=1}^{4} p_i^2$, here $s+t+u = 2m_e^2 + 2m_\mu^2,\ t = q$

$$\Rightarrow\ |\overline{\mathcal{M}}|^2 = \frac{2e^4}{t^2}\left(s^2+u^2-4(s+u)(m_e^2+m_\mu^2)+6(m_e^2+m_\mu^2)^2\right)$$

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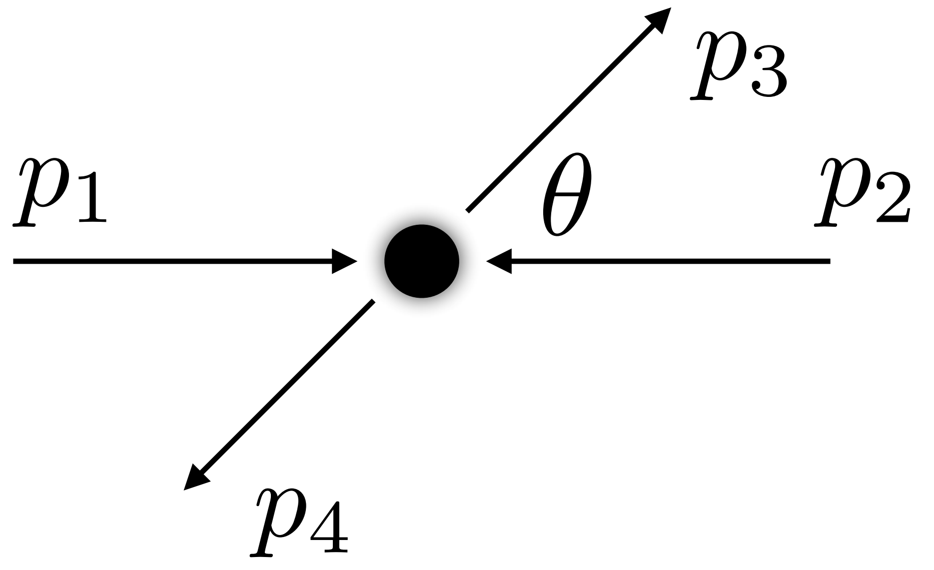
high energy limit: $s \gg m_e^2, m_\mu^2$

$$|\overline{\mathcal{M}}|^2 = 2e^4 \, \frac{s^2+u^2}{t^2}$$

photon-exchange in the t-channel

center-of-mass-system:



definine unit vectors $\vec{n} = (0,0,1)\;,\; \vec{n}' = (0, \sin\theta, \cos\theta)$

$\Rightarrow$ (masses neglected)

$$p_1 = \frac{\sqrt{s}}{2}(1,\vec{n})\;,\; p_2 = \frac{\sqrt{s}}{2}(1,-\vec{n})$$

$$p_3 = \frac{\sqrt{s}}{2}(1,\vec{n}')\;,\; p_4 = \frac{\sqrt{s}}{2}(1,-\vec{n}')$$

$$t = -2p_1\cdot p_3 = -\frac{s}{2}(1-\cos\theta) = -s\,\sin^2\frac{\theta}{2}$$

$$u = -2p_2\cdot p_3 = -\frac{s}{2}(1+\cos\theta) = -s\,\cos^2\frac{\theta}{2}$$

$$\Rightarrow\; |\overline{\mathcal{M}}|^2 = 2e^4\,\frac{1+\cos^4(\frac{\theta}{2})}{\sin^4(\frac{\theta}{2})}$$

2-particle-phase space

cross section: $$d\sigma = \frac{J}{\text{flux}} \cdot |\overline{\mathcal{M}}|^2 \cdot d\Phi_2$$

$d\Phi_2$ 2-particle phase space $p_1 + p_2 \to p_3 + p_4$

$$d\Phi_2 = \frac{1}{(2\pi)^6}\, d^4p_3\delta(p_3^2-m_3^2)\Theta(E_3)\; d^4p_4(2\pi)^4\delta(p_4^2-m_4^2)\Theta(E_4)\; (2\pi)^4\delta^{(4)}\Big(p_1+p_2-p_3-p_4\Big)$$

massless case: $$d^4p_j\delta(p_j^2)\Theta(E_j) = dE_j d^3\vec{p}_j\delta(E_j^2-\vec{p}_j^{\,2})\Theta(E_j) = \frac{1}{2E_j}d^3\vec{p}_j\Big|_{E_j=|\vec{p}_j|}$$

eliminate $\vec{p}_4$

$$d\Phi_2 = \frac{1}{(2\pi)^3}\frac{d^3\vec{p}_3}{2|\vec{p}_3|}\frac{1}{2E_4}\,(2\pi)\,\delta\Big(E_1+E_2-E_3-E_4\Big)\Big|_{E_j=|\vec{p}_j|}$$

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$$d\Phi_2 = \frac{1}{(2\pi)^3}\frac{d^3\vec{p}_3}{2|\vec{p}_3|}\,\frac{1}{2E_4}\,(2\pi)\,\delta\big(E_1+E_2-E_3-E_4\big)\Big|_{E_j=|\vec{p}_j|}$$

(spherical coordinates)

$$= \frac{1}{(2\pi)^3}d\Omega\, d|\vec{p}_3|\,\frac{|\vec{p}_3|^2}{2|\vec{p}_3|}\,\frac{1}{2E_4}\,(2\pi)\,\delta\left(E_{\rm cm}-E_3-E_4\right)\Big|_{E_j=|\vec{p}_j|}$$

$$= \frac{1}{16\pi^2}d\Omega\,\frac{|\vec{p}_3|}{E_{\rm cm}}$$

center-of-mass system (CM): $E_{\rm cm} = E_1 + E_2 = \sqrt{s}\ ,\quad \frac{|\vec{p}_3|}{E_{\rm cm}} = \frac{1}{2}$

$$\Rightarrow\quad \frac{d\sigma}{d\Omega} = \frac{1}{2s}\,\frac{1}{32\pi^2}\,2e^4\,\frac{s^2+u^2}{t^2} = \frac{\alpha^2}{2s}\,\frac{s^2+u^2}{t^2} = \frac{\alpha^2}{2s}\,\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}$$

↑ flux ↑ phase space factor ↑ matrix element

Appendix 2: phase space in D dimensions

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1 to N particle phase space:

$$Q \to p_1 + \ldots + p_N$$

$$\int d\Phi_N^D = (2\pi)^{N-D(N-1)} \int \prod_{j=1}^{N} d^D p_j \, \delta^+(p_j^2 - m_j^2) \delta^{(D)}\Big(Q - \sum_{i=1}^{N} p_i\Big)$$

In the following consider massless case $p_j^2 = 0$. Use for $i = 1, \ldots, N-1$

$$\begin{aligned} \int d^D p_i \delta^+(p_i^2) &\equiv \int d^D p_i \delta(p_i^2)\theta(E_i) = \int d^{D-1}\vec{p}_i \, dE_i \, \delta(E_i^2 - \vec{p}_i^{\,2})\theta(E_i) \\ &= \frac{1}{2E_i} \int d^{D-1}\vec{p}_i \Big|_{E_i = |\vec{p}_i|} \end{aligned}$$

and eliminate p_N by momentum conservation

$$\Rightarrow \quad \int d\Phi_N^D = (2\pi)^{N-D(N-1)} \, 2^{1-N} \int \prod_{j=1}^{N-1} d^{D-1}\vec{p}_j \, \frac{\Theta(E_j)}{E_j} \, \delta^+([Q - \sum_{i=1}^{N-1} p_i]^2)\Big|_{E_j = |\vec{p}_j|}$$

phase space in D dimensions

phase space volume of unit sphere in D dimensions

$$\int d\Omega_{D-1} = V(D) = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \quad , \quad V(D) = \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \ldots \int_0^{\pi} d\theta_{D-1} (\sin\theta_{D-1})^{D-2}$$

$$\frac{\mathrm{d}^{D-1}\vec{p}}{|\vec{p}|} f(|\vec{p}|) = \mathrm{d}\Omega_{D-2}\, \mathrm{d}|\vec{p}|\, |\vec{p}|^{D-3} f(|\vec{p}|)$$

$$\mathrm{d}\Phi_{1\to n} = (2\pi)^{n-D(n-1)} 2^{1-n} \mathrm{d}\Omega_{D-2} \prod_{j=1}^{n-1} \mathrm{d}|\vec{p}_j|\, |\vec{p}_j|^{D-3} \delta\Big((Q - \sum_{j=1}^{n-1} p_j)^2\Big)$$

in the massless case, use $|\vec{p}_j| = E_j$