Karlsruhe Institute of Technology

Collaborative Research Center TRR 257

Particle Physics Phenomenology after the Higgs Discovery

Lecture 3: IR singularities, jets and event shapes, PDFs

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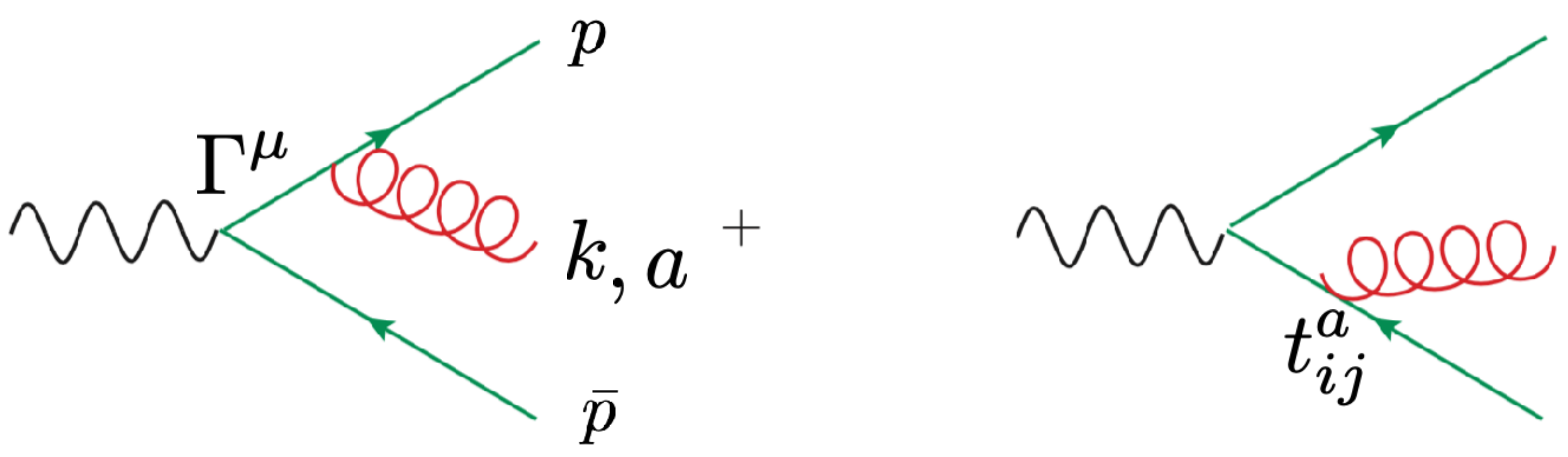
Outline

- universal infrared properties of QCD
- jets and event shapes
- deeply inelastic scattering (DIS)
- parton densities and parton evolution
- hadronic initial states
- parton showers

Soft gluon emission



$$\mathcal{M}^\mu_{Born} = \bar{u}(p)\Gamma^\mu v(\bar{p})$$

$$\mathcal{M}^\mu = t^a_{ij}\, g_s \mu^\epsilon \bar{u}(p)\ \not{\epsilon}(k)\frac{\not{p}+\not{k}}{(p+k)^2}\Gamma^\mu v(\bar{p}) - t^a_{ij}\, g_s \bar{u}(p)\Gamma^\mu \frac{\not{\bar{p}}+\not{k}}{(\bar{p}+k)^2}\ \not{\epsilon}(k) v(\bar{p})$$

soft limit: $k \to 0$ except in linear terms in denominator

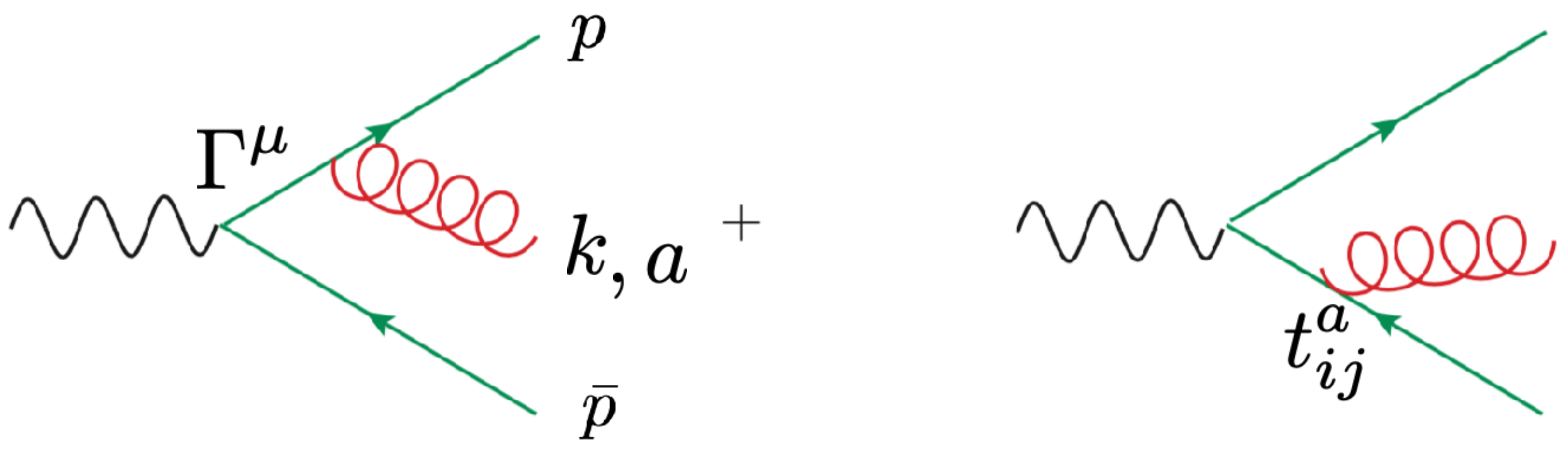
$$\mathcal{M}^\mu_{soft} = g_s\, \mu^\epsilon\, \bar{u}(p)\, \Gamma^\mu \left(t^a_{ij}\, \frac{2\epsilon(k)\cdot p}{2p\cdot k} - t^a_{ij}\, \frac{2\epsilon(k)\cdot \bar{p}}{2\bar{p}\cdot k} \right)\, v(\bar{p})$$

$$= g_s\, \mu^\epsilon\, \mathbf{J}^{a,\nu}(k)\epsilon_\nu(k)\, \mathcal{M}^\mu_{Born}$$

soft current

$$\boxed{\mathbf{J}^{a,\mu}(k) = \sum_{r=p,\bar{p}} \tilde{\mathbf{T}}^a\, \frac{r^\mu}{r\cdot k}}$$

Soft gluon emission



$$\mathcal{M}^\mu_{Born} = \bar{u}(p)\Gamma^\mu v(\bar{p})$$

$$\mathcal{M}^\mu = t^a_{ij}\, g_s \mu^\epsilon \bar{u}(p)\; \not{\epsilon}(k) \frac{\not{p} + \not{k}}{(p+k)^2} \Gamma^\mu v(\bar{p}) - t^a_{ij}\, g_s \bar{u}(p)\Gamma^\mu \frac{\not{\bar{p}} + \not{k}}{(\bar{p}+k)^2}\; \not{\epsilon}(k) v(\bar{p})$$

soft limit: $k \to 0$ except in linear terms in denominator

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$$= g_s\, \mu^\epsilon \mathbf{J}^{a,\nu}(k) \epsilon_\nu(k)\, \mathcal{M}^\mu_{Born}$$

soft current

$$\boxed{\mathbf{J}^{a,\mu}(k) = \sum_{r=p,\bar{p}} \tilde{\mathbf{T}}^a \frac{r^\mu}{r\cdot k}}$$

universal factorisation property

Soft factorisation at amplitude squared level

$$|\mathcal{M}(k, p_1, \ldots, p_m)|^2 \to -g_s^2\, \mu^{2\epsilon}\, 2 \sum_{i,j=1}^{m} S_{ij}(k)\, |\mathcal{M}_{(i,j)}(p_1, \ldots, p_m)|^2$$

m+1 momenta

k soft

m momenta

$$\boxed{S_{ij}(k) = \frac{p_i \cdot p_j}{2\,(p_i \cdot k)(p_j \cdot k)}}$$

Eikonal factor

colour correlated matrix element:

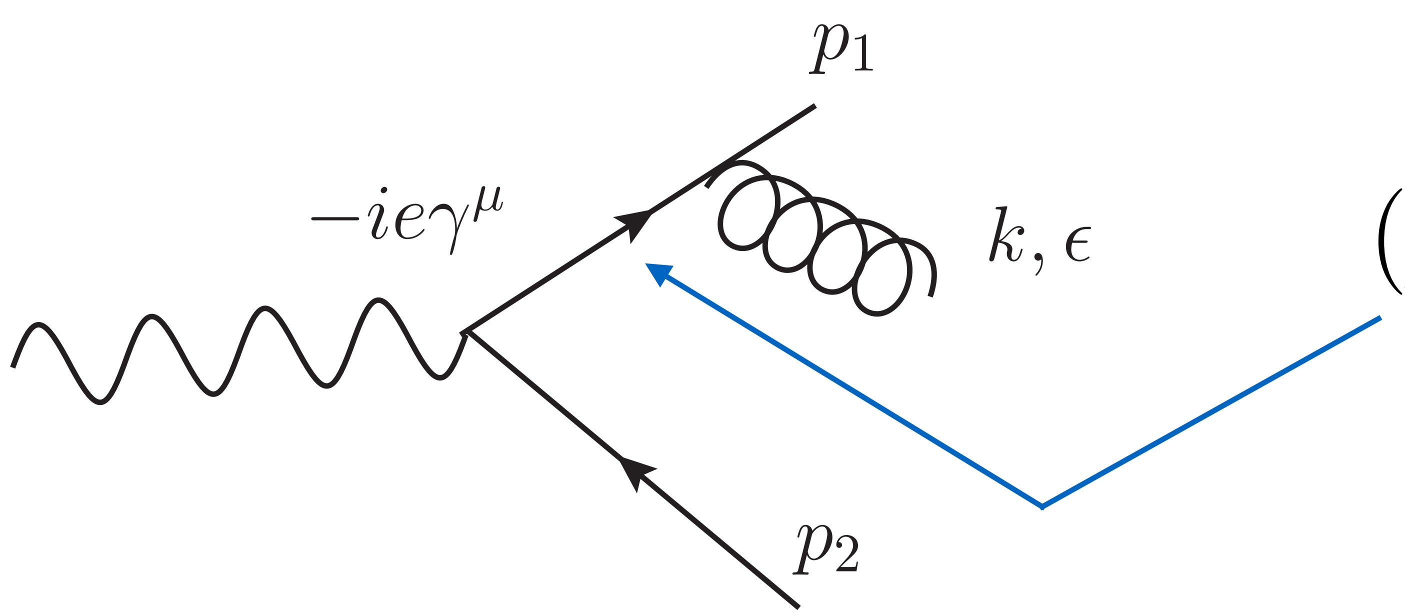
$$|\mathcal{M}_{(i,j)}(p_1, \ldots, p_m)|^2 \equiv \langle\, \mathcal{M}(p_1, \ldots, p_m)\,|\, \mathbf{T}_i \cdot \mathbf{T}_j \,|\, \mathcal{M}(p_1, \ldots, p_m)\rangle$$

$$= \left[\mathcal{M}_{c_1..b_i...b_j...c_m}(p_1, \ldots, p_m)\right]^* \; T^a_{b_i d_i}\, T^a_{b_j d_j}\; \mathcal{M}_{c_1..d_i...d_j...c_m}(p_1, \ldots, p_m)$$

m-parton matrix element

colour correlations

Collinear singularities



$$(p_1 + k)^2 = 2E\,\omega\,(1 - \cos\theta) \to 0 \;\; \text{for} \;\; \theta \to 0$$

convenient parametrisation of momenta: “Sudakov parametrisation”

$$\begin{aligned} p_1 &= z\,p^\mu + k_\perp^\mu - \frac{k_\perp^2}{z}\frac{n^\mu}{2p_1 n} \qquad z = \frac{E_1}{E_1+E_g} \\ k &= (1-z)\,p^\mu - k_\perp^\mu - \frac{k_\perp^2}{1-z}\frac{n^\mu}{2p_1 n} \end{aligned}$$

p^μ collinear direction

n^μ light-like auxiliary vector

$$k_\perp p = k_\perp n = 0$$

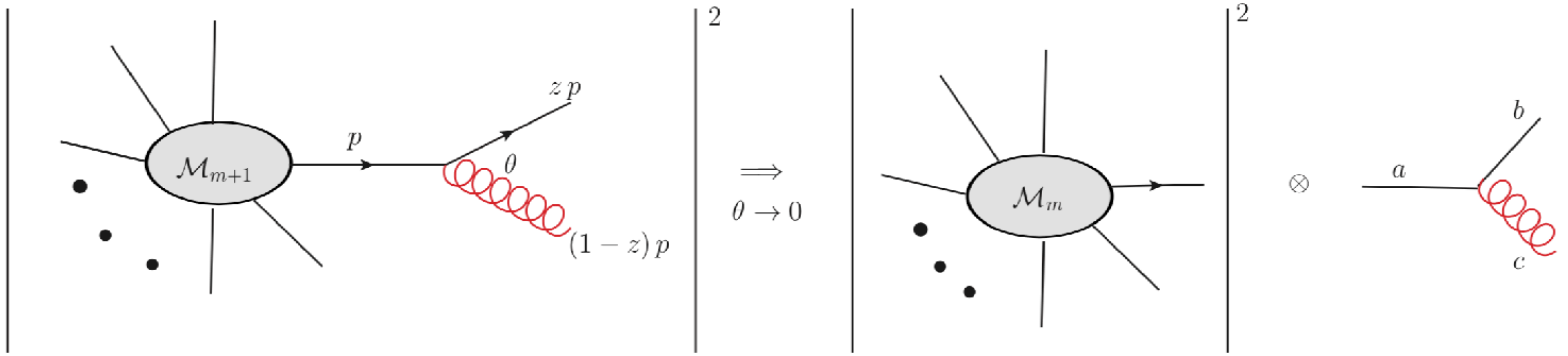
collinear limit in this parametrisation: $k_\perp \to 0$

$$|\mathcal{M}_1(p_1, k, p_2)|^2 \overset{coll}{\to} g^2 \frac{1}{p_1 \cdot k} P_{qq}(z)\; |\mathcal{M}_0(p_1 + k, p_2)|^2$$

$P_{qq}(z)$: splitting functions

Collinear singularities

factorisation property of squared amplitudes in the collinear limit:



$$|\mathcal{M}_{m+1}|^2 \, \mathrm{d}\Phi_{m+1} \to |\mathcal{M}_m|^2 \mathrm{d}\Phi_m \, \frac{\alpha_s}{2\pi} \, \frac{\mathrm{d}k_\perp^2}{k_\perp^2} \, \frac{\mathrm{d}\phi}{2\pi} \, \mathrm{d}z \, \boxed{P_{a\to bc}(z)}$$

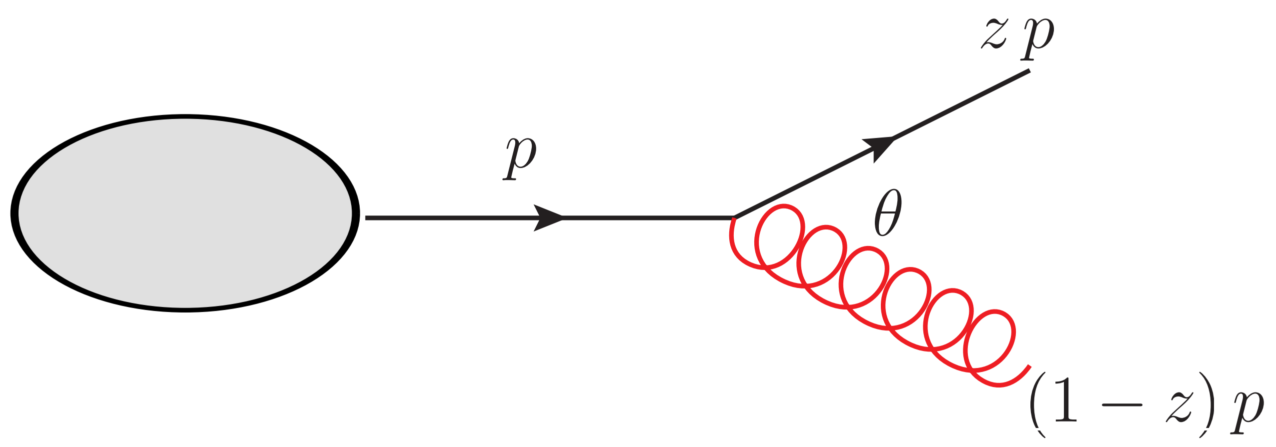
note that the phase space also can be factorised in this limit: $\mathrm{d}\Phi_{m+1} \to \mathrm{d}\Phi_m \otimes \mathrm{d}\Phi_k$

this factorisation does not depend on the details of $\mathcal{M}_m$

splitting functions

Dokshitzer, Gribov, Lipatov, Altarelli, Parisi

factorisation is universal, only depends on the types of splitting partons



$$P_{qq}(z) \equiv P_{q\to qg}(z) = C_F\,\frac{1+z^2}{1-z}$$

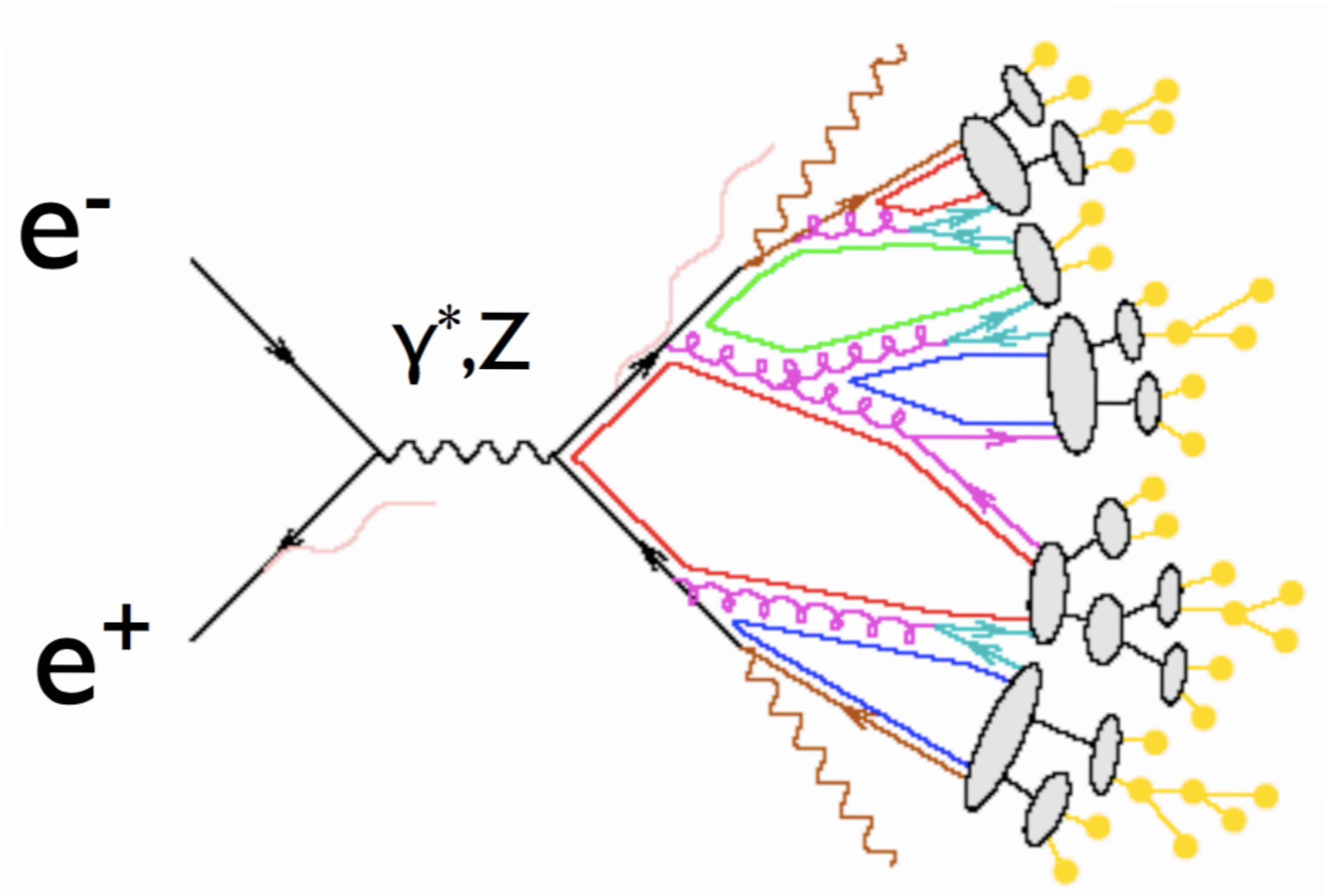
other types:

$$P_{q\to gq}(z) = C_F\,\frac{1+(1-z)^2}{z}$$

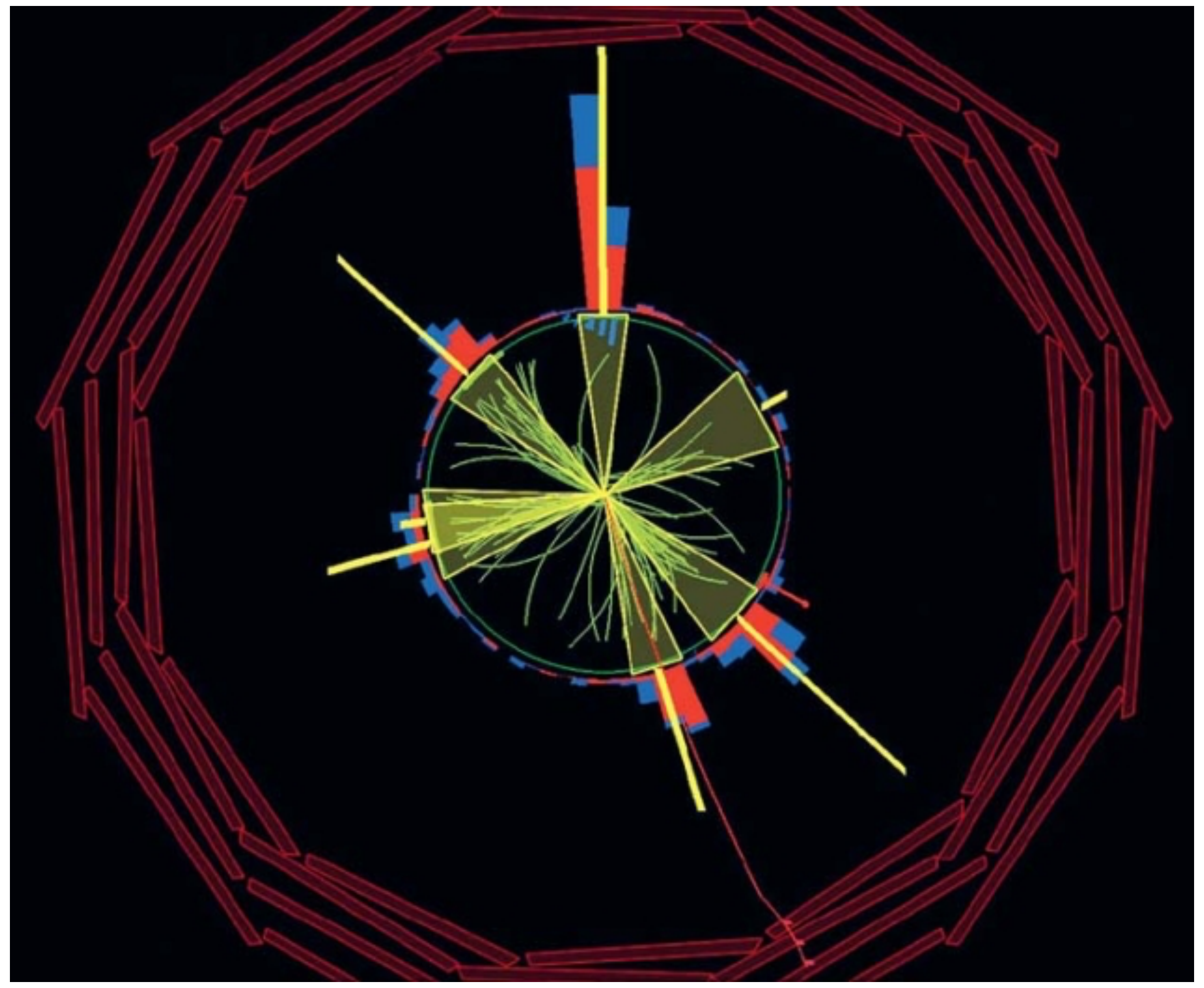
$$P_{g\to q\bar{q}}(z) = T_R\,\left(z^2+(1-z)^2\right)$$

$$P_{g\to gg}(z) = C_A\,\left(z\,(1-z)+\frac{z}{1-z}+\frac{1-z}{z}\right)$$

Jets



[Figure: Fabio Maltoni]



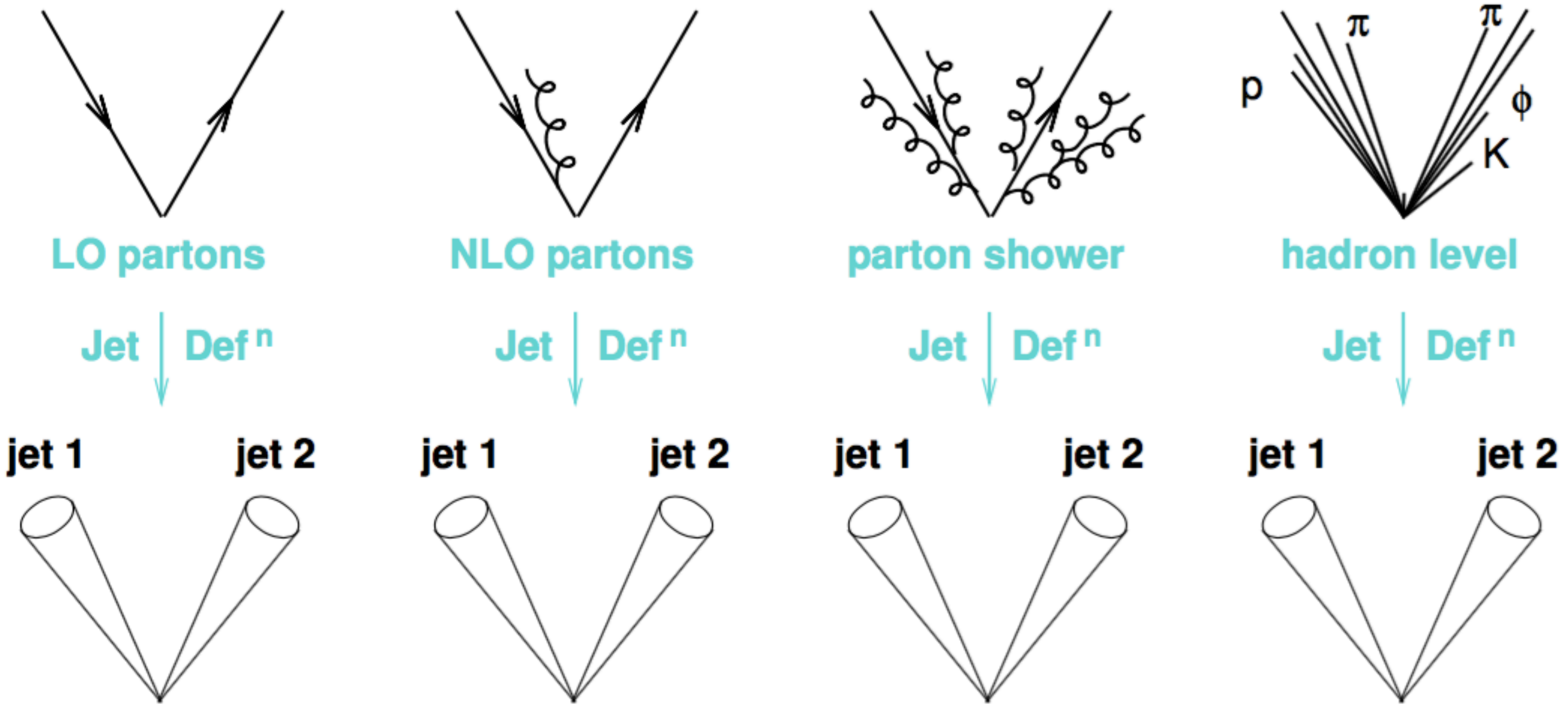
[6-jet event, Figure: CMS]

important: infrared safe jet algorithm

soft or collinear radiation should not yield different jet identification

Jets

considering also a parton shower and hadronisation:



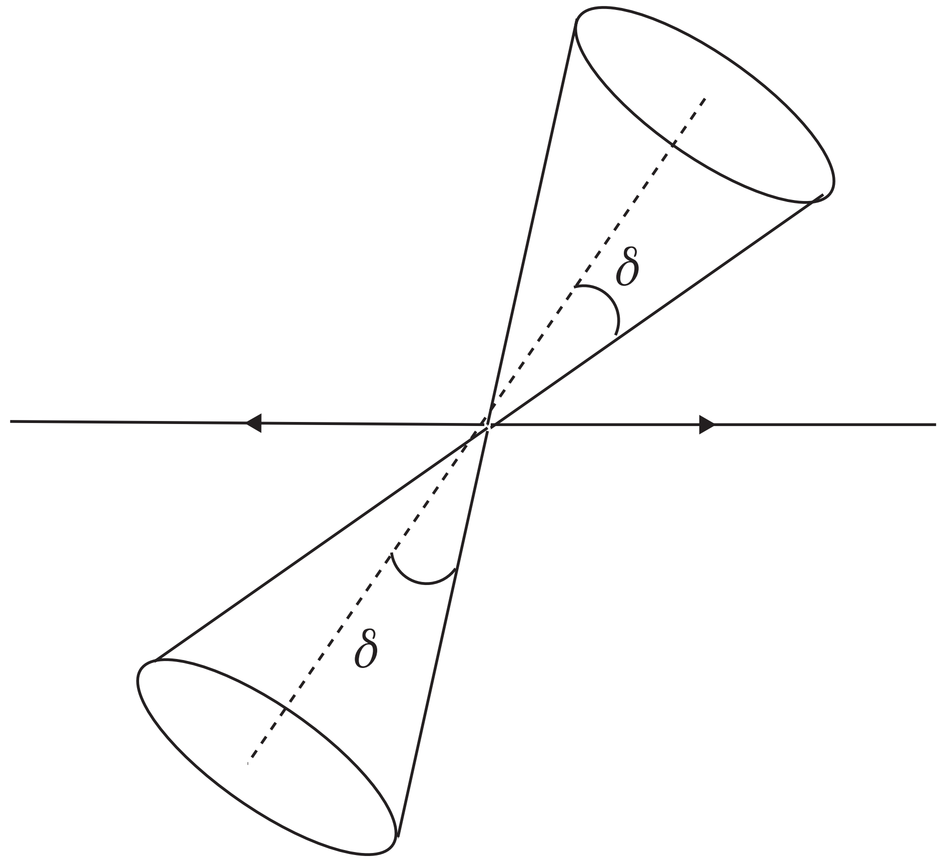
[Figure: Gavin Salam]

- *jet substructure* is taken into account in modern jet analysis
- jet algorithms based on machine learning are increasingly successful

Jets

Sterman and Weinberg 1977:

final state is *two-jet-like* if all but a fraction ε of the total available energy E is contained in two cones of opening angle δ



2-jet cross section:

depends on jet definition, i.e. ε and δ

$$\sigma^{2\,jet} \sim \sigma_0 \left(1 - 4\,C_F\,\frac{\alpha_s}{2\pi}\left[\ln(2\varepsilon)\ln\delta + \frac{3}{4}\ln\delta + \text{ finite}\right]\right)$$

if δ is very large, even radiation at relatively large angles will be clustered into the same jet

if δ is small, very little extra radiation is permitted in the cone

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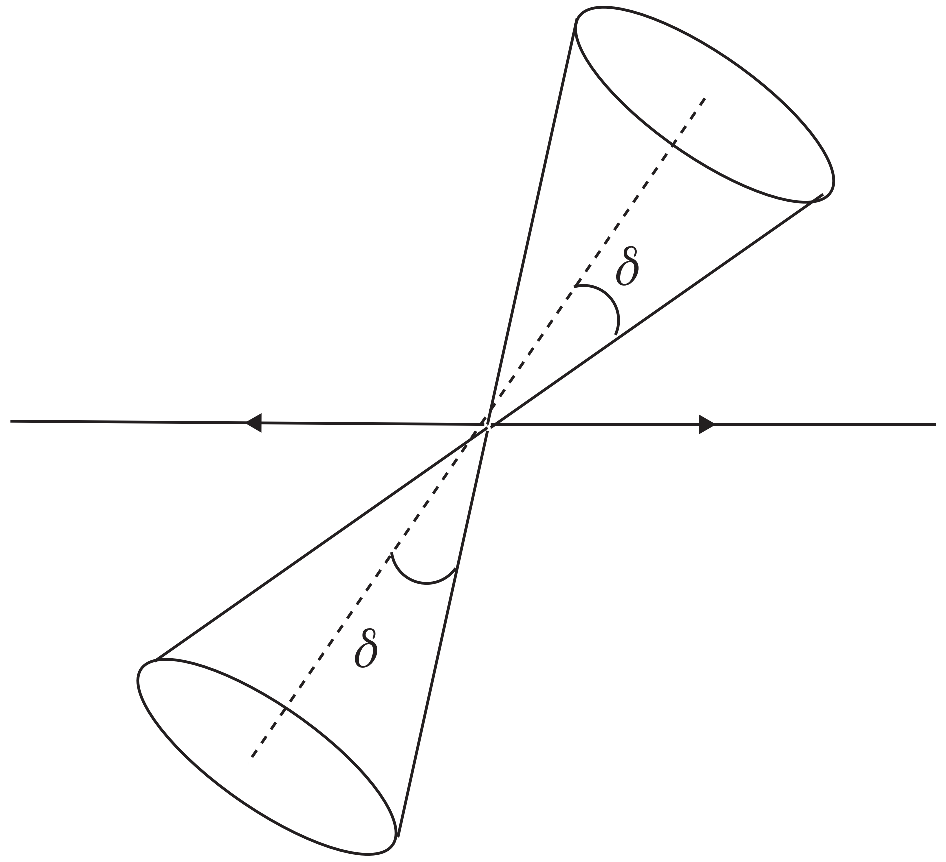

historically 1st, but not useful for multi-jet events

2-jet cross section:

depends on jet definition, i.e. ε and δ

$$\sigma^{2\,jet} \sim \sigma_0 \left(1 - 4\,C_F\,\frac{\alpha_s}{2\pi}\left[\ln(2\varepsilon)\ln\delta + \frac{3}{4}\ln\delta + \text{ finite}\right]\right)$$

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Jet algorithms

typically iterative clustering algorithms:

1. start from n particles, for all pairs i and j calculate $(p_i + p_j)^2$

2. define a jet resolution parameter $y_{\rm cut}$, Q is max. available energy

   if $\rm min(p_i + p_j)^2 < \ y_{cut} \, Q^2$ then define a new pseudo-particle

   $p_J = p_i + p_j$, this decreases $n \to n - 1$

3. if n=1: stop, else repeat step 2.

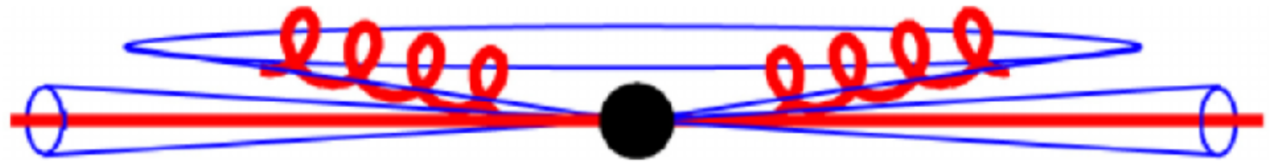
with this algorithm $\sigma^{2\,jet} = \sigma_0 \left(1 - C_F \frac{\alpha_s}{\pi} \left[\ln^2 y_{\rm cut} + \frac{3}{2} \ln y_{\rm cut} + \text{ finite} \right] \right)$

Jet algorithms

- in above jet algorithm example (JADE-algorithm):

cluster into one pseudo-particle if $$y_{ij,\mathrm{Jade}} = \frac{\min(\mathrm{p_i} + \mathrm{p_j})^2}{Q^2} < y_{\mathrm{cut}}$$

can lead to situations like

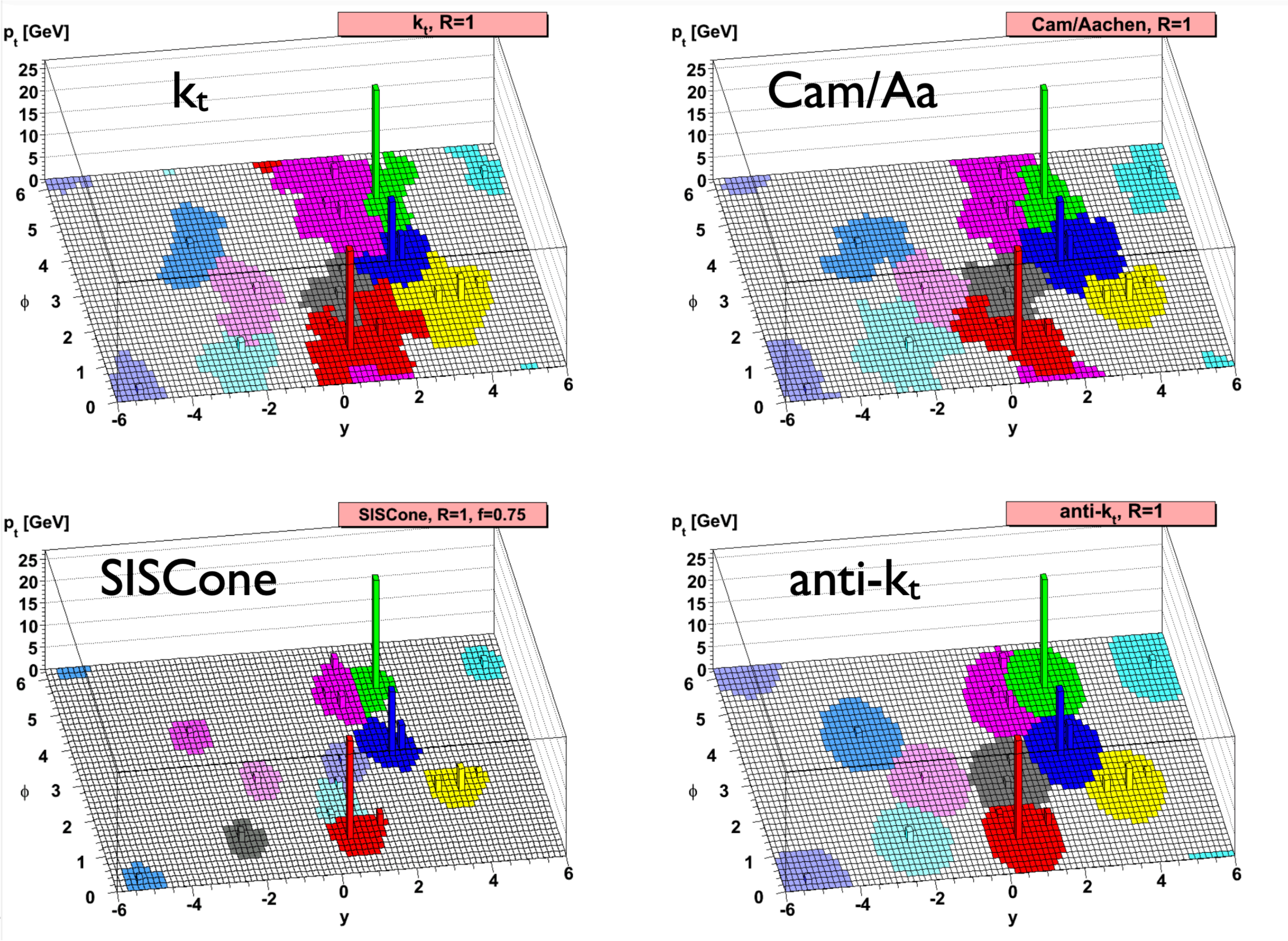


soft/collinear radiation should not change number of jets

- examples of better algorithms: (differ by distance measure)
  - Durham-kT: cluster if $$y_{ij,D} = \frac{2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})}{Q^2} < y_{\mathrm{cut}}$$

  - anti-kT: cluster if $$y_{ij,a} = \frac{1}{8} Q^2 \min\left(\frac{1}{E_i^2}, \frac{1}{E_j^2}\right)(1-\cos\theta_{ij}) < y_{\mathrm{cut}}$$

Jet algorithms



[Figure: Cacciari, Salam, Soyez]

distance measure including jet radius:

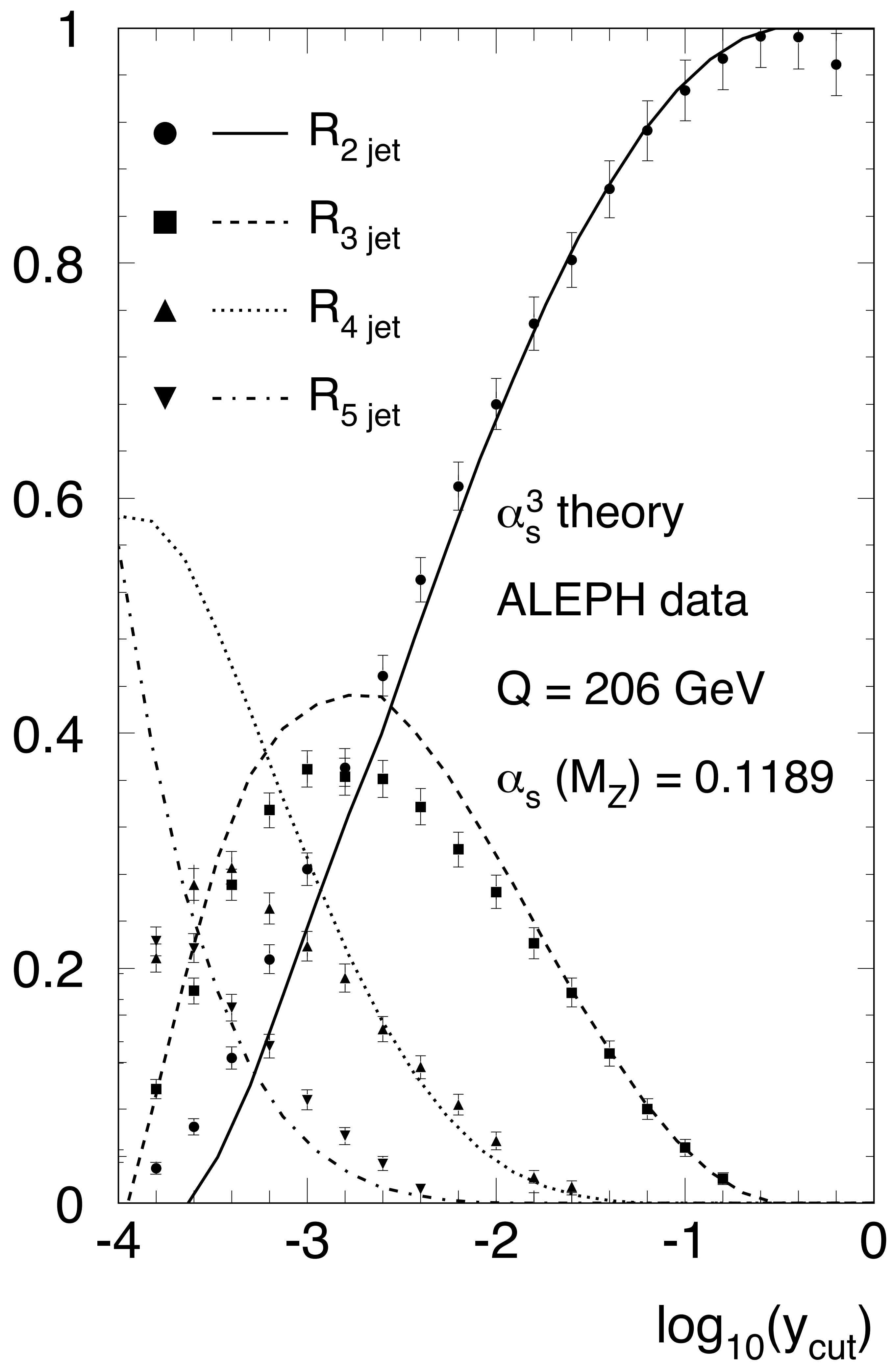
$$d_{ij} = \min(p_{ti}^{2p}, p_{tj}^{2p}) \frac{\Delta y^2 + \Delta\phi^2}{R^2}$$

$p = 1$ kt

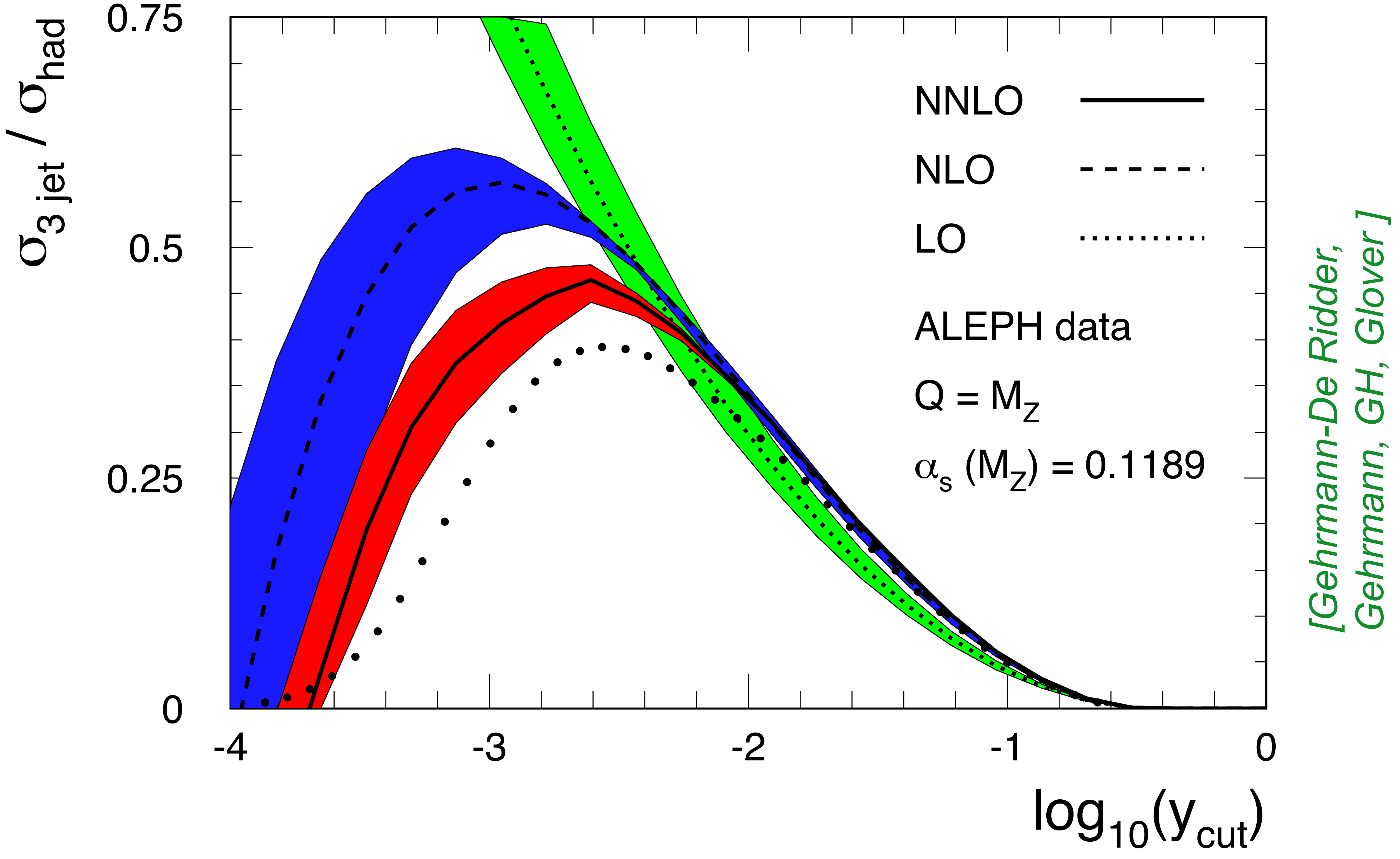
$p = 0$ Cam/Aa

$p = -1$ anti-kt

Jet rates



dependence of jet rates on ycut



3-jet rate at different orders

Event shape observables

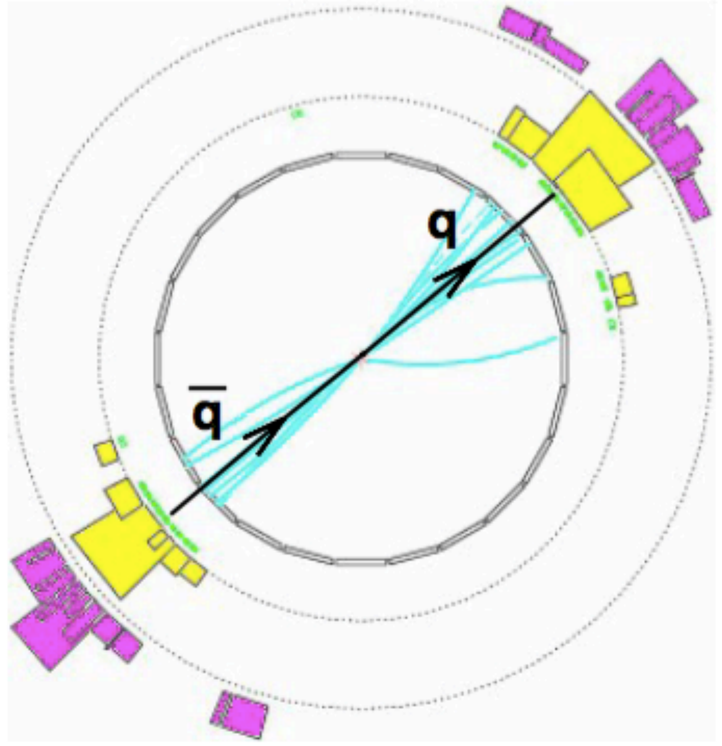
- observables which describe the topology of the final state
- particularly useful in e+e- collisions (Q^2 fixed)
- do not depend on jet algorithm/jet measurement
- examples:

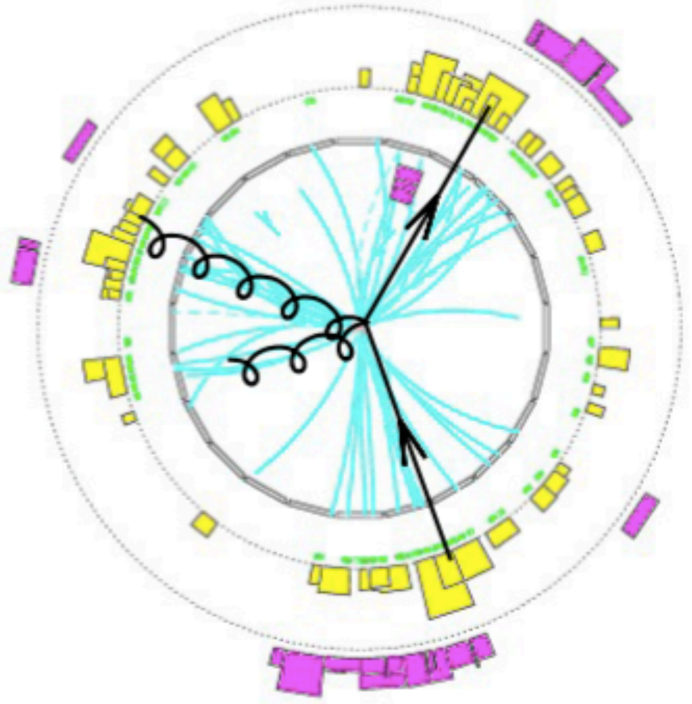
  thrust, C-parameter, jet broadening, heavy hemisphere mass, …

  for definitions see e.g. arXiv:0711.4711

thrust describes how “pencil-like” (2-jet-like) an event is

Thrust





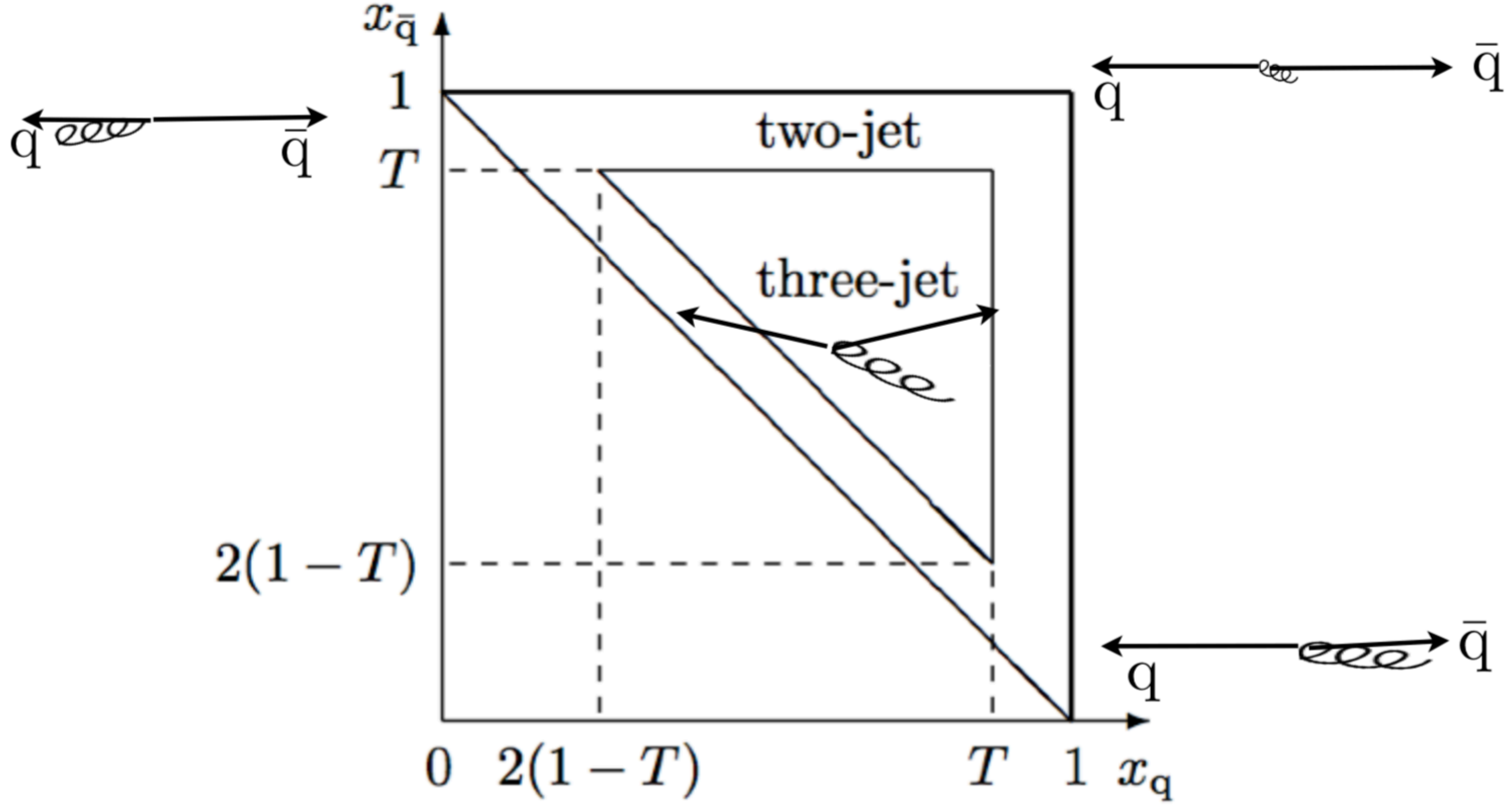
pencil-like → spherical

$T \to 1$ $\quad$ $T \to \frac{1}{2}$

$$T = \max_{\vec{n}} \frac{\sum_{i=1}^{m} |\vec{p}_i \cdot \vec{n}|}{\sum_{i=1}^{m} |\vec{p}_i|}$$

$\vec{n}$: thrust axis along which T is maximal

Thrust



[Figure: G. Dissertori]

$$\begin{aligned}|\overline{\mathcal{M}}_1|^2 &= |\overline{\mathcal{M}}_0|^2\,\frac{2g^2\,C_F}{s}\left(\frac{s_{13}}{s_{23}}+\frac{s_{23}}{s_{13}}+2s\,\frac{s_{12}}{s_{13}s_{23}}\right)\\ &= |\overline{\mathcal{M}}_0|^2\,\frac{2g^2\,C_F}{s}\left(\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}\right)\end{aligned}$$

$$x_1 \equiv x_q = \frac{2E_1}{\sqrt{s}}$$

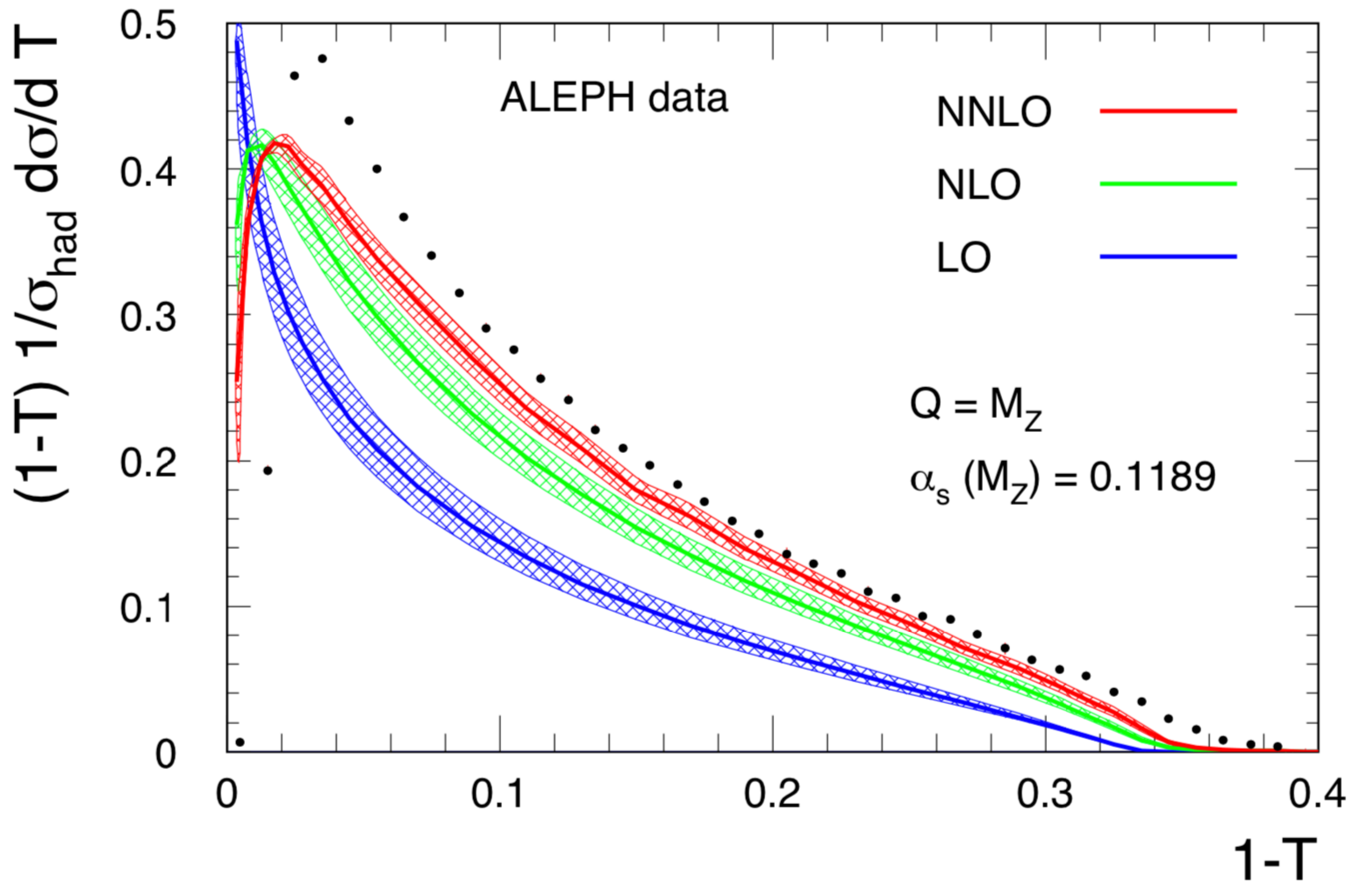
$$x_2 \equiv x_{\bar{q}} = \frac{2E_2}{\sqrt{s}}$$

Thrust

at LO: (with 3 partons)

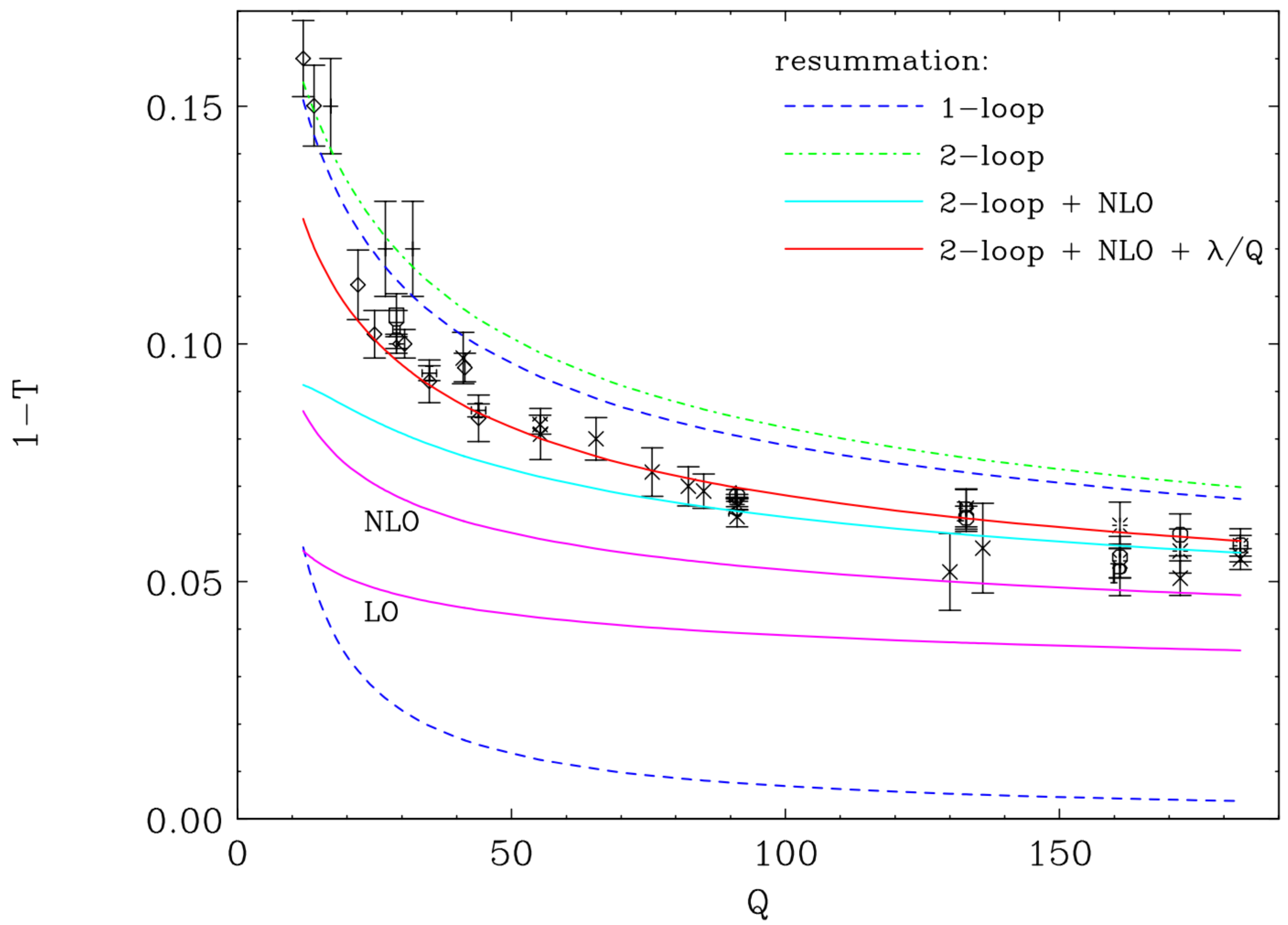
$$\frac{1}{\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}T} = C_\mathrm{F}\frac{\alpha_\mathrm{s}}{2\pi}\left[\frac{2\left(3T^2-3T+2\right)}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right) - 3(3T-2)\frac{2-T}{1-T}\right]$$

divergent for $T \to 1$



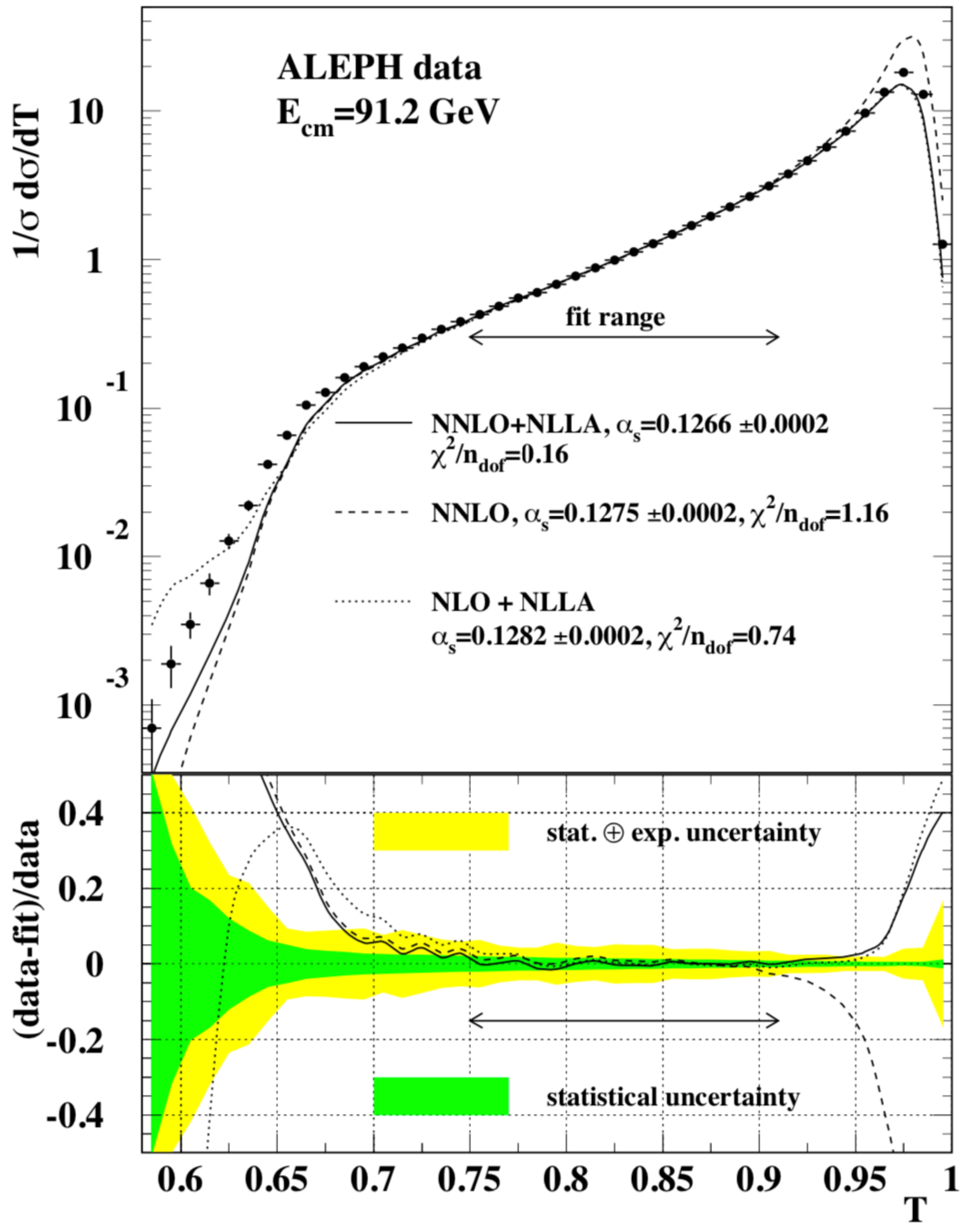
arXiv:0711.4711

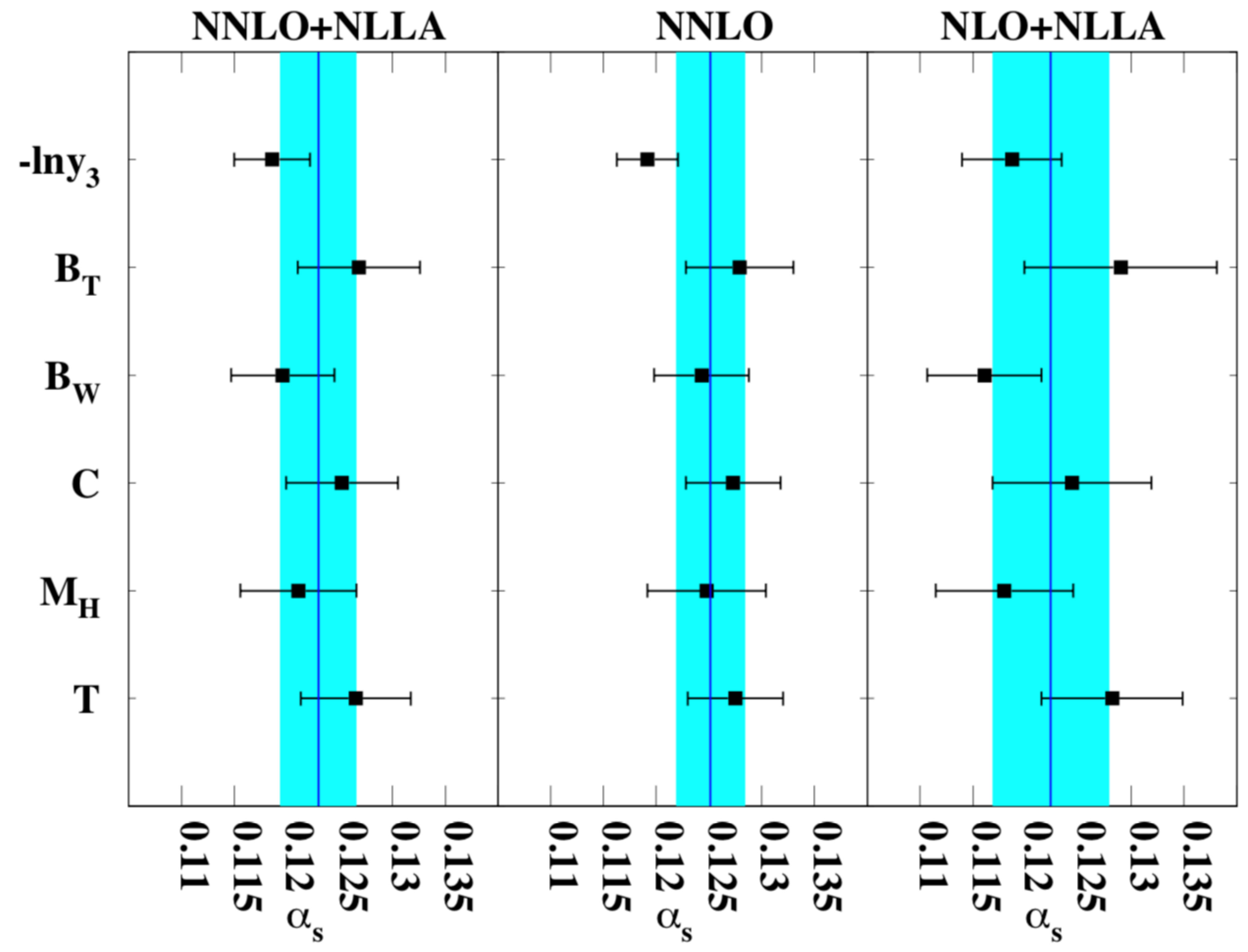
requires resummation of large logarithms of type $\alpha_s^n \frac{1}{1-T}\ln^{2n-1}\left(\frac{1}{1-T}\right)$



hep-ph/9908458

strong coupling from event shapes





arXiv:0910.4283

NLLA: “next-to-leading-log approximation”

(introduces additional scale dependence which is absent at fixed order NNLO)

Parton distribution functions

consider processes with one proton in initial state: DIS *(deeply inelastic scattering)*

$$e(k) + p(P) \to e(k') + X$$

kinematic variables:

$$s = (P+k)^2 \quad [\text{cms energy}]^2$$

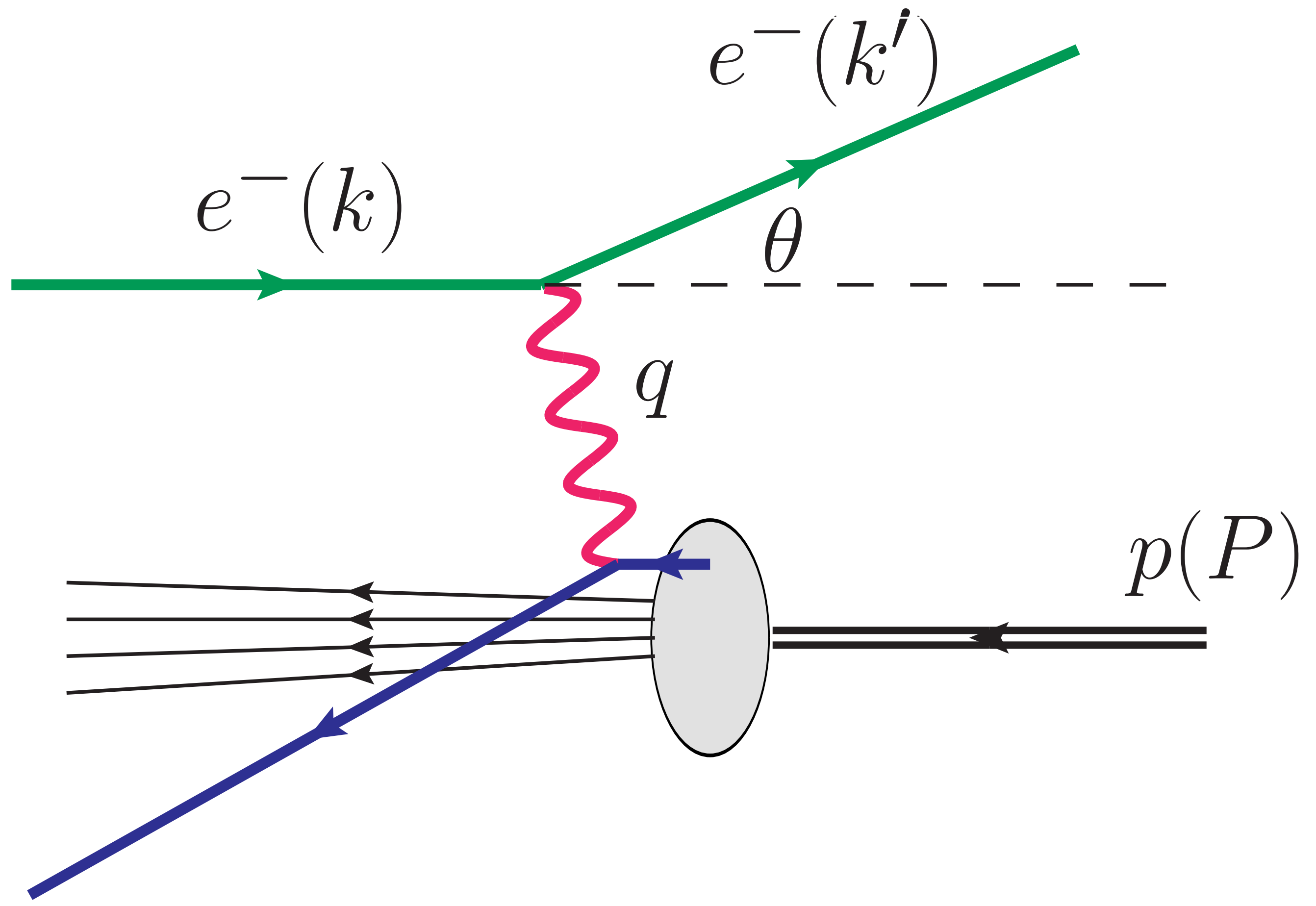
$$q^\mu = k^\mu - k'^\mu \quad [\text{momentum transfer}]$$

$$Q^2 = -q^2 = 2MExy$$

$$x = \frac{Q^2}{2P\cdot q} \quad [\text{scaling variable}]$$

$$\nu = \frac{P\cdot q}{M} = E - E' \quad [\text{energy loss}]$$

$$y = \frac{P\cdot q}{P\cdot k} = 1 - \frac{E'}{E} \quad [\text{relative energy loss}]$$



$$Q^2 = -q^2 > 1\,\text{GeV}$$

DIS cross section

cross section $$d\sigma = \sum_X \frac{1}{4ME} \int d\Phi \, \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2$$

split into leptonic and hadronic part

$$d\Phi = \frac{d^3k'}{(2\pi)^3 2E'} \, d\Phi_X \;, \quad \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{Q^4} L^{\mu\nu} H_{\mu\nu}$$

leptonic hadronic

define $$W_{\mu\nu} = \frac{1}{8\pi} \sum_X \int d\Phi_X H_{\mu\nu}$$

$W_{\mu\nu}$ can only depend on P^μ, q^μ

Ansatz: $$W_{\mu\nu}(P,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x,Q^2) + \left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right) \frac{W_2(x,Q^2)}{P\cdot q}$$

DIS structure functions

leads to $$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{y\,Q^2}\left[y^2 W_1(x,Q^2) + \left(\frac{1-y}{x} - xy\frac{M^2}{Q^2}\right)W_2(x,Q^2)\right]$$

$W_i(x,Q^2)$ dimensionless functions of scaling variable x and momentum transfer

scaling limit: $Q^2 \to \infty$, x fixed

then $M^2/Q^2 \to 0$, rename $W_1 \to -F_1$, $W_2 \to F_2$

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{y\,Q^2}\left[\left(1+(1-y)^2\right)F_1 + \frac{1-y}{x}\left(F_2 - 2xF_1\right)\right]$$

F_1, F_2 : structure functions

DIS structure functions

in the scaling limit:

$$F_2(x,Q^2) \to F_2(x) \quad \text{independent of } \; Q^2$$

$$F_2(x) = 2x\, F_1(x) \qquad \text{Callan-Gross relation}$$

characteristic for scattering at point-like spin-1/2 particles

observation: scaling violations at small x

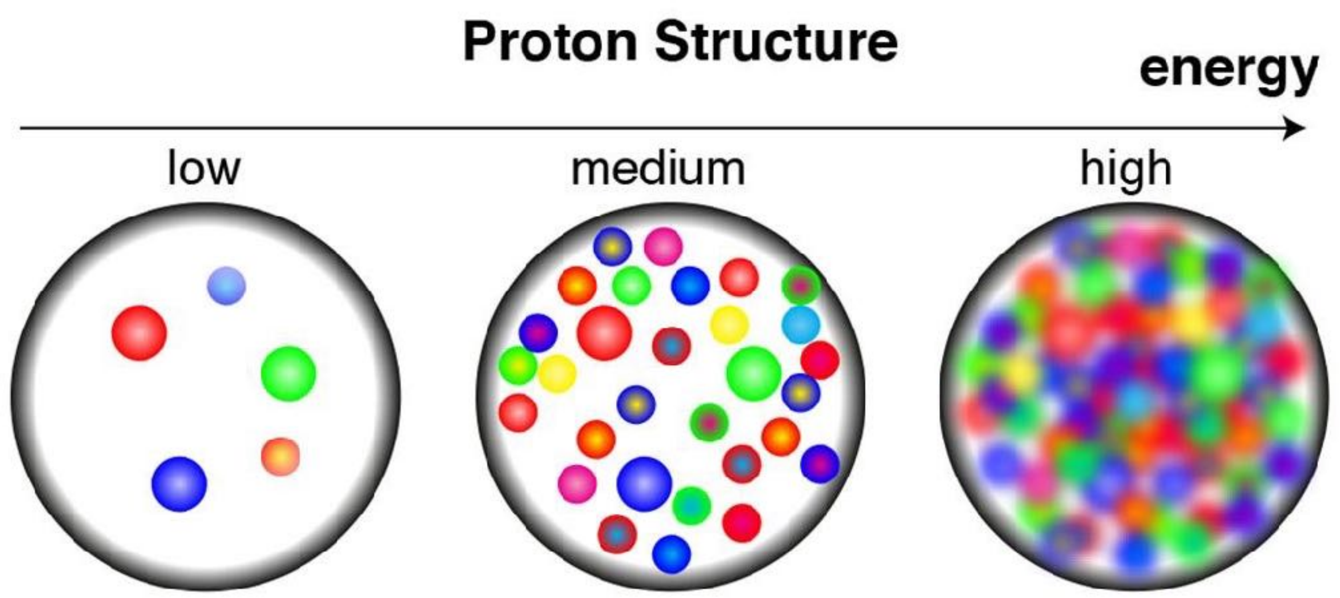
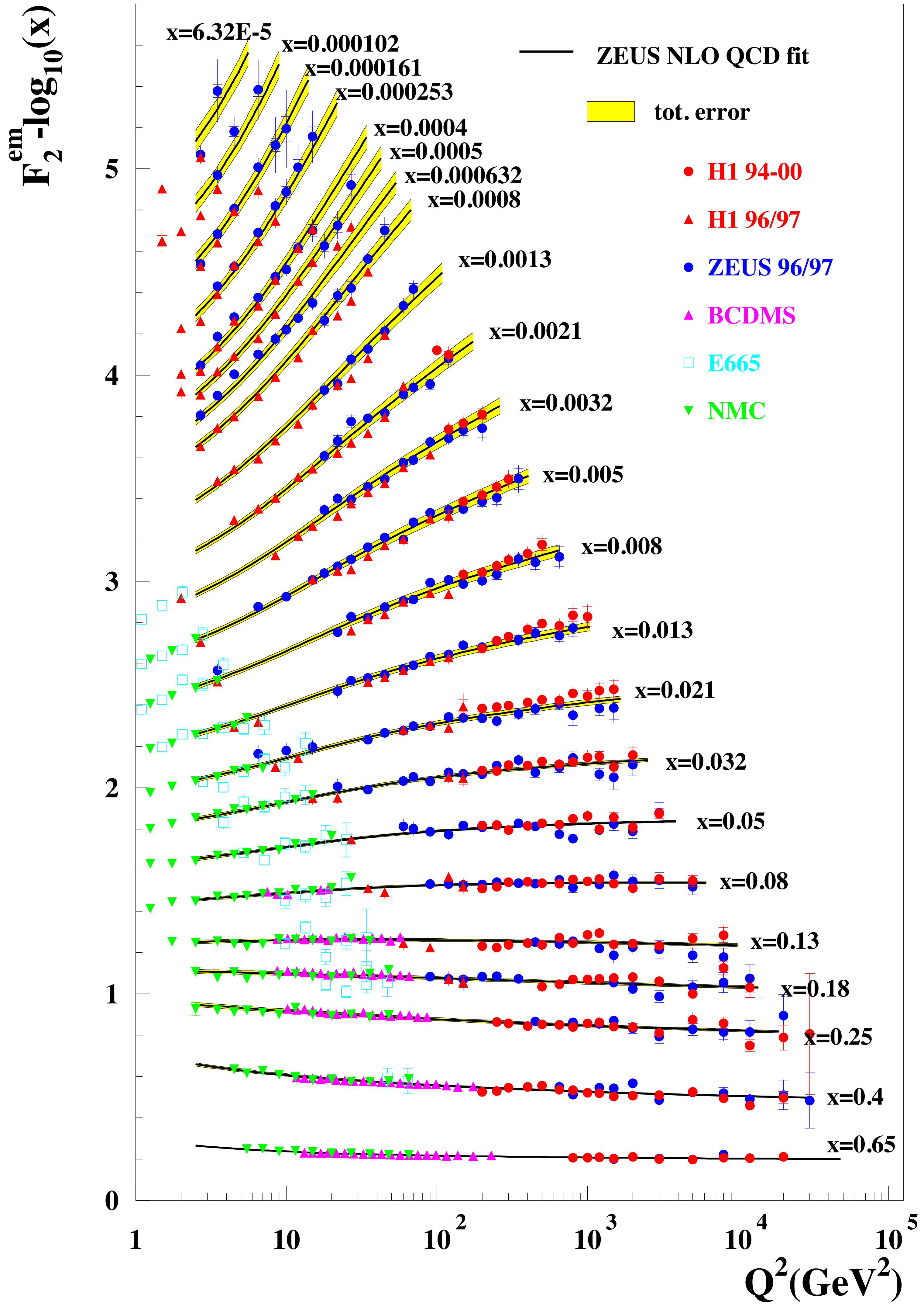


image source: Utrecht University



hep-ex/0212008

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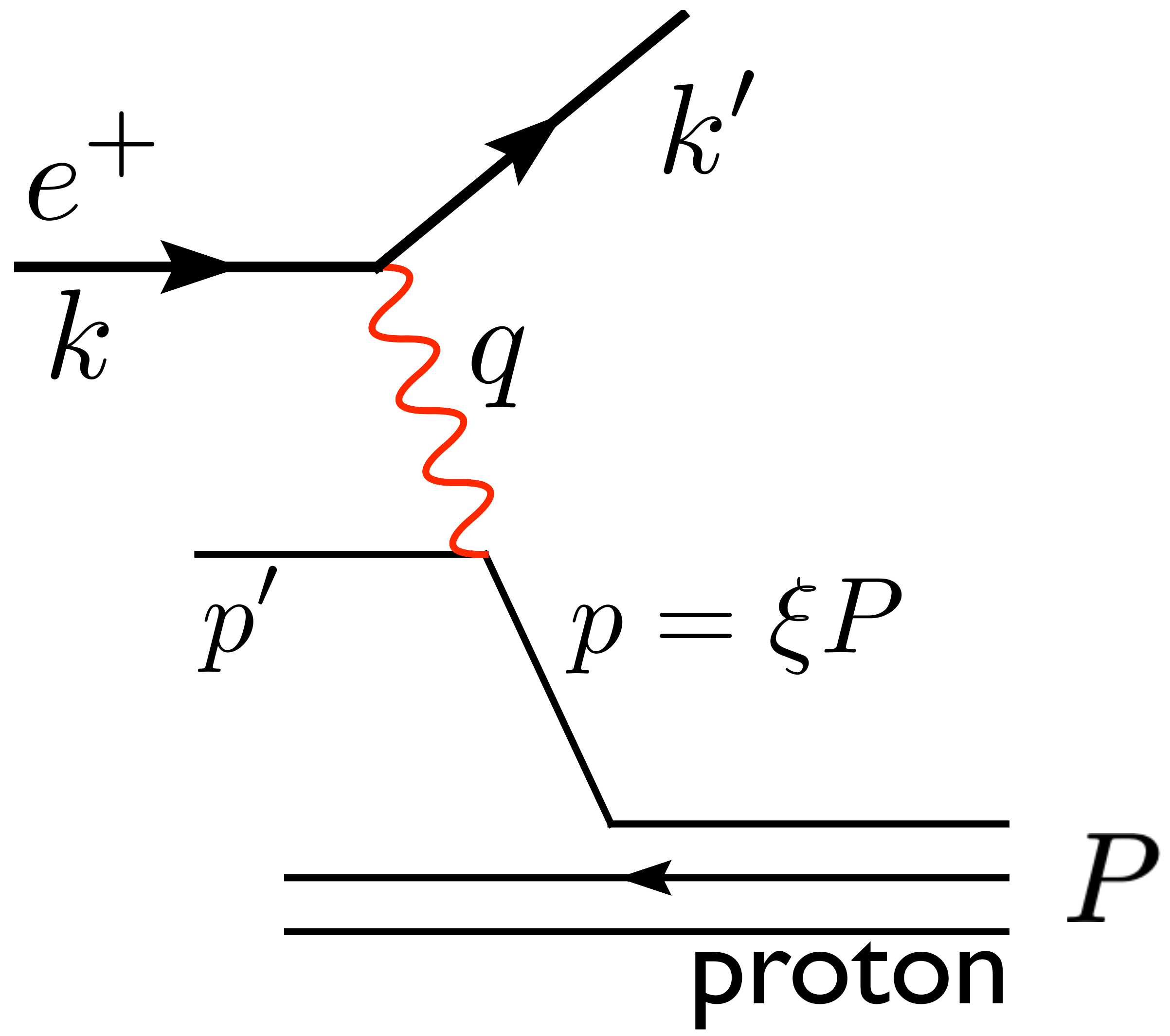
proton structure in the parton model

parton model picture: photon scatters off one quark

quark carries momentum fraction ξ of proton momentum, $p^\mu = \xi P^\mu$

$$e(k) + q_f(p) \to e(k') + q_f(p')$$



for elastic photon-quark scattering:

$$p^2 = p'^{\,2} = (p+q)^2 = p^2 + 2\xi\, P\cdot q - Q^2$$

$$\Rightarrow Q^2 = 2\xi\, P\cdot q \quad \Rightarrow \xi = \frac{Q^2}{2\,P\cdot q} = x$$

partonic cross section:

$$\hat\sigma = \frac{1}{2\hat s}\int d\Phi_2\, \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

$$\boxed{\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e_q^2 e^4}{Q^4} L^{\mu\nu} Q_{\mu\nu}}$$

$$\hat s = (p+k)^2$$
$$Q^2 = -q^2$$

proton structure in the parton model

$$Q_{\mu\nu} = \frac{1}{2} Tr[\not{p}\gamma^{\mu}\ \ \not{p}'\gamma^{\nu}] = p^{\mu}p'^{\nu} + p^{\nu}p'^{\mu} - g^{\mu\nu}p\cdot p'$$

$$\Rightarrow L^{\mu\nu}Q_{\mu\nu} = 2(\hat{s}^2 + \hat{u}^2)$$

$$\hat{u} = (p-k')^2 = -2p\cdot k' \qquad \hat{s} = (p+k)^2$$

$$y = Q^2/\hat{s} \;\; \Rightarrow \;\hat{u}^2 = (1-y)^2\hat{s}^2$$

$$\Rightarrow \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e_q^2 e^4}{Q^4}L^{\mu\nu}Q_{\mu\nu} = 2e_q^2e^4\frac{\hat{s}^2}{Q^4}\left(1+(1-y)^2\right)$$

phase space: $$d\Phi_2 = \frac{d^3k'}{(2\pi)^3 2E'}\,\frac{d^4p'}{(2\pi)^3}\,\delta\left(p'^{\,2}\right)\,(2\pi)^4\,\delta^{(4)}(k+p-k'-p')$$

$$= \frac{d\phi}{2\pi}\,\frac{E'}{4\pi}\,dE'\,d\cos\theta\,\frac{x}{Q^2}\,\delta(\xi - x) \quad = \frac{d\phi}{(4\pi)^2}\,dy\,dx\,\delta(\xi-x)$$

proton structure in the parton model

$$\Rightarrow \quad \frac{d^2\hat{\sigma}}{dx\,dy} = \frac{4\pi\alpha^2}{yQ^2}\left[1+(1-y)^2\right]\frac{1}{2}e_q^2\delta(\xi - x)$$

comparison with expression for structure functions

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\alpha^2}{y\,Q^2}\left[\left(1+(1-y)^2\right)F_1 + \frac{1-y}{x}\left(F_2 - 2xF_1\right)\right]$$

leads to: $\hat{F}_1(x) \sim e_q^2\delta(\xi - x)$, $F_2 - 2xF_1 = 0$

Callan-Gross relation
derived from first principles!

interpretation: a quark constituent of the proton with momentum fraction $\xi = x$ takes part in the hard scattering

parton distribution functions

we can infer $F_2(x) = \sum_i \int_0^1 d\xi\, f_i(\xi)\, x\, e_{q_i}^2\, \delta(x-\xi) = x \sum_i e_{q_i}^2\, f_i(x)$

$f_i(\xi)$ denotes the probability that a parton ($q, \bar{q}, g$) with flavour *i*

carries a momentum fraction of the proton between ξ and $\xi + d\xi$

$f_i(\xi)$: **parton distribution functions (PDFs)**

PDFs are fitted from data, but their energy scale dependence is calculable in perturbation theory

parton distribution functions

hadronic cross section:

$$\boxed{\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} = \int_x^1 \frac{\mathrm{d}\xi}{\xi} \sum_i f_i(\xi) \frac{\mathrm{d}^2\hat{\sigma}}{\mathrm{d}x\,\mathrm{d}Q^2}\left(\frac{x}{\xi}, Q^2\right)}$$

factorisation into a *convolution* of partonic cross section and PDFs

def. convolution:

$$f \otimes_x g \equiv \int_x^1 \frac{d\xi}{\xi} f(\xi)\, g\left(\frac{x}{\xi}\right)$$

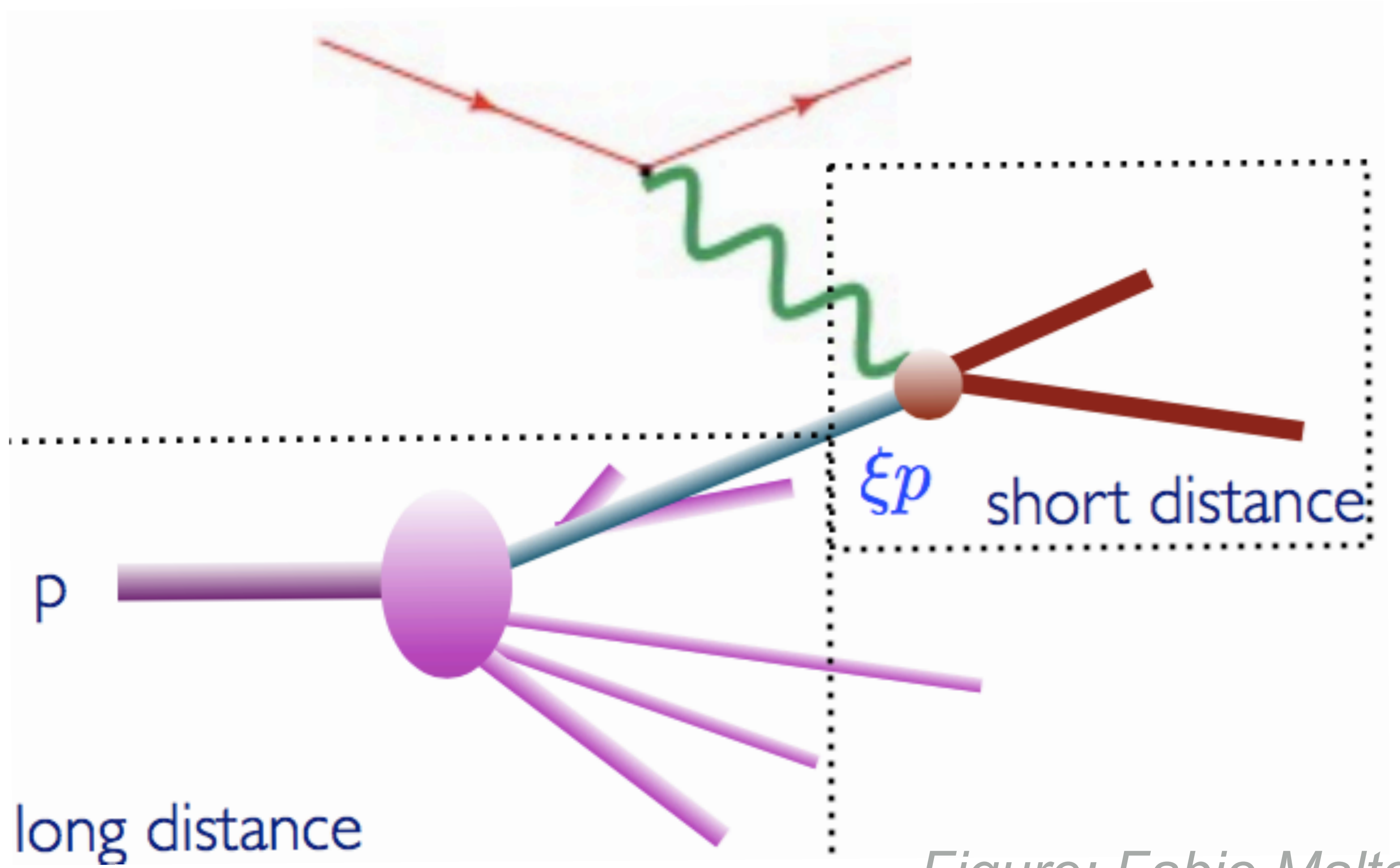


Figure: Fabio Maltoni

parton distribution functions

in the naive parton model:

$$F_2(x) = 2xF_1(x) = \sum_i \int_0^1 d\xi\, f_i(\xi)\, x\, e_{q_i}^2\, \delta(x-\xi) = x \sum_i e_{q_i}^2\, f_i(x)$$

$$\Rightarrow \quad F_2^{\text{proton}}(x) = x\left[\frac{4}{9}\left(u(x)+\bar{u}(x)\right) + \frac{1}{9}\left(d(x)+\bar{d}(x)\right)\right]$$

define valence and sea quarks by

$$u(x) = \underset{\substack{\uparrow \\ \text{valence quark}}}{u_v(x)} + \underset{\substack{\uparrow \\ \text{sea quark}}}{\bar{u}(x)}\ ,\ \ d(x) = d_v(x) + \bar{d}(x)$$

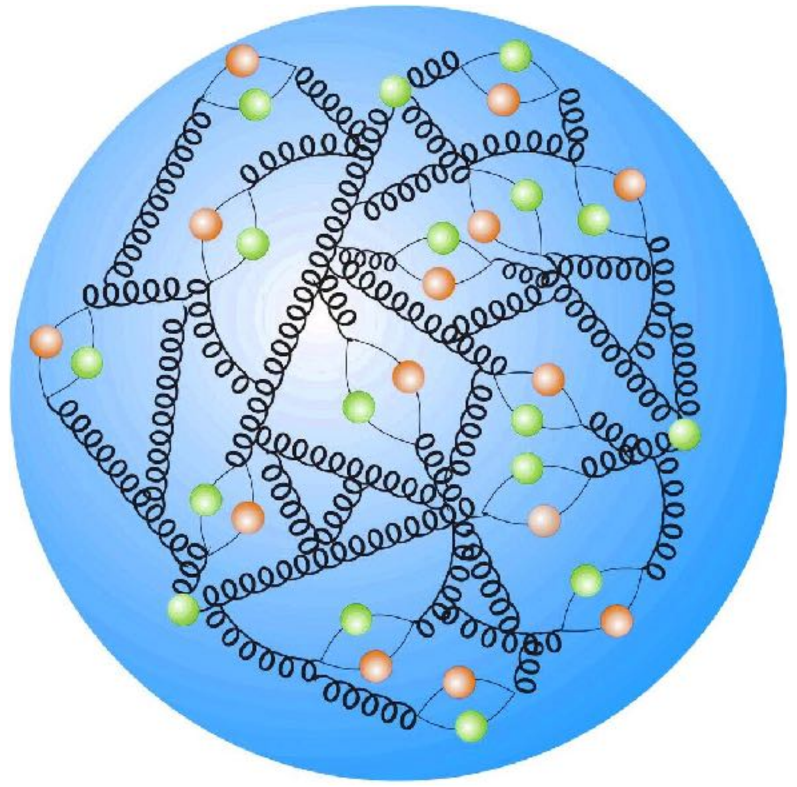
valence quarks define the quantum numbers of the nucleon e.g. proton: uud, charge=+1

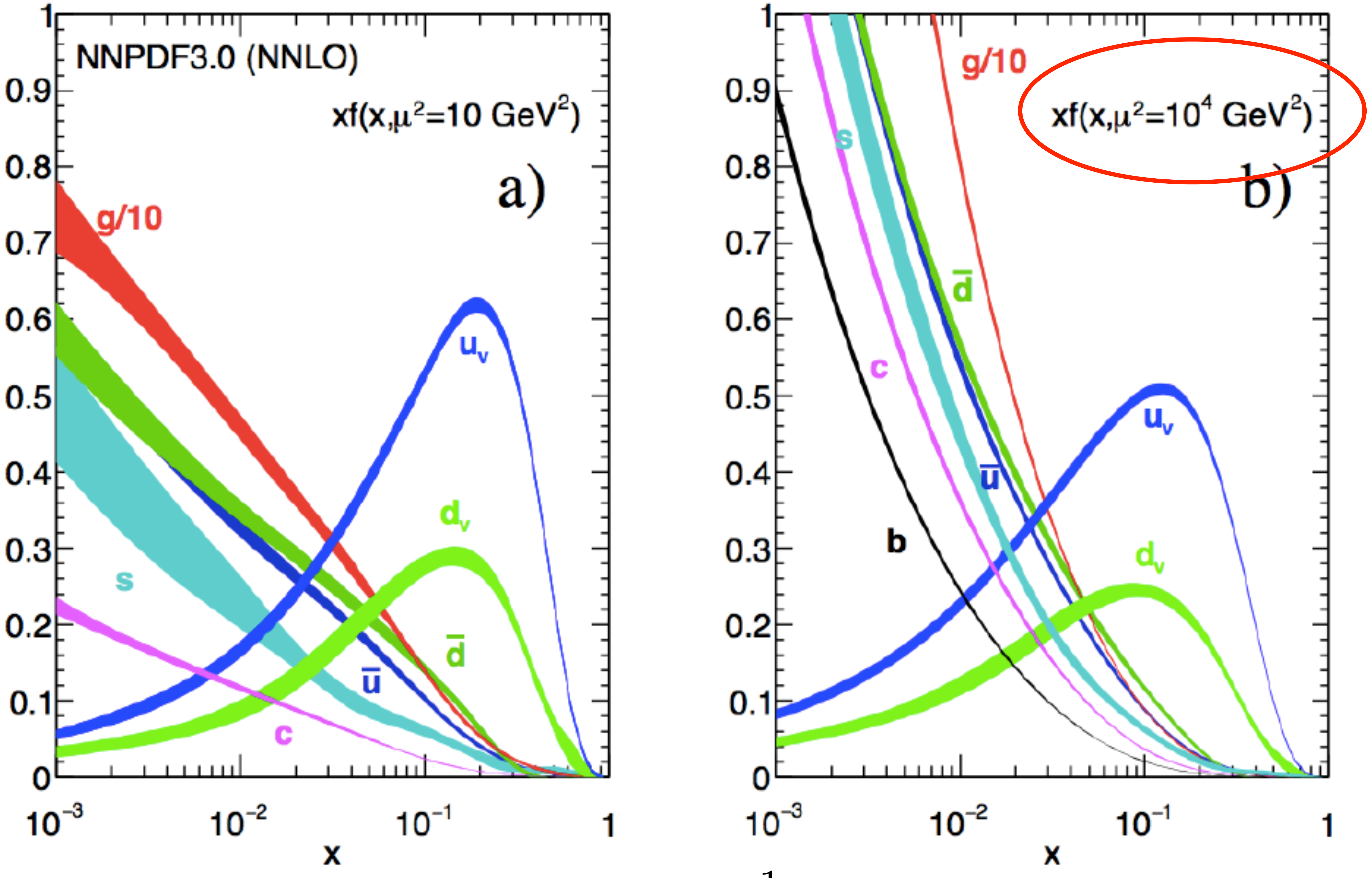
"sum rules": $\int_0^1 dx\, u_v(x) = 2\ ,\ \int_0^1 dx\, d_v(x) = 1\ ,\ \int_0^1 dx\, (s(x) - \bar{s}(x)) = 0$

parton distribution functions (PDFs)

sea quarks and gluons are more important at

- small x
- large Q^2





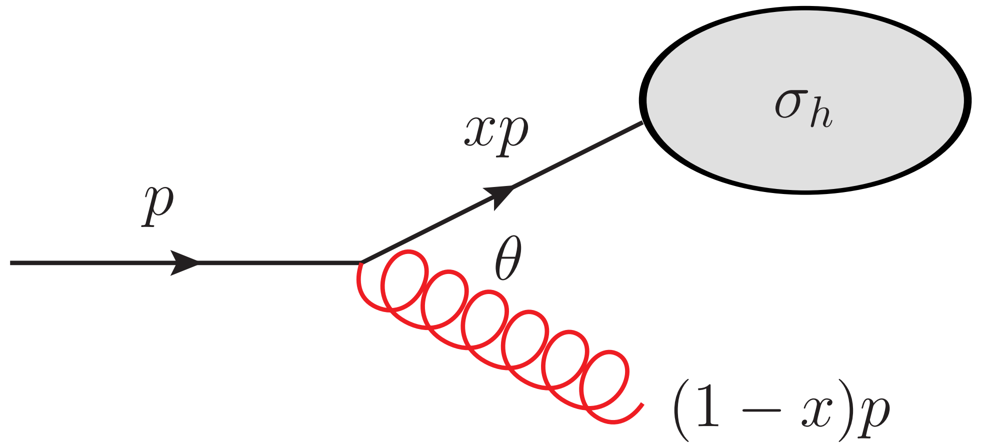
note: quarks carry only about 50% of the proton momentum

$$\sum_i \int_0^1 dx\, x\, [q_i(x) + \bar{q}_i(x)] \simeq 0.5$$

the rest is carried by gluons

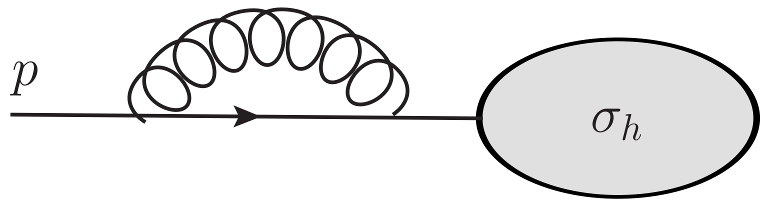
back to IR singularities

consider parton splitting in the **initial** state:



$$\sigma_{h+g}(p) \simeq \sigma_h(xp)\, 2C_F \frac{\alpha_s}{\pi}\frac{\mathrm{d}E}{E}\frac{\mathrm{d}\theta}{\theta} \to \sigma_h(xp)\, C_F \frac{\alpha_s}{\pi}\,\mathrm{d}x\,(1-x)^{-1-\epsilon}\,\mathrm{d}k_\perp^2\,(k_\perp^2)^{-1-\epsilon}$$

virtual corrections:



$$\sigma_{h+V} \simeq -\sigma_h(p)\, C_F \frac{\alpha_s}{\pi}\,\mathrm{d}x\,(1-x)^{-1-\epsilon}\,\mathrm{d}k_\perp^2\,(k_\perp^2)^{-1-\epsilon}$$

cancellation for $x \to 1$ (soft limit) but what about there collinear limit?

initial state singularities

$$\sigma_{h+g} + \sigma_{h+V} \simeq C_F \frac{\alpha_s}{\pi} \int_0^{Q^2} \underbrace{\mathrm{d}k_\perp^2\,(k_\perp^2)}_{\text{uncanceled collinear singularity}}{}^{-1-\epsilon} \mathrm{d}x\, \underbrace{(1-x)^{-1-\epsilon}[\sigma_h(xp) - \sigma_h(p)]}_{\text{finite}}$$

however behaviour is **universal**, does not depend on details of σ_h

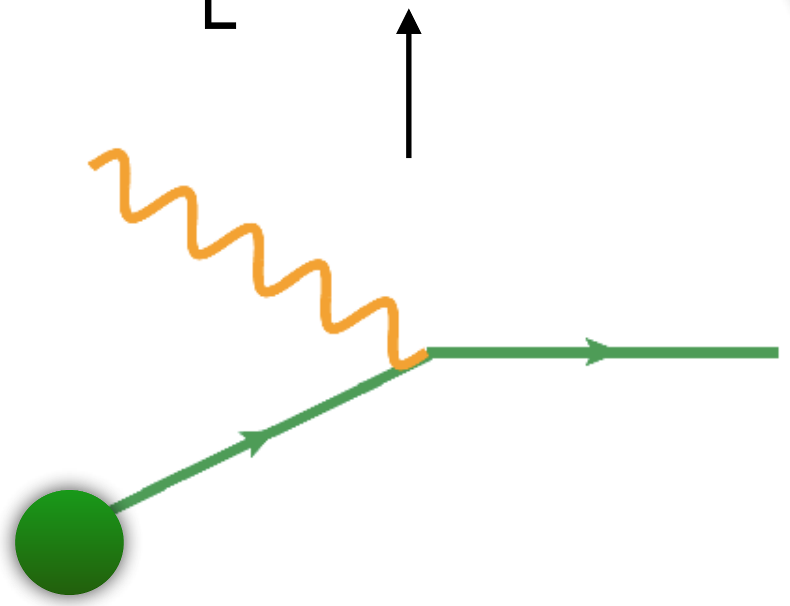
procedure: absorb singularities into “bare” parton densities at some scale μ_f

(similar to renormalisation)

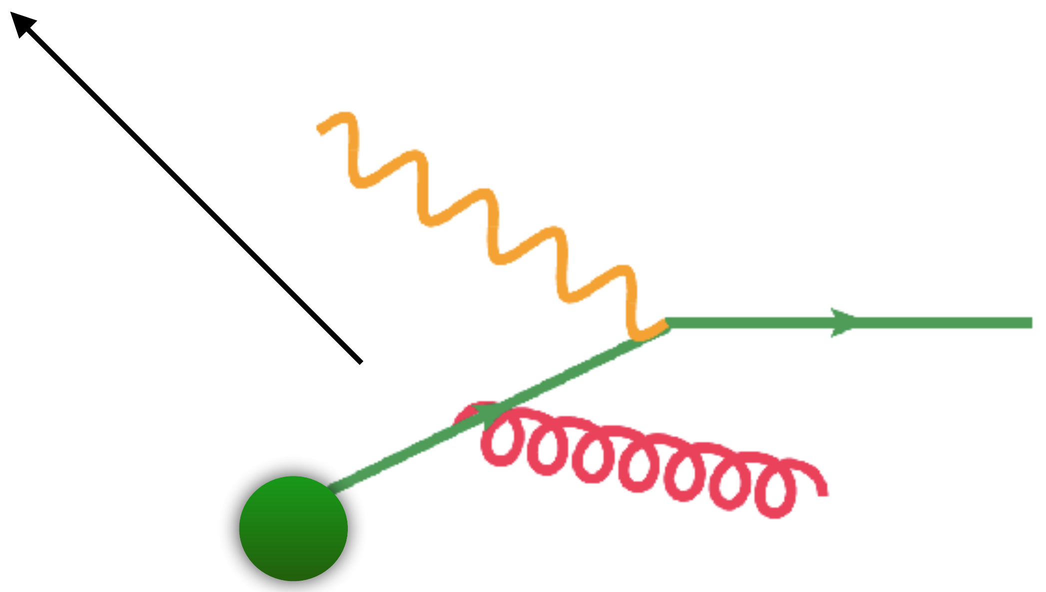
μ_f is called factorisation scale

example structure functions

$$\hat{F}_{2,q}(x) = e_q^2 x \left[\delta(1-x) + \boxed{\frac{\alpha_s}{4\pi}}\left(-\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\frac{1}{\epsilon}P_{q\to qg}(x) + C_2^q(x)\right)\right]$$

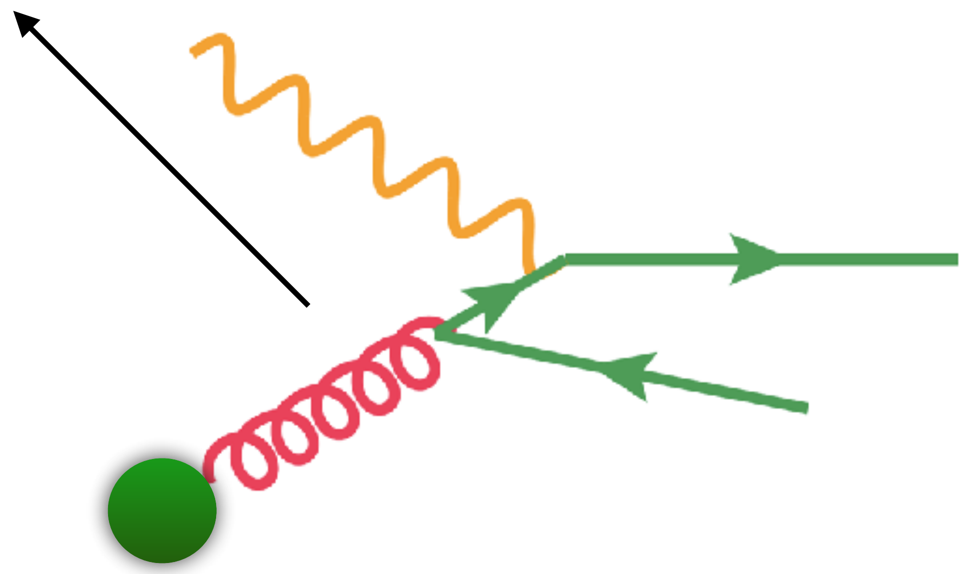


parton model



gluon emission

$$\hat{F}_{2,g}(x) = \sum_q e_q^2 x \left[0 + \boxed{\frac{\alpha_s}{4\pi}}\left(-\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\frac{1}{\epsilon}P_{g\to q\bar{q}}(x) + C_2^g(x)\right)\right]$$



splitting of a gluon into a quark-antiquark pair

initial state singularities

from parton level to hadron level:

$$F_{2,q}(x,Q^2) = x\sum_i e_{q_i}^2 \left[f_i^{(0)}(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi} f_i^{(0)}(\xi)\left(-\left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\frac{1}{\epsilon}P_{q\to qg}\left(\frac{x}{\xi}\right) + C_2^q\left(\frac{x}{\xi}\right)\right)\right]$$

bare parton distribution functions

define physical PDF by

$$\boxed{f_i(x,\mu_f^2) = f_i^{(0)}(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{\mathrm{d}\xi}{\xi}\left\{ f_i^{(0)}(\xi)\left[-\frac{1}{\epsilon}\left(\frac{\mu_f^2}{\mu^2}\right)^{-\epsilon} P_{q\to qg}\left(\frac{x}{\xi}\right) + K_{qq}\right]\right\}}$$

replace $f_i^{(0)}(x)$, expand to order α_s

$$F_{2,q}(x,Q^2) = x\sum_i e_{q_i}^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_f^2)\left\{\delta(1-\frac{x}{\xi}) + \frac{\alpha_s(\mu_r)}{2\pi}\left[P_{q\to qg}\left(\frac{x}{\xi}\right)\boxed{\ln\frac{Q^2}{\mu_f^2}} + (C_2^q - K_{qq})\right]\right\}$$

$$= x\sum_i e_{q_i}^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_f^2)\,\hat{F}_{2,i}(\frac{x}{\xi},Q^2,\mu_r,\mu_f)$$

now dependence on μ_f

factorisation of initial state singularities

$$F_{2,q}(x,Q^2) = x\sum_i e_{q_i}^2 \int_x^1 \frac{d\xi}{\xi} f_i(\xi,\mu_f^2)\,\hat{F}_{2,i}(\frac{x}{\xi},Q^2,\mu_r,\mu_f)$$

use convolution $f \otimes_x g \equiv \int_x^1 \frac{d\xi}{\xi} f(\xi)\, g\left(\frac{x}{\xi}\right)$

$$F_{2,q}(x,Q^2) = x\sum_i e_{q_i}^2\, f_i(\mu_f) \otimes_x \hat{F}_{2,i}(\mu_r,t) \qquad t = \ln\frac{Q^2}{\mu_f^2}$$

left side is a physical quantity, should not depend on μ_f

$\Rightarrow$ we can derive a “renormalisation group equation” which determines how the PDFs evolve with the energy scale

PDF evolution

this is best done in **Mellin space** (convolution turns into simple product)

Mellin transform: $f(N) \equiv \int_0^1 dx\, x^{N-1} f(x) \qquad (N \geq 1)$

factorisation equation in Mellin space:

$$F_{2,q}(N,Q^2) = x \sum_i e_{q_i}^2\, f_i(N,\mu_f^2)\, \hat{F}_{2,i}(N,\mu_r,t)$$

we must have $\boxed{\frac{\mathrm{d}}{\mathrm{d}\mu_f} F_{2,q}(N,Q^2) = 0}$

consider just one quark flavour $i \to q$

$$\Rightarrow \quad \hat{F}_{2,q}(N,t) \frac{df_q(N,\mu_f^2)}{d\mu_f^2} + f_q(N,\mu_f^2) \frac{d\hat{F}_{2,q}(N,t)}{d\mu_f^2} = 0$$

PDF evolution

$$\hat{F}_{2,q}(N,t)\,\frac{df_q(N,\mu_f^2)}{d\mu_f^2}+f_q(N,\mu_f^2)\,\frac{d\hat{F}_{2,q}(N,t)}{d\mu_f^2}=0$$

divide by $f_q\,\hat{F}_{2,q}$

$$\mu_f^2\,\frac{d\ln f_q(N,\mu_f^2)}{d\mu_f^2}=-\mu_f^2\,\frac{d\ln\hat{F}_{2,q}(N,t)}{d\mu_f^2}\equiv\gamma_{qq}(N)\;,\qquad t=\ln\left(Q^2/\mu_f^2\right)$$

$$\boxed{t\,\frac{df_q(N,t)}{dt}=\underset{\uparrow}{\gamma_{qq}}\left(N,\alpha_s(t)\right)\,f_q(N,t)}$$

anomalous dimension

PDF evolution

back from Mellin space to x-space:

$$\boxed{t\frac{\partial}{\partial t} f_{q_i}(x,t) = \int_x^1 \frac{d\xi}{\xi} P_{q_i/q_j}\left(\frac{x}{\xi},\alpha_s(t)\right) f_{q_j}(\xi,t)}$$

DGLAP evolution equation

(Dokshitzer, Gribov, Lipatov, Altarelli, Parisi)

N3LO partially:

2308.07958, 2307.04158, 2302.07593, 2111.15561

$$P_{q_i/q_j}(x,\alpha_s) = \frac{\alpha_s}{2\pi} P_{ij}^{(0)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^2 P_{ij}^{(1)}(x) + \left(\frac{\alpha_s}{2\pi}\right)^3 P_{ij}^{(2)}(x) + \mathcal{O}(\alpha_s^4)$$

LO (1974) NLO (1980) NNLO (2004, Moch, Vermaseren Vogt)

the equation above holds for a single quark flavour or a non-singlet flavour combination

$$q_{\rm ns} = f_{q_i} - f_{q_j}$$

in general it is a matrix equation

DGLAP evolution

$$t\frac{\partial}{\partial t}\begin{pmatrix} f_{q_i}(x,t)\\ f_g(x,t)\end{pmatrix} = \sum_{q_j,\bar{q}_j}\int_x^1 \frac{d\xi}{\xi}\begin{pmatrix} P_{q_i/q_j}(\frac{x}{\xi},\alpha_s(t)) & P_{q_i/g}(\frac{x}{\xi},\alpha_s(t))\\ P_{g/q_j}(\frac{x}{\xi},\alpha_s(t)) & P_{g/g}(\frac{x}{\xi},\alpha_s(t))\end{pmatrix}\begin{pmatrix} f_{q_j}(\xi,t)\\ f_g(\xi,t)\end{pmatrix}$$

pictorially, to 1st order in α_s

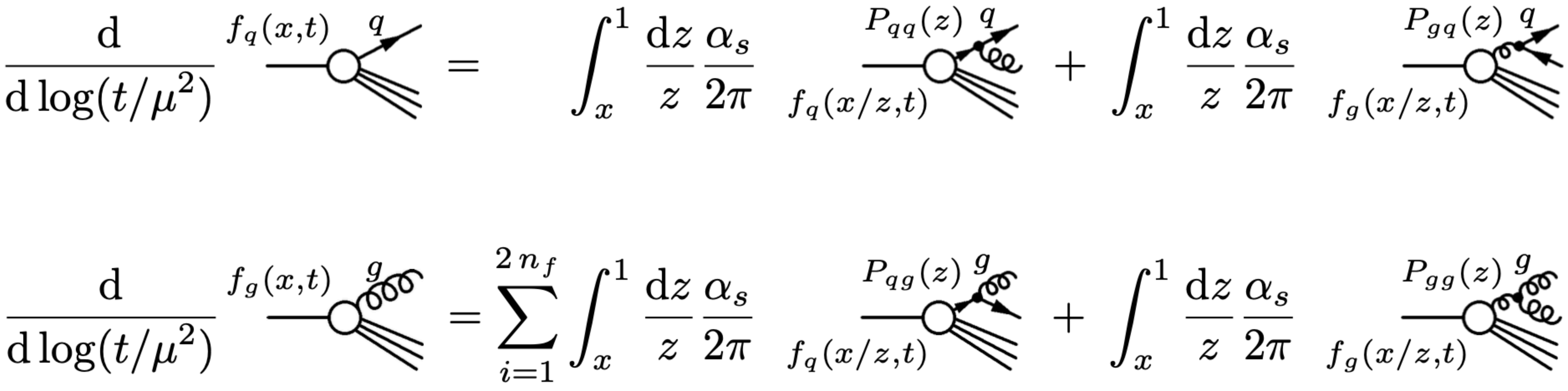
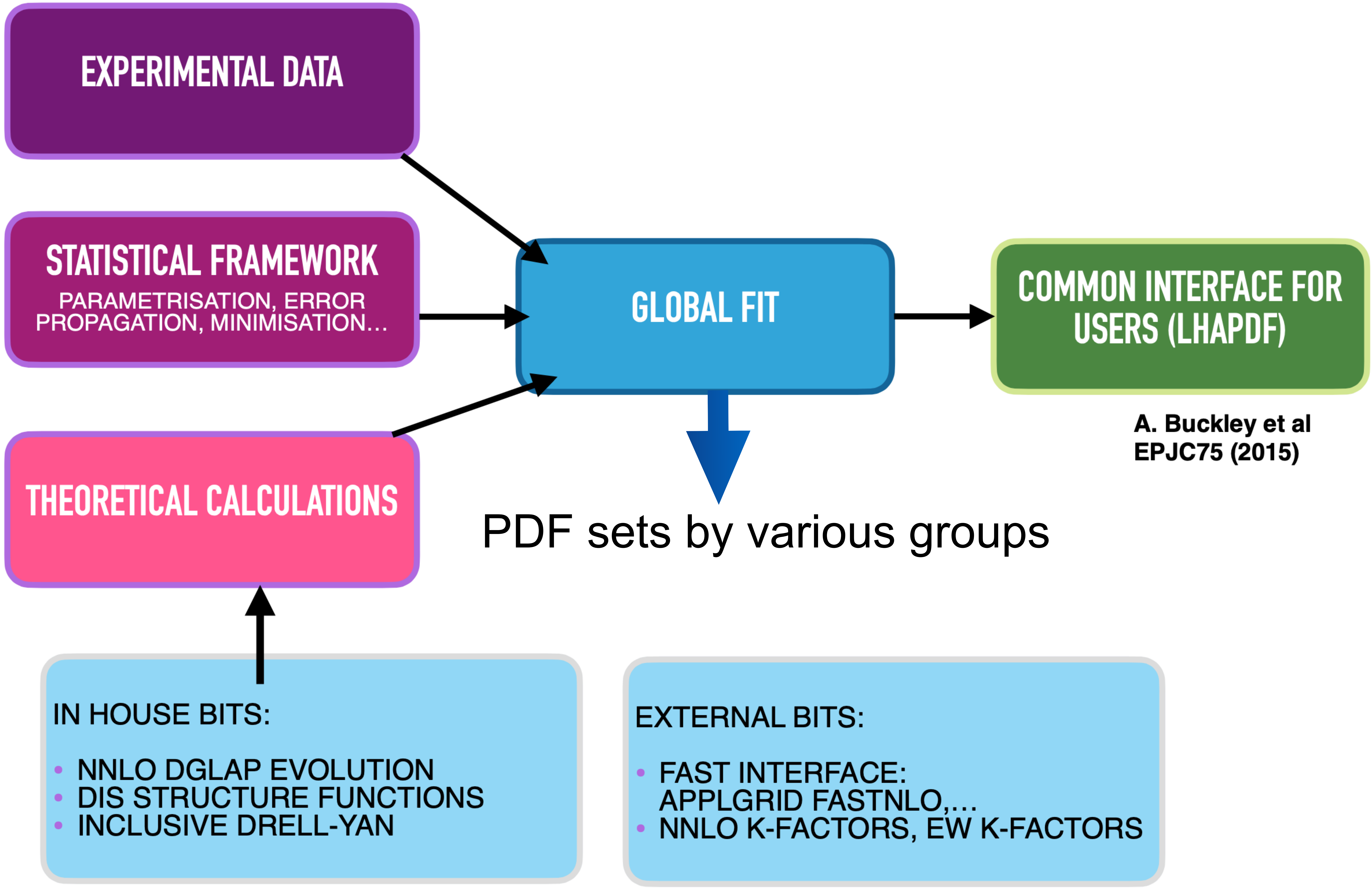


figure:
Stefan Höche
1411.4085

PDF fitting machinery



EXPERIMENTAL DATA

STATISTICAL FRAMEWORK

PARAMETRISATION, ERROR PROPAGATION, MINIMISATION…

THEORETICAL CALCULATIONS

GLOBAL FIT

COMMON INTERFACE FOR USERS (LHAPDF)

A. Buckley et al
EPJC75 (2015)

PDF sets by various groups

IN HOUSE BITS:

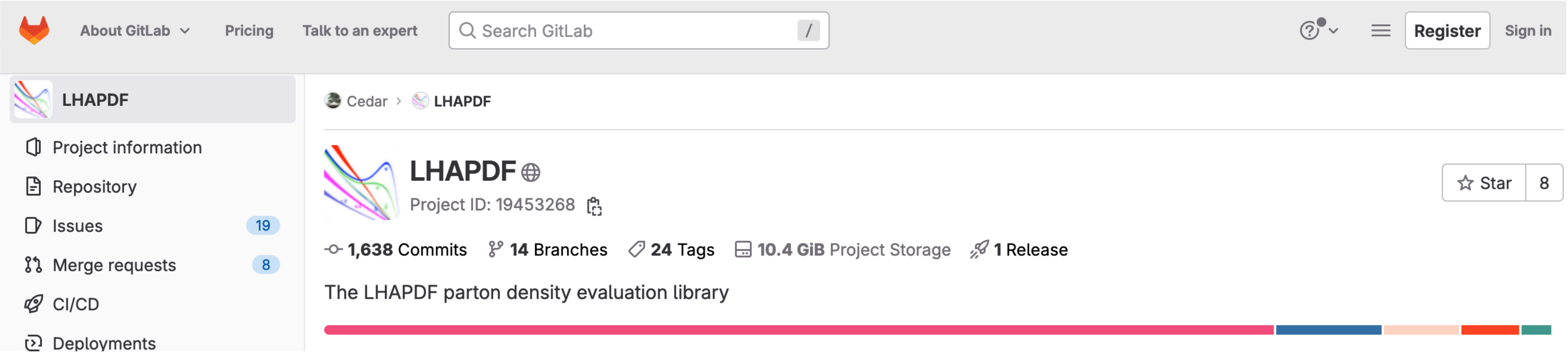
- NNLO DGLAP EVOLUTION
- DIS STRUCTURE FUNCTIONS
- INCLUSIVE DRELL-YAN

EXTERNAL BITS:

- FAST INTERFACE:
  APPLGRID FASTNLO,…
- NNLO K-FACTORS, EW K-FACTORS

extension of a figure by
Maria Ubiali (NNPDF coll.)

PDF sets



LHAPDF

Project ID: 19453268

1,638 Commits · 14 Branches · 24 Tags · 10.4 GiB Project Storage · 1 Release

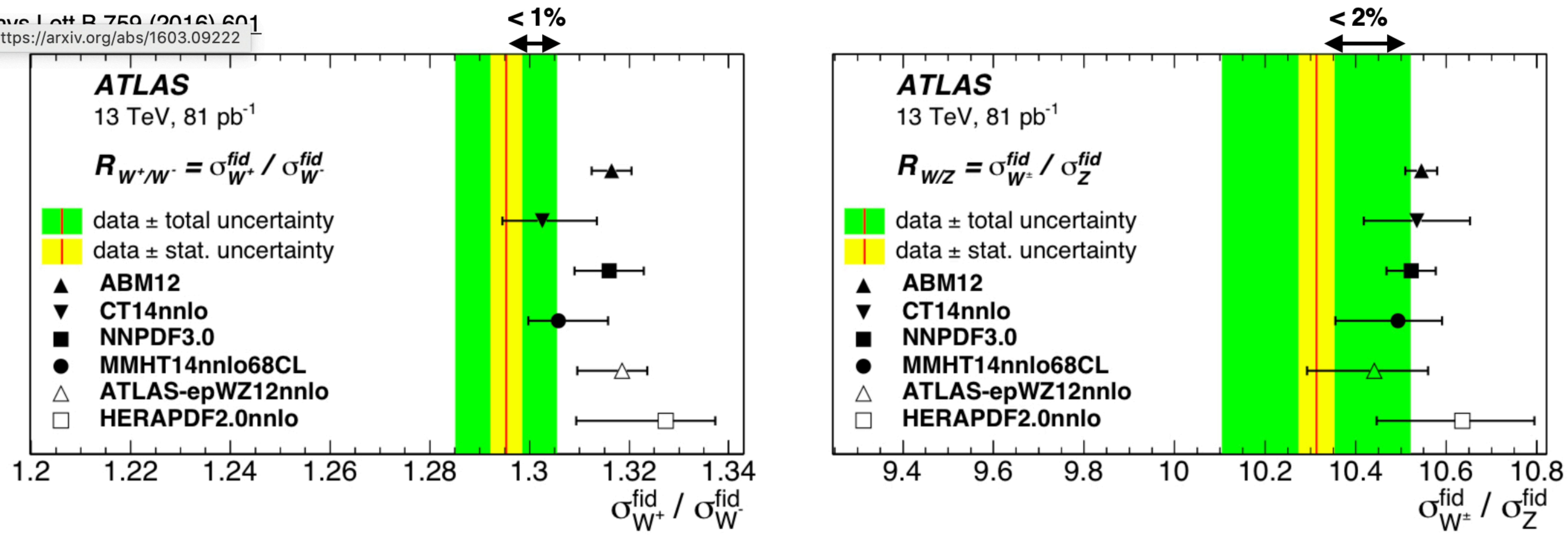
The LHAPDF parton density evaluation library

available on GitLab

(or cernweb, hepforge, …)

Phys.Lett.B 759 (2016) 601

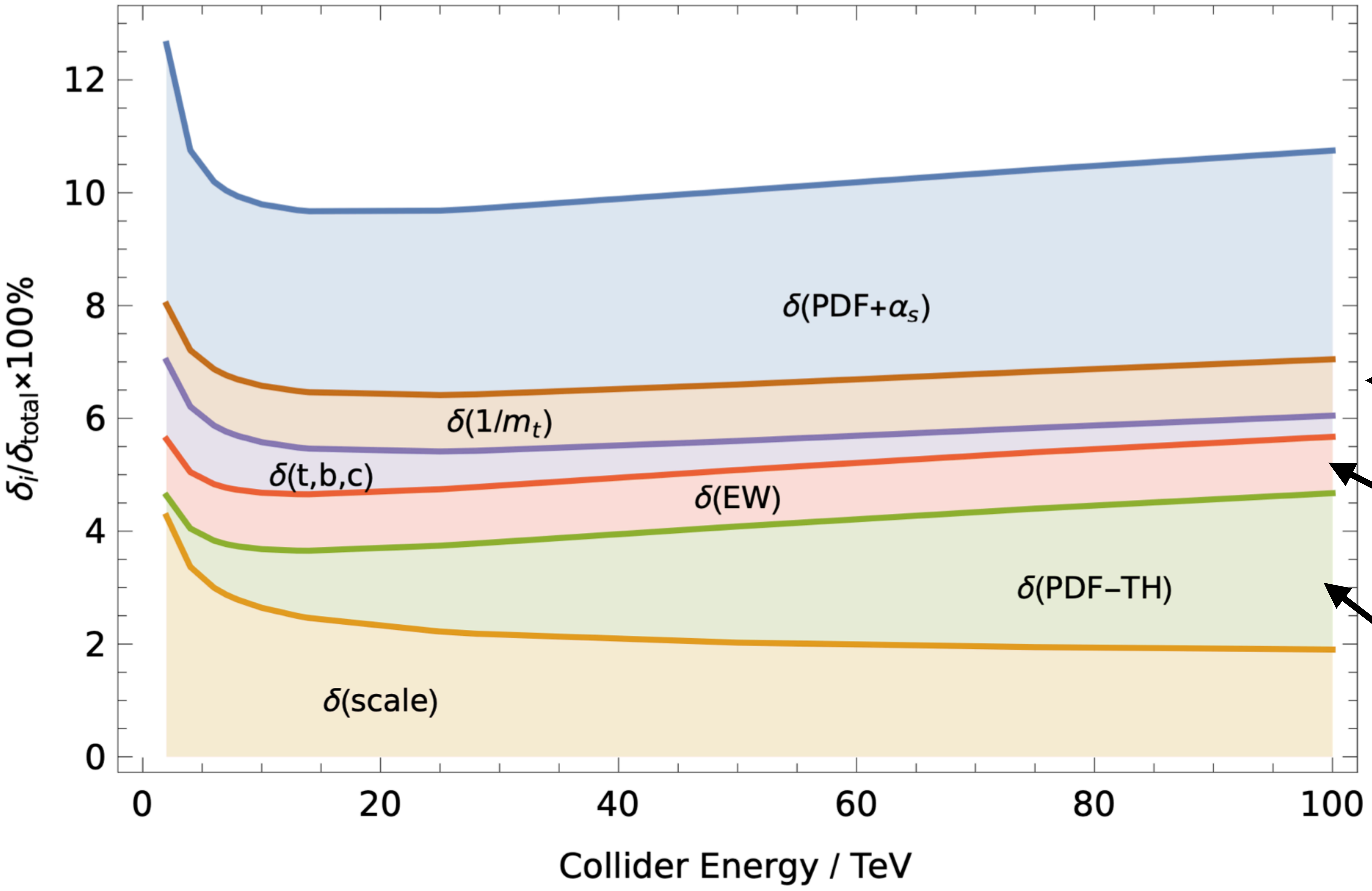
https://arxiv.org/abs/1603.09222



reducing PDF uncertainties is very important!

Higgs production in gluon fusion

uncertainty budget Dulat, Lazopoulos, Mistlberger ‘18



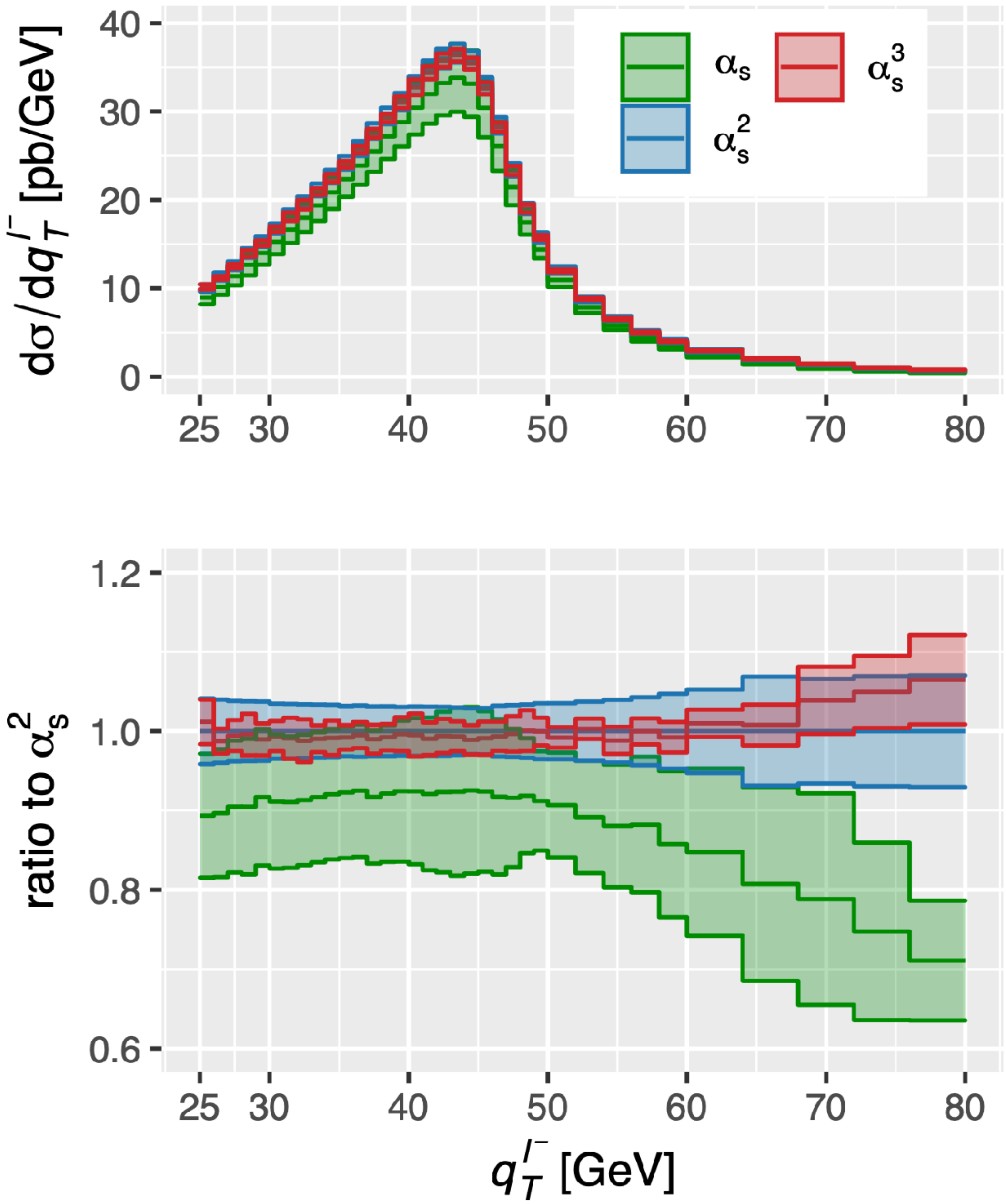
basically removed recently
Czakon, Harlander, Klappert, Niggetiedt ‘20

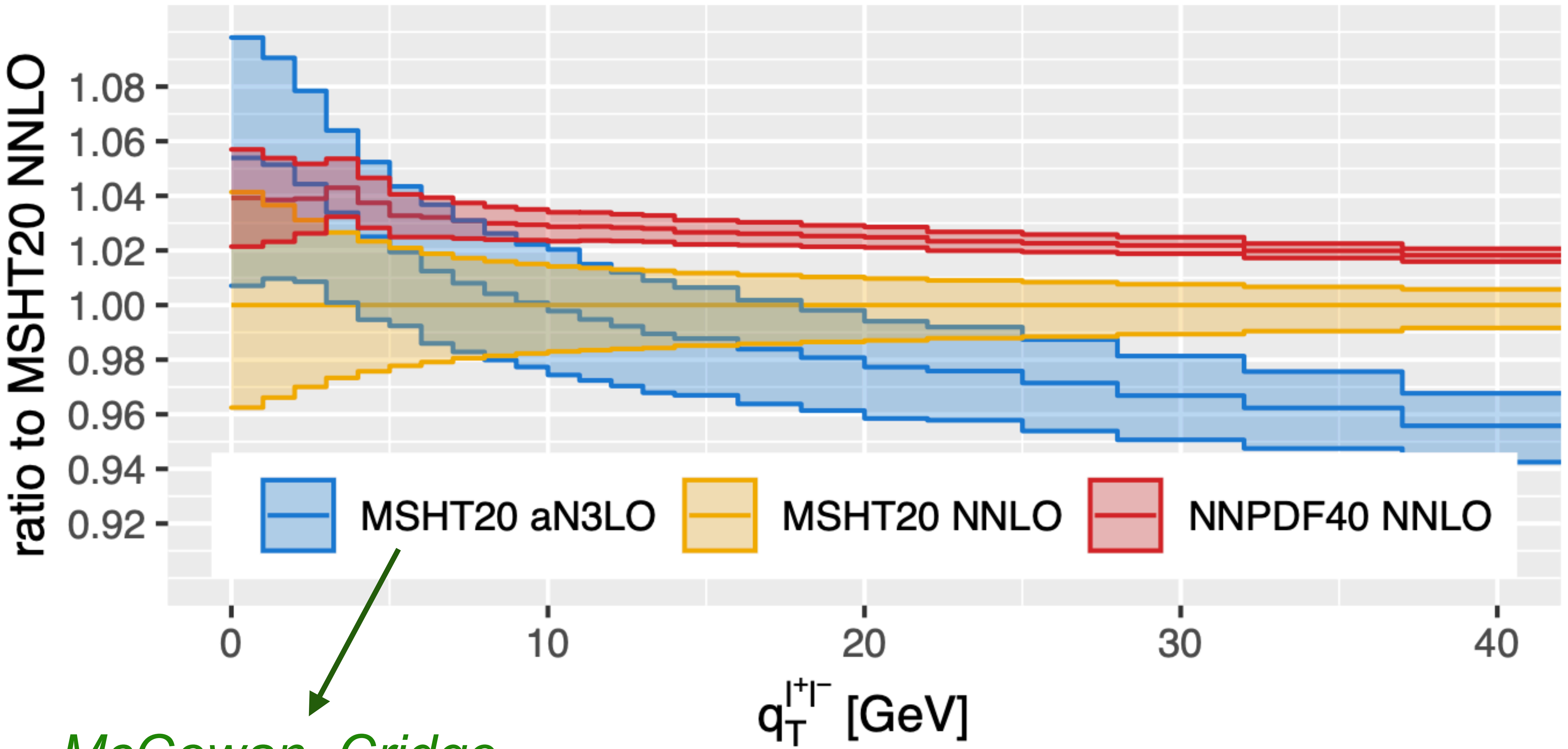
reduced substantially
Bechetti et al ’20, ’21, Bonetti et al ’18, ’20, ‘22

mismatch between PDF (NNLO) and ME (N3LO)
partly addressed by McGowan et al. 2207.04739

Scale uncertainties: Drell-Yan (Z-production)

N4LL+N3LO *Neumann, Campbell 2207.07056*





McGowan, Cridge, Harland-Lang, Thorne 2207.04739

(approximate) N3LO PDFs introduce shape change!

looking into history

from PDF determination “wishlist” 2013

[S.Forte, G.Watt, **1301.6754]**

- The parametrisation should be sufficiently general and unbiased

  e.g. new approach based on deep learning [e.g. S.Carrazza et al. ’19]

- The experimental uncertainties should be understood and carefully propagated

  LHAPDF6: metadata `ErrorType`, `ErrorConfLevel` [A.Buckley et al. ’14]

- PDFs including electroweak corrections will have to be constructed

  QED corrections done (see next slide)

- The strong coupling, in addition to being determined simultaneously with PDFs, should also be decoupled from the PDF determination

  available, see e.g. **PDF4LHC15** J. Butterworth et al. ‘15

- The treatment of heavy quarks will have to include mass-suppressed terms

  in progress, see e.g. Blümlein, Moch et al. ; NNPDF charm study (Nature article)

- An estimate of theoretical uncertainties should be performed together with PDF sets

  in progress, see e.g. *McGowan, Cridge, Harland-Lang, Thorne 2207.04739*

huge progress
in items 1- 4

also:
aN3LO PDFs

2306.15294, 2207.04739

looking into history

Karlsruhe Institute of Technology

from PDF determination “wishlist” 2013

[S.Forte, G.Watt, **1301.6754]**

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huge progress
in items 1- 4

also:
aN3LO PDFs

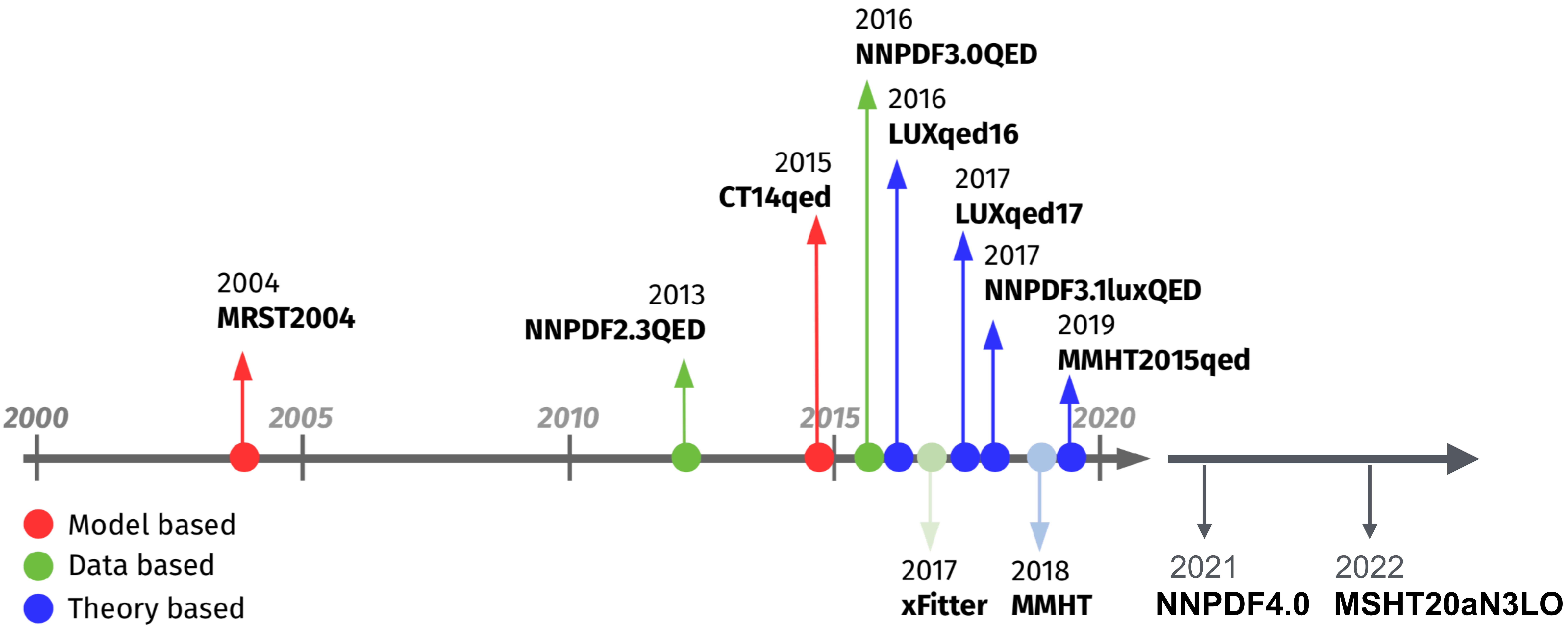
2306.15294, 2207.04739

new: “SMEFT PDF sets”

how to avoid absorbing
new physics effects
in PDF fits?

see e.g. 2307.10370

PDF development (incl. QED corrections)



extended from S.Carrazza, E.Villa et al, 1909.10547

hadron-hadron collisions

- same principle: absorb poles due to initial state collinear radiation into bare parton densities

$$\sigma^{NLO} = \underbrace{\int_{m+1}\left[d\sigma^R - d\sigma^S\right]_{\epsilon=0}}_{\text{numerically}} + \underbrace{\int_m \left[\underbrace{d\sigma^V}_{\text{cancel poles}} + \underbrace{\int_{\mathrm{s}} d\sigma^S}_{\text{analytically}}\right]_{\epsilon=0}}_{\text{numerically}}$$

- at NLO, "IR subtraction" procedure is automated; two main schemes:
  - Catani-Seymour (CS) dipole subtraction *Catani, Seymour '96* (momentum mappings)
  - Frixione-Kunszt-Signer (FKS) subtraction *FKS '95* (partition of phase space according to IR singular regions)

automated IR subtraction

various schemes are used in NLO-capable Monte-Carlo programs, e.g.

Sherpa, Dire, Herwig7: **CS**

MadGraph5_aMC@NLO, Powheg, Whizard: **FKS**

further:

Vincia: antenna subtraction

[P. Skands et al.]

Geneva: n-jettiness, qT (builds on resummation in resolution parameter, extension to NNLO)

[S. Alioli et al.]

… very incomplete list!

parton shower idea in a nutshell

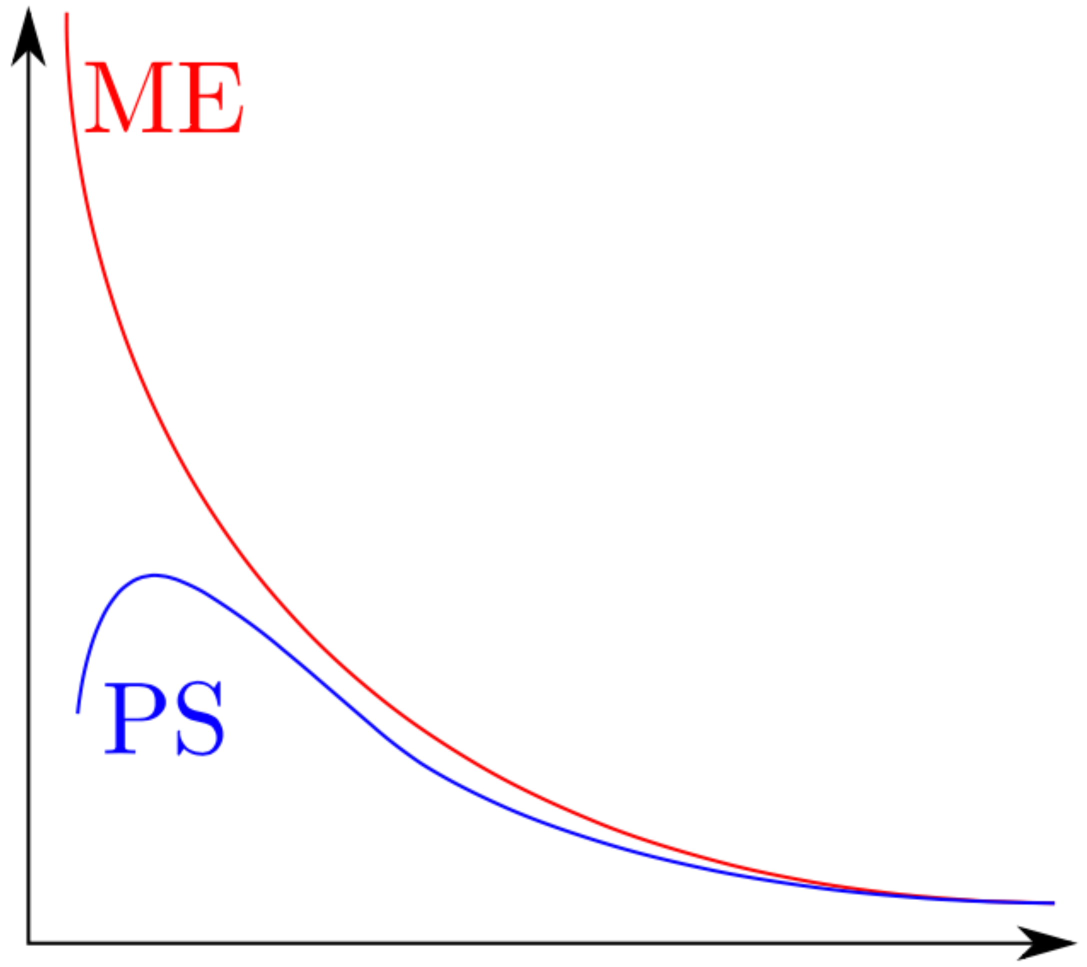
$$\mathcal{P} = \frac{\alpha_s}{2\pi} \int_{p_{T,min}}^{p_{T,max}} \frac{dp_T^2}{p_T^2} \int_{z_{min}}^{z_{max}} dz \, P_{ij}(z)$$

probability to emit a parton with $p_T \in [p_{T,min}\,,\, p_{T,max}]$ and energy fraction *z* of parent

consider successive emissions, ordering variable not necessarily p_T (call it t)

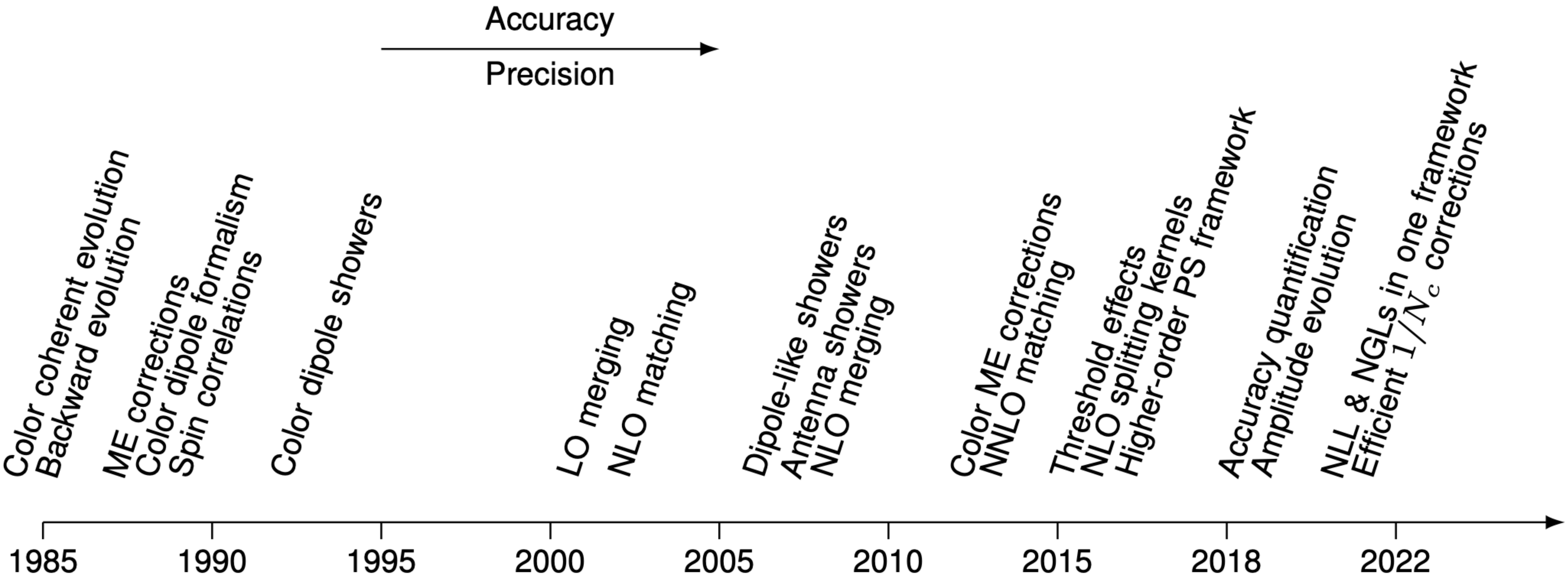
⟶ Sudakov form factor

$$\Delta_i(t,t') = \exp\left\{ - \sum_{j\in\{q,g\}} \frac{\alpha_s}{2\pi} \int_t^{t'} \frac{d\tilde{t}}{\tilde{t}} \int_{z_{min}}^{z_{max}} dz \, P_{ij}(z) \right\}$$



survival probability for a parton not to undergo a branching between t’ and t

Parton shower developments


Accuracy

Precision

Color coherent evolution
Backward evolution

ME corrections
Color dipole formalism
Spin correlations

Color dipole showers

LO merging

NLO matching

Dipole-like showers
Antenna showers
NLO merging

Color ME corrections
NNLO matching

Threshold effects
NLO splitting kernels
Higher-order PS framework

Accuracy quantification
Amplitude evolution

NLL & NGLs in one framework
Efficient $1/N_c$ corrections

Stefan Höche
DESY Theory workshop 2022

Parton shower developments

- Lots of activity in parton shower development ...
  - Logarithmic precision [PanScales,Herwig,Sherpa,...]
  - Higher-order kernels [Vincia,Sherpa,Herwig,...]
  - Interplay w/ NNLL [PanScales,...]
- ... and matching to fixed-order calculations
  - Improvements at NLO [Herwig,Pythia,Sherpa,...]
  - Resummation based [Geneva,MINNLO$_{\mathrm{PS}}$]
  - Fully differential [Vincia,UN^{X}LOPS,TOMTE]

- Still, many questions remain [Campbell et al.] arXiv:2203.11110
  - Systematic treatment of kinematic edge effects
  - Massive quark production & evolution
  - Interplay with hadronization
  - ...

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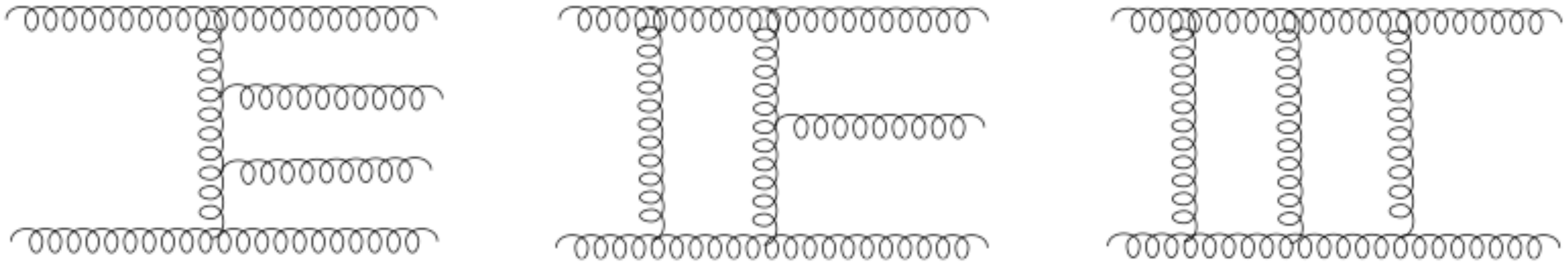
DESY Theory workshop 2022

NNLO building blocks

example 2-jet production
(gluon channel)



double real

1-loop virtual
$\otimes$ single real

2-loop virtual

↓ implicit IR poles
(phase space integration)

↓ explicit and implicit poles

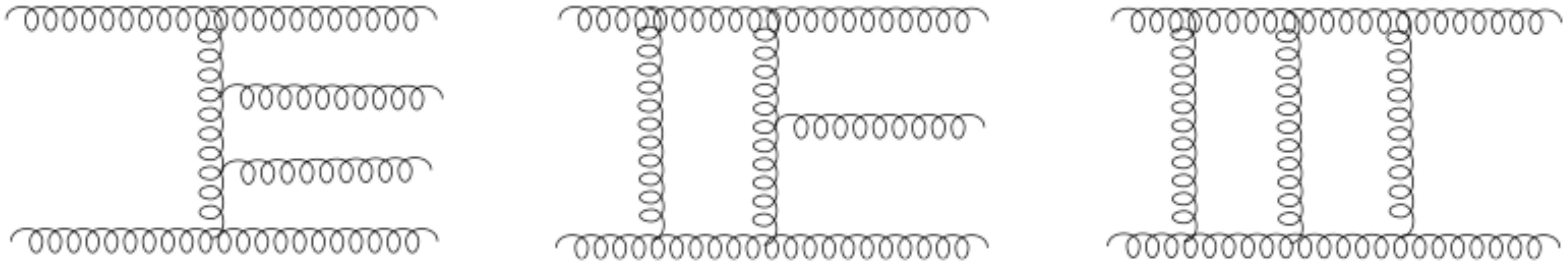
↓ explicit poles $1/\epsilon^{2L}$ $\quad (D = 4 - 2\epsilon)$

bottlenecks: IR subtraction

(multi)-loop integrals

NNLO building blocks

example 2-jet production
(gluon channel)



double real

1-loop virtual
$\otimes$ single real

2-loop virtual

implicit IR poles
(phase space integration)

explicit and implicit poles

explicit poles $1/\epsilon^{2L}$ $\quad (D = 4 - 2\epsilon)$

bottlenecks: IR subtraction

(multi)-loop integrals

current frontiers:

- NNLO automation
- N3LO coloured

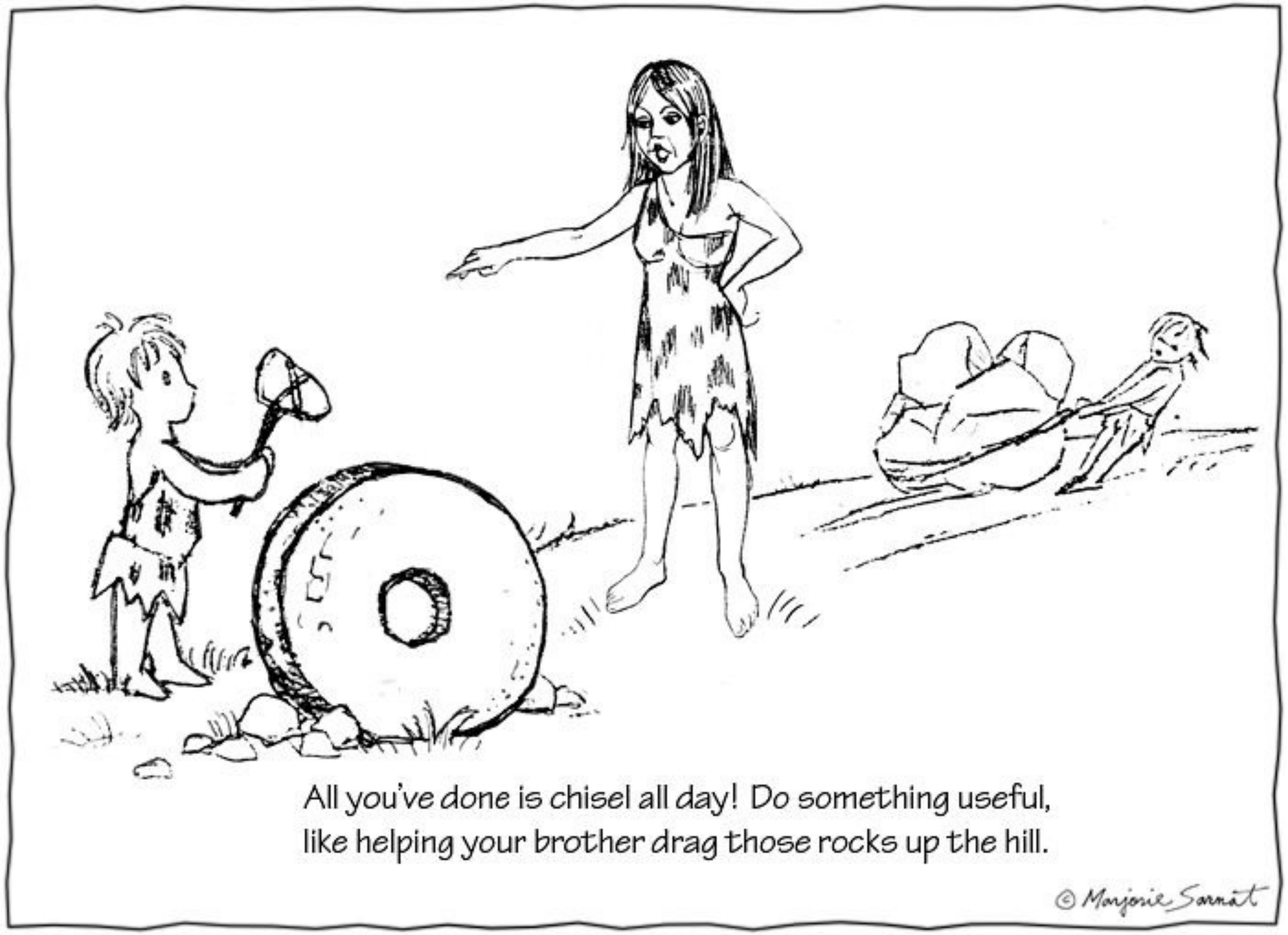
- 2 loops, 4 legs with several mass scales
- 2 loops, 5 legs
- more than 2 loops

Summary

- The universal infrared properties of QCD are important to
  - factorise and cancel/subtract poles in fixed order calculations
  - come up with well-defined observables (e.g. jets, event shapes)
  - derive the evolution of parton densities with energy
  - construct parton showers

The Standard Model is great,
but a lot is still to be discovered

New ideas wanted,
the future is yours!



All you've done is chisel all day! Do something useful, like helping your brother drag those rocks up the hill.

Appendix

Colour basis for matrix elements

convenient notation for amplitudes with m partons:

[Catani, Seymour ’96; Catani, Grazzini ’00]

label colour matrices such that it is clear which parton emitted a gluon

introduce abstract basis in colour space $\{|c_1 \ldots c_m\rangle\}$ with

$$\mathcal{M}_{c_1 \ldots c_m}(p_1, \ldots, p_m) \equiv \langle c_1 \ldots c_m | \mathcal{M}(p_1, \ldots, p_m)\rangle$$

such that $|\mathcal{M}(p_1, \ldots, p_m)|^2 = \langle \mathcal{M}(p_1, \ldots, p_m) \,|\mathcal{M}(p_1, \ldots, p_m)\,\rangle$

define **colour charge operator** for emission of a gluon from parton *i* by

$$\boxed{\mathbf{T}_i \equiv \langle a|\, T_i^a}$$

such that

$$\langle a_1, \ldots, a_i, \ldots, a_m, a\,|\boldsymbol{T}_i|b_1, \ldots, b_i, \ldots, b_m\rangle = \delta_{a_1 b_1} \ldots T^a_{a_i b_i} \ldots \delta_{a_m b_m}$$

Colour basis for matrix elements

$$\langle a_1, \ldots, a_i, \ldots, a_m, a\,|\boldsymbol{T}_i|b_1, \ldots, b_i, \ldots, b_m\rangle = \delta_{a_1 b_1} \ldots . T^a_{a_i b_i} \ldots \delta_{a_m b_m}$$

if emitting particle is a quark: $T^a_{a_i b_i} \equiv t^a_{a_i b_i} \;\; a_i, b_i \in \{1,2,3\}$

antiquark: $T^a_{a_i b_i} \equiv -t^a_{b_i a_i} \;\; a_i, b_i \in \{1,2,3\}$

gluon: $T^a_{bc} \equiv -i f_{abc} \quad a, b, c \in \{1, \ldots, 8\}$

universal behaviour in the soft limit:

$$\langle a\,|\mathcal{M}(k, p_1, \ldots, p_m)\rangle \rightarrow g_s\, \mu^\epsilon \varepsilon^\mu(k)\, J^a_\mu(k)\,|\,\mathcal{M}(p_1, \ldots, p_m)\rangle$$

$$\mathbf{J}^\mu(k) = \sum_{i=1}^{m} \mathbf{T}_i \, \frac{p_i^\mu}{p_i \cdot k}$$ soft gluon current