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DIFFERENTIATING SIGNAL FROM NOISE: TOWARDS A MULTIVARIATE SPECTRAL ANALYSIS FOR LISA

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Data analysis challenges for stochastic gravitational wave backgrounds

1. Problem statement
2. Multivariate time series model
3. Bayesian inference
4. Revisiting null channels

Stochastic GW background generated with Midjourney

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  → Instrumental stochastic processes must be very precisely accounted for to allow for a detection

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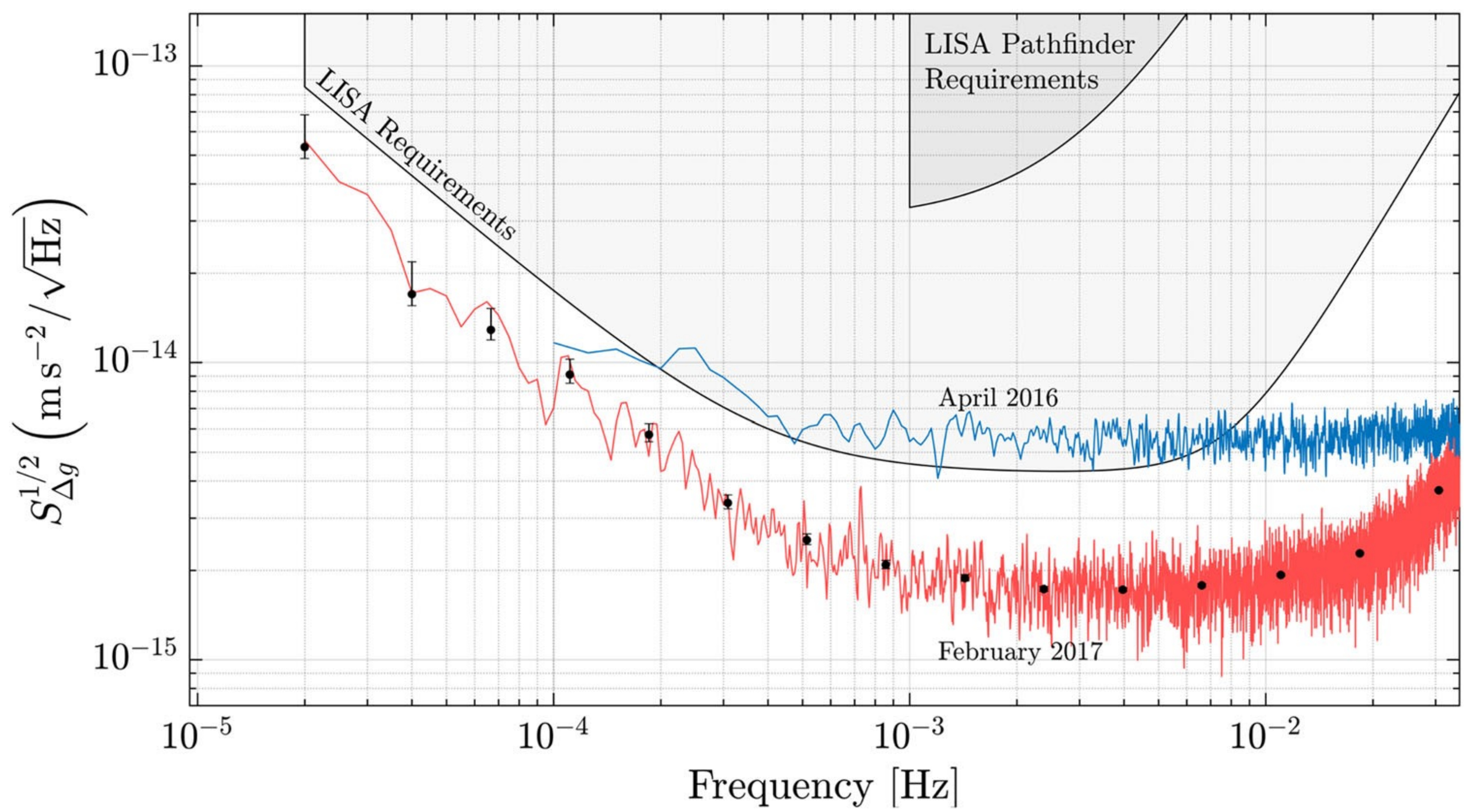

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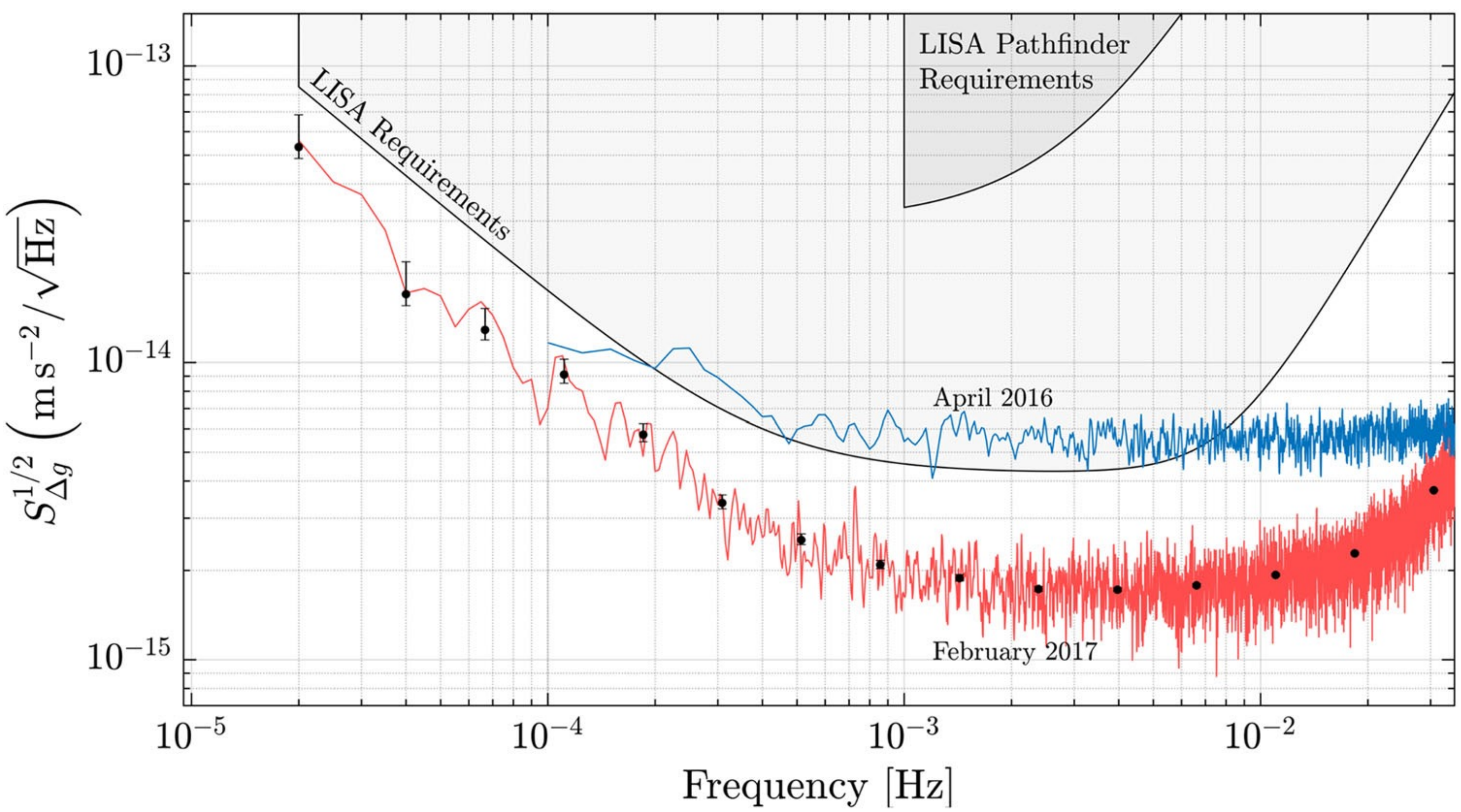
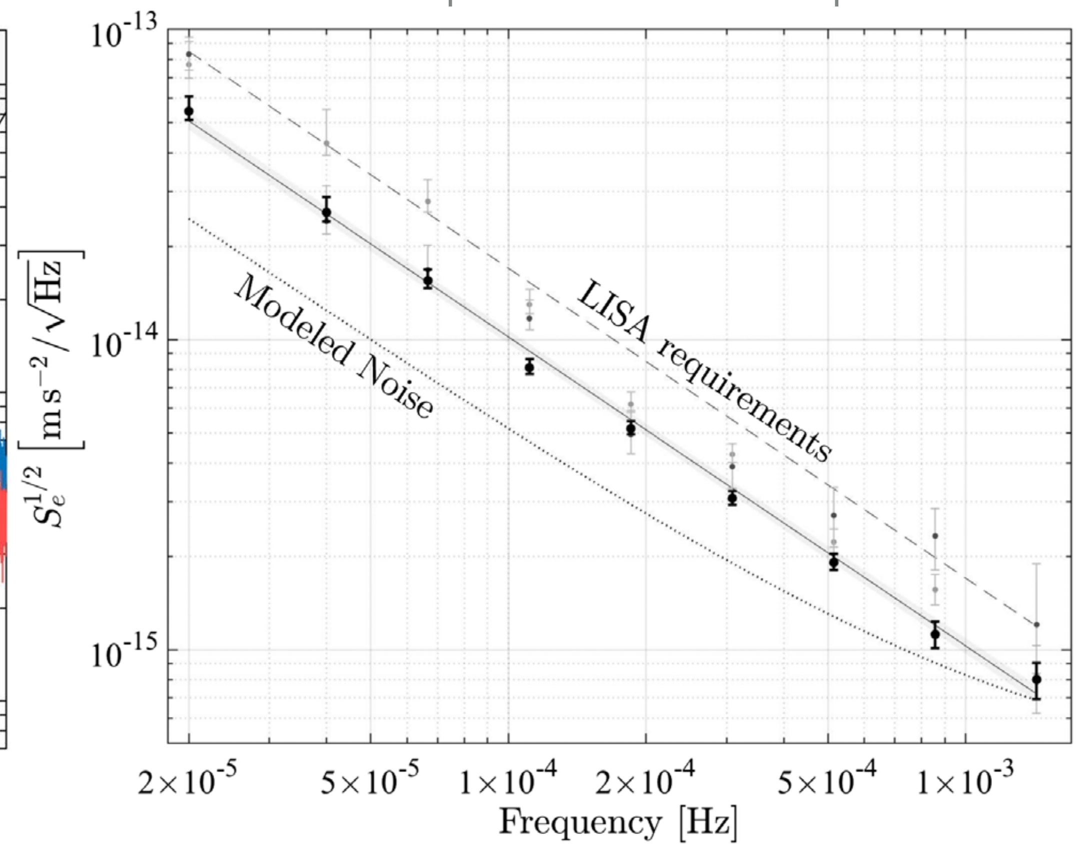


Excess of power at low frequencies



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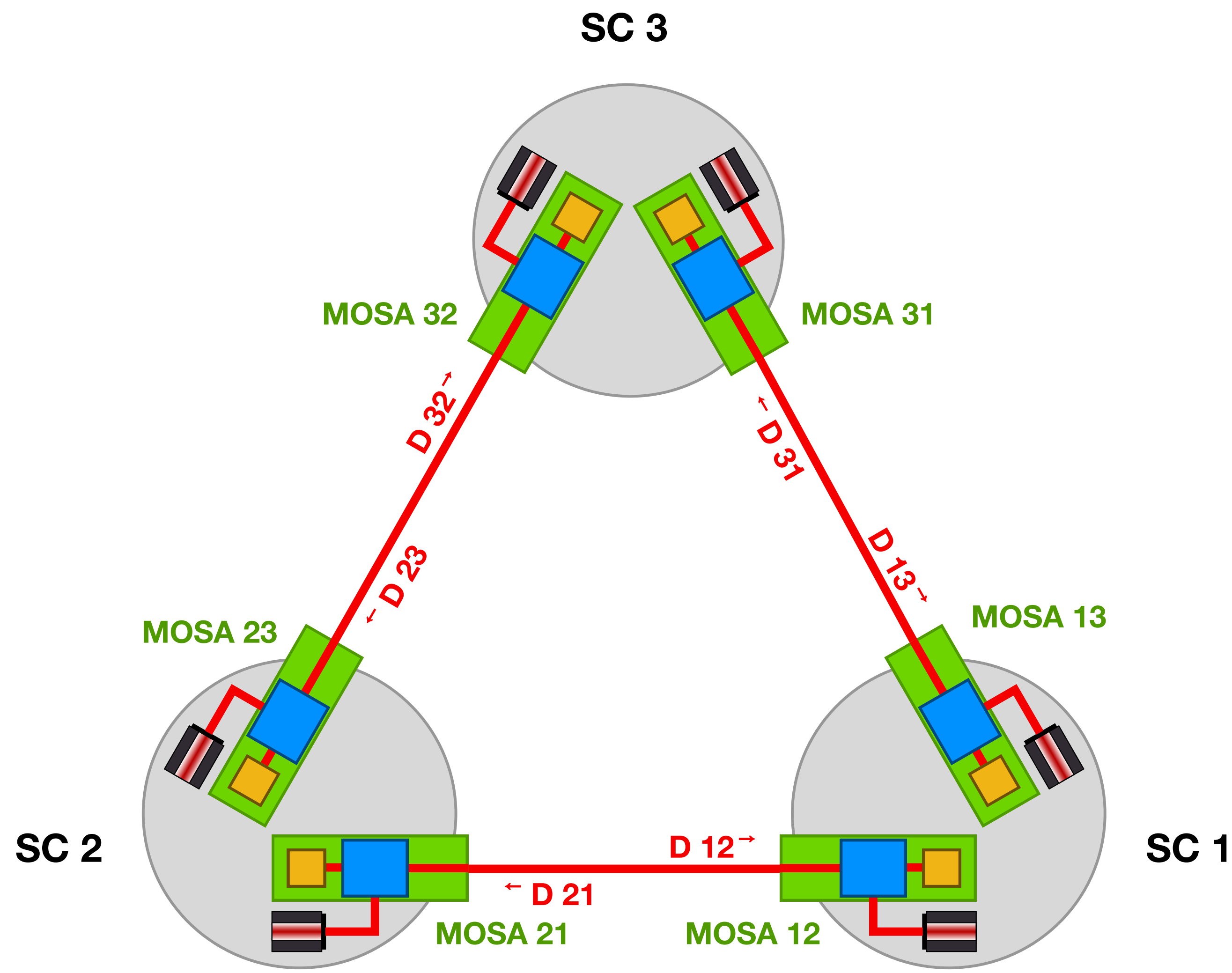
  - Isotropic, stationary SGWB
  - Noise PSD is smooth on mHz scales
  - All resolvable GW sources have been removed (!!!!)

- We start from interferometric measurements
- After some combinations to suppress S/C motion and half of the laser noise, we obtain 6 intermediate variables η

$$\boldsymbol{\eta} = \begin{pmatrix} \eta_{12} \\ \eta_{23} \\ \eta_{31} \\ \eta_{13} \\ \eta_{32} \\ \eta_{21} \end{pmatrix}$$



- They are affected by noise and GW signal:

$$\boldsymbol{\eta} = \mathbf{h} + \mathbf{p} + \mathbf{n}$$

GW signal

laser frequency noise

secondary noises

- To cancel laser frequency noise we form TDI variables through a linear operation:

$$\mathbf{d} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \boxed{\mathbf{M}}\boldsymbol{\eta} \quad \rightarrow \text{reduces to 3 effective variables}$$

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TDI transformation matrix

- We obtain a 3-dimensional multivariate time series $\mathbf{d}$
- We can take their discrete Fourier transform (DFT): $\tilde{\boldsymbol{d}} = \mathrm{DFT}\left(\boldsymbol{d}\right)$
- Stationarity assumption implies:
  - ✦ DFT components at frequencies f_k approximately uncorrelated
  - ✦ Each frequency bin is characterised by a **spectrum matrix**

$$\tilde{\Sigma}_d(f) \equiv \mathrm{E}\left[\tilde{\mathbf{d}}(f)\tilde{\mathbf{d}}(f)^{\dagger}\right]$$

Multivariate time series model

- We can split the spectrum matrix into 2 independent components: signal and noise

$$\tilde{\mathbf{\Sigma}}_d = \tilde{\mathbf{\Sigma}}_{\mathrm{noise}} + \tilde{\mathbf{\Sigma}}_{\mathrm{GW}}$$

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Sky-averaged TDI response matrix

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- What helps distinguishing between noise and signal:

   1. **Differences in correlation structure** between $\tilde{\mathbf{\Sigma}}_{\text{noise}}$ and $\tilde{\mathbf{\Sigma}}_{\text{GW}}$
   2. **Spectral features** of the SGWB template $S_h(f)$
   3. **Priors** on noise and signal parameters

Bayesian inference

- We recently conducted a study with additional assumptions:
  - All interferometric noises have the same transfer function
  - All interferometric noises have the same PSD
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- Power-law template to model the SGWB PSD:

$$S_h(f) = \Omega_{\mathrm{GW}}(f)\frac{3H_0^2}{4\pi^2 f^3} \qquad \Omega_{\mathrm{GW}}(f) = \Omega_{\mathrm{GW},0}\left(\frac{f}{f_0}\right)^n$$

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Energy density   Power-law index

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Modelling the spectrum matrix

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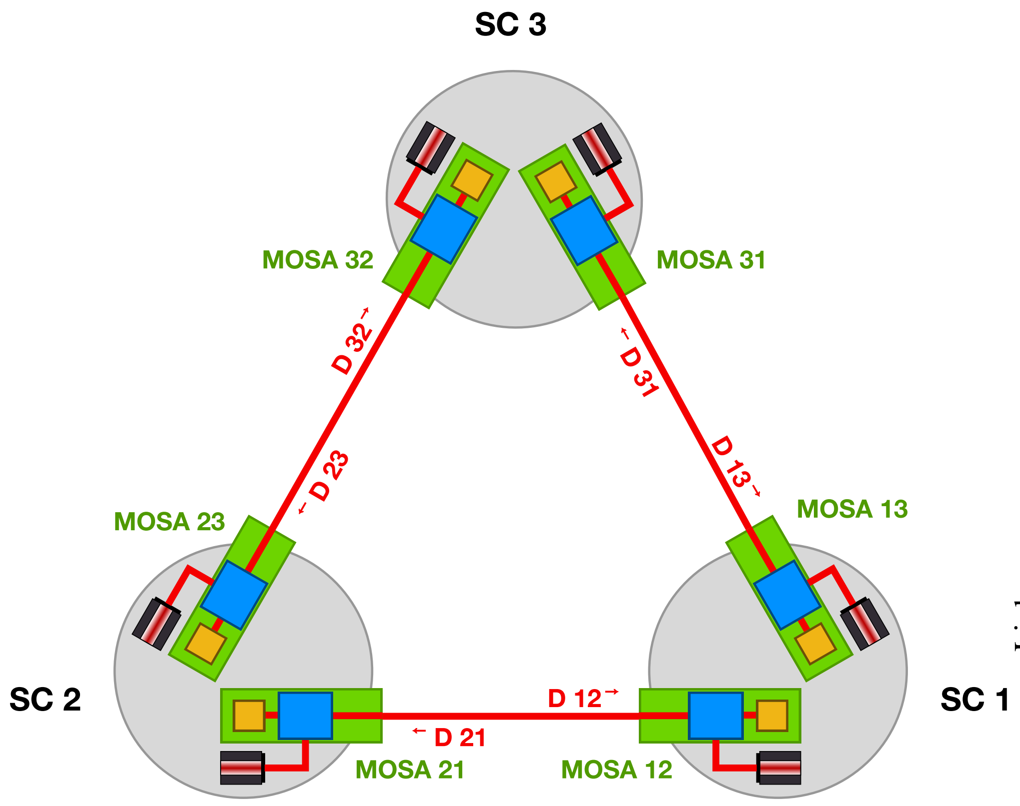
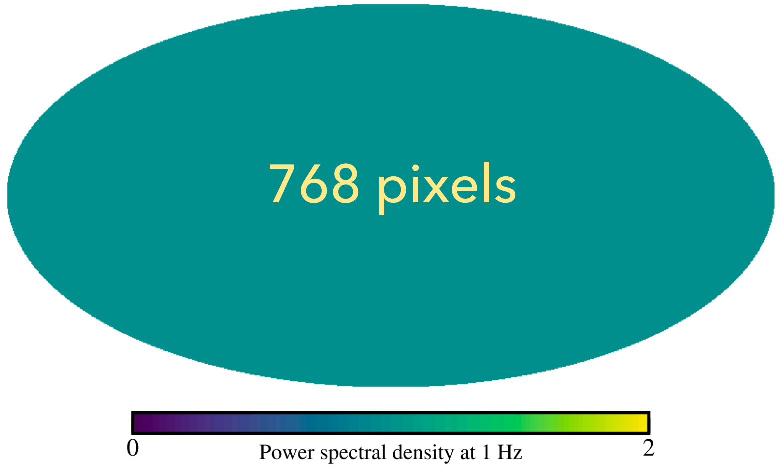
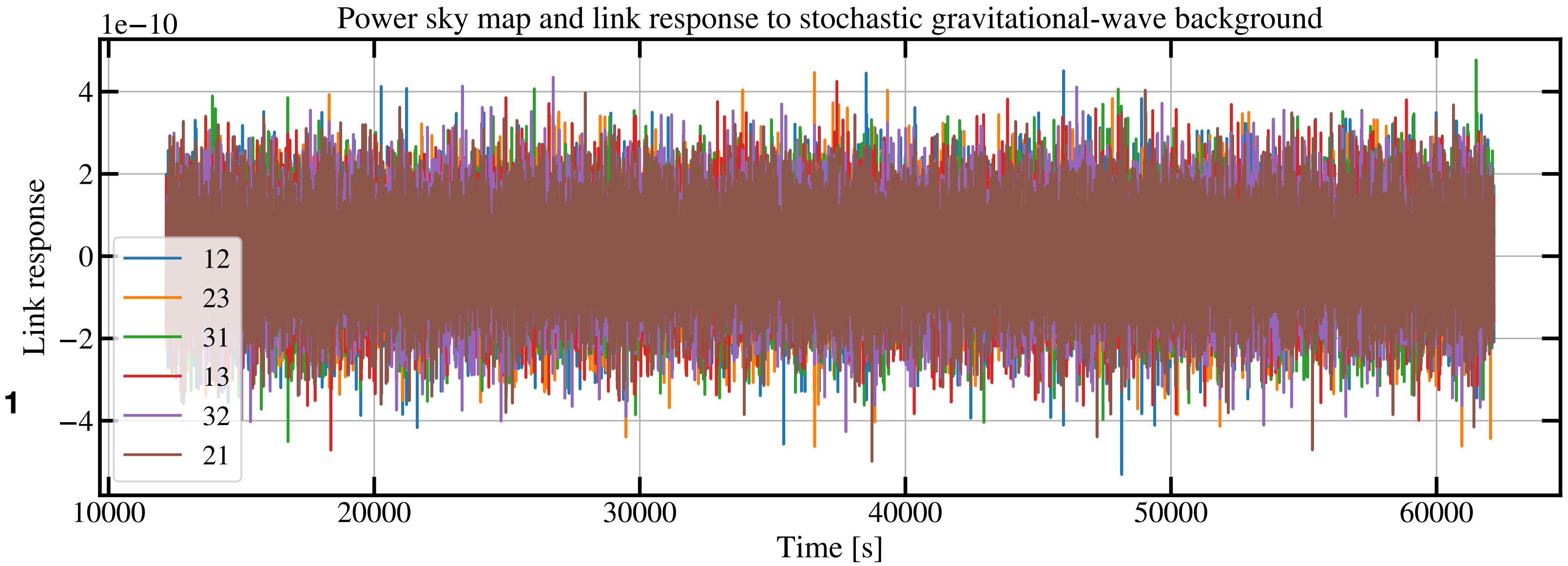
- We generate single-link measurements using LISA GW Response [Bayle, Baghi, Renzini, Le Jeune 2022]
- Obtain 6 science interferometer time series
- Add instrumental noise from prescribed PSD







- Process data through time-delay interferometry (TDI) using the pyTDI code [Staab, Bayle, Hartwig 2022]
- Time-varying arm lengths → second-generation TDI

Bayesian inference

- Bayesian data analysis :
  - Sampling posterior distributions with parallel tempered Markov chain Monte Carlo (MCMC)
  - Uniform priors on GW parameters $\Omega_{m0} \in [10^{-16}, 10^{-14}]$ and $n \in [-5, 7]$
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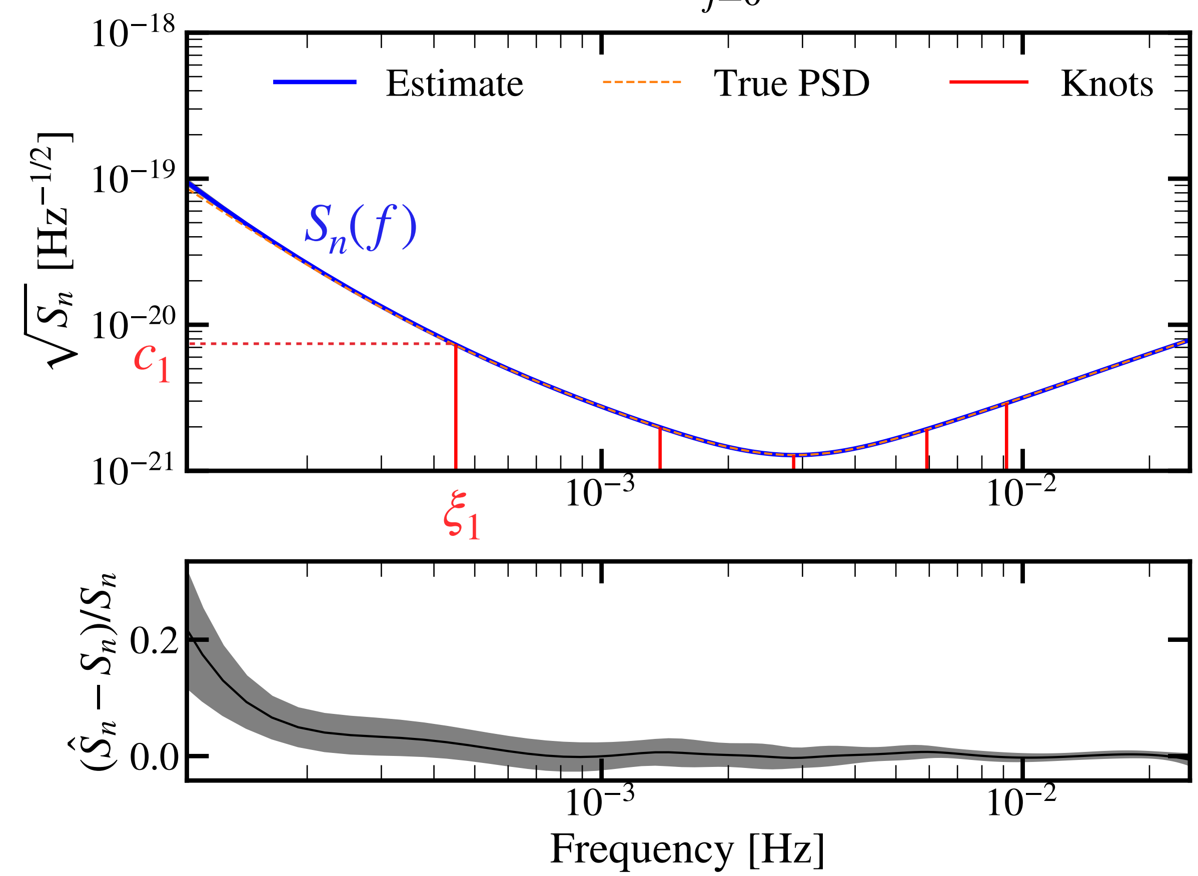

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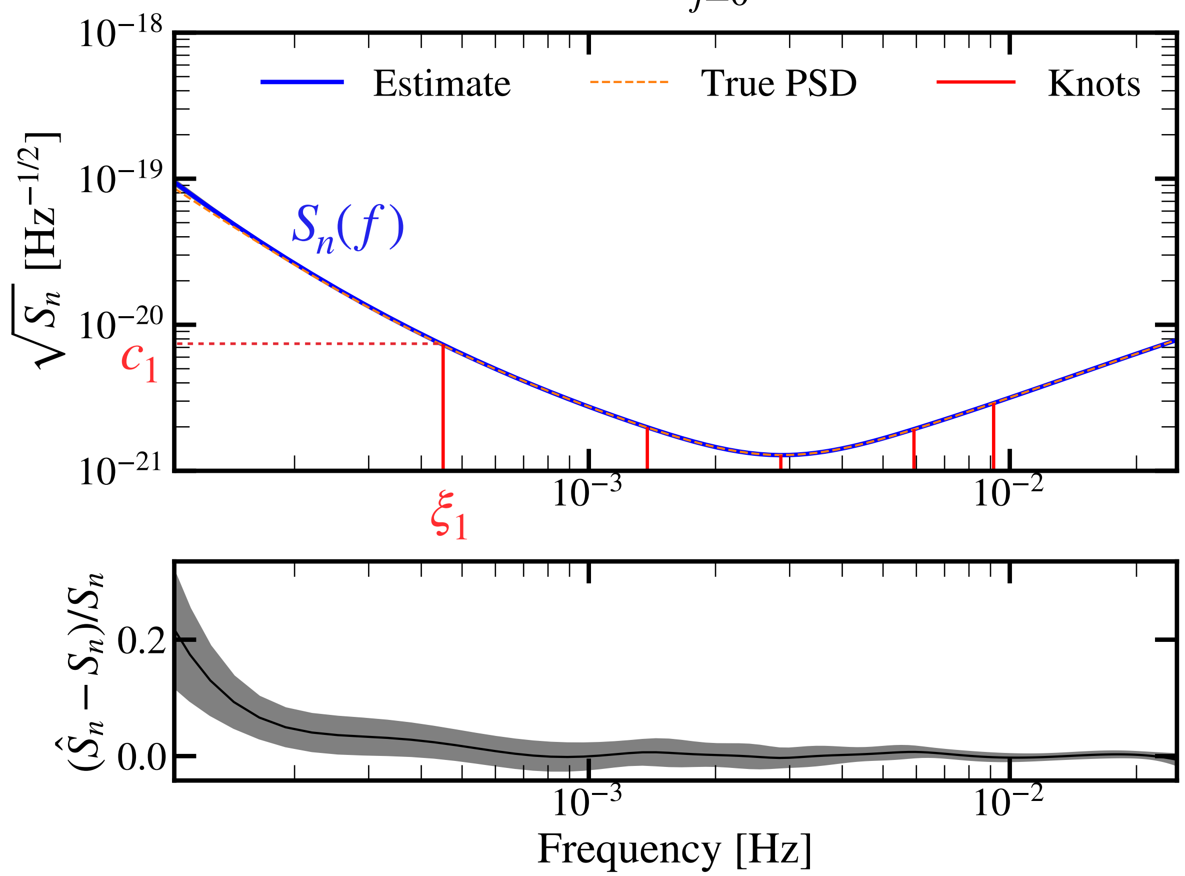

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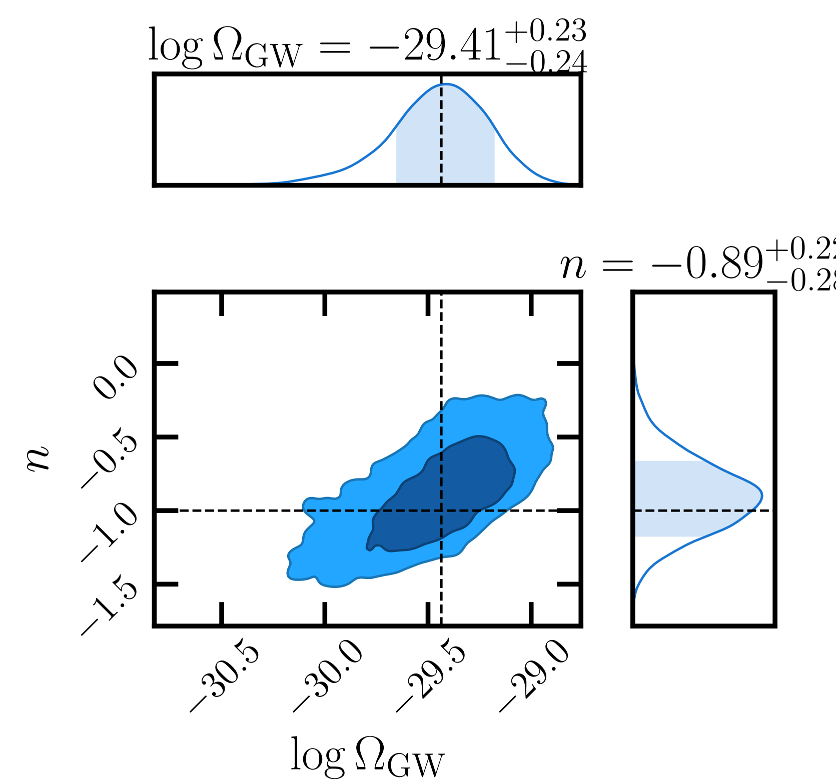

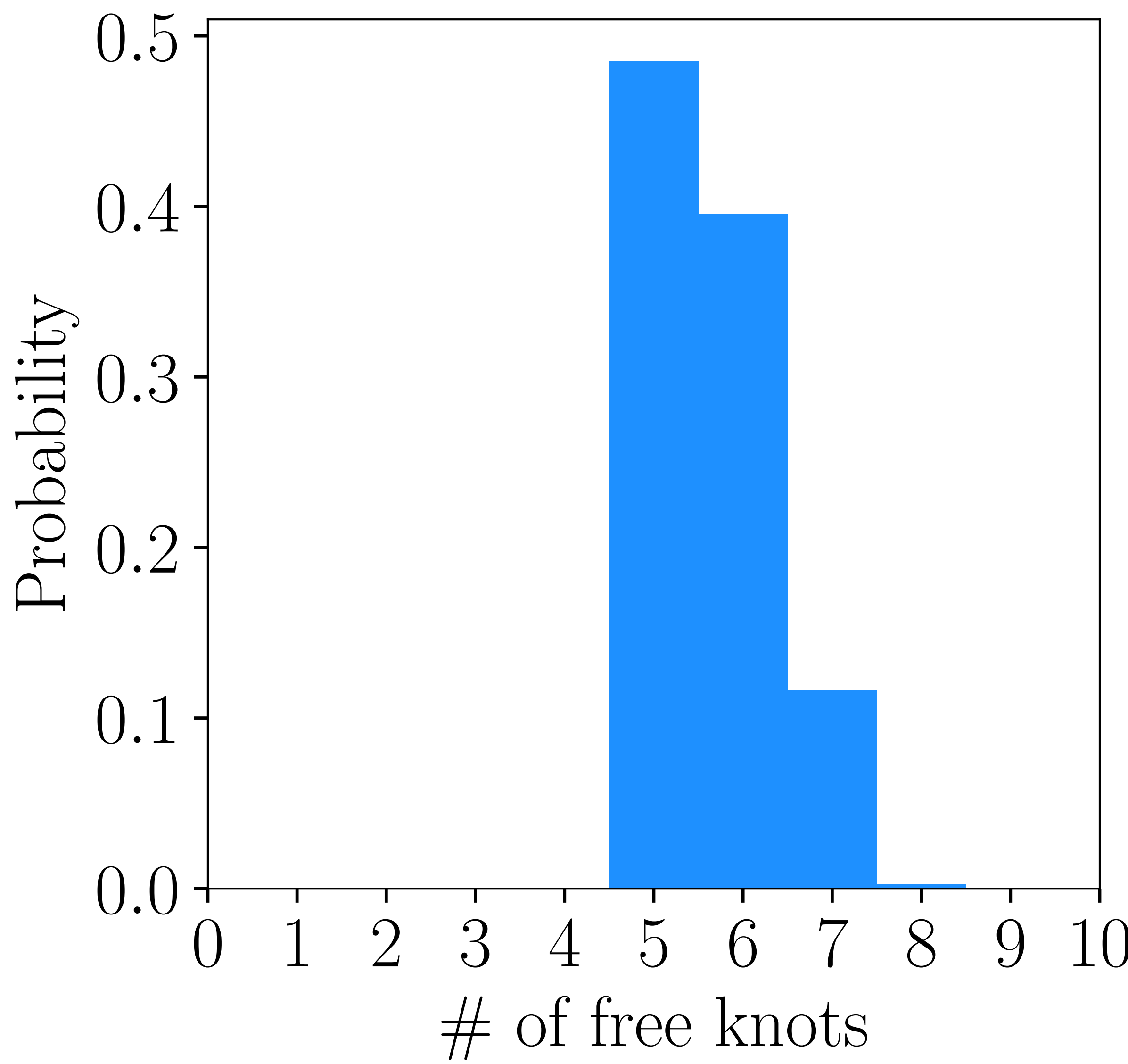

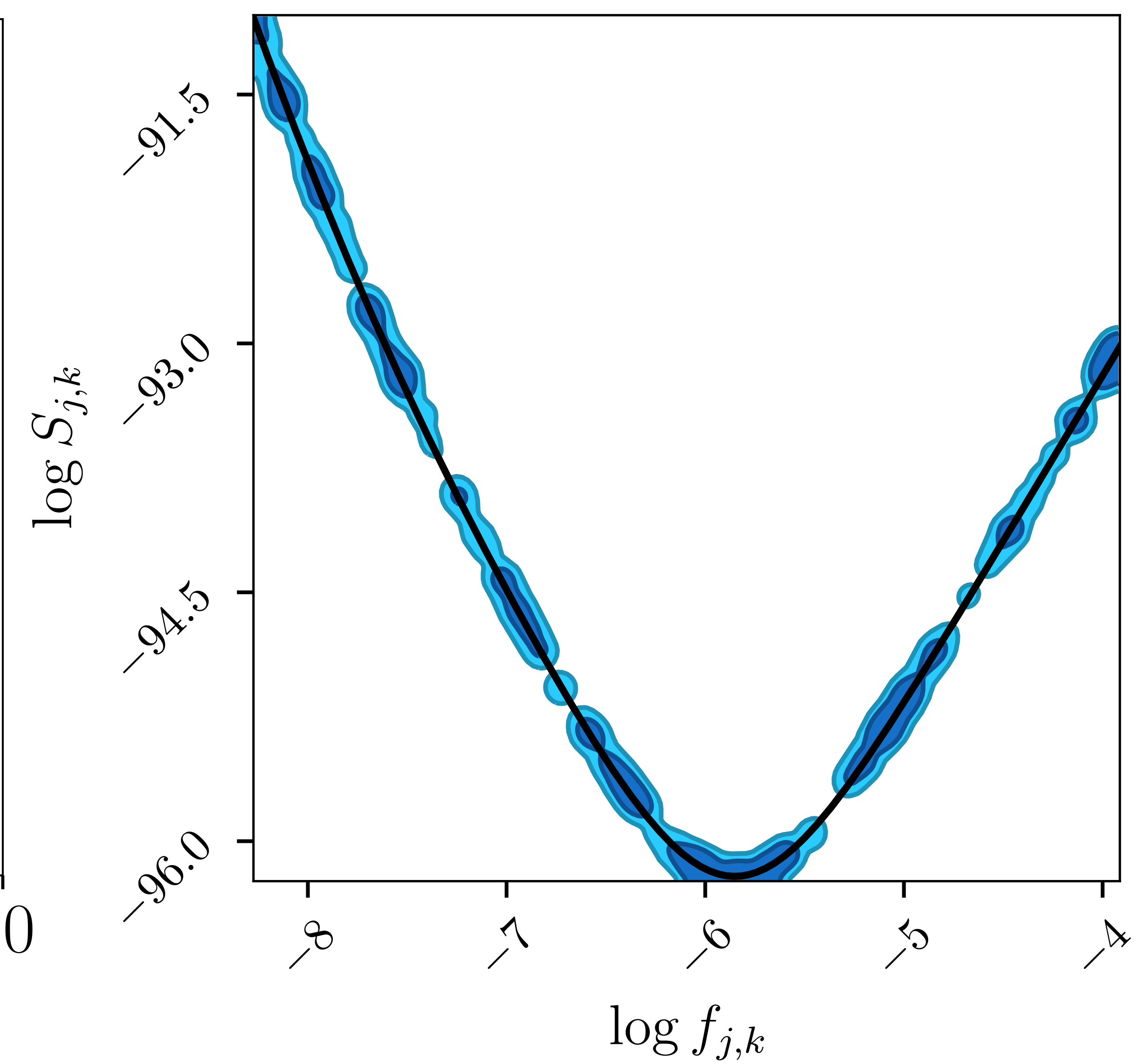

[Done with Eryn: Karnesis et al., 2023]

Bayesian inference

- Detection using Bayesian model comparison
  - Hypothesis H_0: only noise in the data $\quad \tilde{d} = M\tilde{n}$
  - Hypothesis H_1: presence of a SGWB $\quad \tilde{d} = M\left(\tilde{h} + \tilde{n}\right)$

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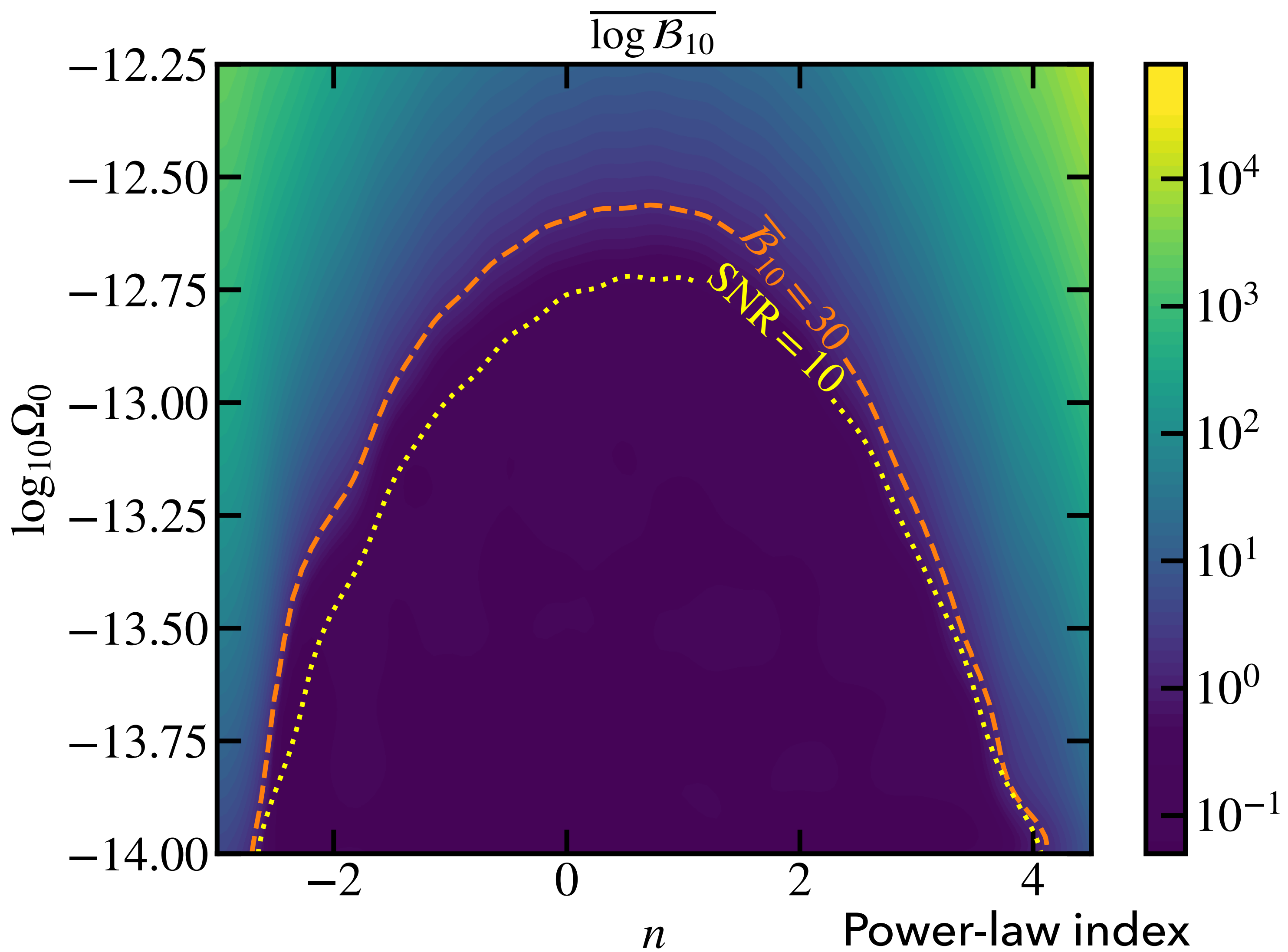

Log-energy density

Power-law index

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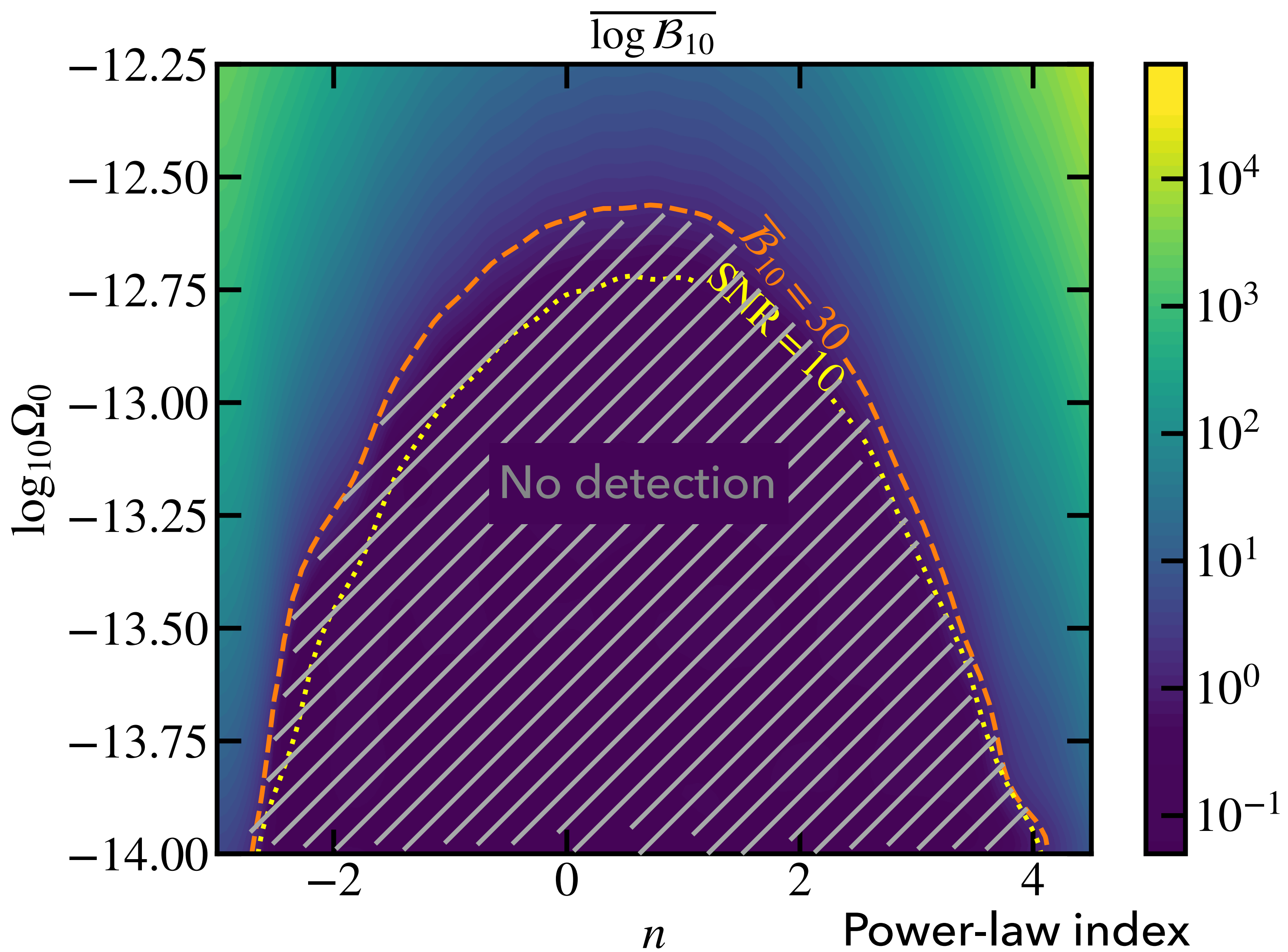

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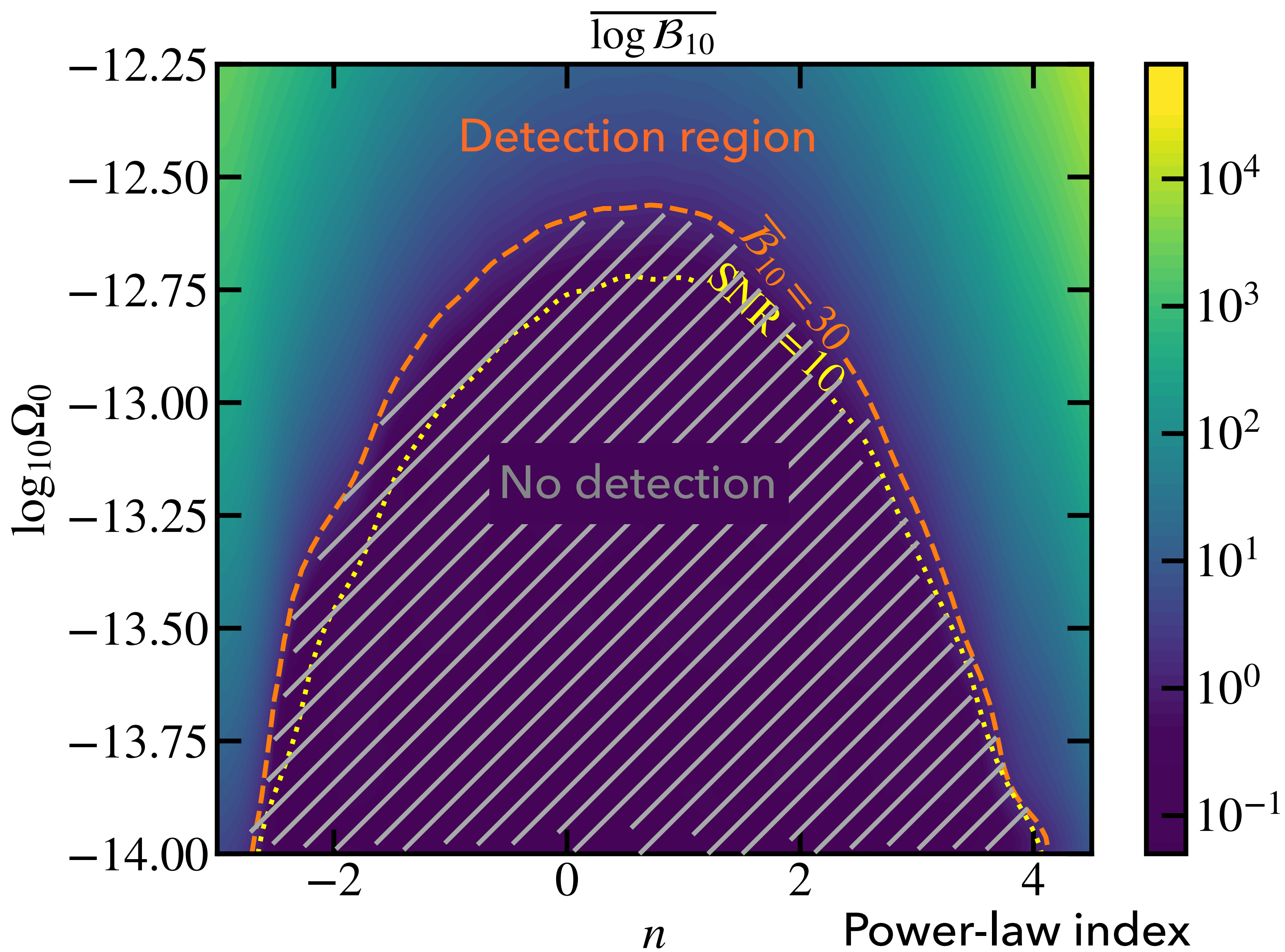

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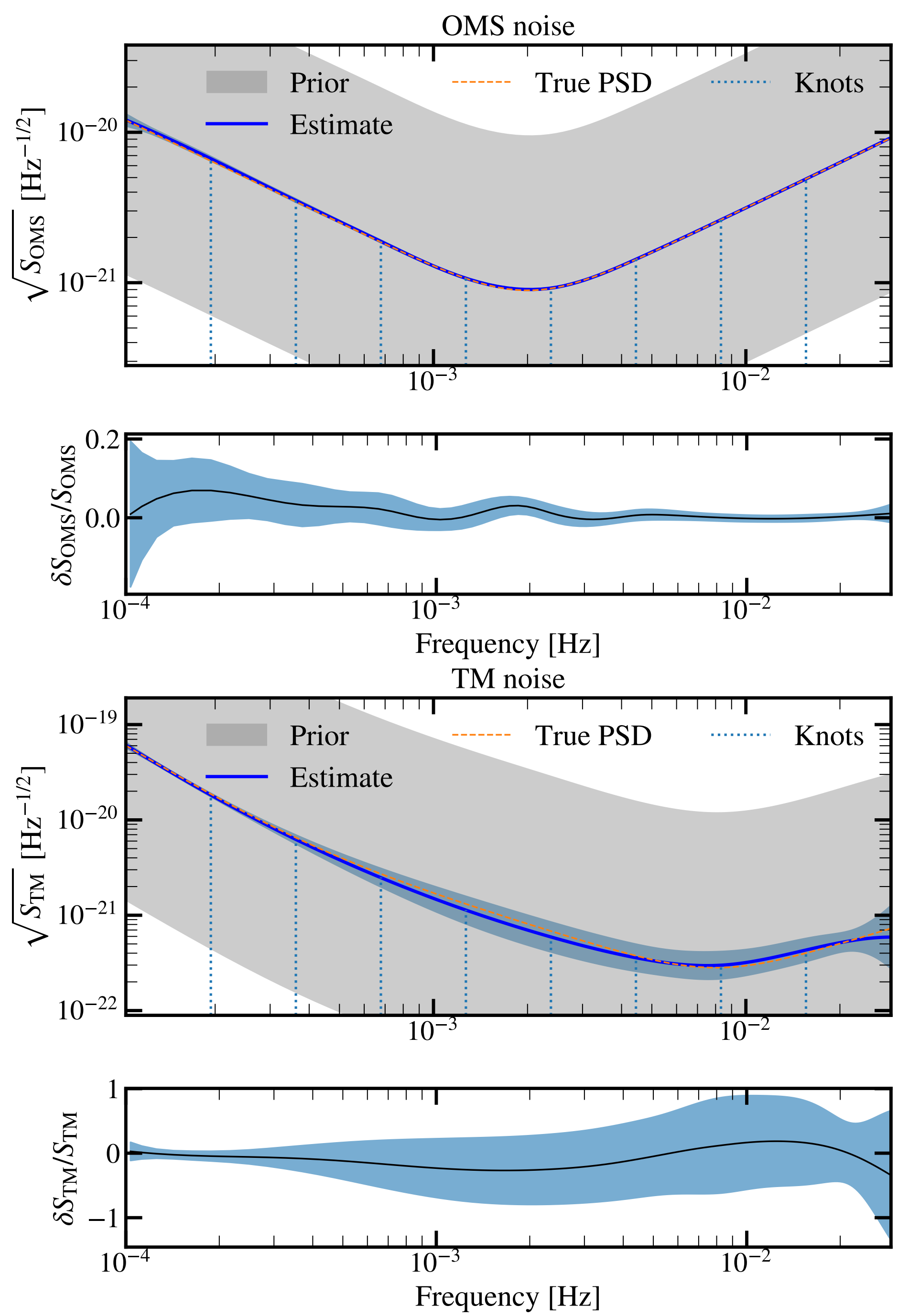
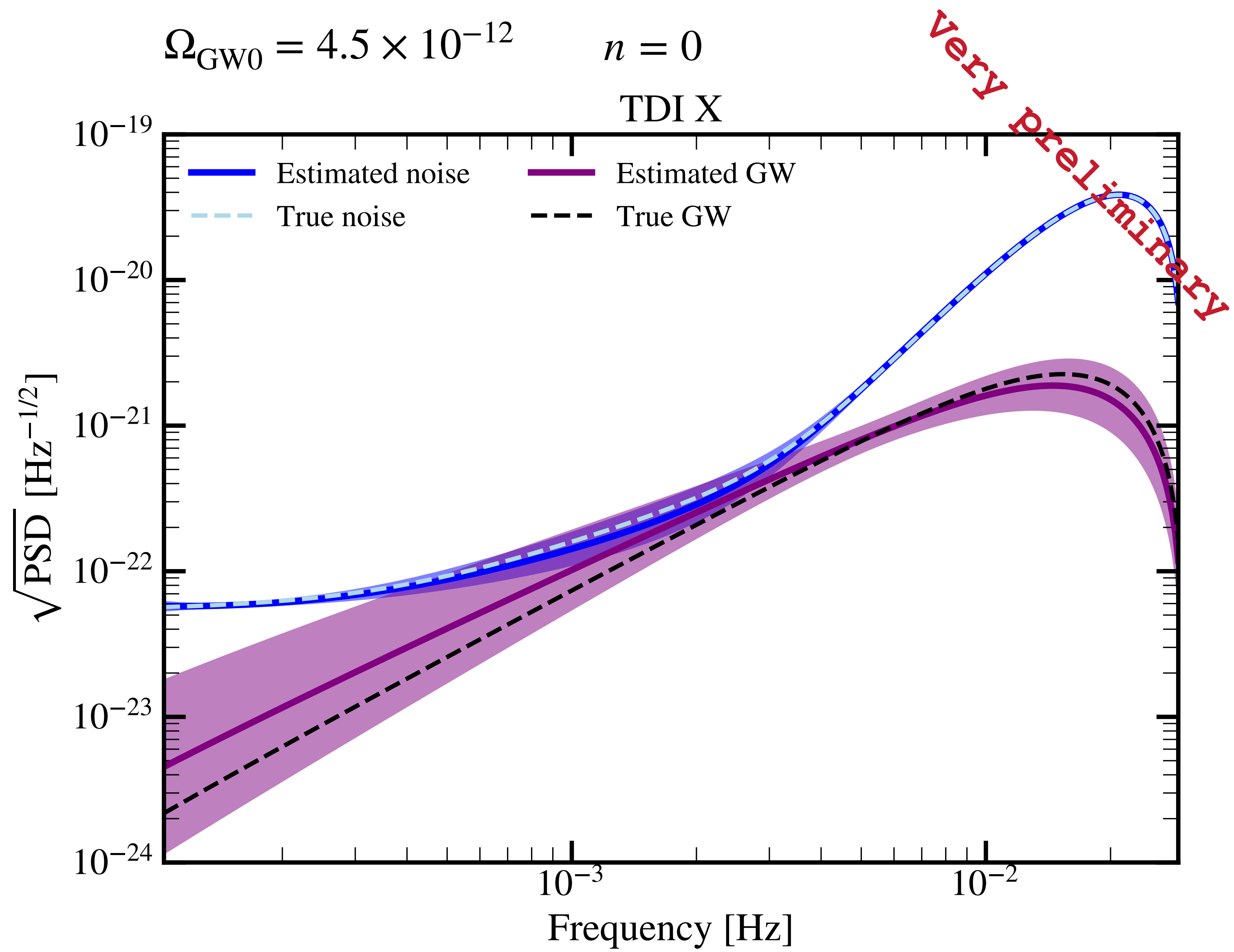
- But we only have 9 observable degrees of freedom, so we might rather directly fit for them…?

$$\tilde{\mathbf{\Sigma}}_{\text{noise}}(f) = \begin{pmatrix} S_{XX}(f) & S_{XY}(f) & S_{XZ}(f) \\ S_{YX}(f) & S_{YY}(f) & S_{YZ}(f) \\ S_{ZX}(f) & S_{ZY}(f) & S_{ZZ}(f) \end{pmatrix}$$

- Attempt with 2 noise components: optical metrology system (OMS) and test-mass (TM) noises

$$\tilde{\mathbf{\Sigma}}_{\mathrm{noise}}(f) = \tilde{\mathbf{R}}_{\mathrm{OMS}}(f)S_{\mathrm{OMS}}(f) + \tilde{\mathbf{R}}_{\mathrm{TM}}(f)S_{\mathrm{TM}}(f)$$





- Going beyond: can the concept of null channel help?

Revisiting null channels

- **Classic definition following Prince et al.** (2002). We can look for a combination e of TDI variables that maximises signal-to-noise ratio (SNR) of a deterministic signal $\mathbf{s}$:

$$e = a_1 X + a_2 Y + a_3 Z \qquad \mathrm{SNR} = \int_{f_l}^{f_u} \frac{\mathbf{a}^\dagger \tilde{\boldsymbol{A}}_{\mathrm{GW}} \mathbf{a}}{\mathbf{a}^\dagger \tilde{\boldsymbol{\Sigma}}_{\mathrm{noise}} \mathbf{a}} df \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Where $\tilde{\boldsymbol{A}}_{\mathrm{GW}}$ is the matrix of cross-products of the signal vector $\tilde{\mathbf{s}}$: $\tilde{\boldsymbol{A}}_{\mathrm{GW}} = \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\dagger$

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- Which from Rayleigh's principle is equivalent to the generalised eigenvalue problem [Borloz & Xerri 2005]:

$$\tilde{A}_{\mathrm{GW}} \mathbf{a} = \lambda \tilde{\mathbf{\Sigma}}_{\mathrm{noise}} \mathbf{a}$$

- **Classic definition following Prince et al.** (2002). We can look for a combination e of TDI variables that maximises signal-to-noise ratio (SNR) of a deterministic signal $\mathbf{s}$:

$$e = a_1 X + a_2 Y + a_3 Z \qquad \mathrm{SNR} = \int_{f_l}^{f_u} \frac{\mathbf{a}^\dagger \tilde{\boldsymbol{A}}_{\mathrm{GW}} \mathbf{a}}{\mathbf{a}^\dagger \tilde{\boldsymbol{\Sigma}}_{\mathrm{noise}} \mathbf{a}} df \qquad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Where $\tilde{\boldsymbol{A}}_{\mathrm{GW}}$ is the matrix of cross-products of the signal vector $\tilde{\mathbf{s}}$: $\tilde{\boldsymbol{A}}_{\mathrm{GW}} = \tilde{\mathbf{s}}\tilde{\mathbf{s}}^\dagger$

- Which from Rayleigh's principle is equivalent to the generalised eigenvalue problem [Borloz & Xerri 2005]:

$$\tilde{\boldsymbol{A}}_{\mathrm{GW}} \mathbf{a} = \lambda \tilde{\boldsymbol{\Sigma}}_{\mathrm{noise}} \mathbf{a}$$

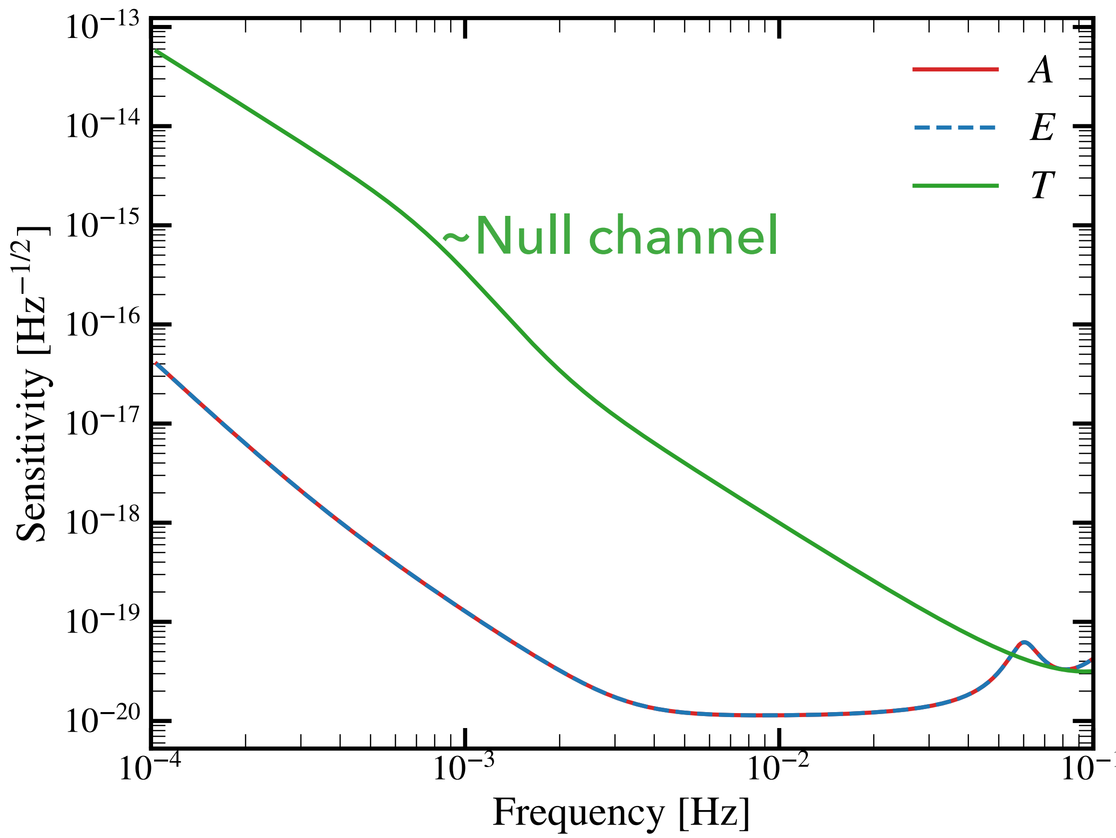
- For deterministic signals $\tilde{\boldsymbol{A}}_{\mathrm{GW}}$ is of rank 1 so it is equivalent to performing the **eigendecomposition of the noise covariance matrix $\tilde{\boldsymbol{\Sigma}}_{\mathrm{noise}}$**

$$\tilde{\boldsymbol{\Sigma}}_{\mathrm{noise}}(f) = \begin{pmatrix} S_{XX}(f) & S_{XY}(f) & S_{XZ}(f) \\ S_{YX}(f) & S_{YY}(f) & S_{YZ}(f) \\ S_{ZX}(f) & S_{ZY}(f) & S_{ZZ}(f) \end{pmatrix}$$

- That means we compute $\tilde{\mathbf{\Sigma}}_{\text{noise}}(f) = \mathbf{V}(f)\mathbf{\Lambda}(f)\mathbf{V}(f)^{\dagger}$ so that the transformation $\tilde{\mathbf{e}} = \mathbf{V}^{\dagger}\tilde{\mathbf{d}}$ forms a set of orthogonal channels
- **The null channel corresponds to the eigenvector with smallest SNR**
- Under specific assumptions
  - ✦ Fixed arm lengths → first-generation TDI
  - ✦ Equal armlengths
  - ✦ Equal interferometric noises → all η_{ij} have the same PSDs
- The diagonalisation is independent of frequency and noise levels → A, E, T

$$\mathbf{V} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$\propto \mathrm{SNR}^{-1}$



Both noise and signal are orthogonal

But unrealistic assumptions!

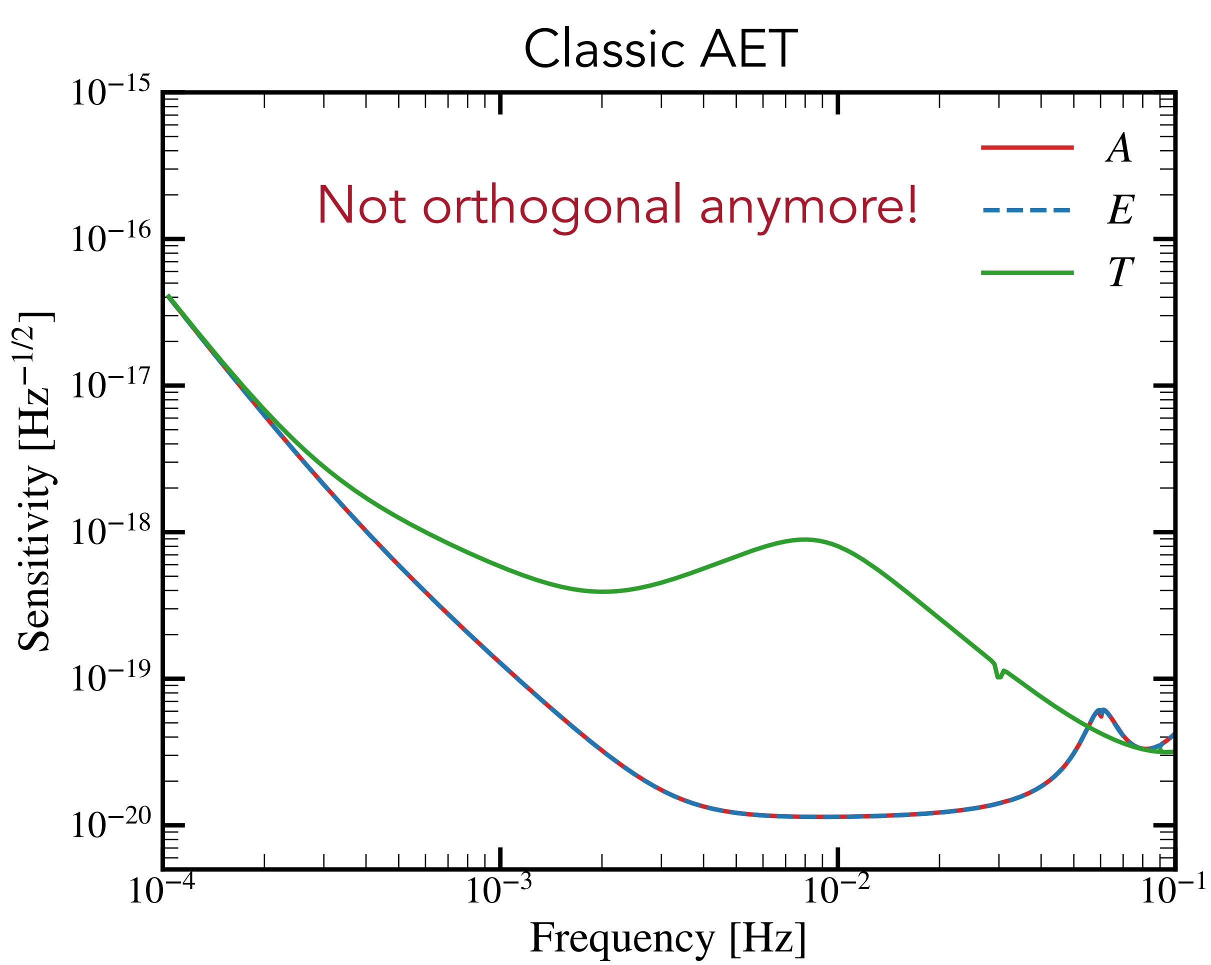
- Now assume

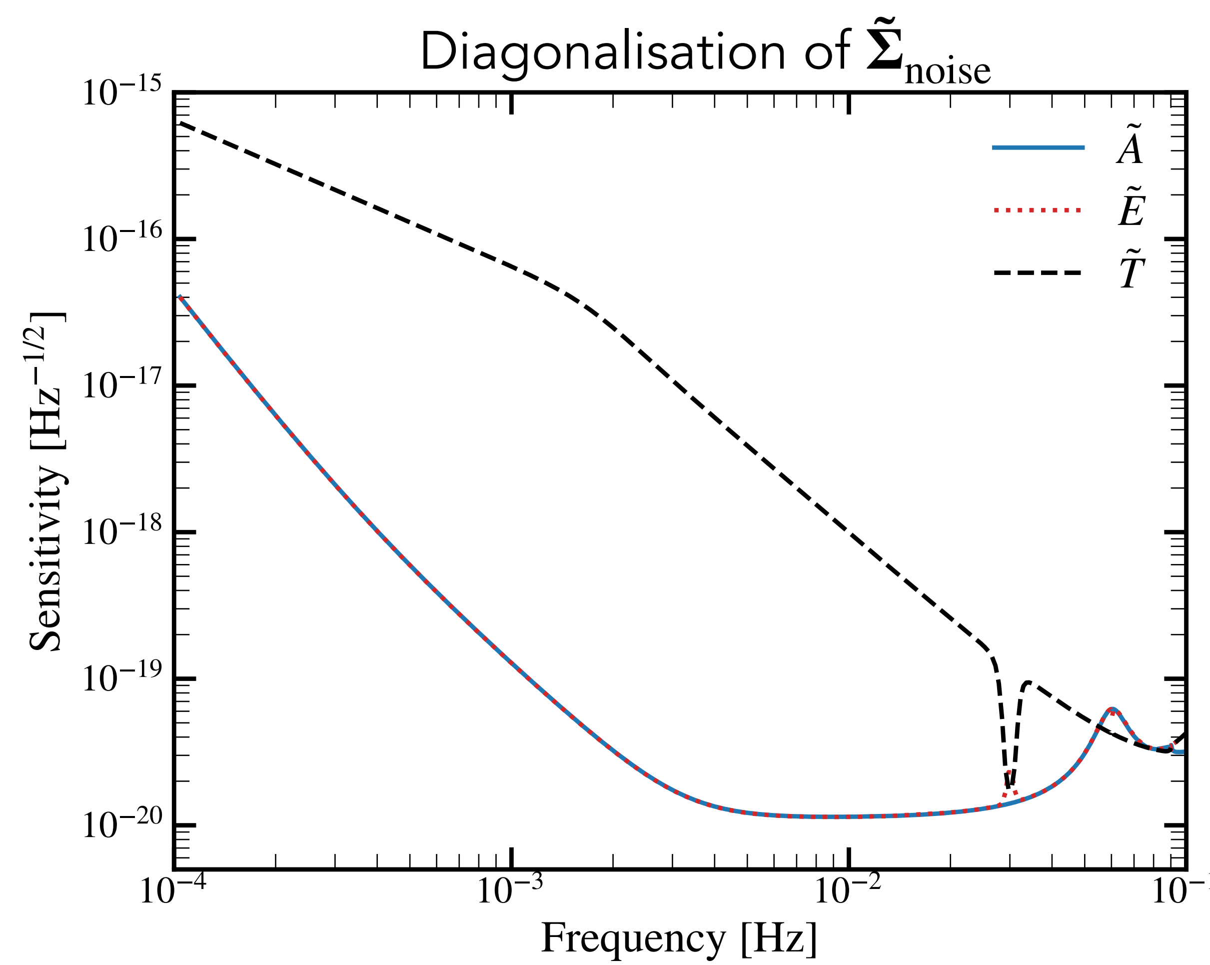
    ✦ **Flexing armlenghts** → second-generation TDI

    ✦ **Unequal armlengths**

    ✦ Equal interferometer noises

- The eigenvectors $\mathbf{V}(f)$ of $\tilde{\mathbf{\Sigma}}_{\text{noise}}(f)$ are now frequency dependent!





- But we are looking for stochastic signals. What is a null channel in this case?
- The definition of the optimal SNR needs to account for SGWB correlations:

$$\mathrm{SNR} = \int_{f_l}^{f_u} \frac{\mathbf{a}^{\dagger}\tilde{\mathbf{\Sigma}}_{\mathrm{GW}}\mathbf{a}}{\mathbf{a}^{\dagger}\tilde{\mathbf{\Sigma}}_{\mathrm{noise}}\mathbf{a}} df \qquad \tilde{\mathbf{\Sigma}}_{\mathrm{GW}} = \mathrm{E}\left[\mathbf{s}\mathbf{s}^{\dagger}\right]$$

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$$\tilde{\boldsymbol{\Sigma}}_{\mathrm{GW}} \mathbf{a} = \lambda \tilde{\boldsymbol{\Sigma}}_{\mathrm{noise}} \mathbf{a} \qquad \tilde{\boldsymbol{\Sigma}}_{\mathrm{GW}}(f) = \tilde{\mathbf{R}}_{\mathrm{GW}}(f) S_h(f)$$

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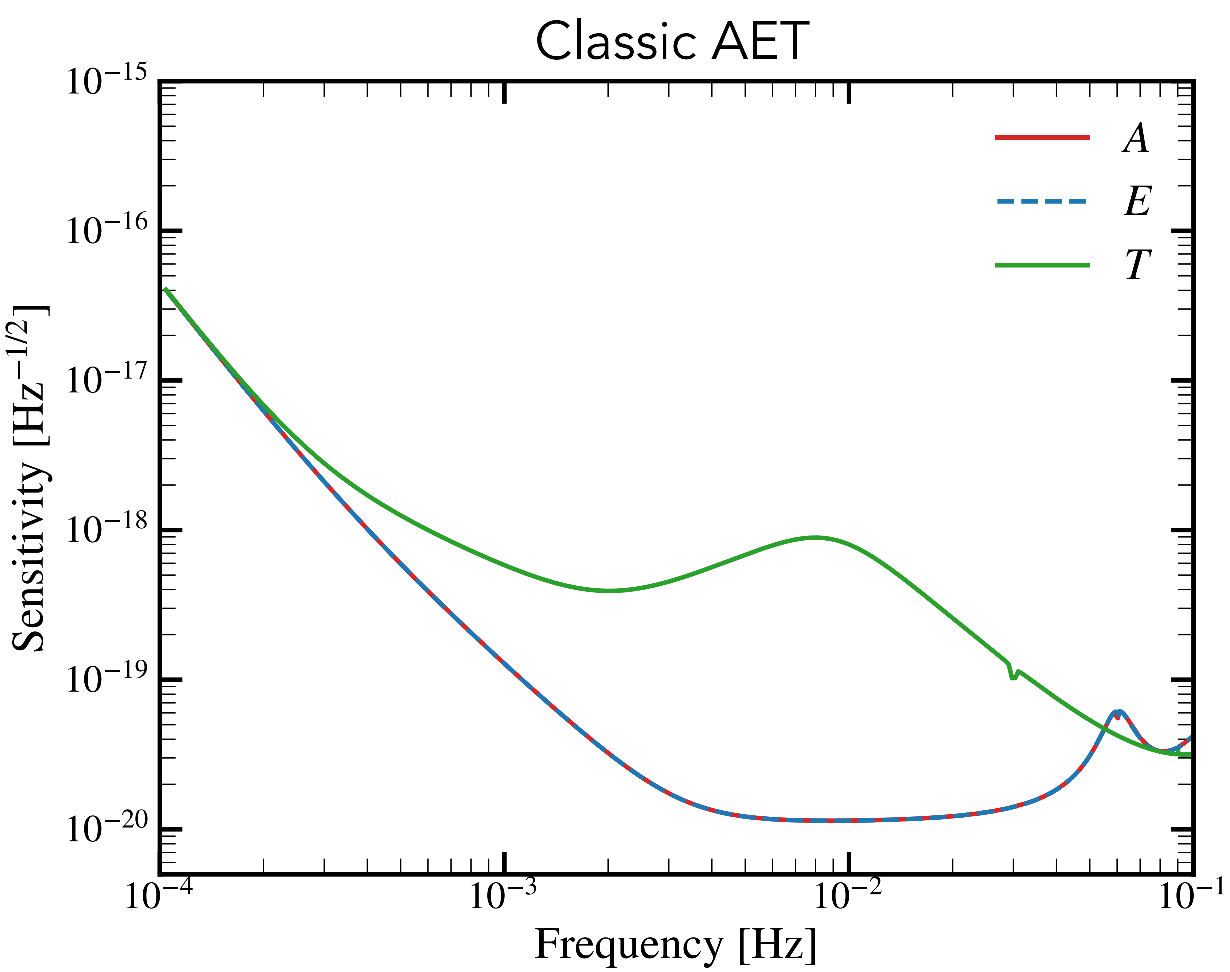
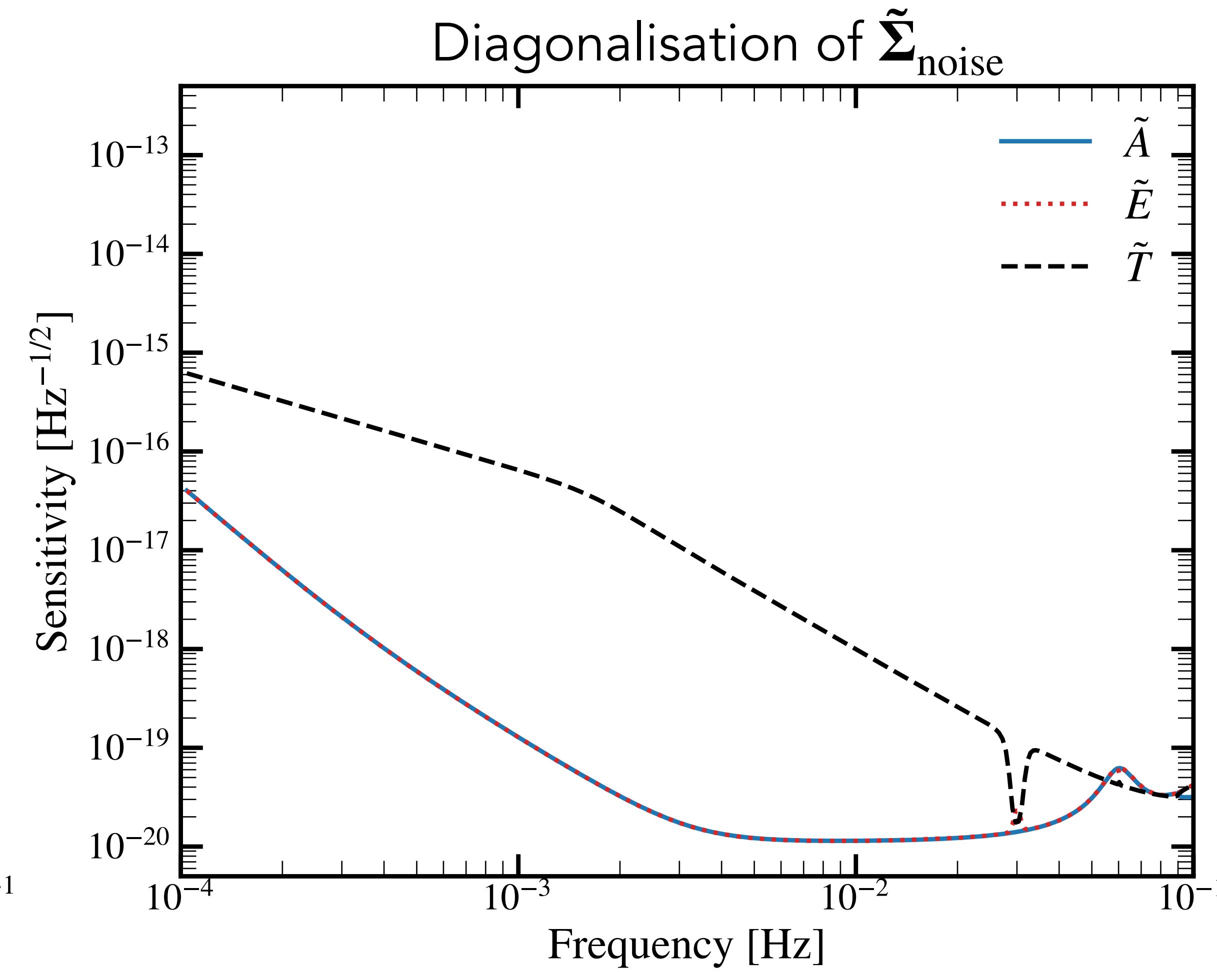
$$\mathbf{B}\mathbf{z} = \tilde{\lambda}\mathbf{z}$$

Where we set $$\mathbf{B} = \mathbf{L}^{-1} \tilde{\mathbf{\Sigma}}_{\mathrm{noise}} \mathbf{L}^{\dagger -1} \qquad \mathbf{z} = \mathbf{L}^\dagger \mathbf{a}$$

With $\mathbf{L}$ the Cholesky decomposition of the SGWB response matrix: $\tilde{\mathbf{R}}_{\mathrm{GW}}(f) = \mathbf{L}\mathbf{L}^\dagger$

- Let us revisit the null channel with this new definition of the optimal SNR:





Very similar to ζ channel, see Martina's talk!

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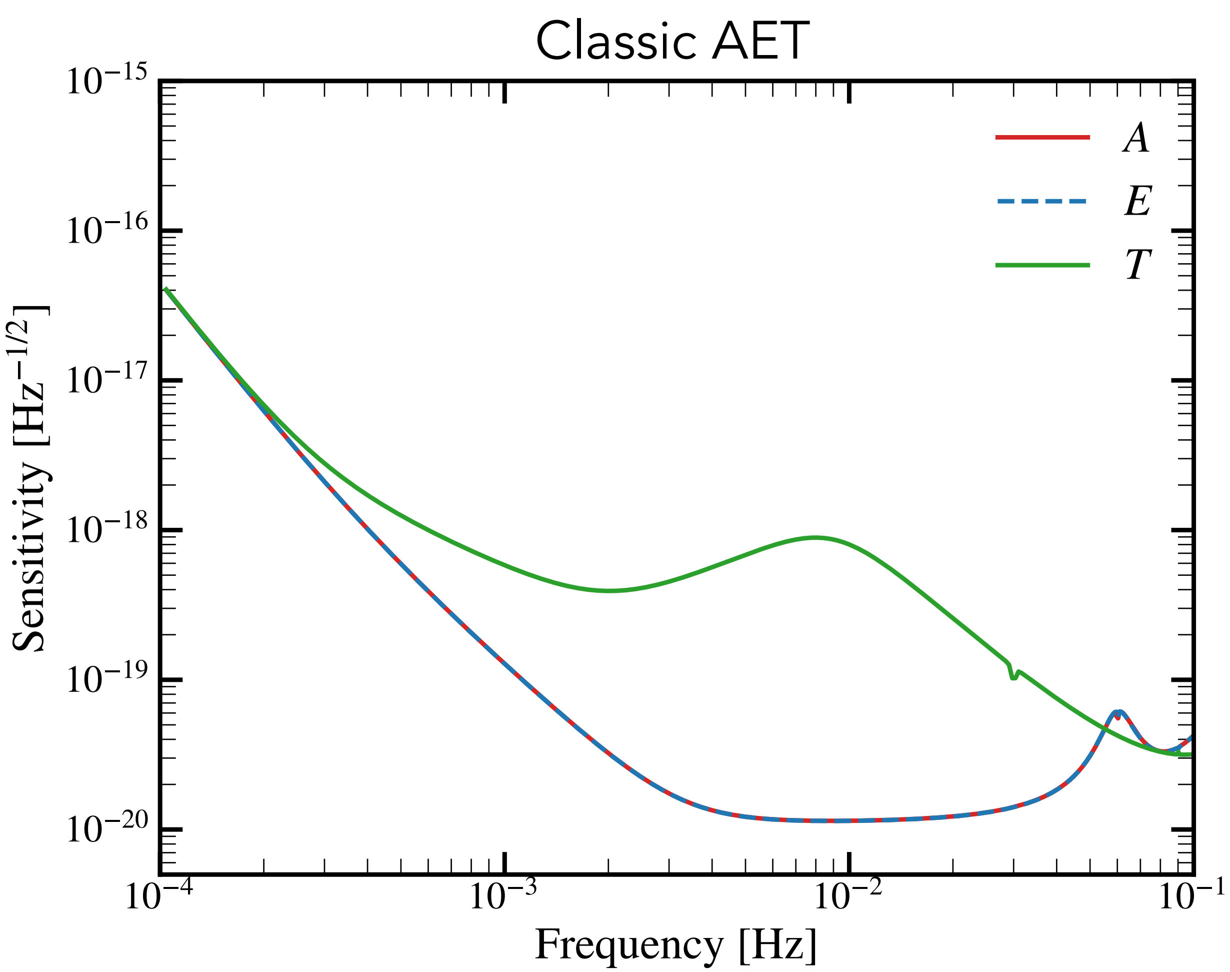

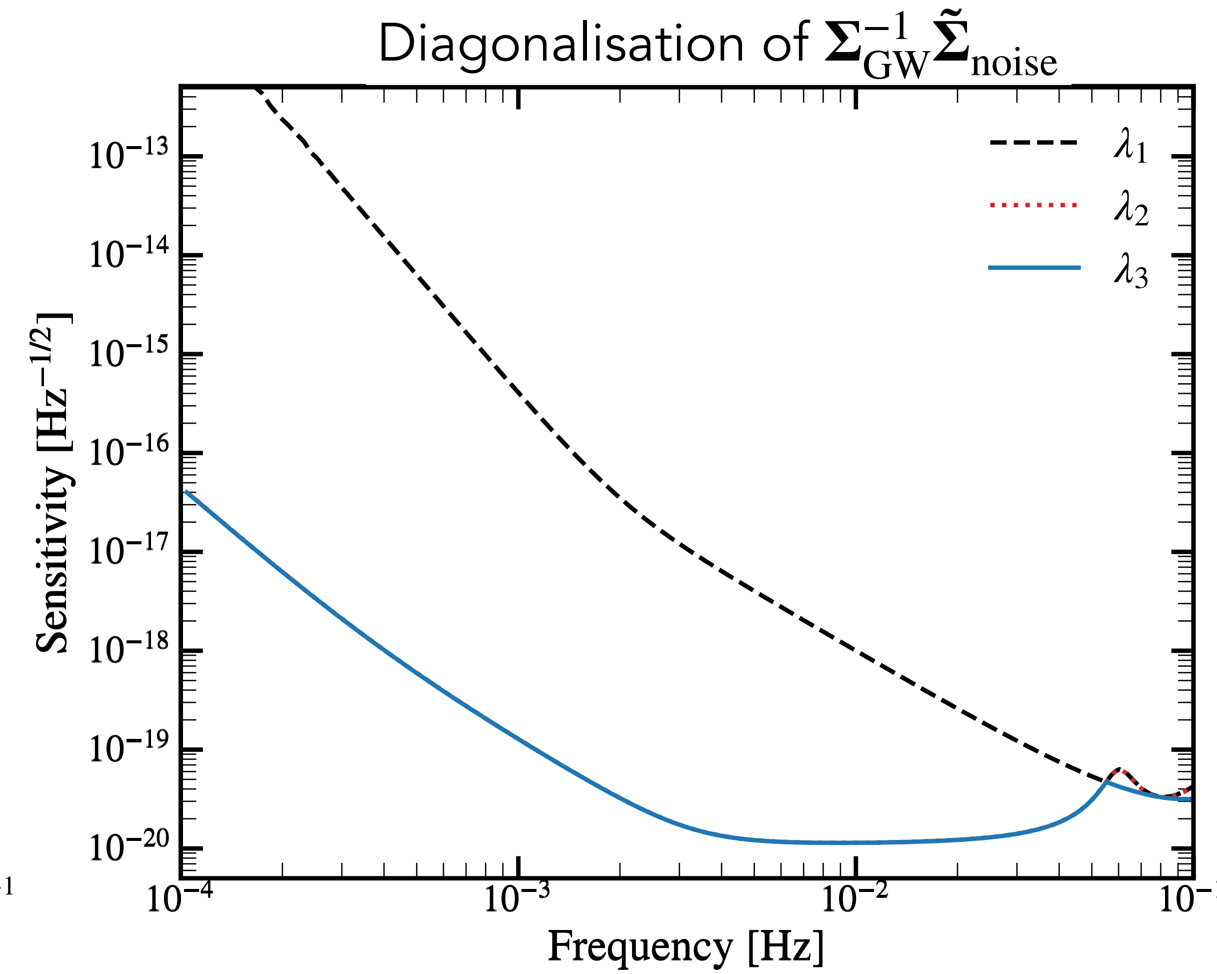

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Prospects

- **We need a non-parametric spectral estimation for the full TDI** covariance matrix
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Thank you for your attention !