Lecture 2: Effective Hamiltonian

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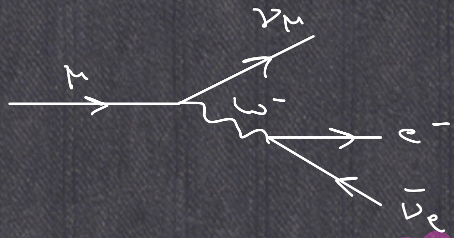
- hep-ph/9806471 Les Houches
- hep-ph/9512380 Review

weak decays: e.g. μ-decay



lifetime

$$\tau_\mu = \frac{1}{\Gamma_\mu}$$ ← total decay rate

Result: $$\Gamma_\mu = \frac{G_F^2\, m_\mu^5}{192\pi^3}\, f\!\left(\frac{m_e}{m_\mu}\right)$$

assume: $m_\nu = 0$

- Fermi-constant G_F

$$\frac{g_2^2}{2}\,\frac{1}{k^2-M_W^2} \simeq -\frac{g_2^2}{2M_W^2}$$

$$G_F = \frac{g_2^2}{4\sqrt{2}\,M_W^2}$$

- $f(x)$ is phase space function

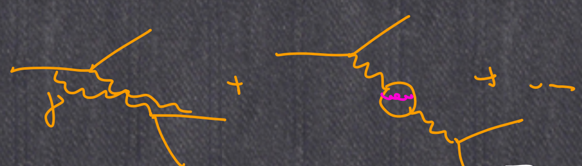
$$\boxed{f(x) = 1-8x^2+8x^6-x^8-24x^4\ln x}$$

$$x = \frac{m_e}{m_\mu} = \frac{0.511\,\text{MeV}}{105\,\text{MeV}} \ll 1 \qquad f(x) \simeq 1$$

Exp: $$\tau_\mu^{\text{Exp}} = 2.1969811(22)\cdot 10^{-6}\,\text{s}$$

Theory: $$\tau_\mu^{\text{Theo}} = 2.18776 \cdot 10^{-6}\,\text{s}$$

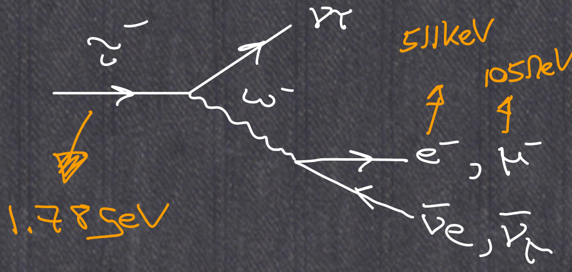
+ higher orders

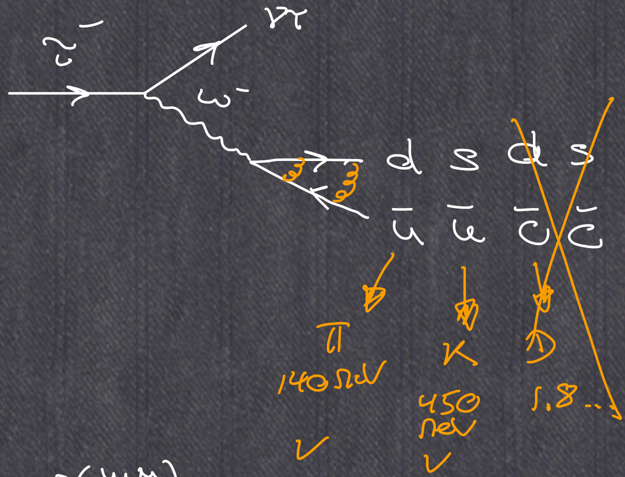


$$f(x) \xrightarrow{\text{add loops}} f(x)\left[1 + \frac{\alpha_{QED}}{4\pi}\cdot 2\left(\frac{25}{4}-\pi^2\right)\right]$$

$$\hookrightarrow \tau_\mu^{\text{Theo 2}} = 2.19699 \cdot 10^{-6}\,\text{s}$$

Tau-decay:





$$\Gamma_\tau = \frac{G_F^2\, m_\tau^5}{192\pi^3}\Big[f\left(\frac{m_e}{m_\tau}\right) + f\left(\frac{m_\mu}{m_\tau}\right)$$
$$+ |V_{ud}|^2 N_c\, g\left(\frac{m_d}{m_\tau}, \frac{m_u}{m_\tau}\right)$$
$$+ |V_{us}|^2 N_c\, g\left(\frac{m_s}{m_\tau}, \frac{m_u}{m_\tau}\right)\Big]$$

$$m_e, m_\mu, m_d, m_u, m_s \ll m_\tau \Rightarrow f \simeq 1 \simeq g$$

$$[\ldots] = 1 + 1 + N_c\,\big(\underbrace{|V_{ud}|^2 + |V_{us}|^2}_{= 1 - |V_{ub}|^2 \simeq 1}\big) = \underline{5}$$

$$\frac{\tau_\tau}{\tau_\mu} = \frac{\Gamma^\mu}{\Gamma^\tau} = \left(\frac{m_\mu}{m_\tau}\right)^5 \cdot \frac{1}{5}$$

Exp: $\tau_\tau^{\text{Exp}} = 2.906(1) \cdot 10^{-13}\,\text{s}$

↳ $\underline{\text{Theory}}$: $\tau_\tau^{\text{Theory}} = 3.267 \cdot 10^{-13}\,\text{s}$

τ_μ^{Exp}

$\underline{\text{QCD-effects}}$: compare $\alpha_{QED} = 1/137$

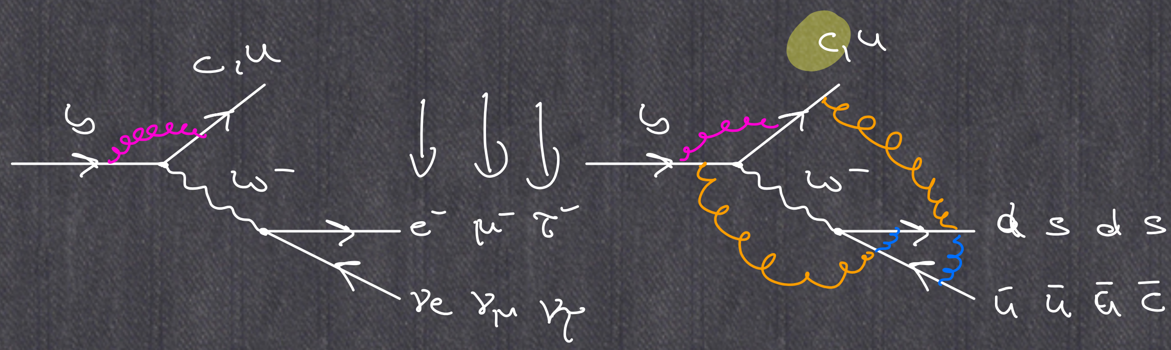
$$\alpha_{QCD}(m_Z) = 0.1$$

$$\alpha_{QCD}(m_b) \simeq 0.2$$

$$\alpha_{QCD}(m_\tau) \simeq 0.3$$

$\underline{\text{currently}}$: $\tau_\mu^{\text{Exp}} \Longrightarrow \alpha_s(m_\tau)$

b-quark decay



$$\Gamma^b = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 \Big\{ g\left(\frac{m_c}{m_b}, \frac{m_e}{m_b}\right) + g\left(\frac{m_c}{m_b}, \frac{m_\mu}{m_b}\right) + g\left(\frac{m_c}{m_b}, \frac{m_\tau}{m_b}\right)$$

$$+ N_c \Big[|V_{ud}|^2 \, h\left(\frac{m_c}{m_b}, \frac{m_d}{m_b}, \frac{m_u}{m_b}\right) + \ldots$$

$$+ |V_{cs}|^2 \, h\left(\frac{m_c}{m_b}, \frac{m_s}{m_b}, \frac{m_c}{m_b}\right) + \ldots \Big]$$

$$+ \frac{|V_{ub}|^2}{|V_{cb}|^2} \Big[\ldots\ldots \Big] \Big\}$$

Exp: $\tau_b^{Exp} \simeq 1.5\,\mathrm{ps}$

$\tau_b^{Theo} \simeq 0.9 \dots 3.7\,\mathrm{ps}$

Try to determine QCD-effects

(1) Def. of a quark mass
- Pole mass
- $\overline{MS}$ mass
- ...

(2) pure QCD corrections

Tree-level $\mathcal{O}(1)$

1-loop-level

naive exp.: $\mathcal{O}(\alpha_s(m_b)) \simeq \mathcal{O}(0.2)$

Real calculation: $\underbrace{\alpha_s(m_b)}_{0.2} \dots + \underbrace{\alpha_s(m_b) \cdot \ln\frac{m_b^2}{m_W^2}}_{1.2} \dots$

general structure:

Tree-level	1			
1-loop	$\alpha_s \ln$	α_s		
2-loop	$\alpha_s^2 \ln^2$	$\alpha_s^2 \ln$	α_s^2	
3-loop	$\alpha_s^3 \ln^3$	$\alpha_s^3 \ln^2$	$\alpha_s^3 \ln$	α_s^3

Theoretical tool to sum up logs to all orders!

The diagrams which are creating the leading logs are simple

$\alpha_s^3 \ln^3$

$\alpha_s^4 \ln^4$

and this can be summed like a geometric series

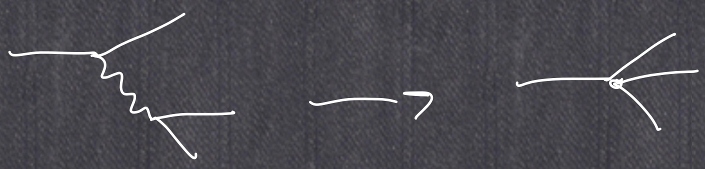
$\alpha_s^3 \ln^{<3}$

This procedure is formalised in the derivation of the effective Hamiltonian

1. start

2. "Integrate out the W-boson"

$$\frac{1}{k^2 - M_W^2} \xrightarrow{k^2 \ll M_W^2} -\frac{1}{M_W^2}$$



Feynman rules

$$\bar c \, \frac{i}{2}\frac{g_2}{\sqrt{2}} V_{cb}^* \gamma_\mu (1-\gamma_5) b \cdot \frac{1}{k^2 - M_W^2} \cdot \bar d \, \frac{i}{2}\frac{g_2}{\sqrt{2}} V_{ud} \gamma^\mu (1-\gamma_5) u$$

$k^2 \ll M_W^2$

$$\mathcal{H}_{eff} = \underbrace{\left(\frac{g_2}{2\sqrt{2}}\right)^2 \frac{1}{M_W^2}}_{\frac{G_F}{\sqrt{2}}} \cdot \underbrace{V_{cb}^* V_{ud}}_{V_{CKM}} \cdot \underbrace{1}_{C_2} \cdot \underbrace{\bar c \gamma_\mu (1-\gamma_5) b \cdot \bar d \gamma^\mu (1-\gamma_5) u}_{Q_2}$$

C_2 ⇓ wilson koefficient

Q_2: 4-quark operator

Add QCD:

① new operators

colour indices

$Q_2: \; (\bar c^{\alpha} b^{\alpha})_{V-A} \cdot (\bar d^{\beta} u^{\beta})_{V-A}$ colour-singlett operator

$Q_1: \; (\bar c^{\alpha} b^{\beta})_{V-A} \cdot (\bar d^{\beta} u^{\alpha})_{V-A}$ colour-rearranged operator

$$\boxed{\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{CKM} \left(C_1 Q_1 + C_2 Q_2 \right)}$$

(2) no QCD:

$c_1 = 0$

$c_2 = 1$

add QCD:

$c_1 = 0 + \mathcal{O}(\alpha_s) \simeq -0.2$

$c_2 = 1 + \mathcal{O}(\alpha_s) \simeq 1.1$

(3) Sum up all the logs

Renormalisation group

(4) $\mathcal{H}_{eff}$ separates scales

short distance physics is in $C_i(\mu)$

long distance physics is in $Q_i(\mu)$

$$\boxed{\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} V_{CKM} \left(C_1 Q_1 + C_2 Q_2 \right)}$$

$$\left(\ln \frac{m_b^2}{m_W^2} = \ln \frac{m_b^2}{\mu^2} + \ln \frac{\mu^2}{m_W^2} \right)$$

Penguin operators

QCD penguin operators

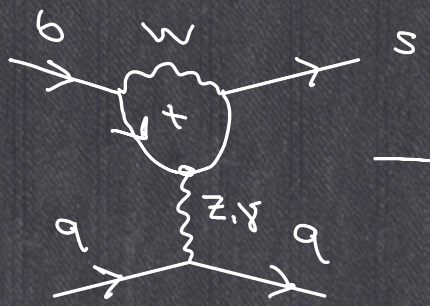
$$Q_3 = (\bar{s}\, b)_{V-A} \sum_{q=u,\dots,b} (\bar{q}\, q)_{V-A}$$

$$Q_4 = (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\dots,b} (\bar{q}^{\beta} q^{\alpha})_{V-A}$$

$$Q_5 = (\bar{s}\, b)_{V-A} \sum_{q=u,\dots,b} (\bar{q}\, q)_{V+A}$$

$$Q_6 = (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\dots,b} (\bar{q}^{\beta} q^{\alpha})_{V+A}$$

Electro-weak penguin operators



$$Q_7 = \frac{3}{2} (\bar{s}\, b)_{V-A} \sum_{q=u,\dots,b} e_q (\bar{q}\, q)_{V+A}$$

$$Q_8 = \frac{3}{2} (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\dots,b} e_q (\bar{q}^{\beta} q^{\alpha})_{V+A}$$

$$Q_9 = \frac{3}{2} (\bar{s}\, b)_{V-A} \sum_{q=u,\dots,b} e_q (\bar{q}\, q)_{V-A}$$

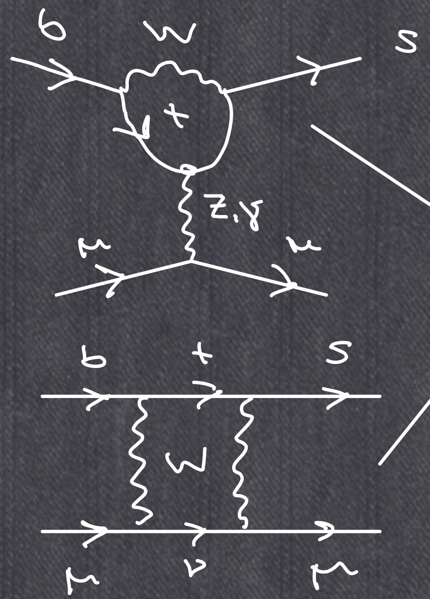
$$Q_{10} = \frac{3}{2} (\bar{s}^{\alpha} b^{\beta})_{V-A} \sum_{q=u,\dots,b} e_q (\bar{q}^{\beta} q^{\alpha})_{V-A}$$

Magnetic penguin operators

$$Q_{7\gamma} = \frac{e}{8\pi^2} m_b \bar{s}_{\alpha} \sigma^{\mu\nu} (1+\gamma_5)\, b_{\alpha} F_{\mu\nu}$$

$$Q_{8g} = \frac{g}{8\pi^2} m_b \bar{s}_{\alpha} \sigma^{\mu\nu} (1+\gamma_5) T^a_{\alpha\beta} b_{\beta} G^a_{\mu\nu}$$

Semileptonic penguin operators



q^2

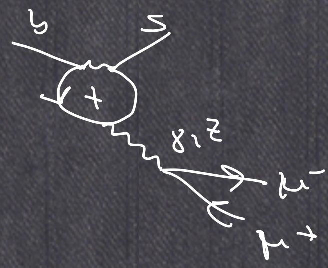
$$Q_{9V} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_V$$

$$Q_{10A} = (\bar{s}b)_{V-A}(\bar{\mu}\mu)_A$$

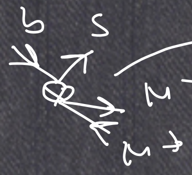
$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}}\left[\sum_{\substack{q=u,c\\ q'=u,c\\ q''=d,s}} V_c \left\{ C_1(\mu) Q_1^{qq'q''} + C_2(\mu) Q_2^{qq'q''} \right\} - V_p \sum_{j\geq 3} C_j(\mu) Q_j \right]$$

How to apply $\mathcal{H}_{eff}$ for $B_s \to \phi \mu\mu$

in SM: $b \to s \mu\mu$

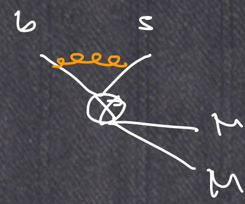


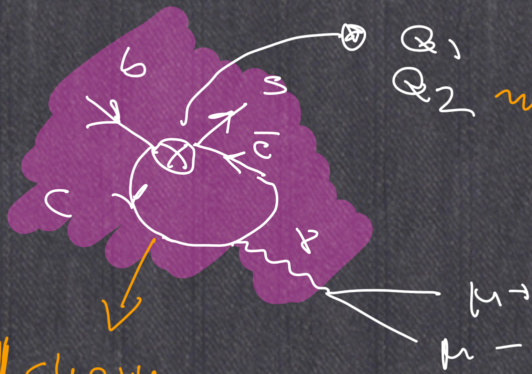
$\mathcal{H}_{eff}$:



$\to Q_{9V}$
Q_{10A}

$+$





$\to Q_1$, Q_2 my most important wilson coefficient

charm loop