Theory of cosmological stochastic gravitational wave background sources

Chiara Caprini
University of Geneva & CERN

Outline

- Generalities of cosmological GW

  - Free wave equation & general solutions
  - Characteristic frequency & discovery potential
  - Reasons for SGWB production
  - Examples: two notable solutions of the SGWB equation

- Example of sources

  - First oder phase transition: more detailed
  - Inflation & cosmic strings: just the basic picture
  - Examples: EPTA parameter estimation

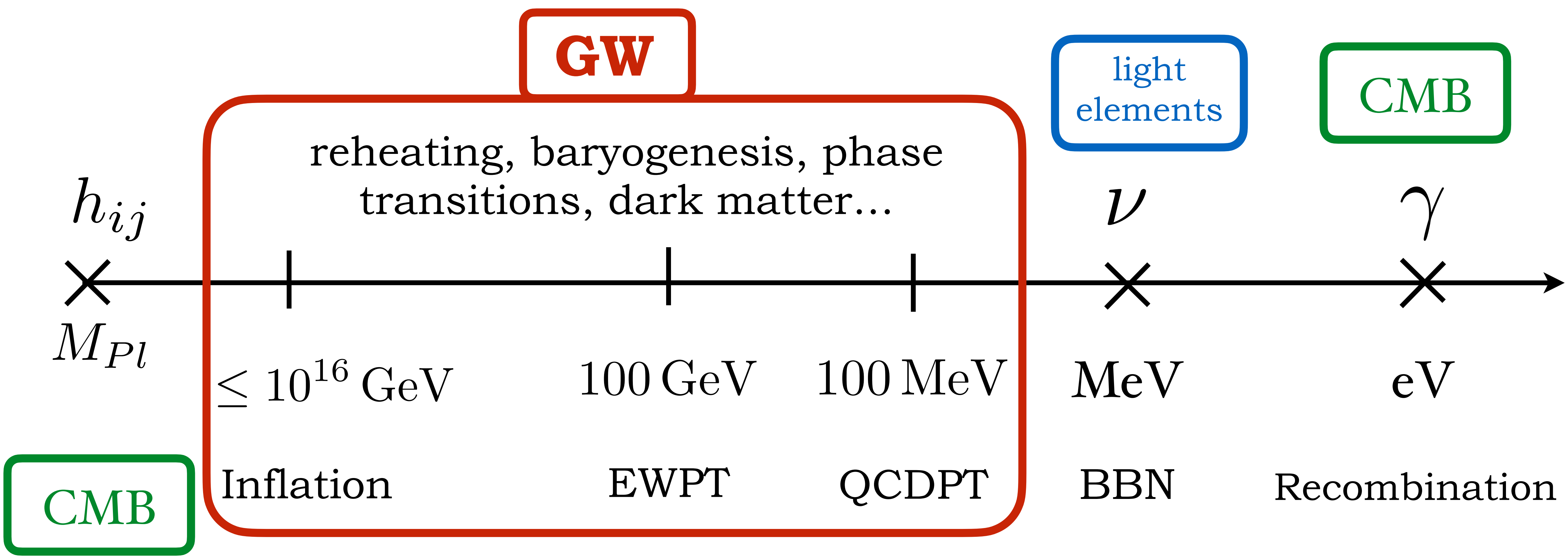
Cosmological gravitational waves

- because of the weakness of the gravitational interaction the universe is transparent to GW

$$\frac{\Gamma(T)}{H(T)} \sim \frac{G^2\, T^5}{T^2/M_{Pl}} \sim \left(\frac{T}{M_{Pl}}\right)^3 < 1$$

- GW emission processes in the early universe form a **fossil radiation**, whose detection would bring *direct information from very early stages of the universe evolution*, to which we have no access through em radiation

- amazing discovery potential, linked to high energy physics



Cosmological gravitational waves

- GWs emerge naturally in General Relativity (Newtonian theory + special relativity = a causal theory of gravitation)

  1. take a background space-time metric (the gravitational field)
  2. define a small perturbation over this background metric
  3. insert it into Einstein equations, which describe the space-time dynamics
  4. (if everything goes well) one finds a dynamical solution for the perturbation which is propagating as a wave -> GWs!

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \delta g_{\mu\nu}(x)\,, \qquad |\delta g_{\mu\nu}(x)| \ll 1$$

- How to decide what is the background and what is the fluctuation?

  - Either the background space-time has a clear symmetry (Minkowski, static, FLRW…)
  - Or it is possible to resort to a clear separation of scales/frequencies

FLRW spacetime is "easy" since
the hypersurfaces of constant time are homogeneous and isotropic:
No need for a separation of scales, e.g. Inflation

Cosmological gravitational waves

1. Exploit the invariance of FLRW space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations (scalar, vector, tensor) -> the dynamical equations decouple
2. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations -> easy for the tensor, invariant

$$\delta g_{00} = -2\phi$$

$$\delta g_{0i} = \delta g_{i0} = (\partial_i B + S_i)$$

$$\delta g_{ij} = -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + h_{ij} \qquad \partial_i h_{ij} = h_{ii} = 0$$

Perfect fluid Perturbation

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

$$\delta T_{ij} = \bar{p}\,\delta g_{ij} + a^2[\delta p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + 2\partial_{(i}v_{j)} + \Pi_{ij}]$$

$$(\partial_i v_i = 0,\ \partial_i \Pi_{ij} = 0,\ \Pi_{ii} = 0)$$

Cosmological gravitational waves

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3. Find the metric perturbation variable that obeys a wave equation *(independently on the presence of the background fluid)* -> GWs!

Evolution equation for the tensor mode:

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\boxed{\Pi_{ij}(\mathbf{x},t)}$$

Source: tensor anisotropic stress

translational invariance -> F.T. in space

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,h_r(\mathbf{k},t)\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,e^r_{ij}(\hat{\mathbf{k}})$$

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

Cosmological gravitational waves

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Source: tensor anisotropic stress

According to GR, any form of energy contributes to space-time curvature

Are GWs a source of space-time curvature?

$$T_{\mu\nu}^{GW} = \frac{1}{32\pi G}\langle\nabla_\mu h_{\alpha\beta}\nabla_\nu h^{\alpha\beta}\rangle \qquad \rho_{GW} = \frac{\langle\dot{h}_{ij}\dot{h}^{ij}\rangle}{32\pi G}$$

Solution of the homogeneous equation

$$H_r(\mathbf{k},\eta) = a\, h_r(\mathbf{k},\eta) \qquad H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = 0$$

Power-law scale factor (it covers matter and radiation domination, and De Sitter inflation) $\quad a''/a \simeq \mathcal{H}^2$

CASE 1: *Sub-Hubble modes*, relevant for propagation after the source stops

$$k^2 \gg \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\, e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\, e^{-ik\eta}$$

In this limit, GWs are plane waves with redshifting amplitude

CASE 2: *Super-Hubble modes*, relevant for inflationary tensor perturbations

$$k^2 \ll \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = A_r(\mathbf{k}) + B_r(\mathbf{k}) \int^{\eta} \frac{d\eta'}{a^2(\eta')}$$

the constant mode is a solution

Solution of the sourced equation

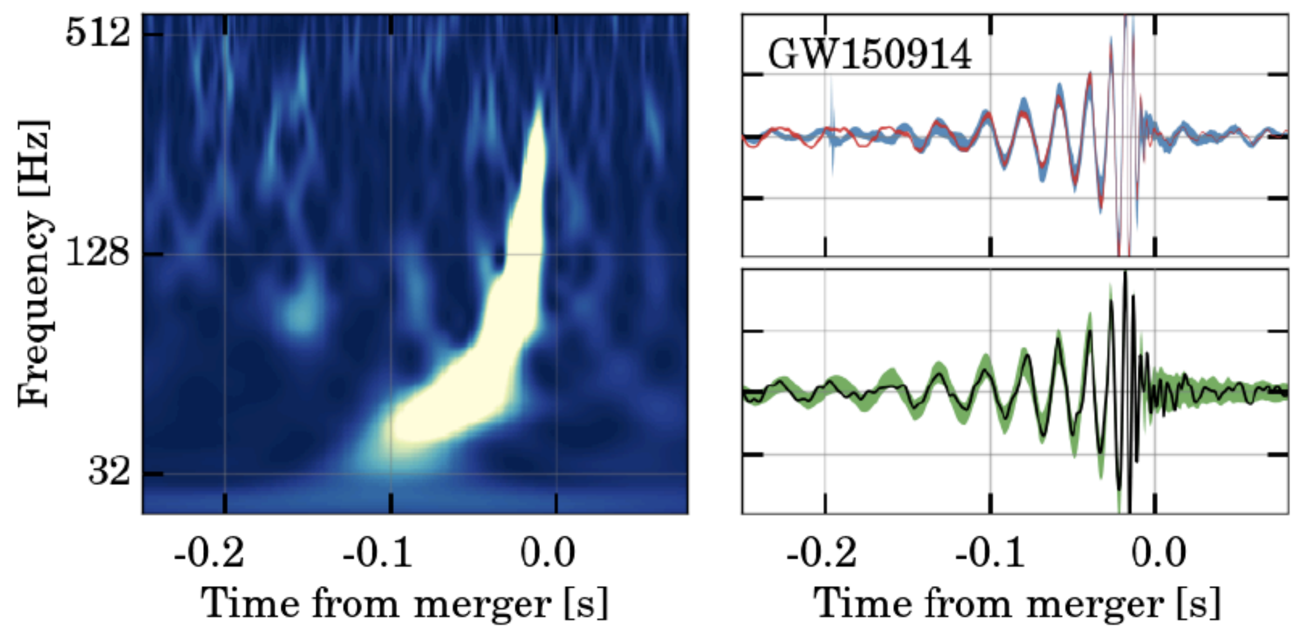
- GWs from astrophysical binaries: frequency set (more or less) by Kepler's law

LIGO/Virgo arXiv:1811.12907

$$f(\tau) = \frac{1}{\pi}\left(\frac{G\,M_c}{c^3}\right)^{-5/8}\left(\frac{5}{256\,\tau}\right)^{3/8}$$

Chirp mass

Time to coalescence



- GWs from the early universe: frequency set (more or less) by the *Hubble scale*

the characteristic length/time scale of the GW generating process cannot be larger than the causal horizon at the generation time

$$\ell_* \leq H_*^{-1}$$

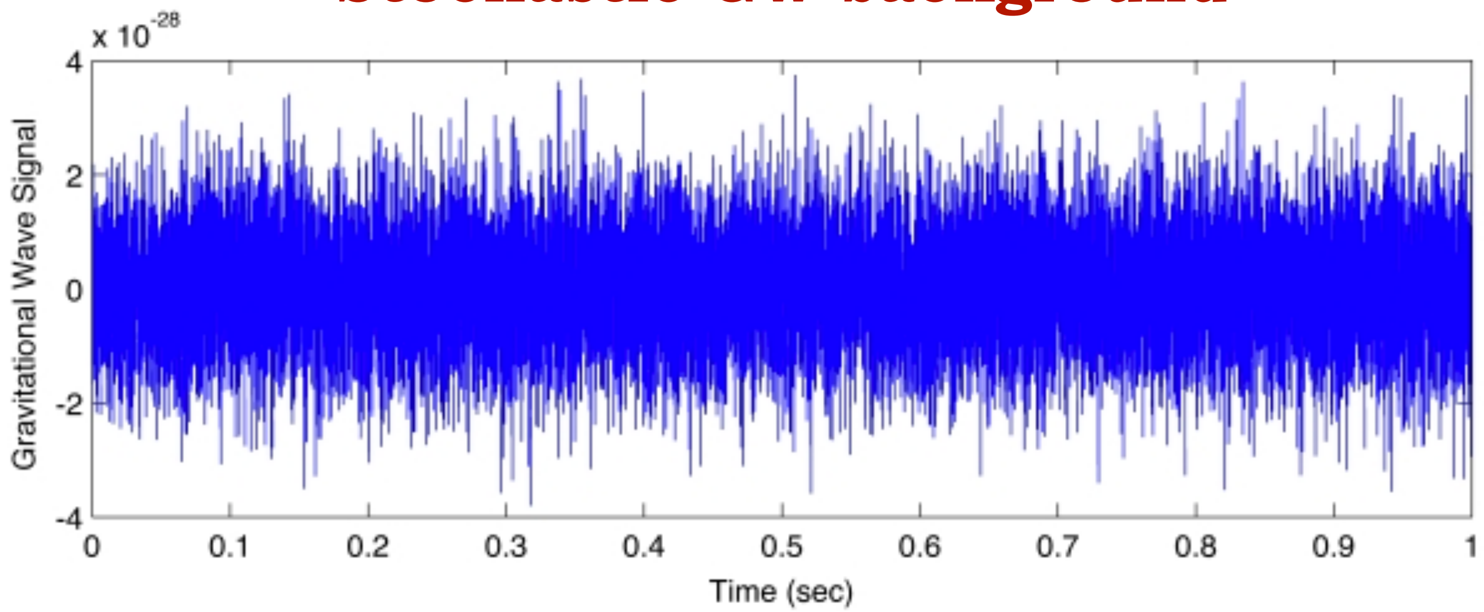
GW frequency $\quad f_* \simeq \dfrac{1}{\ell_*} \geq H_*$

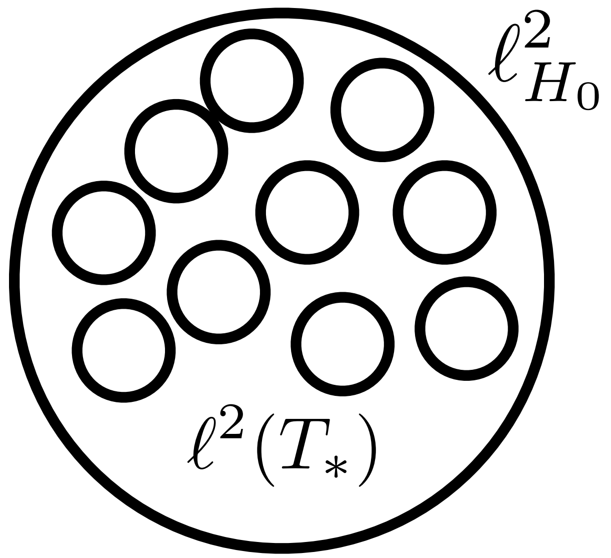
Two consequences:

1. GW signals from the primordial universe have too small correlation scale with respect to the detector resolution -> *only the statistical properties of the signal can be accessed*

$$\Theta(T_* = 100\,\mathrm{GeV}) \simeq 10^{-12}\mathrm{deg}$$

Stochastic GW background





2. The *specific frequency range* of a GW detector allows it to probe GW generating processes occurring at *specific energy scales*

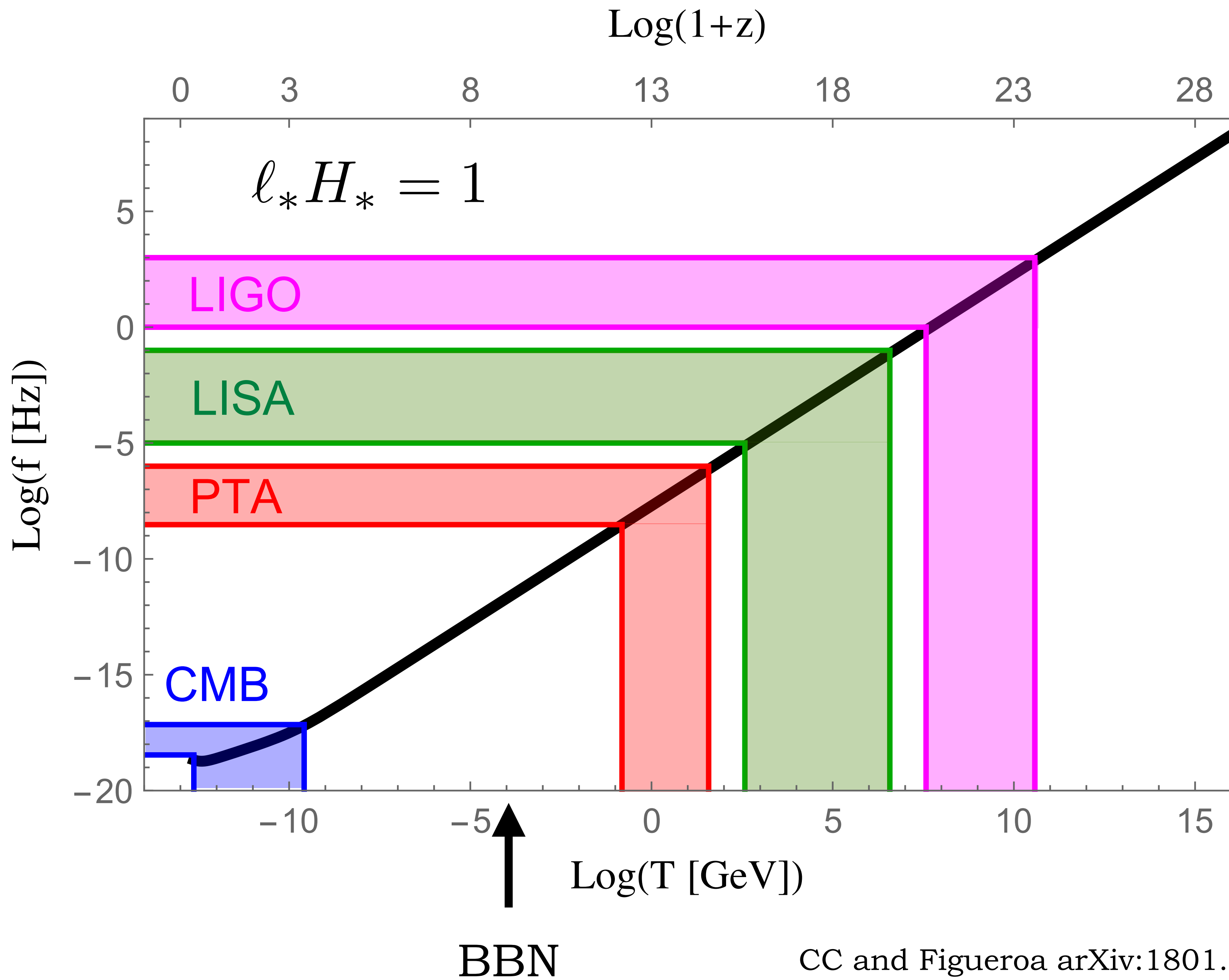
$$f_* \simeq \frac{1}{\ell_*} \geq H_*$$

after redshift:

$$f = f_* \frac{a_*}{a_0} = \frac{1.65 \times 10^{-5}}{\ell_* H_*} \left(\frac{g(T_*)}{100}\right)^{1/6} \frac{T_*}{100\mathrm{GeV}}\ \mathrm{Hz}$$

Two consequences:

SGWB in cosmology: probe of the early universe and high energy physics



CC and Figueroa arXiv:1801.04268

$T_{\rm QCD} \sim 100\,{\rm MeV}$ $\quad \ell_* H_* \sim 1$ $\longrightarrow$ $f \sim 10\,{\rm nHz}$ PTA

$T_{\rm EW} \sim 100\,{\rm GeV}$ $\quad \ell_* H_* \sim 0.01$ $\longrightarrow$ $f \sim {\rm mHz}$ LISA

Stochastic gravitational wave background

GENERAL: the incoherent superposition of sources that cannot be individually resolved

- Astrophysical: binaries too numerous and with too low SNR to be identified
- Cosmological: GW signal from many independent horizon volumes: $h_{ij}(\mathbf{x},t)$ must be treated as a random variable
- Notable exception *SGWB from inflation*: intrinsic quantum fluctuations that become squeezed (effectively classical) and stochastic outside the horizon
- From the "a-causal" initial conditions from inflation, use the ergodic hypothesis: the ensemble average can be substituted with a volume average, and identified with the one necessary to define the GW energy momentum tensor

$$\rho_{GW} = \frac{\langle \dot{h}_{ij} \dot{h}^{ij} \rangle}{32\pi G}$$

Stochastic gravitational wave background

The SGWB is in general homogenous and isotropic, unpolarised and gaussian

As the FLRW space-time

$$\langle h_{ij}(\mathbf{x},\eta_1)\, h_{lm}(\mathbf{y},\eta_2)\rangle = F_{ijlm}(|\mathbf{x}-\mathbf{y}|,\eta_1,\eta_2)$$

Certainly some *induced anisotropy*, e.g. the dipole with respect to the cosmological frame

More challenging to detect than the "monopole"
-> see talk by Carlo, Jishnu

If the sourcing process preserves parity

$$\langle h_{+2}(\mathbf{k},\eta)h_{+2}(\mathbf{k},\eta) - h_{-2}(\mathbf{k},\eta)h_{-2}(\mathbf{k},\eta)\rangle = \langle h_{+}(\mathbf{k},\eta)h_{\times}(\mathbf{k},\eta)\rangle = 0$$

Helicity basis $e_{ij}^{\pm 2} = \frac{e_{ij}^{+} \pm i\, e_{ij}^{\times}}{2}$

There are exceptions!

Central limit theorem: the signal comes from the superposition of many independent regions

Stochastic gravitational wave background

Power spectrum of the GW energy density

$$\rho_{\rm GW} = \frac{\langle \dot{h}_{ij}(\mathbf{x},t)\,\dot{h}_{ij}(\mathbf{x},t)\rangle}{32\pi G} = \int_0^{+\infty} \frac{dk}{k}\,\frac{d\rho_{\rm GW}}{d\log k}$$

- The source has stopped operating -> *freely propagating sub-Hubble modes*

$$h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\,e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\,e^{-ik\eta}$$

- The oscillations can be *time averaged*

$$\langle h_r(\mathbf{k},\eta)\,h_p^*(\mathbf{q},\eta)\rangle = \frac{1}{a^2(\eta)}[\langle A_r(\mathbf{k})\,A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\,B_p^*(\mathbf{q})\rangle]$$

Coefficients from the source

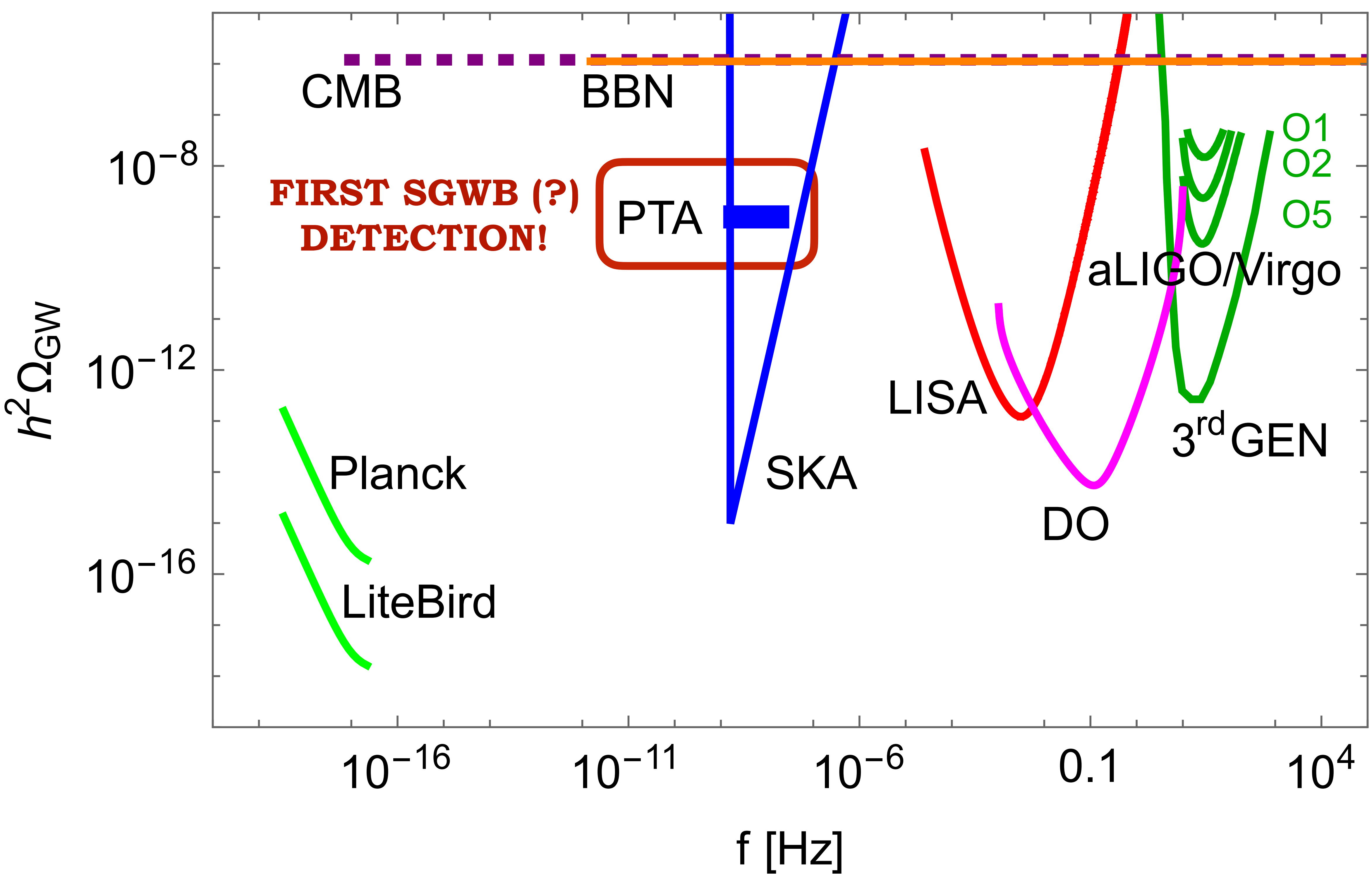
- GW energy density scales like radiation (free massless particles) $\rho_{\rm GW} \propto a(\eta)^{-4}$

- The expansion of the universe is negligible over the time of the measurement so that the SGWB appears *stationary in time*

- One can F.T. in time as well $f = \frac{1}{2\pi}\frac{k}{a_0}$

$$\Omega_{\rm GW}(f) = \frac{4\pi^2}{3H_0^2}\,f^2\,h_c^2(f)$$

What is/will be known about a stochastic GW background:



Are there primordial SGWBs to populate this diagram?

Examples of GW sources in the early universe :

- irreducible SGWB from inflation
  - also sourced by second order scalar perturbations
- beyond the irreducible SGWB from inflation
  - particle production during inflation (scalar, gauge fields… coupled to the inflaton)
  - spectator fields
  - breaking symmetries (space-dependent inflaton, massive graviton)
  - modified gravity during inflation (massive GWs with $c \neq 1$)
  - primordial black holes
  - …
- preheating and non-perturbative phenomena
  - parametric amplification of bosons/fermions
  - symmetry breaking in hybrid inflation
  - decay of flat directions
  - oscillons
  - …
- first order phase transition
  - true vacuum bubble collision
  - sound waves
  - (M)HD turbulence
  - …
- cosmic topological defects
  - irreducible SGWB from topological defect networks
  - decay of cosmic string loops
  - …

CC and Figueroa arXiv:1801.04268

SGWB from a stochastic source in the radiation era

$$H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = 16\pi G\, a^3\, \Pi_r(\mathbf{k},\eta)$$

Possible sources of tensor anisotropic stress in the early universe:

- Scalar field gradients $\Pi_{ij} \sim [\partial_i \phi \partial_j \phi]^{TT}$
- Bulk fluid motion $\Pi_{ij} \sim [\gamma^2 (\rho + p) v_i v_j]^{TT}$
- Gauge fields $\Pi_{ij} \sim [-E_i E_j - B_i B_j]^{TT}$
- Second order scalar perturbations, Π_{ij} from a combination of $\partial_i \Psi, \partial_i \Phi$
- ...

SGWB from a **FAST** stochastic source in the radiation era

Typical example: first order phase transition

Suppose the source operates in a time interval $\eta_{\rm fin}$ - $\eta_{\rm in}$ in the radiation dominated era

$$H_r^{\rm rad}(\mathbf{k},\eta<\eta_{\rm fin}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)^3\, \sin[k(\eta-\tau)]\, \Pi_r(\mathbf{k},\tau)$$

Matching at $\eta_{\rm fin}$ with the homogeneous solution to find the GW signal today

$$H_r^{\rm rad}(\mathbf{k},\eta>\eta_{\rm fin}) = A_r^{\rm rad}(\mathbf{k})\cos(k\eta) + B_r^{\rm rad}(\mathbf{k})\sin(k\eta)$$

$$A_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \sin(-k\tau)\, \Pi_r(\mathbf{k},\tau),$$

$$B_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \cos(k\tau)\, \Pi_r(\mathbf{k},\tau)$$

SGWB from a **FAST** stochastic source in the radiation era

GW amplitude power spectrum today for modes $k\eta_0 \gg 1$

$$\langle h_r(\mathbf{k},\eta_0)\, h_p^*(\mathbf{q},\eta_0)\rangle = \frac{1}{a_0^2}[\langle A_r(\mathbf{k})\, A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\, B_p^*(\mathbf{q})\rangle]$$

freely propagating sub-Hubble modes

GW energy density power spectrum today

$$\frac{d\rho_{\mathrm{GW}}}{d\log k}(k,\eta_0) \sim \frac{G}{a_0^4}\, k^3 \int_{\eta_{\mathrm{in}}}^{\eta_{\mathrm{fin}}} d\tau\, a^3(\tau) \int_{\eta_{\mathrm{in}}}^{\eta_{\mathrm{fin}}} d\zeta\, a^3(\zeta)\, \cos[k(\tau-\zeta)]\, \Pi(k,\tau,\zeta)$$

Anisotropic stress power spectral density *at unequal time*

$$\langle \Pi_r(\mathbf{k},\tau)\, \Pi_p^*(\mathbf{q},\zeta)\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{q})\, \delta_{rp}\, \Pi(k,\tau,\zeta)$$

SGWB from a **FAST** stochastic source in the radiation era

GW amplitude power spectrum today for modes $k\eta_0 \gg 1$

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a_*^3 $\quad a_*^3 \quad \simeq 1 \quad \Pi(k)$

SUPPOSE:

$\Delta\eta = \eta_{\rm fin} - \eta_{\rm in} \ll \mathcal{H}_*^{-1}$ $\qquad k\eta_{\rm in} \ll 1$ $\qquad \Pi(k,\tau,\eta)$ constant over $\Delta\eta$

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) \sim h^2\Omega^0_{\rm rad}\left(\frac{g_0}{g_*}\right)^{\frac{1}{3}}(\Delta\eta\mathcal{H}_*)^2\left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2(k\ell_*)^3\tilde{P}_{\rm GW}(k)$$

↓

From the time integrals

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) \sim h^2\Omega^0_{\rm rad}\left(\frac{g_0}{g_*}\right)^{\frac{1}{3}}(\Delta\eta\mathcal{H}_*)^2\left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2(k\ell_*)^3\tilde{P}_{\rm GW}(k)$$

$\underbrace{h^2\Omega_{\rm GW}}$: $\mathcal{O}(10^{-9})$ — Value detected at PTA

$\underbrace{h^2\Omega^0_{\rm rad}}$: $\mathcal{O}(10^{-6})$ — Factor depending slightly on the generation epoch through the number of relativistic d.o.f.

$\underbrace{(\Delta\eta\mathcal{H}_*)^2}$: $\mathcal{O}(10^{-3})$

Value for detection at LISA

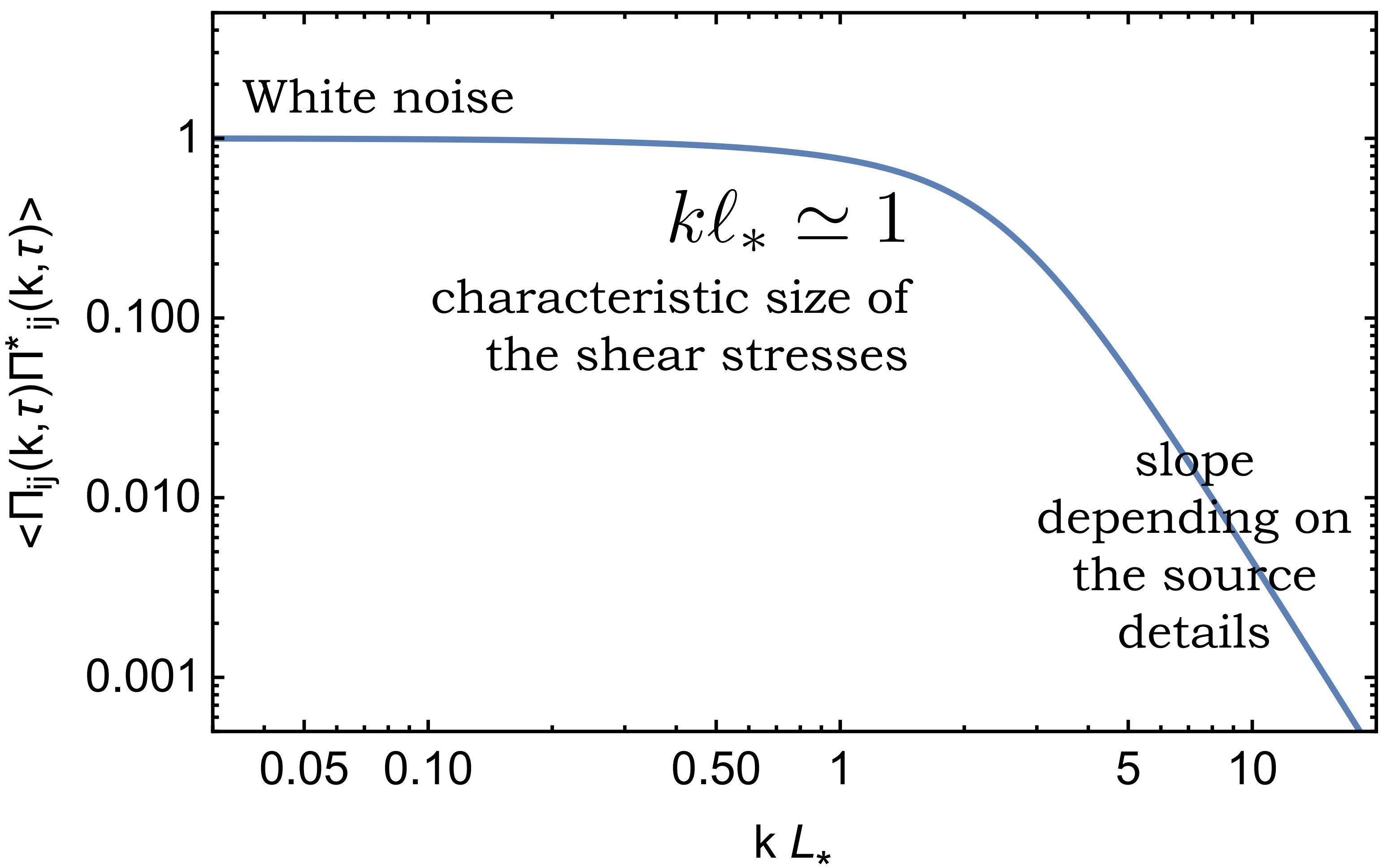
$\mathcal{O}(10^{-11})$ $\mathcal{O}(10^{-6})$ $\mathcal{O}(10^{-5})$

Only slow, very anisotropic processes have the chance to generate detectable SGWB signals for sub-Hubble sources

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) \sim h^2\Omega^0_{\rm rad}\left(\frac{g_0}{g_*}\right)^{\frac{1}{3}}(\Delta\eta\mathcal{H}_*)^2\left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2(k\ell_*)^3\tilde{P}_{\rm GW}(k)$$



Independent on k for large enough scales (uncorrelated)

$$\ell_* \leq H_*^{-1}$$

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) \sim h^2\Omega^0_{\rm rad}\left(\frac{g_0}{g_*}\right)^{\frac{1}{3}}(\Delta\eta\mathcal{H}_*)^2\left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2(k\ell_*)^3\tilde{P}_{\rm GW}(k)$$

$$1/\eta_0 \ll \underbrace{k \ll \mathcal{H}_*}_{\text{Range of validity of the solution}} \ll \underbrace{1/(a_*\ell_*)}_{\text{Causality of the sourcing process}}$$

Range of validity of the solution

Causality of the sourcing process

$$\Omega_{\rm GW}(k) \propto (k\ell_*)^3$$

SGWB from a **FAST** stochastic source in the radiation era

- Characteristic time of the source evolution $$\delta t_c = \frac{\ell_*}{v_{\rm rms}}$$
- Characteristic time of the GW production from the Green's function: $$\delta t_{\rm gw} \sim \frac{1}{k}$$
- GW production goes faster than source evolution for all relevant wave-numbers including the spectrum peak $$k > \frac{v_{\rm rms}}{\ell_*}$$
- One assumes that the source is constant in time for a finite time interval (which can be larger than the Hubble time) $$\delta t_{\rm fin} \sim \mathcal{N} \delta t_c$$
- One can then easily integrate to find the GW spectrum

$$h^2 \Omega_{\rm GW}(k, \eta_0) \propto h^2 \Omega_{\rm rad}^0 \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} \left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2 (k\ell_*)^3 \tilde{P}_{\rm GW}(k) \begin{cases} \ln^2[1 + \mathcal{H}_* \delta t_{\rm fin}] & \text{if } k\,\delta t_{\rm fin} < 1 \\ \ln^2[1 + (k/\mathcal{H}_*)^{-1}] & \text{if } k\,\delta t_{\rm fin} \geq 1 \end{cases}$$

A. Roper Pol et al, arXiv:2201.05630

SGWB from a stochastic source in the radiation era: example (phase transition)

$$k_{\text{peak}} \simeq 4\pi/\ell_*$$

$$\Omega_{\text{gw, peak}} \propto \left(\frac{\rho_\Pi}{\rho_{\text{rad}}}\right)^2 (\mathcal{H}_* \ell_*)^2$$

Transition from k^3 to k^1 at $k \simeq 1/\delta t_{\text{fin}}$

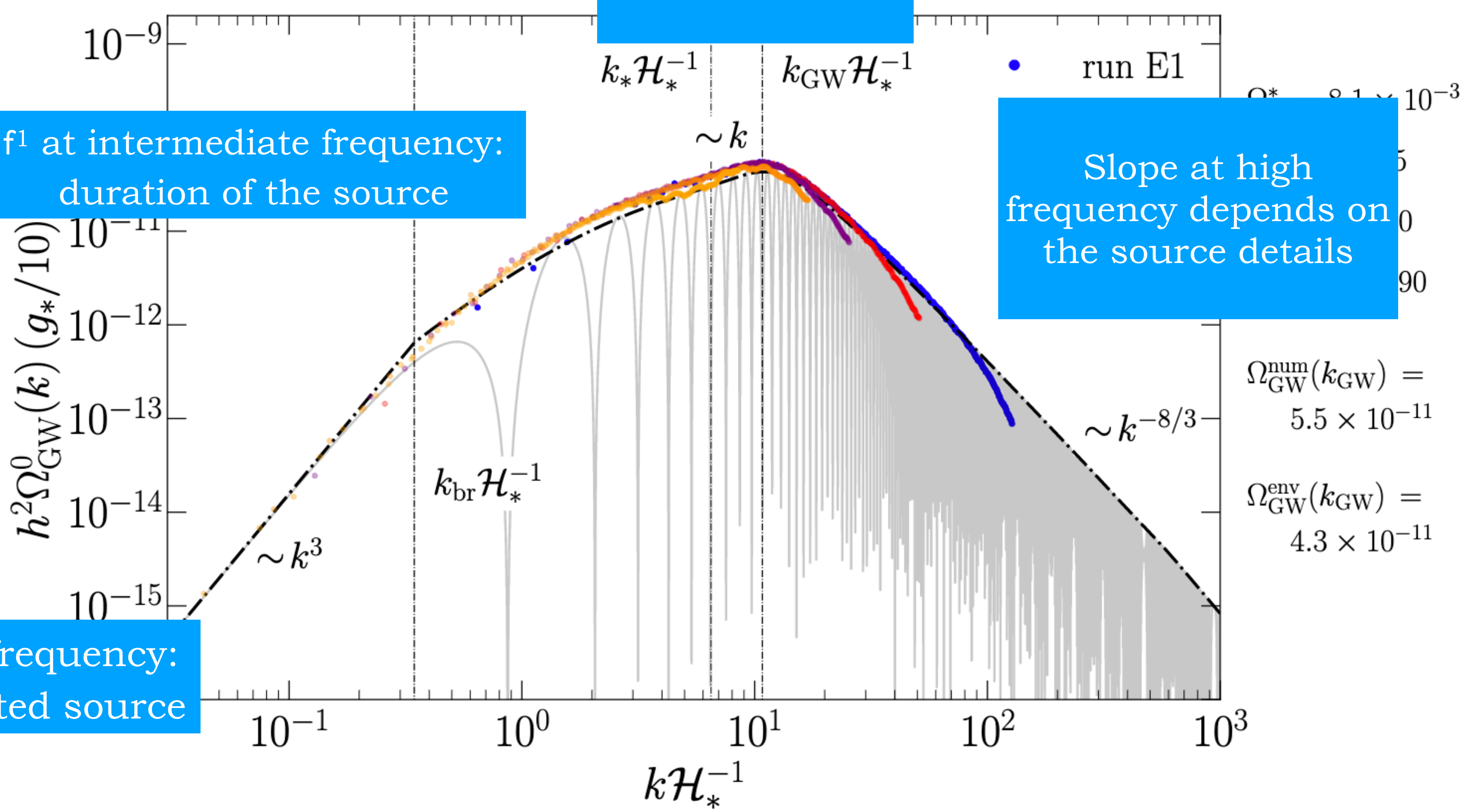
Can be smoother if $\delta t_{\text{fin}} > 1/\mathcal{H}_*$

Peak at f ~ $1/\ell_*$

f[1] at intermediate frequency: duration of the source

Slope at high frequency depends on the source details

f[3] at low frequency: uncorrelated source



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SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects

Suppose the source is operating continuously in the radiation dominated era

- No matching at the end time of the source
- No *free* sub-Hubble modes

$$H_r^{\rm rad}(\mathbf{k},\eta) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)^3\,\sin[k(\eta-\tau)]\,\Pi_r(\mathbf{k},\tau)$$

$$H_r(\mathbf{k},\eta) = a\,h_r(\mathbf{k},\eta)$$

$$\rho_{\rm GW} = \frac{\langle \dot{h}_{ij}(\mathbf{x},t)\,\dot{h}_{ij}(\mathbf{x},t)\rangle}{32\pi G} = \int_0^{+\infty}\frac{dk}{k}\,\frac{d\rho_{\rm GW}}{d\log k}$$

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta) \sim \frac{G}{a^4}\,k^3\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)\int_{\eta_{\rm in}}^{\eta} d\zeta\, a(\zeta)\,\mathcal{G}(k,\eta,\tau,\zeta)\,\Pi(k,\tau,\zeta)$$

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects

Suppose the source is operating continuously in the radiation dominated era

- Scaling (property of the topological defects network)
- Decays very fast in off-diagonal $k\tau \neq k\zeta$
- Decays as a power law on the diagonal $k\tau = k\zeta$

D. Figueroa et al, arXiv:1212.5458

$$\Pi(k,\tau,\zeta) \sim \frac{v^4}{\sqrt{\tau\zeta}}\,\mathcal{U}(k\tau,k\zeta)$$

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta) \sim \frac{G}{a^4}\,k^3 \int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau) \int_{\eta_{\rm in}}^{\eta} d\zeta\, a(\zeta)\, \mathcal{G}(k,\eta,\tau,\zeta)\, \Pi(k,\tau,\zeta)$$

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects

Suppose the source is operating continuously in the radiation dominated era

$$h^2\Omega_{\rm GW}(f) \sim h^2\Omega_{\rm rad}\left(\frac{v}{M_{\rm Pl}}\right)^4 F_{\rm RD}^{[\mathcal{U}]}(\infty)$$

TODAY FLAT SPECTRUM AT SUB-HORIZON MODES IN THE RADIATION ERA

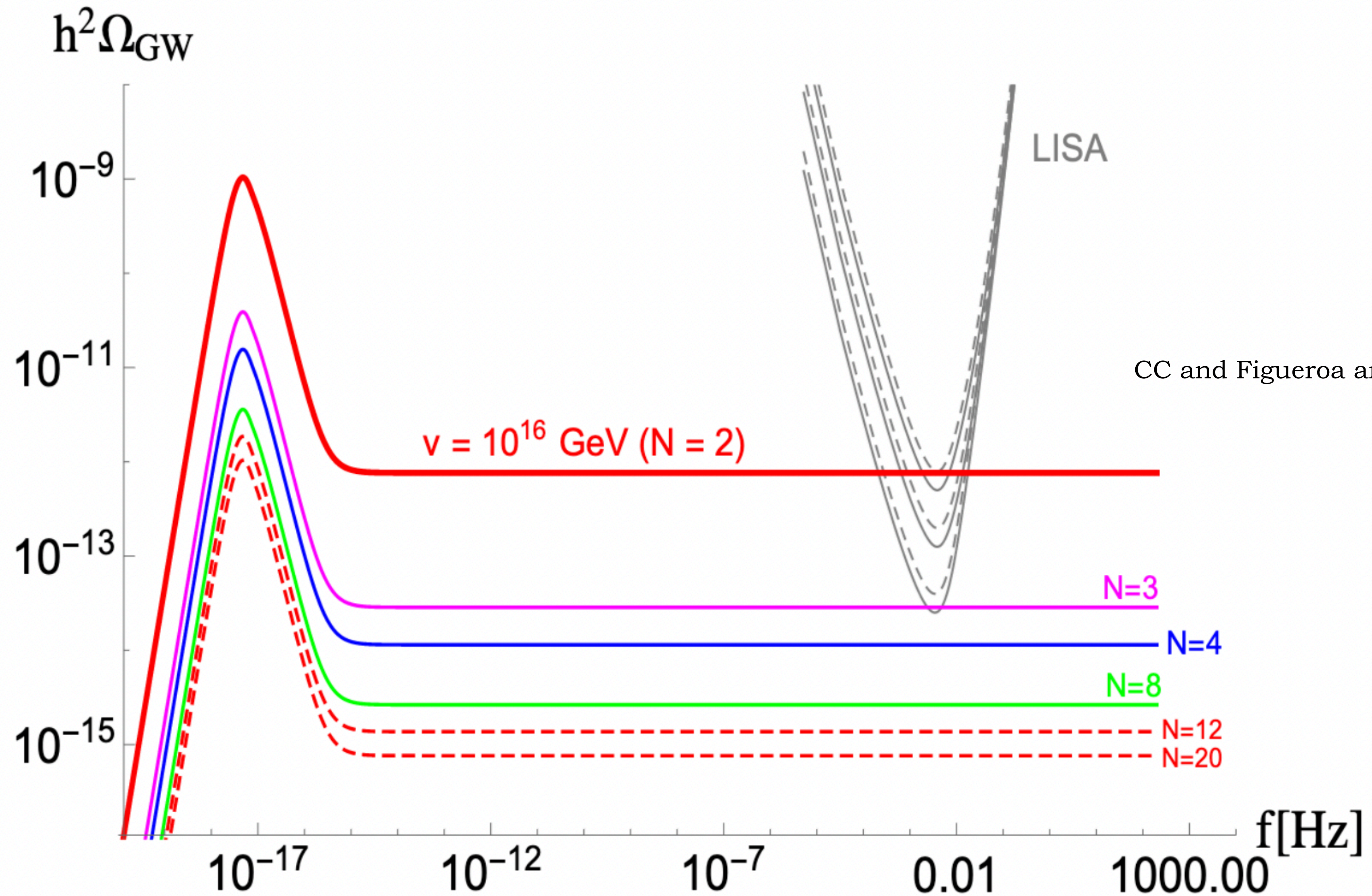
D. Figueroa et al, arXiv:1212.5458

Progressively independent on the upper bound

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta) \sim \Omega_{\rm rad}\frac{\rho_c}{a^4}\left(\frac{v}{M_{\rm Pl}}\right)^4 \int_{x_{\rm in}}^{x} dx_1 \int_{x_{\rm in}}^{x} dx_2\,\sqrt{x_1 x_2}\,\mathcal{G}(x,x_1,x_2)\,\mathcal{U}(x_1,x_2)$$

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects



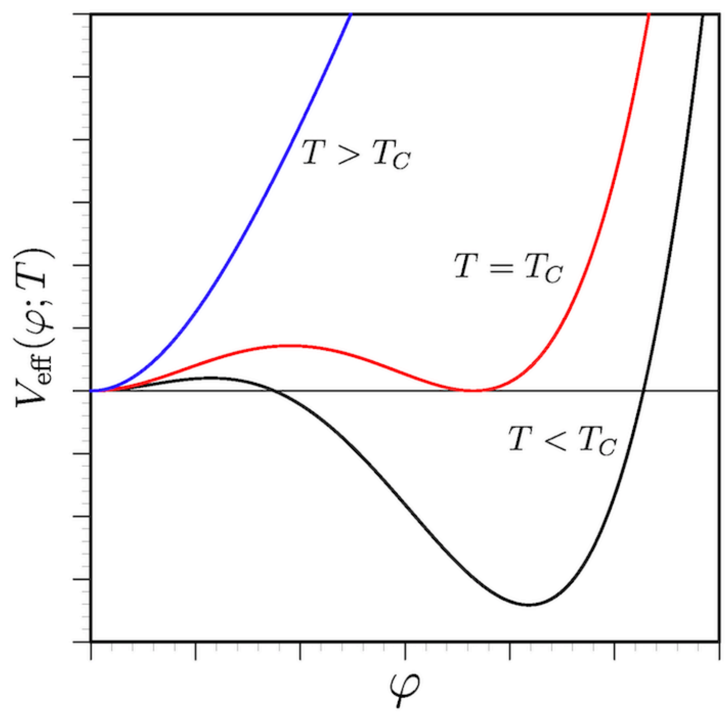
CC and Figueroa arXiv:1801.04268

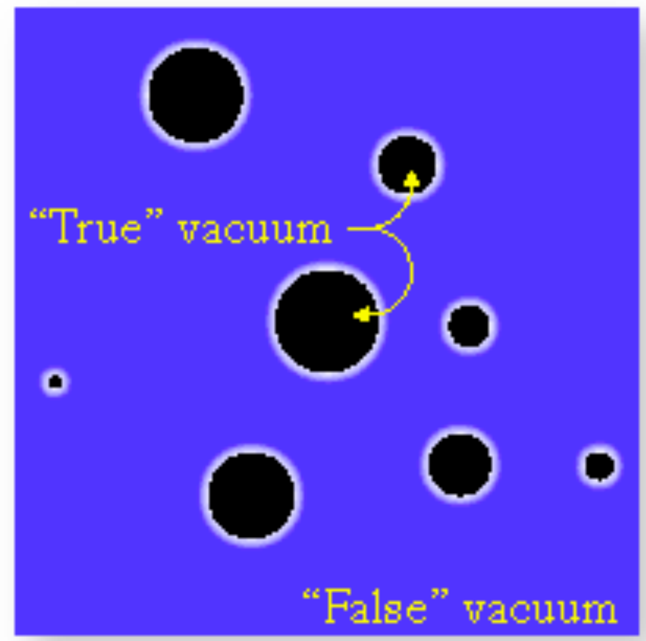
Examples of signals

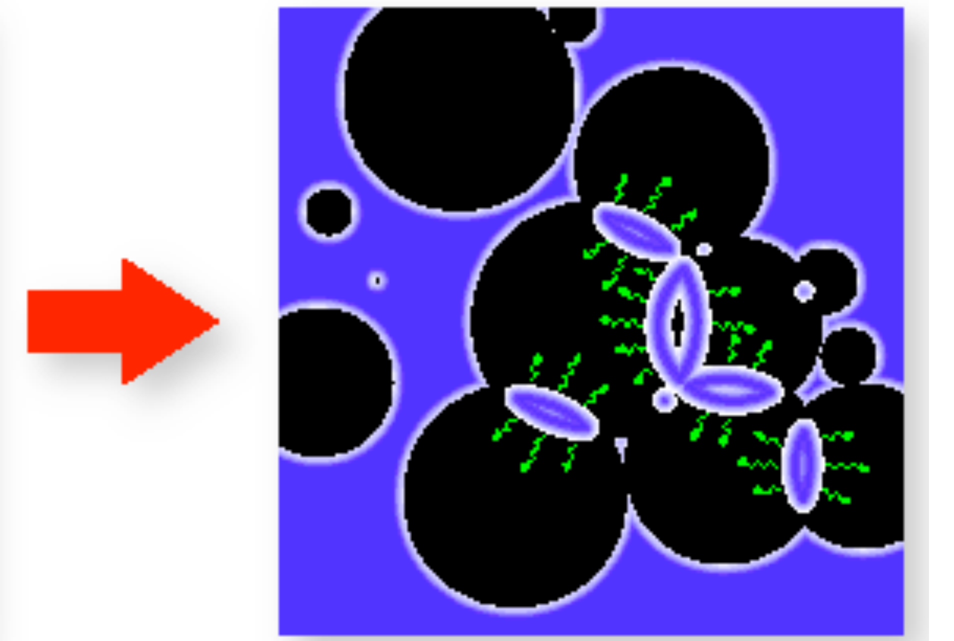
- First oder phase transitions

Sources of tensor anisotropic stress at a first order phase transition:

GW sourcing process $$\ddot{h}_{ij} + 3H\,\dot{h}_{ij} + k^2\,h_{ij} = 16\pi G\,\Pi_{ij}^{TT}$$





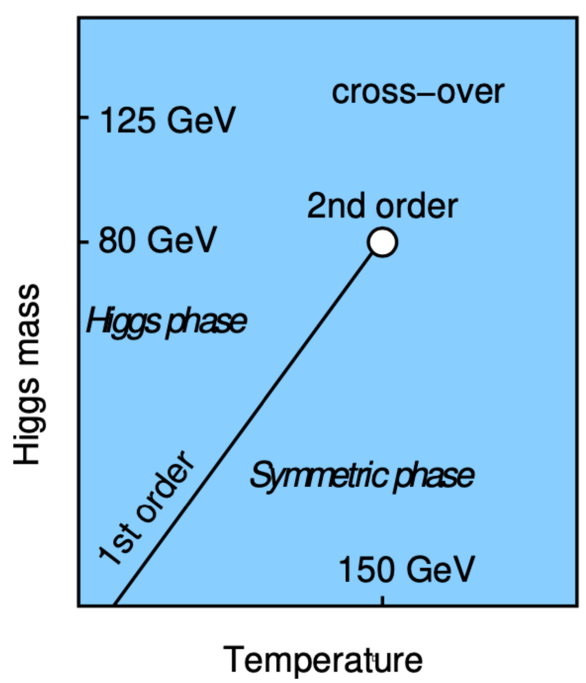


Several processes, rich phenomenology!

- Bubble collision (scalar field gradients) $$\Pi_{ij}^{TT} \sim [\partial_i\phi\partial_j\phi]^{TT}$$
- Bulk fluid motion $$\Pi_{ij}^{TT} \sim [\gamma^2(\rho+p)v_i v_j]^{TT}$$
- Electromagnetic fields $$\Pi_{ij}^{TT} \sim [-E_iE_j - B_iB_j]^{TT}$$

Electroweak phase transition: phase transition of the Higgs field, driven by the temperature decrease as the universe expands

Standard Model
of particle physics:
Cross-over
Negligible GW production



M. Hindmarsh et al, arXiv:2008.09136

Beyond the Standard Model:
First order phase transition
Possibly observable GW production

Examples of scenarios leading to observable signals:

- singlet/multiplet extensions of SM or MSSM (SUSY motivated or not)
- SM plus dimension six operator (EFT approach)
- Dark Matter sector uncoupled to the SM
- Warped extra dimensions
- …

CC et al arXvi:1512.06239

The SM plasma in thermal equilibrium generates a GW background, but very weak and peaking at the GHz: not observable in the near future

Ghiglieri and Laine arXiv:1504.02569

Parameters entering the GW signal

$$\Omega^*_{\rm GW}(f) = \tilde{\Omega}\,(\ell_* H_*)^2 \left(\frac{\rho_\Pi}{\rho^*_{\rm tot}}\right)^2 S(f)$$

$T_*,\ \alpha,\ \dfrac{\beta}{H_*}$ → Determined by the effective potential

$v_w(\alpha,\eta),\ \kappa(\alpha,\eta)$ → Determined by the bubble expansion hydrodynamics

If the PT is strong and non-linearities in the bulk fluid develop, another parameter adds: the fraction of kinetic energy which is in turbulent motions

$$\varepsilon = \frac{\kappa_{\rm turb}}{\kappa_v}$$

Most of these parameters are known (at least in principle) given a PT model

Parameters entering the GW signal

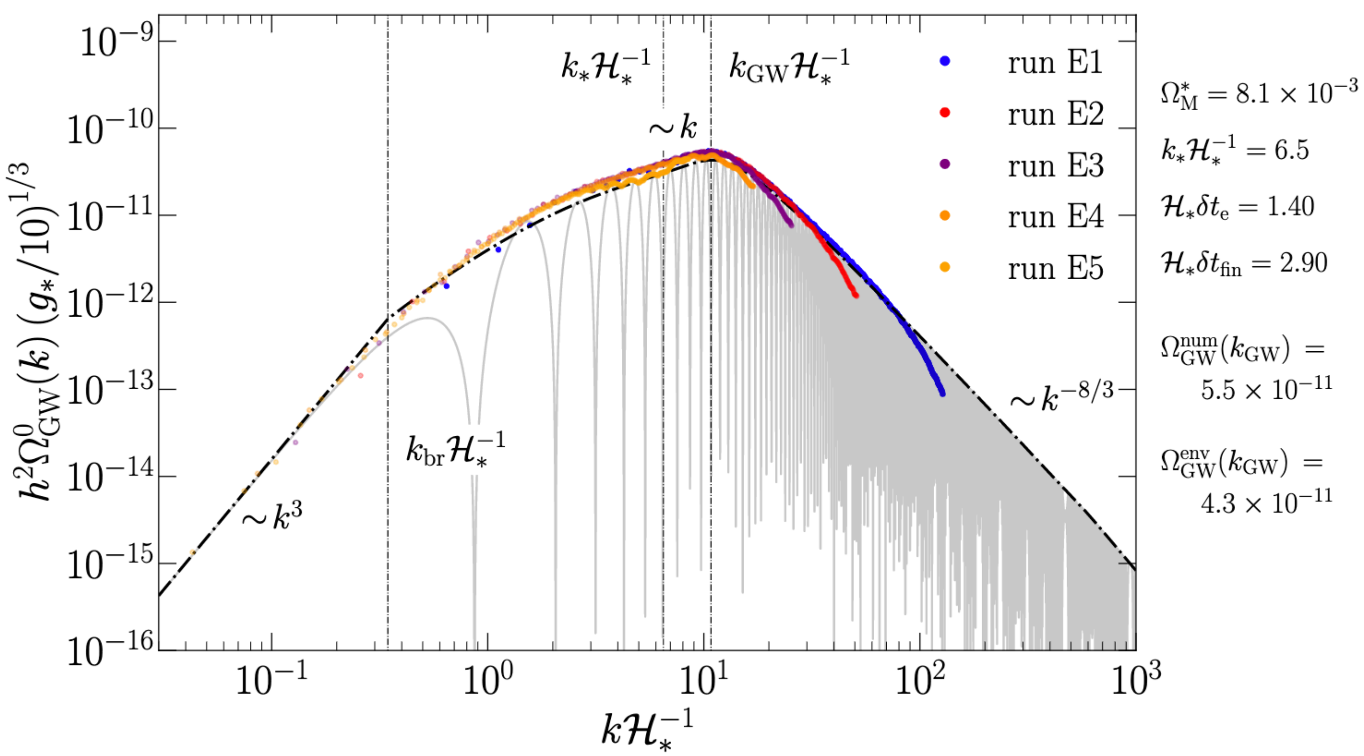
$$\Omega^*_{\rm GW}(f) = \tilde{\Omega}\,(\ell_* H_*)^2 \left(\frac{\rho_\Pi}{\rho^*_{\rm tot}}\right)^2 S(f)$$

How much kinetic energy is in anisotropies stresses?

What is the spectral shape of the GW signal as a function of frequency?

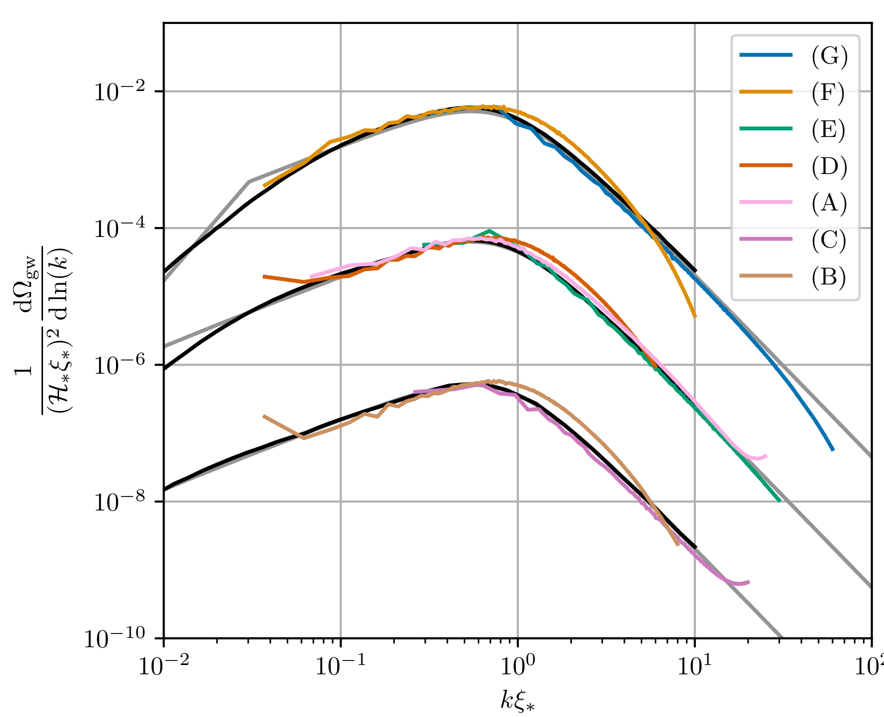
Determining both quantities often requires **numerical simulations**

GW signal from MHD turbulence from a simulation with the Pencil code, together with the analytical evaluation



A. Roper Pol et al, arXiv:2201.05630

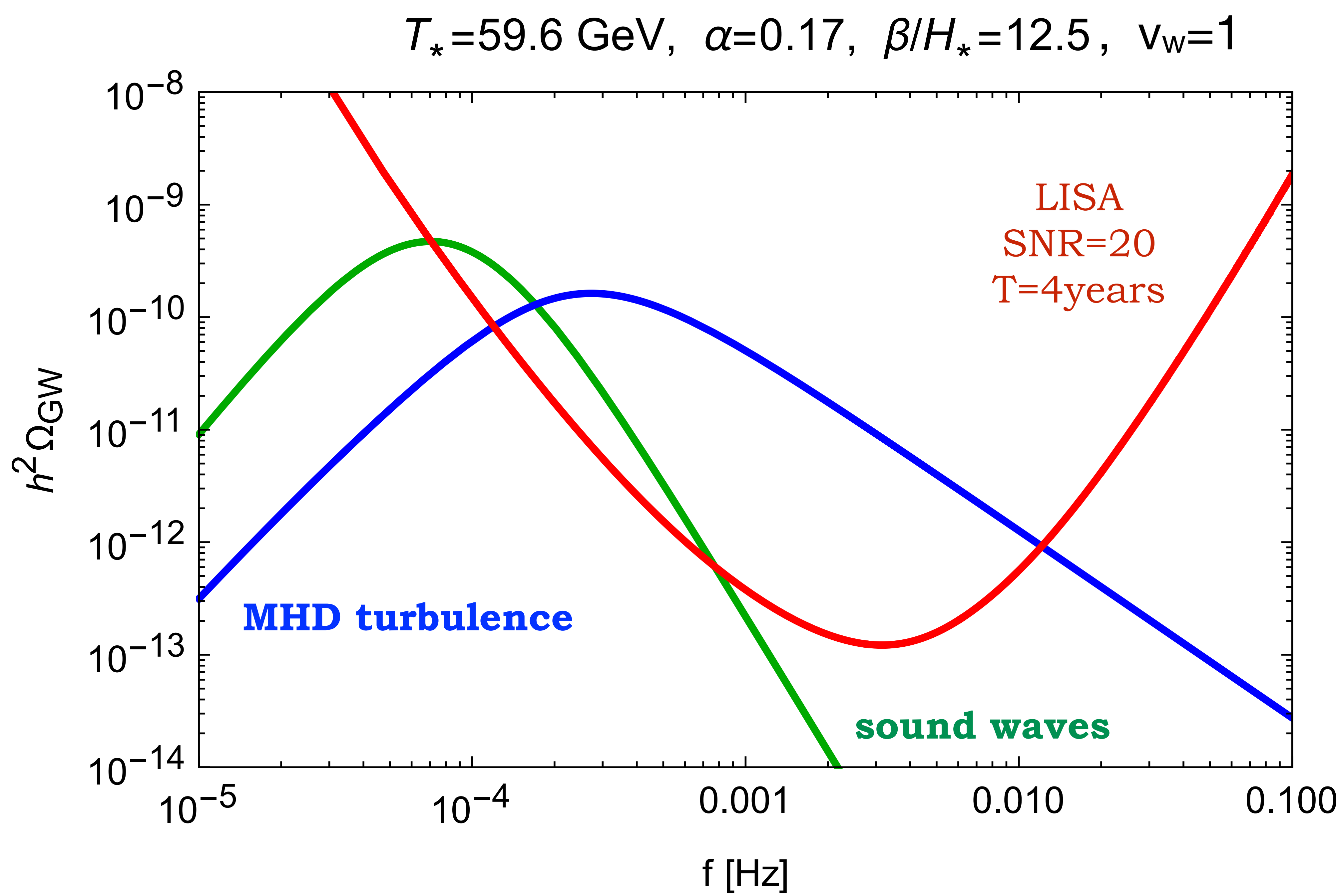
GW signal from kinetic turbulence from a simulation with the SCOTTS code, together with the analytical evaluation



P. Auclair et al, arXiv:2205.02588

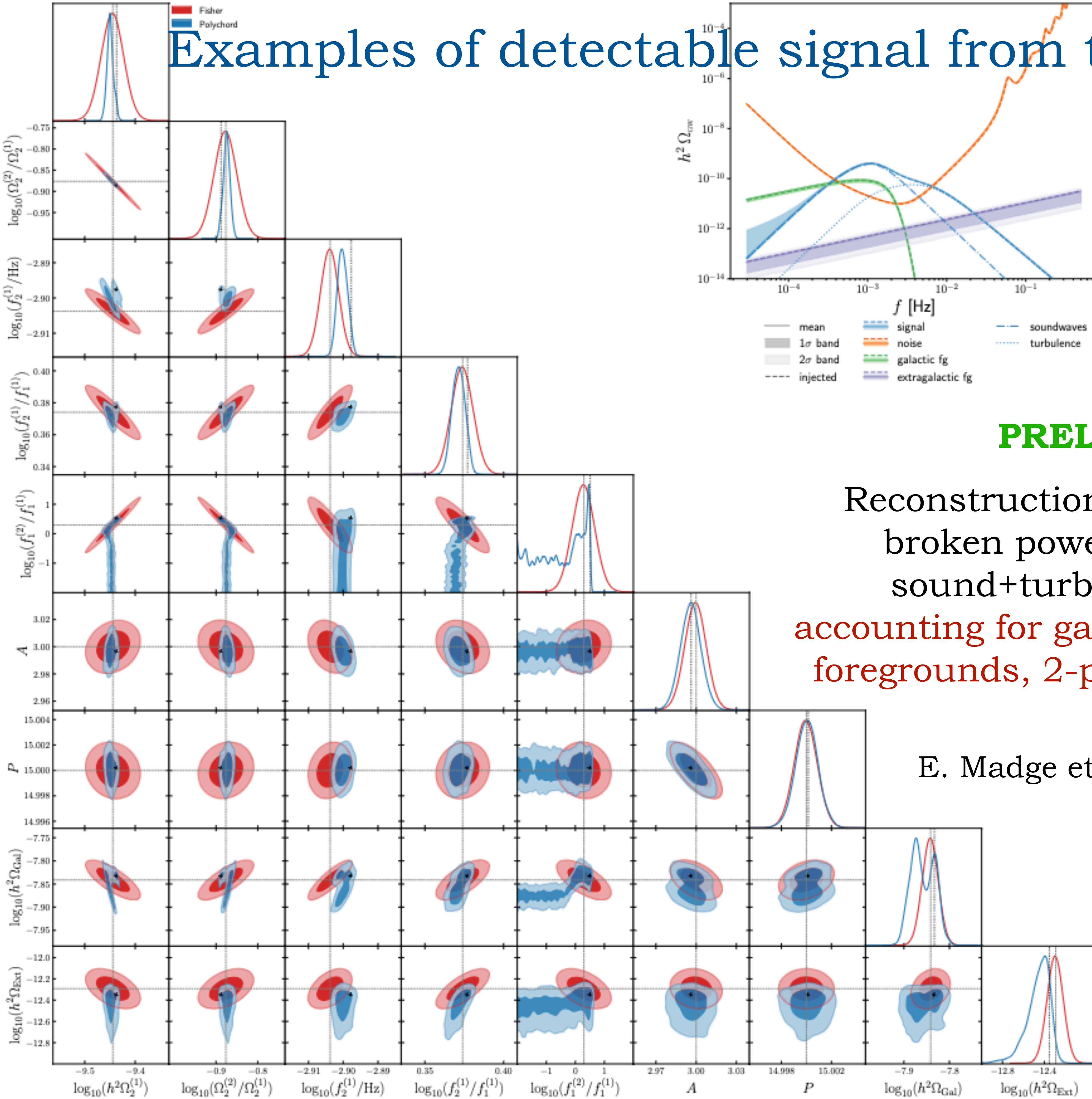
Examples of detectable signal from the EWPT

Just indicative: benchmark point from CC et al arXvi:1512.06239, singlet SM extension



$$\tau_{\rm nl} \sim \frac{\ell_*}{v_{\rm rms}} = \frac{0.54}{\mathcal{H}_*} \qquad \langle v^2 \rangle_{\rm turb} = 0.25 \, \langle v^2 \rangle_{\rm sound}$$

Examples of detectable signal from the EWPT

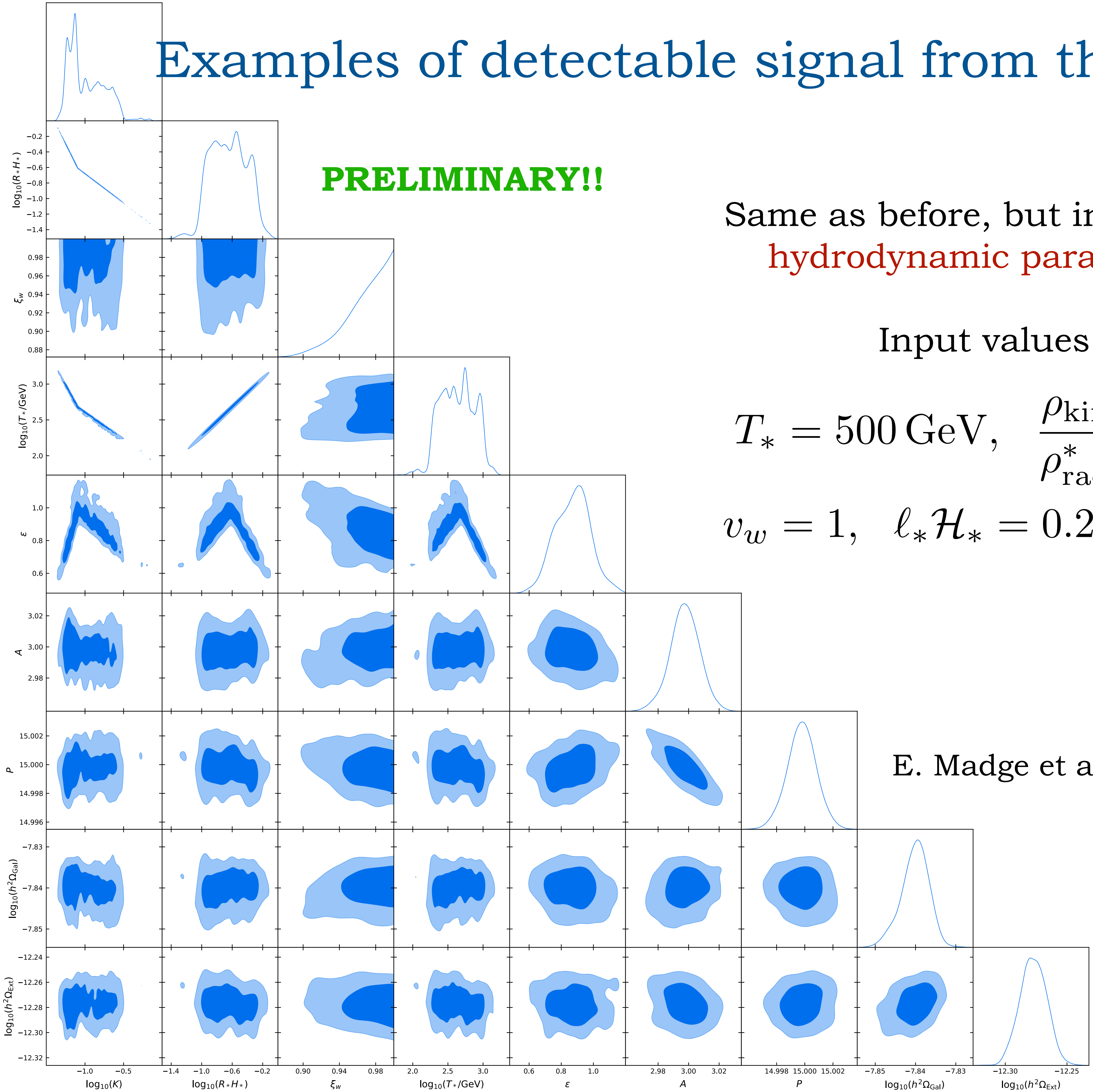


PRELIMINARY!!

Reconstruction of a generic double broken power law (inspired by sound+turbulence) with LISA accounting for galactic and extragalactic foregrounds, 2-parameter noise model

E. Madge et al, in preparation

Examples of detectable signal from the EWPT



PRELIMINARY!!

Same as before, but in terms of hydrodynamic parameters

Input values:

$$T_* = 500\,\mathrm{GeV}, \quad \frac{\rho_{\mathrm{kin}}}{\rho^*_{\mathrm{rad}}} = 0.08,$$

$$v_w = 1, \quad \ell_* \mathcal{H}_* = 0.25, \quad \varepsilon = 1$$

E. Madge et al, in preparation

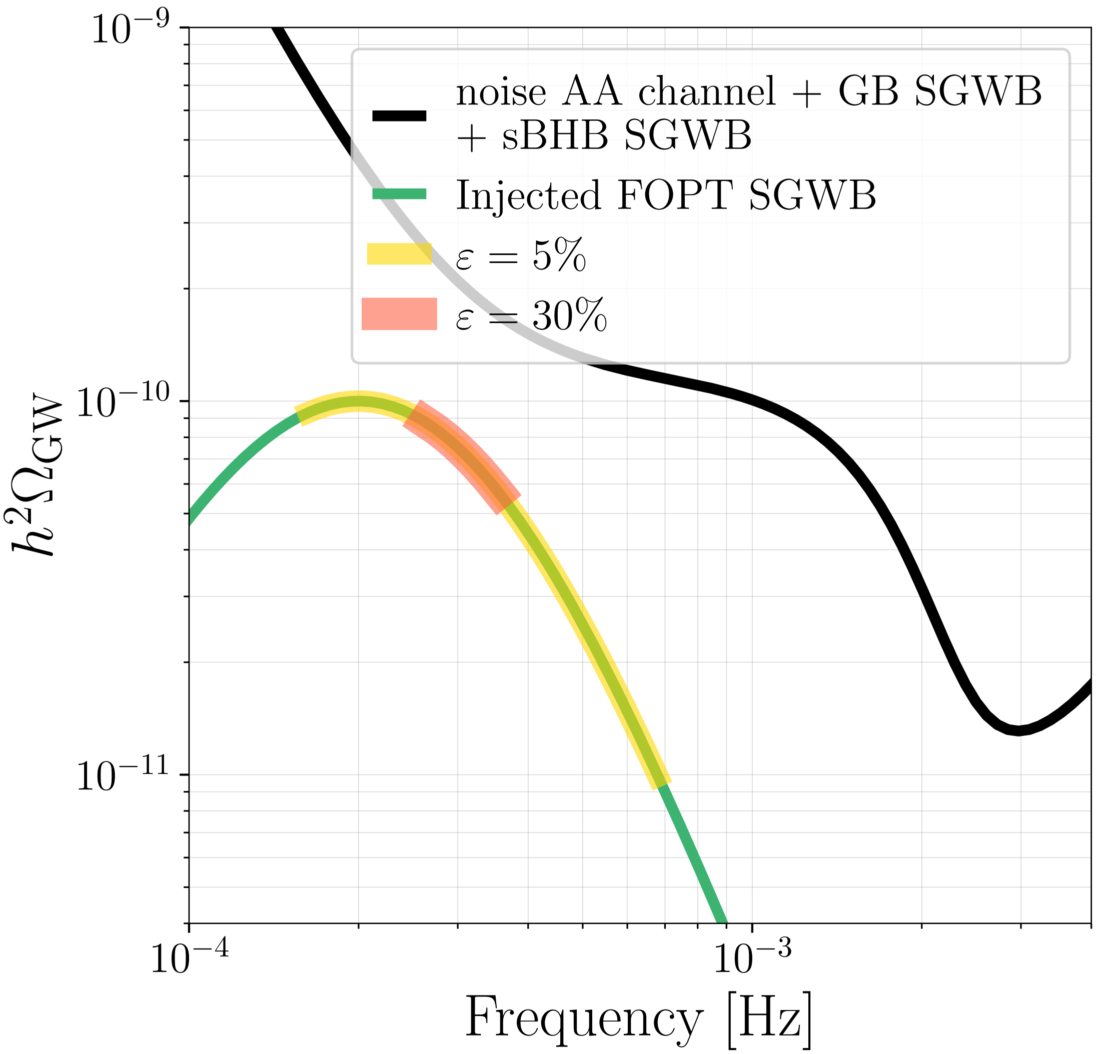
Examples of detectable signal from the EWPT

Is it possible to reconstruct the GW signal spectral shape, to identify that the source is a FOPT?

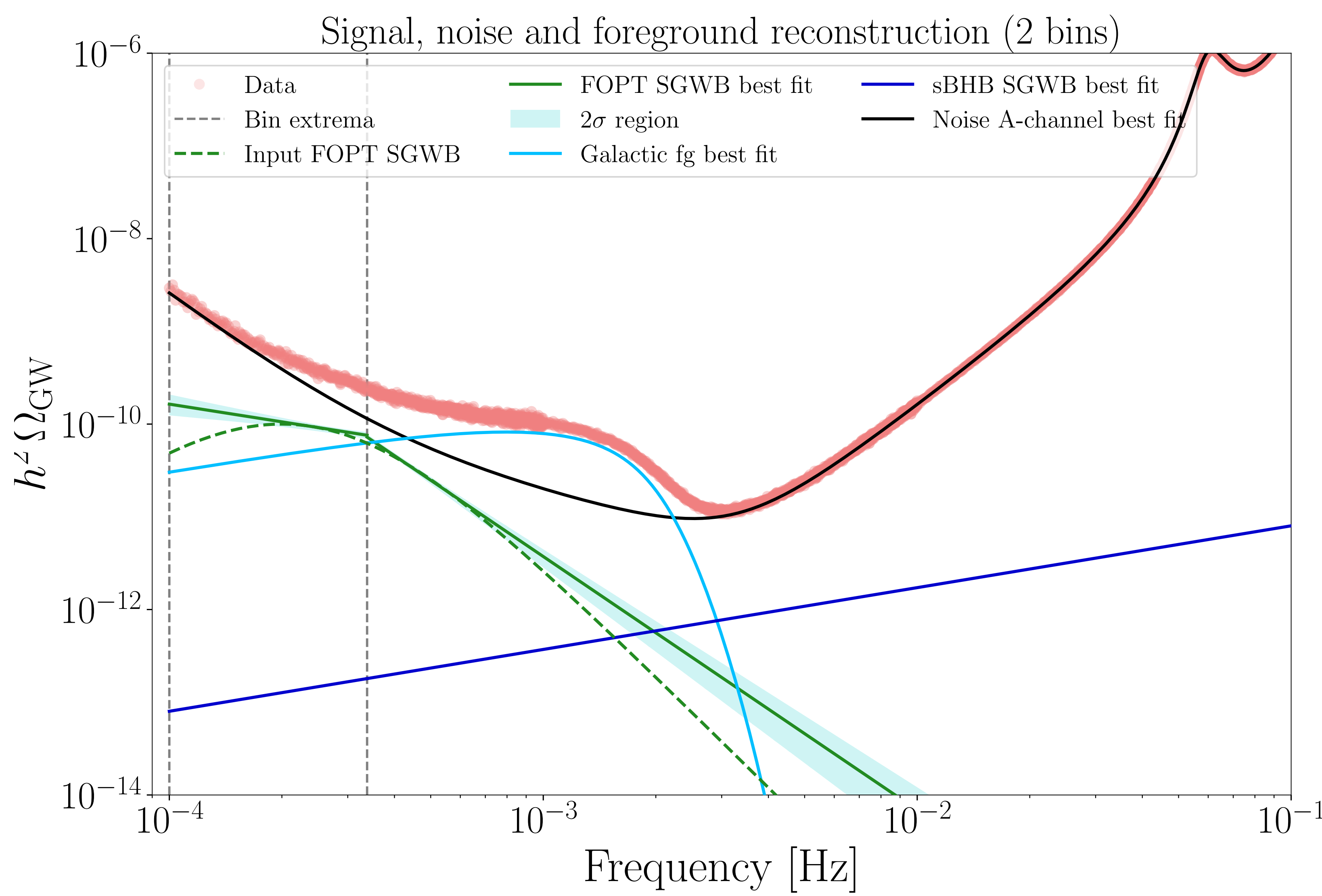
Signal from a singlet extension of SM setting

$$m_s = 0.94\,\text{GeV},\ \lambda_s = 1,\ \lambda_{hs} = 0.92$$

N. Karnesis, arXiv:1906.09027



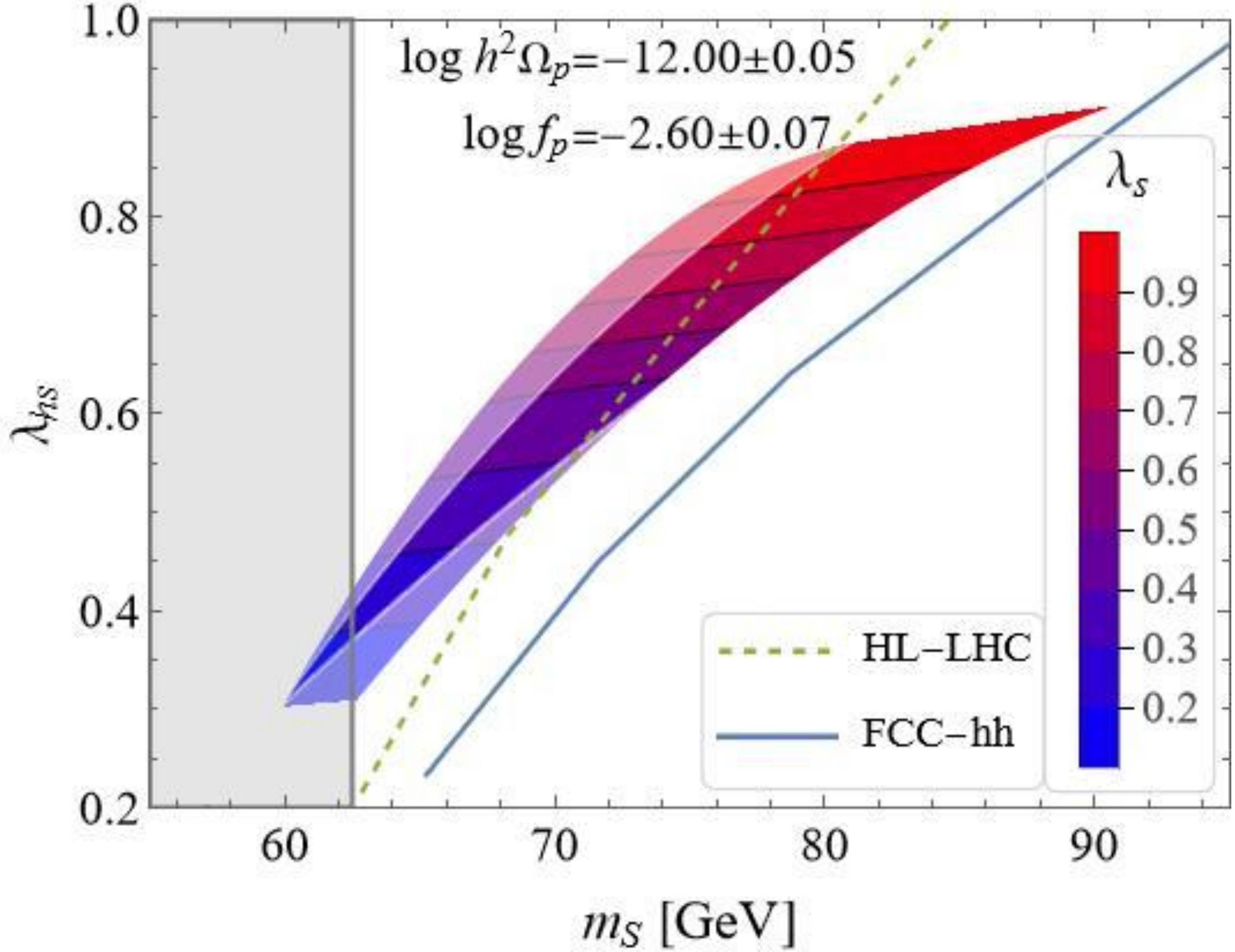
CC et al, arXiv:1906.09244, Flauger et al arXiv:2009.11845



Examples of detectable signal from the EWPT

Several model parameter values can correspond to the same GW signal, here assumed to be a single broken power law

E. Madge et al, in preparation

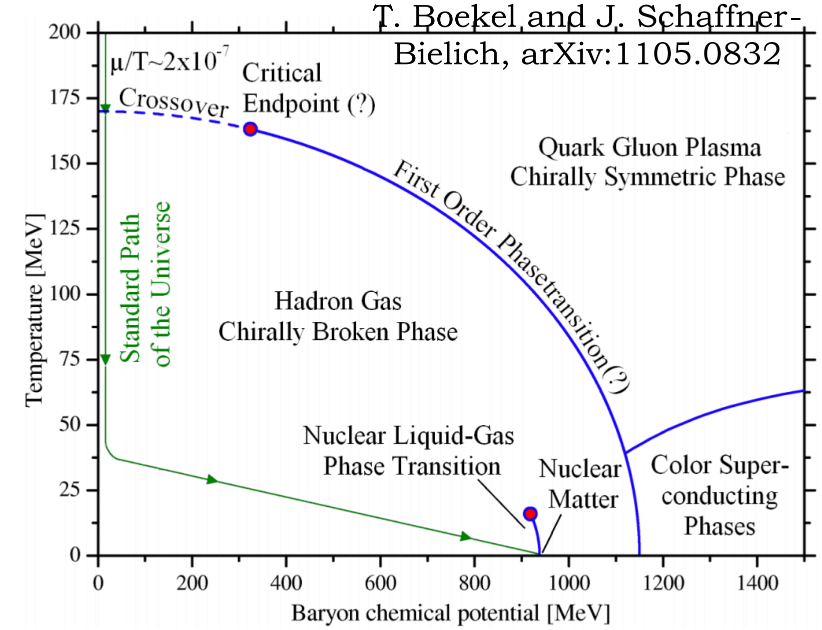


An example of possible detection at PTA?

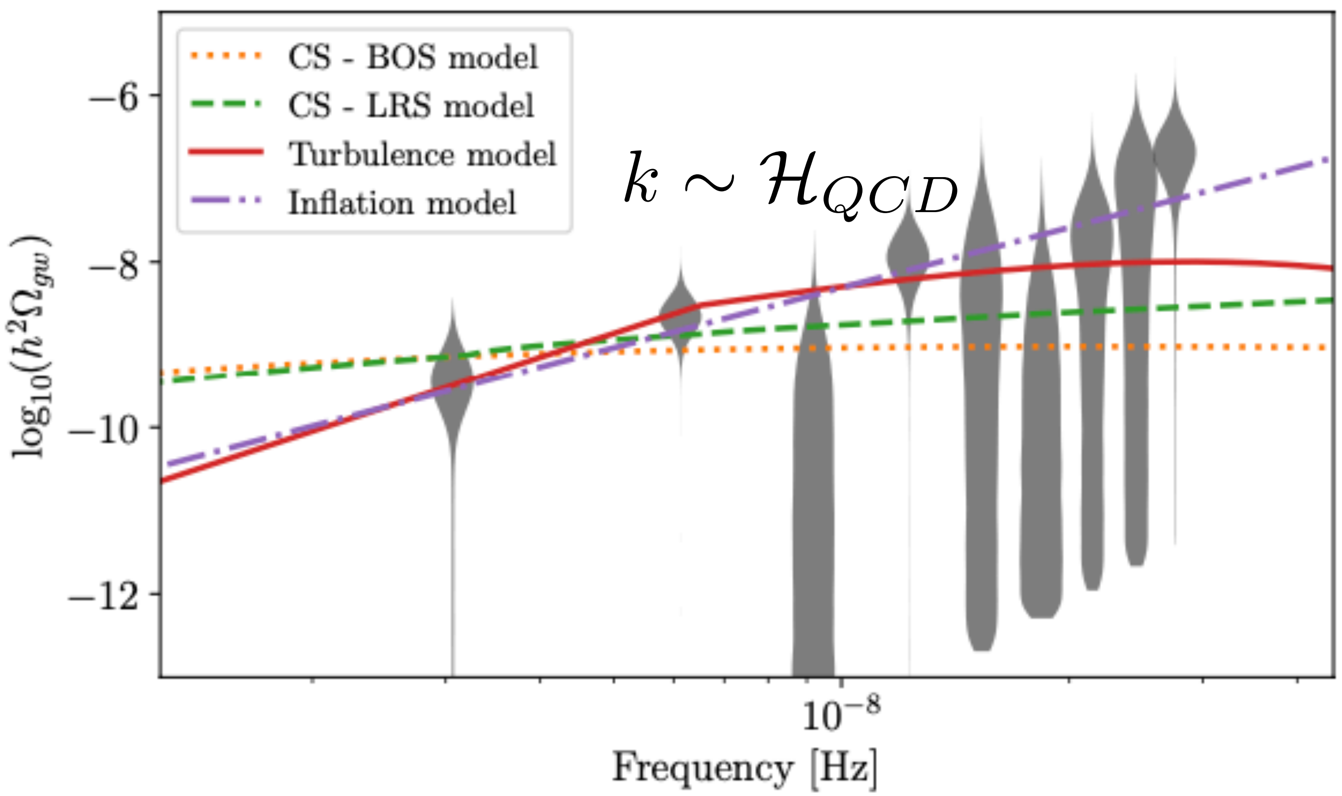
- In the Standard Model at zero baryon chemical potential **QCDPT** is a cross-over, negligible GW production
- It depends on the (uncertain) conditions of the early universe

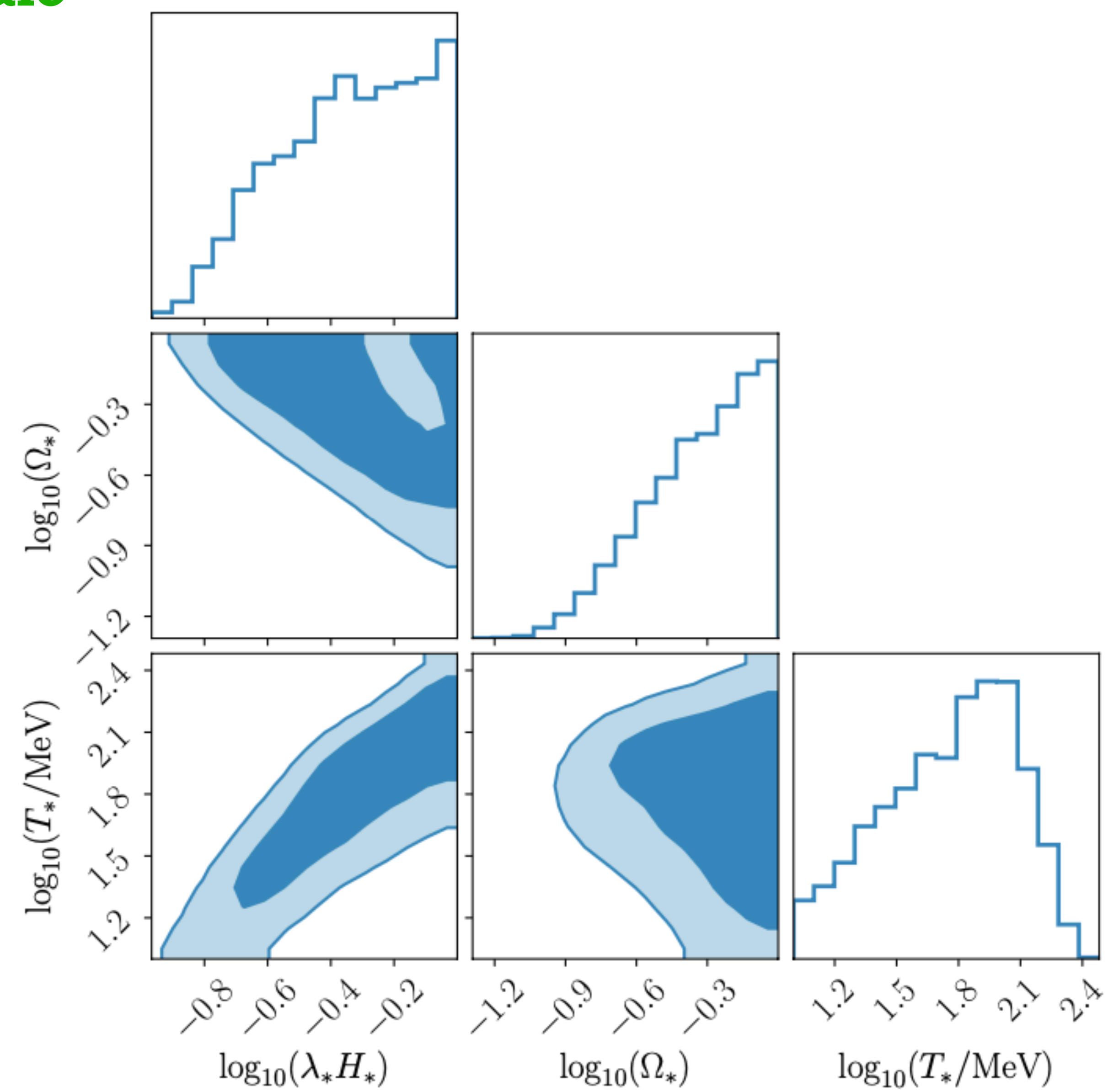
D. Schwarz and Stuke, arXiv:0906.3434
M. Middeldorf-Wygas et al, arXiv:2009.00036

T. Boekel and J. Schaffner-Bielich, arXiv:1105.0832



The PTA signal is compatible with GWs generated by MHD turbulence at the QCD scale





J. Antoniadis et al, arXiv:2306.16227
A. Neronov et al, arXiv:2009.14174
A. Roper Pol et al, arXiv:2201.05630

Examples of signals

- Inflation

GW signal from inflation

Amplification of tensor metric vacuum fluctuations by the exponential expansion

$$h''_r(\mathbf{k},\eta) + 2\,\mathcal{H}\,h'_r(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

- ✓ canonically normalised free field $v_\pm = a\,M_{Pl}\,h_\pm$
- ✓ quantisation
- ✓ homogeneous wave equation: harmonic oscillator with *time dependent* frequency

$$v''_\pm(t) + (k^2 - a^2 H^2) v_\pm(t) = 0$$

$k \gg a\,H$ sub-Hubble modes

$$\omega^2(t) = k^2$$

free field in vacuum
zero occupation number

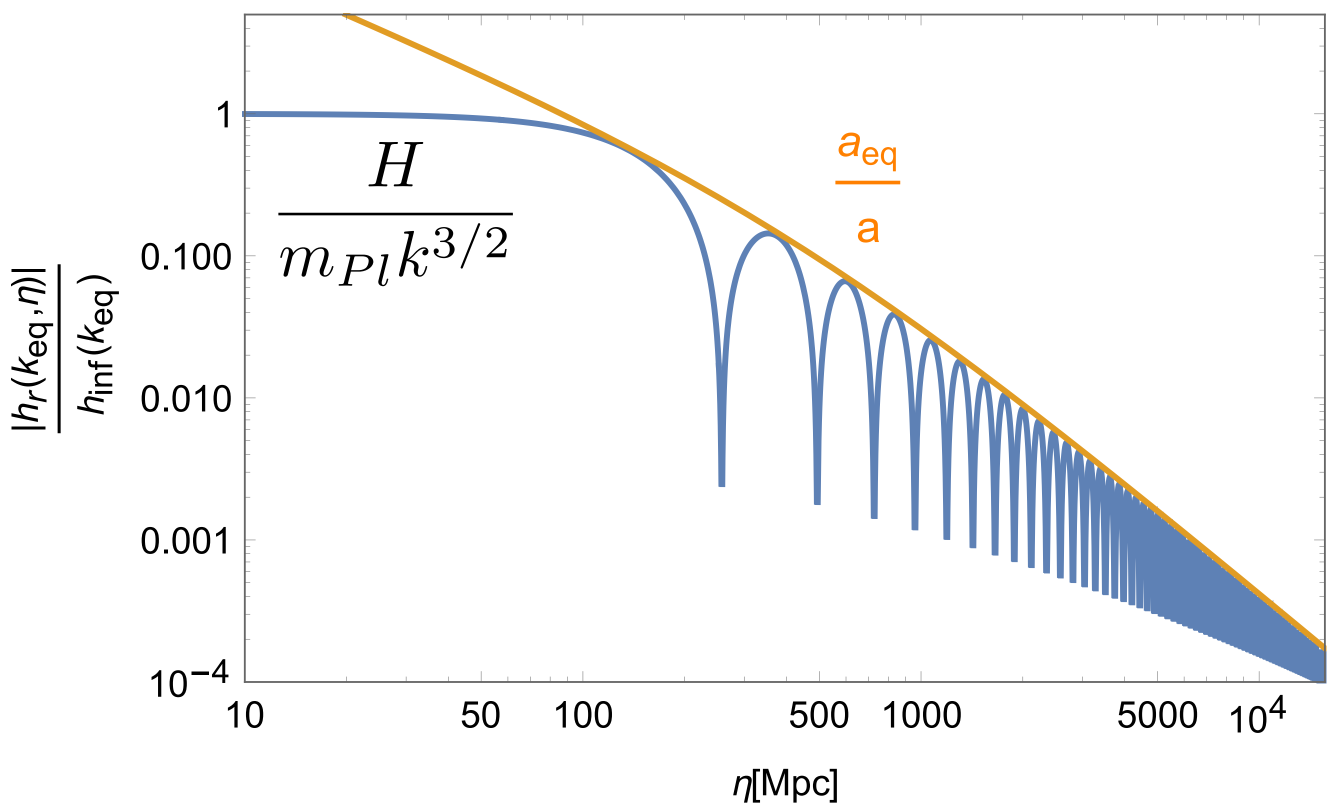
$k \ll a\,H$ super-Hubble modes

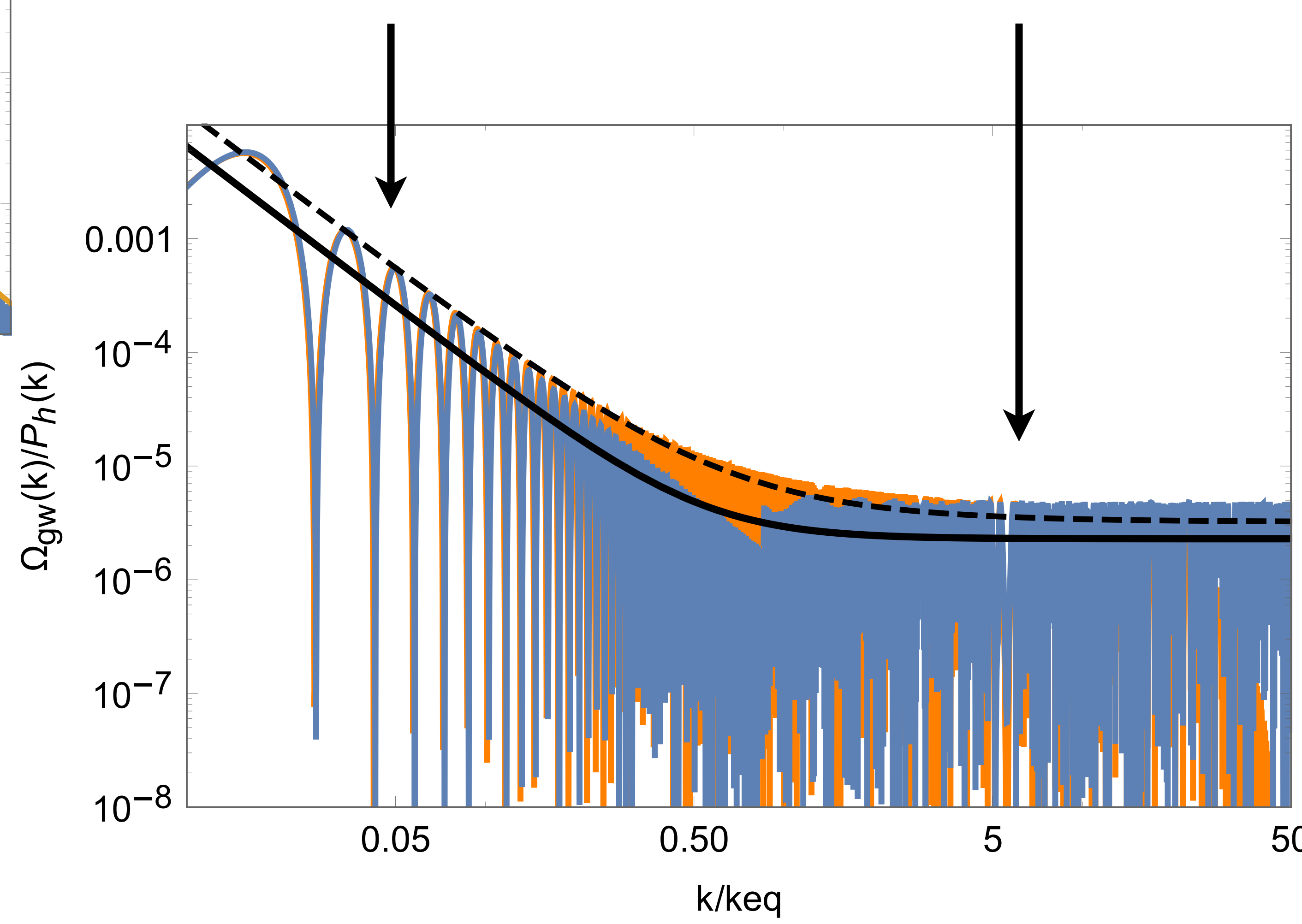
$$\omega^2(t) = -a^2 H^2$$

super-Hubble modes have very large occupation number

GW signal from (slow roll) inflation

- tensor spectrum $\mathcal{P}_h = \frac{2}{\pi}\frac{H^2}{m_{Pl}^2}\left(\frac{k}{aH}\right)^{-2\epsilon}$ $\quad \epsilon \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1$
- transfer function from inflation to today, as modes re-enter the Hubble horizon





$$\Omega_{\rm GW}(k,\eta_0) = \frac{[T'(k,\eta_0)]^2}{12 a_0^2 H_0^2}\mathcal{P}_h(k)$$

GW signal from (slow roll) inflation

- tensor spectrum $\mathcal{P}_h = \frac{2}{\pi}\frac{H^2}{m_{Pl}^2}\left(\frac{k}{aH}\right)^{-2\epsilon}$ $\epsilon \equiv \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \ll 1$

$$\Omega_{\rm GW}(f) = \frac{3}{128}\,\Omega_{\rm rad}\, r\, \mathcal{P}_\mathcal{R}^*\left(\frac{f}{f_*}\right)^{n_T}\left[\frac{1}{2}\left(\frac{f_{\rm eq}}{f}\right)^2 + \frac{16}{9}\right]$$

- tensor to scalar ratio $r = \mathcal{P}_h/\mathcal{P}_\mathcal{R}$
- scalar amplitude at CMB pivot scale $\mathcal{P}_\mathcal{R}^* \simeq 2\cdot 10^{-9}$ $k_* = \frac{0.05}{\rm Mpc}$
- GW signal extended in frequency: $H_0 \leq f \leq H_{\rm inf}$

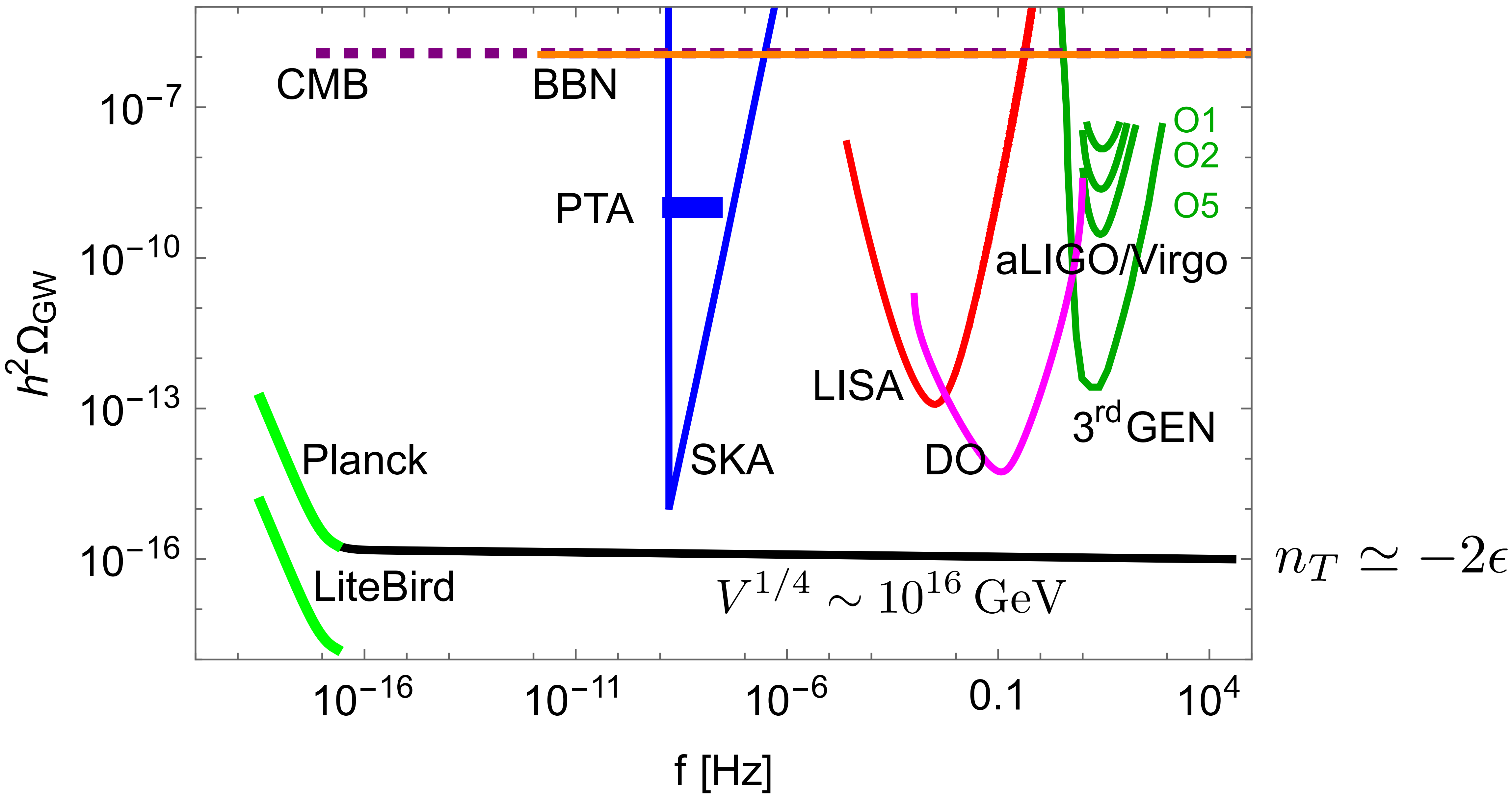
continuous sourcing of GW as modes re-enter the Hubble horizon

GW signal from (slow roll) inflation

Gw detectors offer the amazing opportunity to probe the inflationary power spectrum (and the model of inflation) down to the tiniest scales

BUT! The signal in the standard slow roll scenario is too low

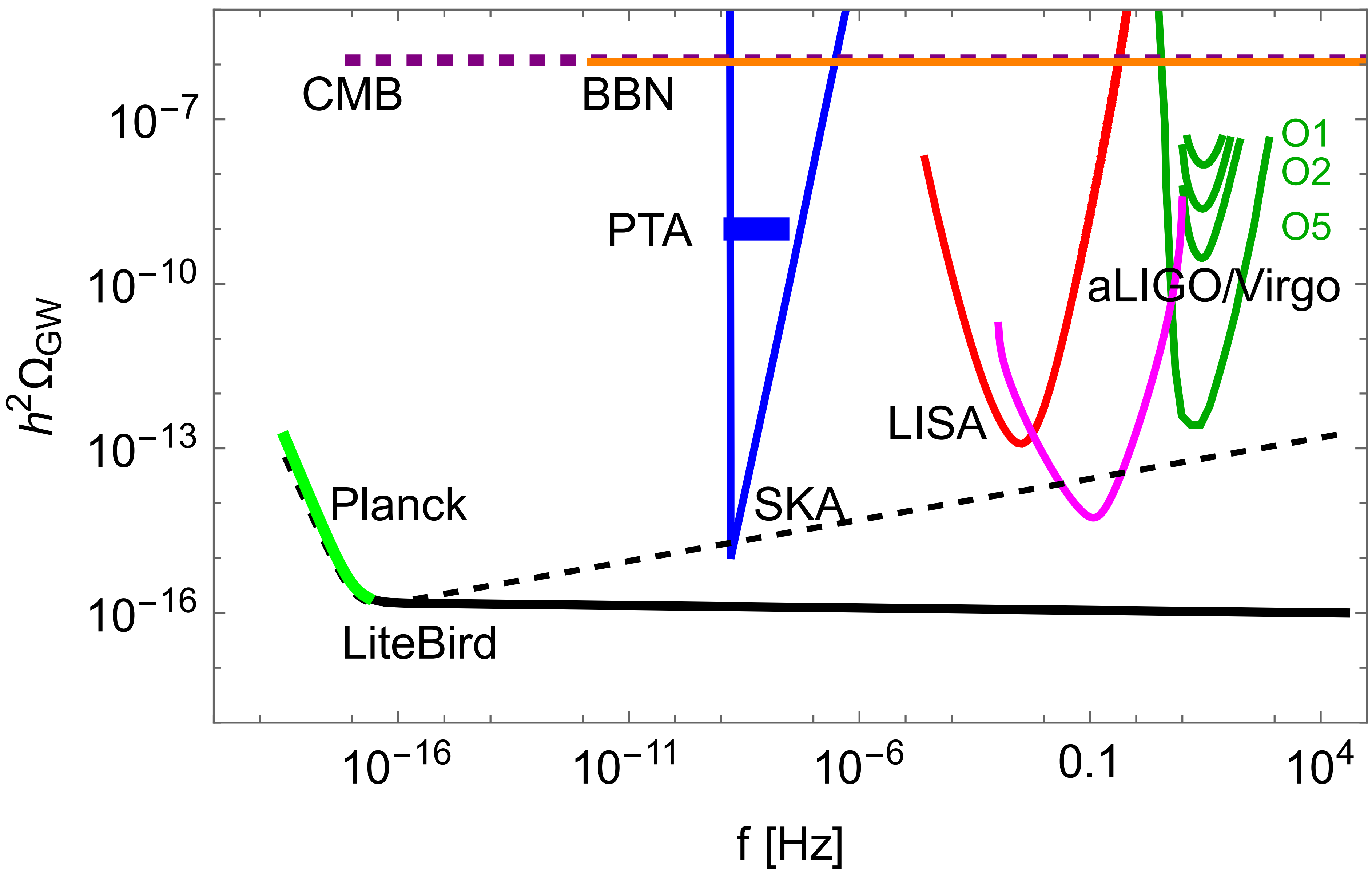
(P)reheating generates a signal, but unfortunately at very high frequencies



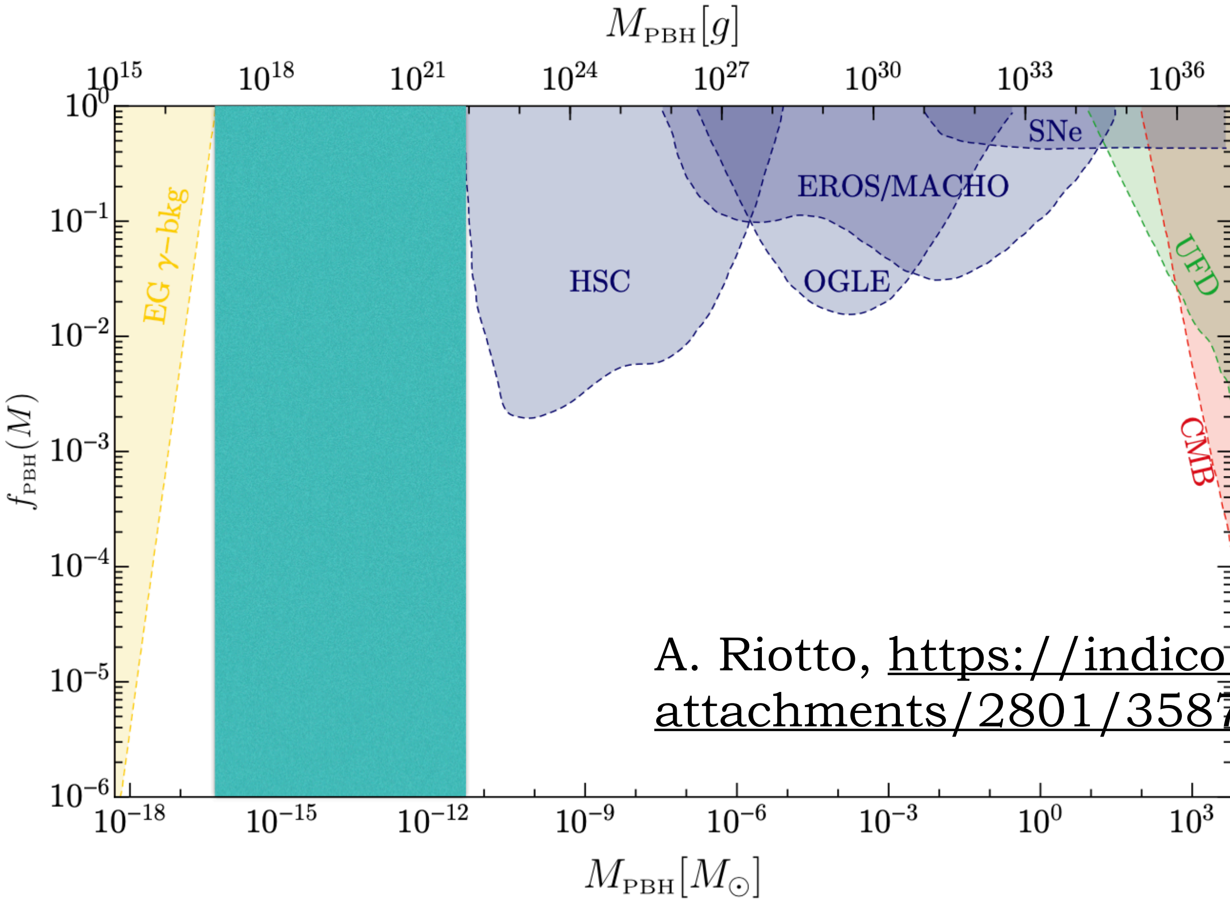
GW signal from (non-standard) inflation

There is the possibility to enhance the signal going beyond the standard inflationary scenario: adding extra fields, modifying the inflaton potential, modifying the gravitational interaction, adding a phase with stiff equation of state...

$$H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = \boxed{16\pi G\, a^3\, \Pi_r(\mathbf{k},\eta)}$$



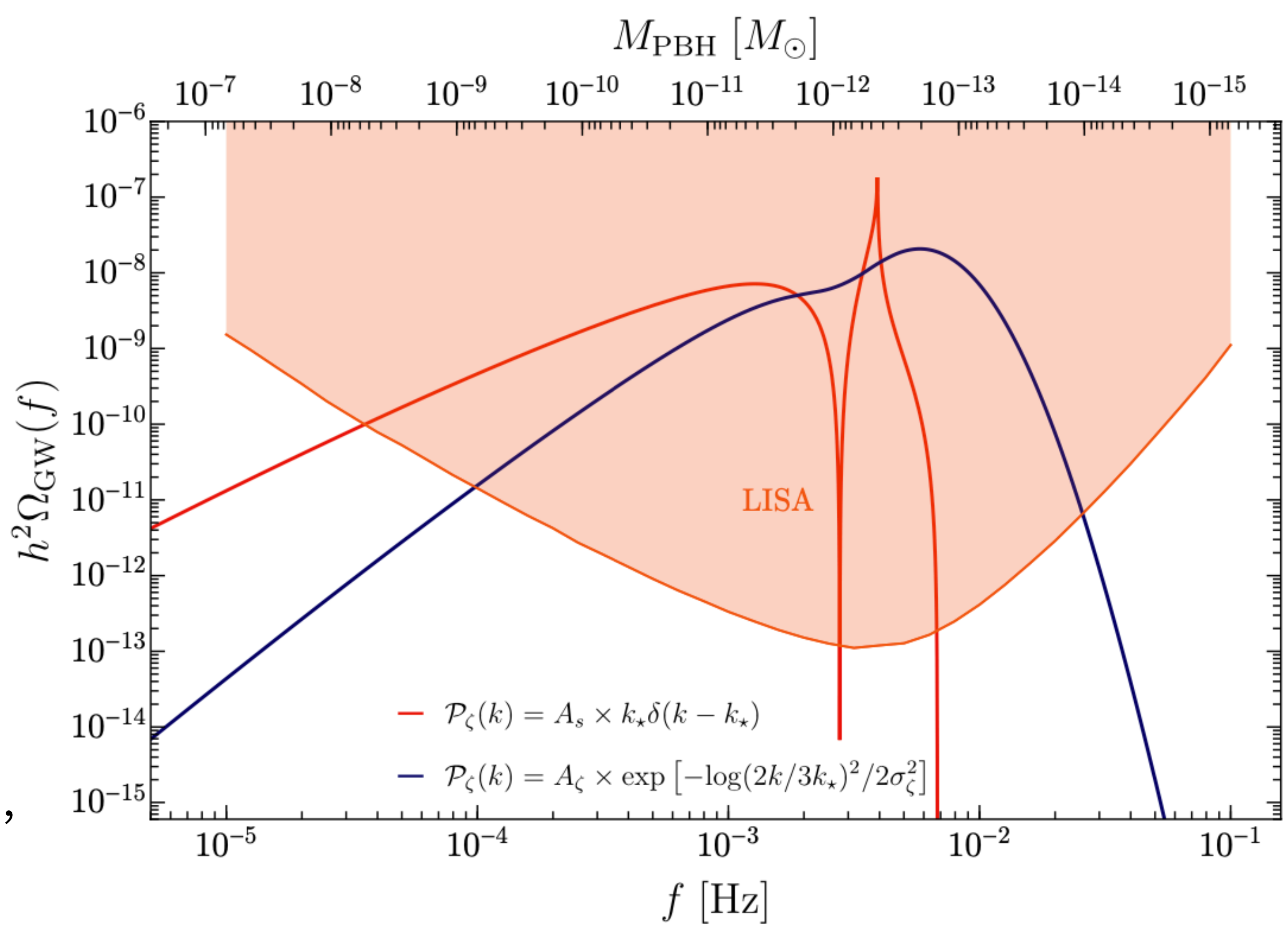
One example: GW signal from second order scalar perturbations, PBH and LISA



There is a mass window for which PBH can still constitute the whole of the dark matter

A. Riotto, https://indico.math.cnrs.fr/event/5766/contributions/5153/attachments/2801/3587/Paris2021.pdf

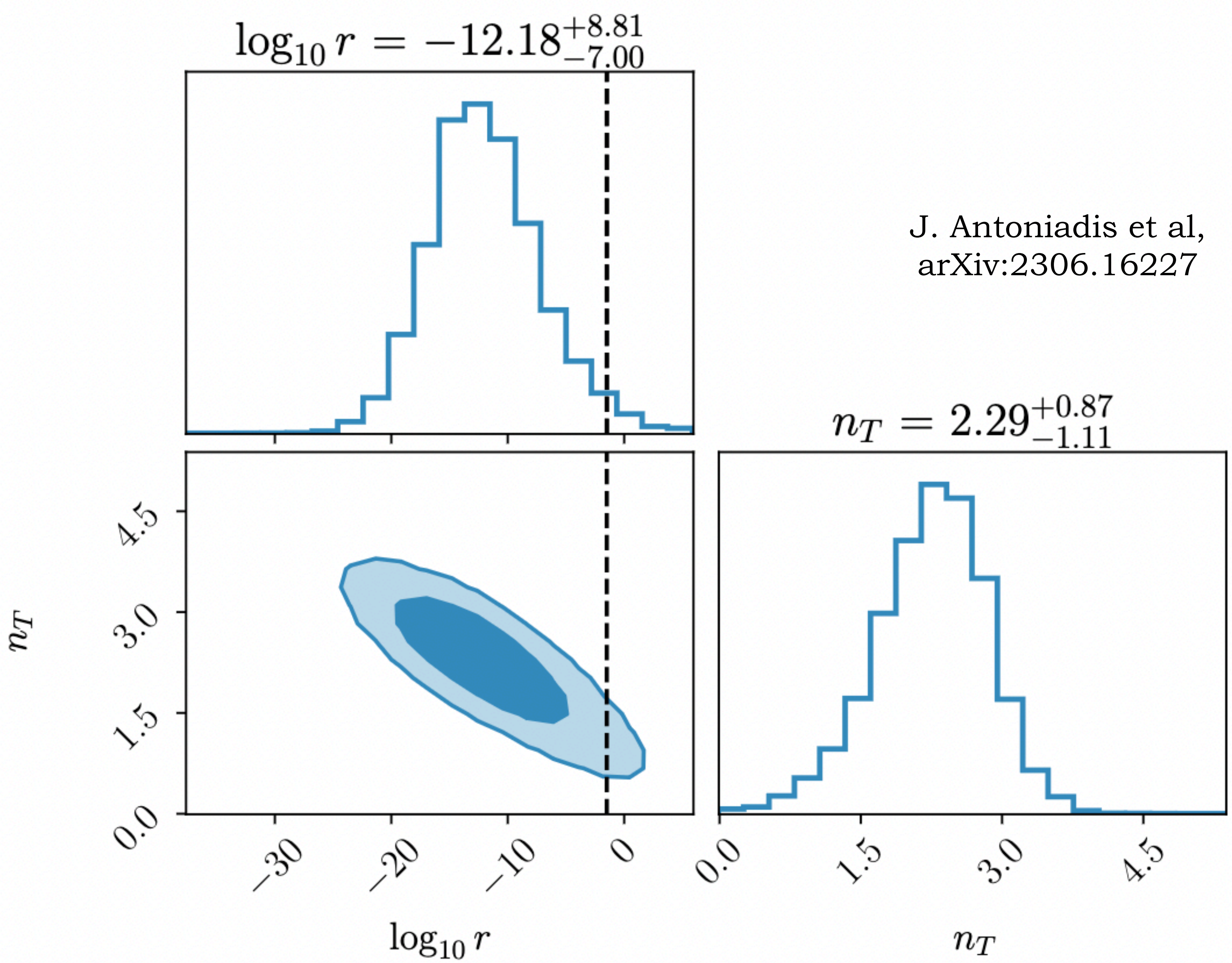
If one wants to produce PBH in this mass range, one also has an observable SGWB in LISA by second order scalar perturbations



N. Bartolo et al, arXiv:1810.12218, arXiv:1810.12224

An example of possible detection at PTA?

$$\Omega_{\rm GW}(f) = \frac{3}{128}\,\Omega_{\rm rad}\, r\, \mathcal{P}_{\mathcal{R}}^{*} \left(\frac{f}{f_*}\right)^{n_T} \left[\frac{1}{2}\left(\frac{f_{\rm eq}}{f}\right)^2 + \frac{16}{9}\right] \times \left(\frac{f}{f_{\rm RD}}\right)^{\frac{2(3w-1)}{3w+1}}$$



J. Antoniadis et al, arXiv:2306.16227

Would this be compatible with slow roll and a stiff equation of state? Marginally!

$$\gamma = 5 - n_T + \frac{2(1-3w)}{3w+1}$$

$$\gamma_{\text{best fit}} \simeq 2.7$$

↓

$$n_T \gtrsim 0.3$$

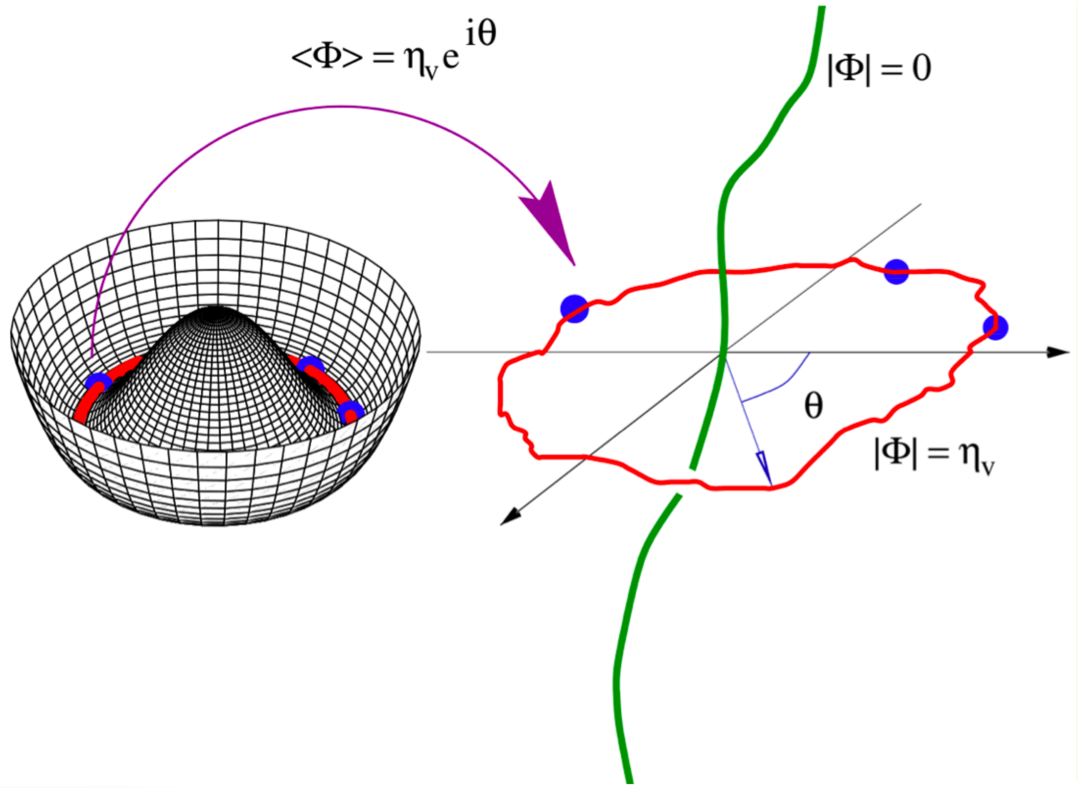
Examples of signals

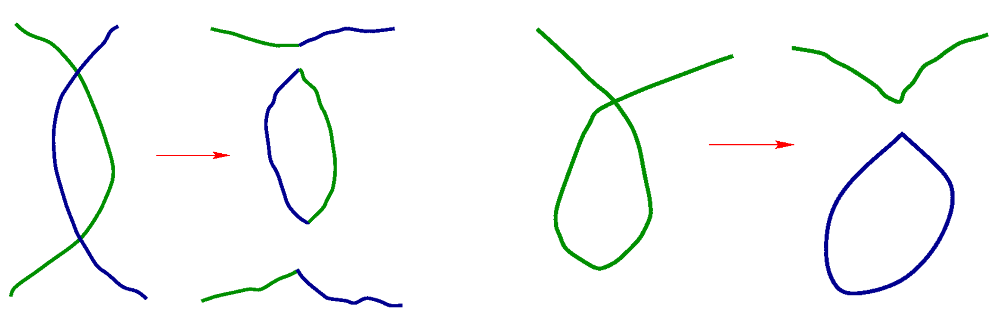
- Cosmic strings

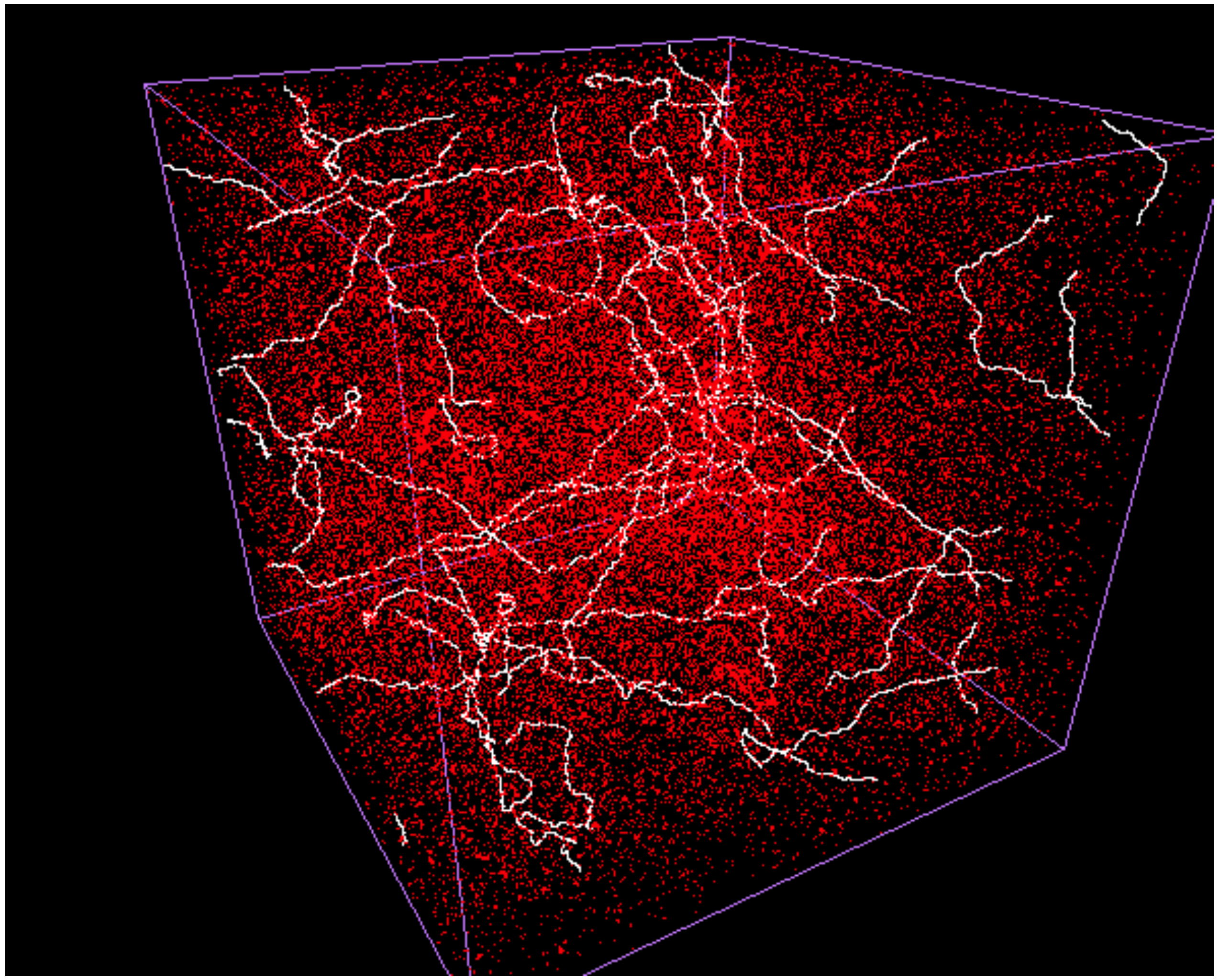
GW signal from cosmic strings

Cosmic strings (or other kind of topological defects) are non-trivial field configurations left-over after the phase transition has completed

A network of cosmic strings emits GWs
(though the results are very model dependent)

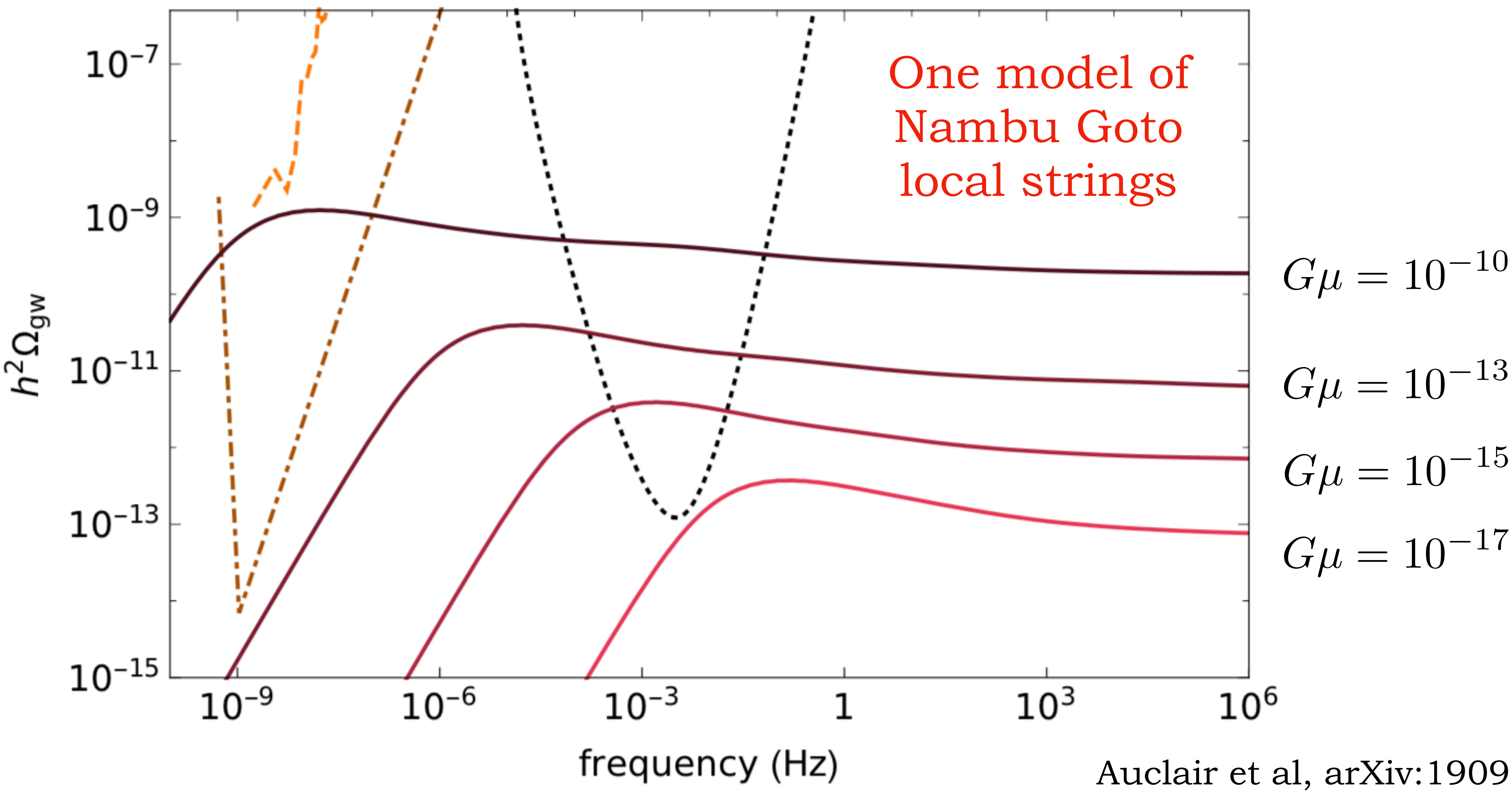






https://curl.irmp.ucl.ac.be/~chris/strings.html

GW signal from cosmic strings

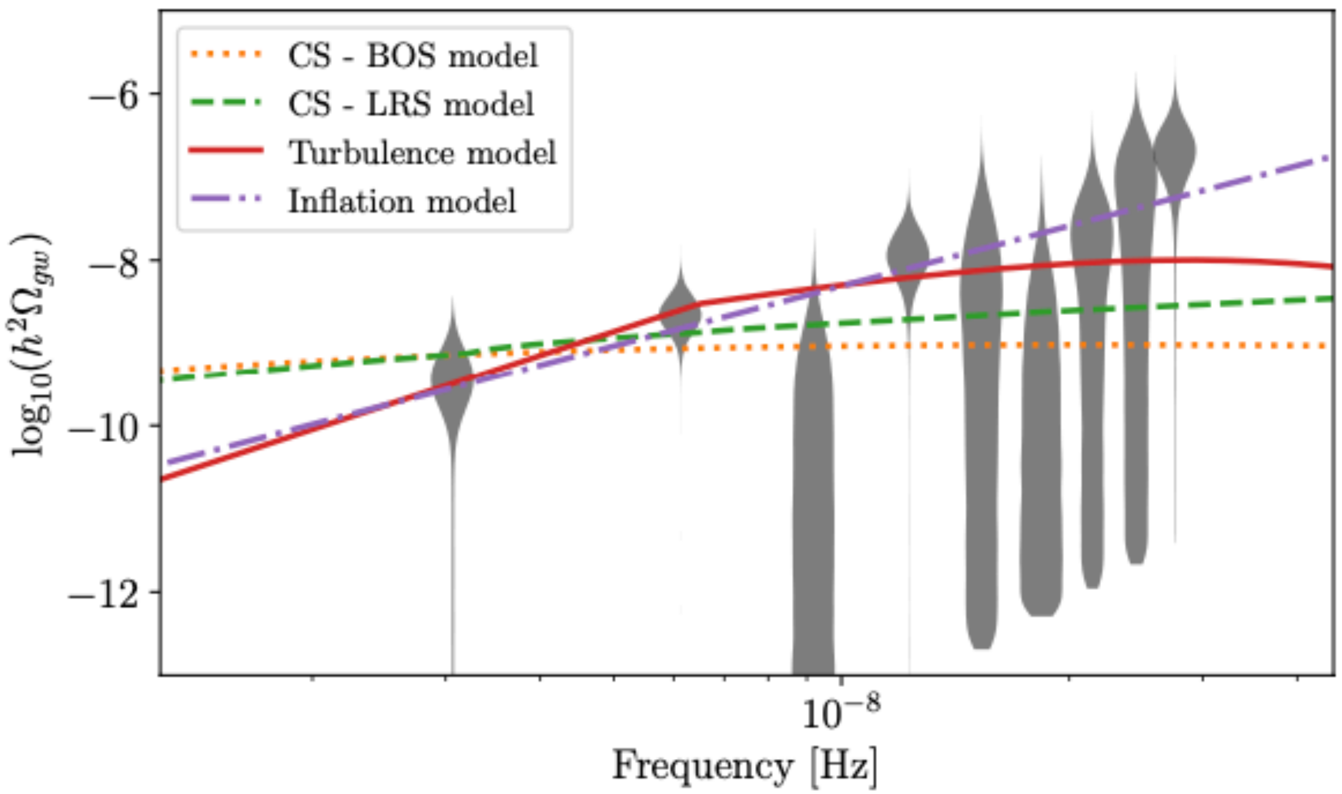


Auclair et al, arXiv:1909.00819

- The signal extends over many frequencies since the GW production is continuous throughout the universe evolution

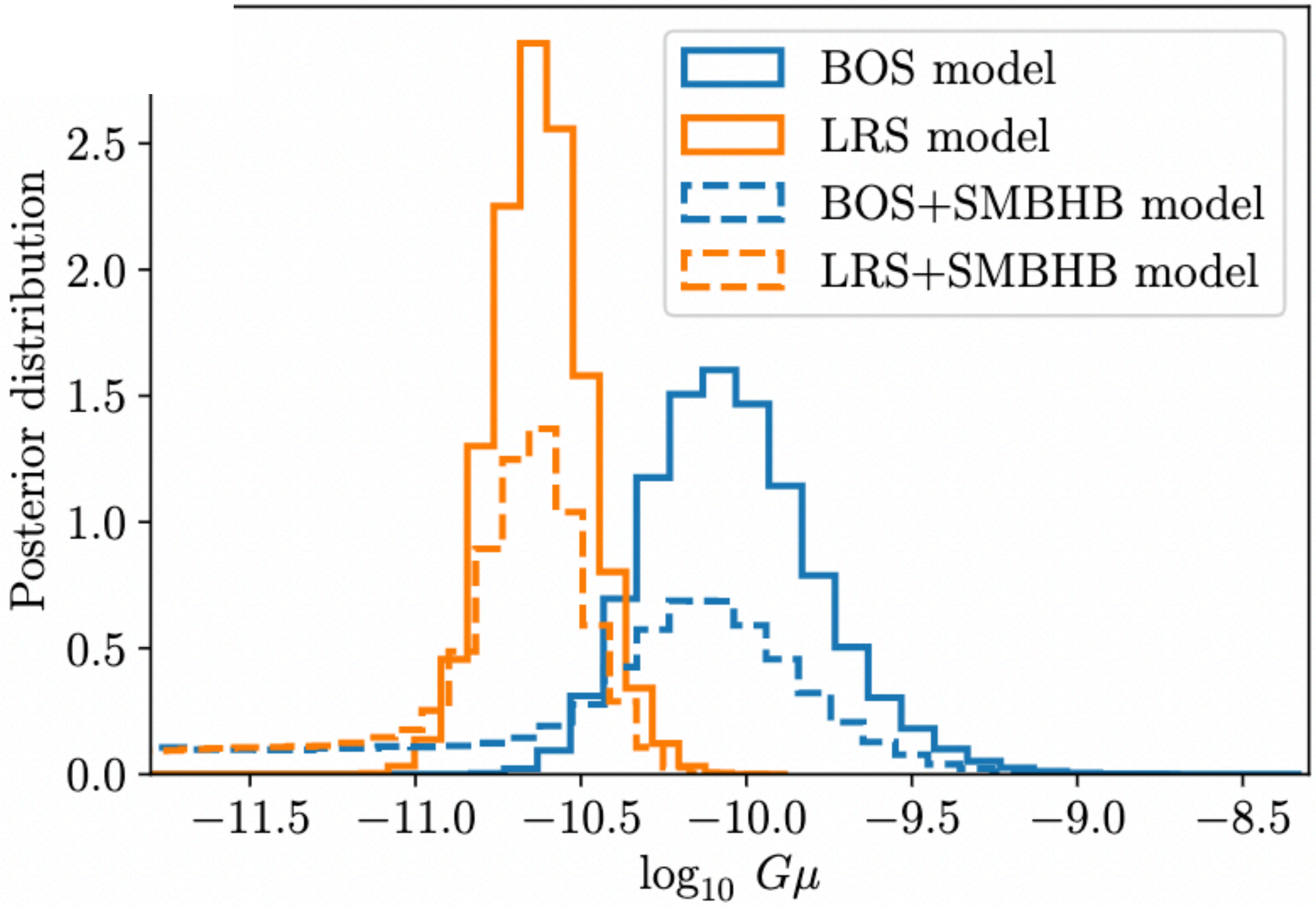
- The energy density of the cosmic string network is a constant fraction of the universe's one

An example of possible detection at PTA?



J. Antoniadis et al, arXiv:2306.16227

$$\eta \sim M_{\rm Pl}\sqrt{G\mu} \sim 10^{14}\,{\rm GeV}$$



To summarise:

- SGWB might reveal a powerful tool to probe the early universe and high energy physics
- The spectral shape must be predicted with good accuracy in order to disentangle the different sources (and also for foregrounds)
- General considerations about the characteristics of the spectral shape are possible in some cases, to pin down at least the class of SGWB sources
- Electroweak PT: at the limit of tested physics, GW signal can be accessed/ constrained by LISA only for models beyond the standard model of particle physics
- QCD PT: tested physics but difficult to predict, GW signal can be accessed/ constrained by PTA only for models beyond the standard model of particle physics
- Inflation: new physics but observationally compelling, extended GW signal in frequency, only accessible by CMB unless one goes beyond the standard slow roll scenario (there are well motivated scenarios!)
- Cosmic strings: amazing potential to probe high energy theory, but need to account for GW signal model dependent
- **SGWBs from the primordial universe might seem speculative but their potential to probe fundamental physics is great and amazing discoveries can be around the corner, especially after the PTA results!**