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Higgs & Beyond | **Homework 2** | **Date: 16.09.2023**

Exercice 1: Star Wars Death Star

In the movie, we learn that the Death Star has a radius of about one-tenth of the Endor planet. Endor is very comparable to the Earth (life develops, humans experience gravitational potential as on Earth, there is a breathable atmosphere).
a) Estimate the size and the mass of the Death Star.
b) The Death Star is a weapon that can destroy planets similar to the Earth, like Alderande. Using dimensional arguments, estimate the energy needed to destroy Alderande, i.e. compute the gravitational potential energy of Alderande. For comparison, the total amount of energy produced on Earth in one year is of the order of 10^{20} J. Given that the Death Star needs about 3 days to produce/store this energy, compute the power of the source of energy.
c) In practise, the Death Star is doing more than destroying Alderande: each little fragment of the planet is expelled with a velocity of about 10^4 km/s. Compute the energy needed by the Death Star to achieve such a destructive action. What is the power of the source of energy? Assuming that this energy is produced by burning oil, compute the volume of oil needed. How many power-plants are needed to reach the same power?
d) Assuming that the energy of the Death Star is produced by the annihilation of matter and antimatter into energy, what would be the amount of anti-matter needed?

Exercice 2: Chirality matrix

In 4d space-time, the four 4×4 Dirac matrices obey the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_4.$$

It is useful to define the *chirality* matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$

a) Show that the chirality matrix anti-commutes with the four Dirac matrices.
b) Compute the square of the chirality matrix.
c) Conclude that $P_L = (1_4 + \gamma^5)/2$ and $P_R = (1_4 - \gamma^5)/2$ are two projector operators.

Exercice 3: Dirac, Weyl and Majorana representations of the Dirac matrices

In the lecture, it was stressed that the representation of the Dirac matrices is not unique. Three standard representations are the following (σ^i, for $i = 1,2,3$, are the three 2×2 Pauli matrices):

$$\text{Dirac: } \gamma^0 = \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}.$$

$$\text{Weyl: } \gamma^0 = \begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix}, \ \gamma^i = \begin{pmatrix} & \sigma^i \\ -\sigma^i & \end{pmatrix}.$$

$$\text{Majorana: } \gamma^0 = \begin{pmatrix} & \sigma^2 \\ \sigma^2 & \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} i\sigma^3 & \\ & i\sigma^3 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} & -\sigma^2 \\ \sigma^2 & \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} -i\sigma^1 & \\ & -i\sigma^1 \end{pmatrix}.$$

a) Verify that these representations indeed satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_4$.
b) Compute the chirality matrix, γ^5, in these 3 representations.
c) In each of these representations, find the charge conjugation matrix B such that $B\gamma^{\mu *} = -\gamma^\mu B$.

Exercice 4: Weyl and Majorana fermions

a) A Weyl or chiral spinor is defined to be an eigenstate of the chirality matrix γ^5:

$$\text{Weyl fermion:} \quad \gamma^5 \psi = \pm\psi.$$

Show that this condition is Lorentz-invariant (you'll consider, here and in the rest of the exercise, *small* Lorentz transformations only: $\psi(x) \to \psi'(x) = (1_4 + [\gamma^\mu, \gamma^\nu]\omega_{\mu\nu}/8)\psi(x)$, with $\omega_{\mu\nu} = -\omega_{\nu\mu}$).
b) Is the condition $\psi^* = \psi$, that would naively define a *real* spinor, Lorentz-invariant?
c) For any spinor ψ, we define the charge-conjugated spinor ψ_C as

$$\psi_C = B\psi^*,$$

where B is the charge conjugation matrix that satisfies $B\gamma^{\mu *} = -\gamma^\mu B$. Show that ψ and ψ_C transform in the way under Lorentz transformations. Conclude that the Majorana reality condition:

$$\text{Majorana fermion:} \quad \psi_C = \psi,$$

is Lorentz-invariant.
d) Show that if a spinor satisfies both Weyl and Majorana conditions at the same time, it has to vanish.
e) In the lecture, we have said that the Dirac mass term

$$\mathcal{L}_{\text{Dirac mass}} = m\psi^\dagger \gamma^0 \psi$$

is Lorentz-invariant. Write explicitly this Dirac mass operator in terms of the two chiralities of the fermion $\psi = \psi_L + \psi_R$, where $\psi_{L/R} = (1_4 \pm \gamma^5)\psi/2$.
f) From the result of **c)**, argue that the Majorana mass term

$$\mathcal{L}_{\text{Majoran mass}} = m\psi_C^\dagger \gamma^0 \psi$$

is Lorentz-invariant.
g) Write the Majorana mass operator in terms of the two chiralities of the fermion $\psi = \psi_L + \psi_R$.

Exercice 5: Scalar, pseudo-scalar, vector and pseudo-vector fermion currents

For two generic spinors $\psi_{1,2}$, show that under under Lorentz transformation and space-parity

$$\bar{\psi}_1\psi_2 \text{ transforms as a scalar,}$$
$$\bar{\psi}_1\gamma^5\psi_2 \text{ transforms as a pseudo-scalar,}$$
$$\bar{\psi}_1\gamma^\mu\psi_2 \text{ transforms as a vector,}$$
$$\bar{\psi}_1\gamma^\mu\gamma^5\psi_2 \text{ transforms as a pseudo-vector.}$$

We recall that $\bar{\psi} = \psi^\dagger\gamma^0$.